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Quaternion Regular Functions and Domains of Regularity.

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Sunto. - *In questo lavoro si sviluppa la teoria delle funzioni regolari (nel senso di Fueter) di più variabili quaternioniche. Fra l'altro, si dimostrano teoremi di tipo Cartan-Thullen e di tipo Hans Lewy.*

Introduction.

In this paper we develop the theory of regular functions of several quaternion variables. The definition of regular functions of quaternion variable was given, in 1935, by R. Fueter and the theory was developed by him and his school. Recently, the theory has been rediscovered and considerably extended: for references see, for instance [S], [N], [P1], [P2], [P3], [P4] and [G-M-T]. However, we refer to [P1] for a complete bibliography.

In the present work we study essentially the properties of domains of regularity of $\mathbb{H}^n$. Among other results, we prove a theorem of Cartan-Thullen type, which asserts that every domain of regularity of $\mathbb{H}^n$ is regularly convex, and we give a differential geometric condition on the boundary of a domain of regularity (cf. Theorem 4). In the last section we study the problem of the local regular extendibility of functions defined on hypersurfaces of $\mathbb{H}^n$ and we prove a theorem of Hans Lewy type.

The conditions that we give, in order to have the local regular extension, are essentially the same as those which we introduced in [P4].

1. - Preliminaries.

1. - We denote by 1, i, j, k a basis of the algebra $\mathbb{H}$ of quaternions. We recall that we have $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$.

If $q = (q_1, \ldots, q_n)$ is an element of $\mathbb{H}^n$, we shall write $q_h = x_0^{(h)} +$

$ix_1^{(h)} + jx_2^{(h)} + kx_3^{(h)}$, for $h = 1, \ldots, n$, where $x_0^{(h)}$, $x_1^{(h)}$, $x_2^{(h)}$, $x_3^{(h)}$, are the real components of q_h. If we set $\bar{q}_h = x_0^{(h)} - ix_1^{(h)} - jx_2^{(h)} - kx_3^{(h)}$, the norm $|q|$ of $q \in \mathbb{H}^n$ is defined by the following equation:

$$|q| = \sqrt{\sum_1^n {}_h q_h \bar{q}_h} = \sqrt{\sum_1^n {}_h \bar{q}_h q_h}\,.$$

In this paper $\mathbb{C}^{2n}$ will be identified with $\mathbb{H}^n$ by the map

$$(z_1^{(1)}, z_2^{(1)}, \ldots, z_1^{(n)}, z_2^{(n)}) \to (z_1^{(1)} + jz_2^{(1)}, \ldots, z_1^{(n)} + jz_2^{(n)})$$

(cf. [P1], p. 68).

2. – We consider now a function of class C^1 $F: A \to \mathbb{H}$, where A is an open set of $\mathbb{H}^n$. We say that F is *left-regular* (or simply *regular*) in A if

$$\frac{\partial F}{\partial \bar{q}_h} = \frac{\partial F}{\partial x_0^{(h)}} + i\,\frac{\partial F}{\partial x_1^{(h)}} + j\,\frac{\partial F}{\partial x_2^{(h)}} + k\,\frac{\partial F}{\partial x_3^{(h)}} = 0 \quad \text{in } A, \text{ for } h = 1, \ldots, n.$$

We denote by $\mathcal{R}(A)$ the set of the left-regular functions on A. We know that $\mathcal{R}(A)$ is a right $\mathbb{H}$-vector space, but it is not an algebra on $\mathbb{H}$ ([P1]).

If we define

$$\frac{\partial}{\partial q_h} = \frac{\partial}{\partial x_0^{(h)}} - i\,\frac{\partial}{\partial x_1^{(h)}} - j\,\frac{\partial}{\partial x_2^{(h)}} - k\,\frac{\partial}{\partial x_3^{(h)}}\,,$$

we have

$$\frac{\partial}{\partial \bar{q}_h}\,\frac{\partial}{\partial q_h} = \frac{\partial}{\partial q_h}\,\frac{\partial}{\partial \bar{q}_h} = \Delta_h\,,$$

where Δ_h is the partial Laplacian realized with respect to real coordinates of q_h. Since the usual Laplacian Δ of $\mathbb{H}^n$ is equal to $\sum_1^n {}_h \Delta_h$, we deduce that every regular function is harmonic.

3. – We can give, for the regular functions, a representation formula of Bochner-Martinelli type ([P2]), that is there exists a kernel $\Omega_{q_0}(q)$ such that, if U is a bounded open set with boundary of class C^1 such that $\bar{U} \subseteq A$, and $f \in \mathcal{R}(A)$, then, for every $q_0 \in U$, we have

$$f(q_0) = \int_{\partial U} \Omega_{q_0}(q) f(q)\,. \tag{1}$$

If $n = 1$ the kernel $\Omega_{q_0}(q)$ reduces to $(1/2\pi^2)\,G(q - q_0)\,Dq$, where G is the regular function on $\mathbb{H} - \{0\}$ defined by $G(q) = \bar{q}/|q|^4$ and Dq

is the following $\mathbb{H}$-valued 3-form:

$$Dq = dx_1 \wedge dx_2 \wedge dx_3 - i\, dx_0 \wedge dx_2 \wedge dx_3 + j\, dx_0 \wedge dx_1 \wedge dx_3 - k\, dx_0 \wedge dx_1 \wedge dx_2 .$$

In this case the formula (1) reduces to the classical Cauchy-Fueter formula (cf. [S]):

$$f(q_0) = \frac{1}{2\pi^2} \int_{\partial U} G(q - q_0)\, Dq f(q) . \tag{2}$$

This formula can be extended to dimension n in other way too. Consider a regular function f: $A \subseteq \mathbb{H}^n \to \mathbb{H}$ and $q_0 = (q_1^0, \ldots, q_n^0) \in A$. Let $\varepsilon_1, \ldots, \varepsilon_n$ be positive numbers such that

$$\{q = (q_1, \ldots, q_n) \in \mathbb{H}^n \colon |q_h - q_h^0| \leq \varepsilon_h,\ h = 1, \ldots, n\} \subseteq A .$$

Then it is easy to show, from (2), that, if $q = (q_1, \ldots, q_n) \in A$ and $|q_h - q_h^0| < \varepsilon_h$ for $h = 1, \ldots, n$, we have

$$f(q_1, \ldots, q_n) = \frac{1}{2^n \pi^{2n}} \int_{|\xi_1 - q_1^0| = \varepsilon_1} \cdots \int_{|\xi_n - q_n^0| = \varepsilon_n} G(\xi_1 - q_1)\, D\xi_1 \ldots G(\xi_n - q_n)\, D\xi_n\, f(\xi_1, \ldots, \xi_n) . \tag{3}$$

4. – In general the composition of regular functions is not regular. Actually the only functions preserving regularity are the affine right linear maps, that is the functions g: $\mathbb{H}^n \to \mathbb{H}^n$ defined by $g(q) = Aq + B$, where $B \in \mathbb{H}^n$ and A is matrix of type $n \times n$, which entries are in $\mathbb{H}$. In fact the following Proposition holds true:

PROPOSITION 1 ([P1], p. 70). – *Let g: $A \to \mathbb{H}^n$ be a function of class C^1 on any open set A of $\mathbb{H}^n$. The following conditions are equivalent*:

a) g is an affine right linear map;

b) $f \circ g$ is regular in A, for every function f, which is regular in a neighbourhood of $g(A)$.

2. – Domains of regularity of $\mathbb{H}^n$.

1. – Let D, $\widehat{D}$ be two domains of $\mathbb{H}^n$ such that $D \subseteq \widehat{D}$. We say that $\widehat{D}$ is a *regular completion of D with respect to a subset S of $\mathcal{R}(D)$*, if, for every $f \in S$, there exists a (unique) function $\widehat{f} \in \mathcal{R}(\widehat{D})$ such that $\widehat{f}\,|_D = f$.

An open subset A of $\mathbb{H}^n$ is said to be a *domain of regularity* if, for every domain $D \subsetneq A$, every regular completion of D with respect to $\mathcal{R}(A)|_D = \{f|_D : f \in \mathcal{R}(A)\}$ is contained in A.

As in the complex case, we can easily show the following propositions:

PROPOSITION 2. – *An open subset A of $\mathbb{H}^n$ is a domain of regularity if and only if every connected component of A is a domain of regularity.*

PROPOSITION 3. – *Let A be an open subset of $\mathbb{H}^n$. Suppose that, for every sequence $\{q_h\}_{h \in \mathbb{N}}$ in A, which have no limit points in A, there exists $f \in \mathcal{R}(A)$ such that $\sup_{h \in \mathbb{N}} |f(q_h)| = +\infty$. Then A is a domain of regularity.*

PROPOSITION 4. – *Let A be an open subset of $\mathbb{H}^n$. Suppose that, for every $q_0 \in \partial A$, there exists $f \in \mathcal{R}(A)$ such that $\lim_{\substack{q \to q_0 \\ q \in A}} |f(q)| = +\infty$. Then A is a domain of regularity.*

From Proposition 4 we deduce that every open subset A of $\mathbb{H}$ is a domain of regularity. Indeed it is sufficient to consider, for every $q_0 \in \partial A$, the regular function $G(q - q_0) = (\bar{q} - \bar{q}_0) / |q - q_0|^4$.

2. – If $n > 1$, there exist open subsets of $\mathbb{H}^n$, which are not domain of regularity. Indeed we have shown in [P2] a theorem of Hartogs type. This theorem asserts that, if A is a connected open subset of $\mathbb{H}^n$ $(n > 1)$, and if K is a compact subset of A such that $A - K$ is connected, then A is a regular completion of $A - K$ with respect to $\mathcal{R}(A - K)$. Then $A - K$ is not a domain of regularity.

Another theorem of Hartogs type is the following:

THEOREM 1. – *Let A, B be non-empty connected open subsets of $\mathbb{H}^{n-1}$ $(n > 1)$, such that $A \subsetneq B$. Then let a, b be real numbers such that $0 < a < b$. If we set*

$$\omega_1 = \{(q_1, \ldots, q_n) \in \mathbb{H}^n : |q_1| < b,\ (q_2, \ldots, q_n) \in A\},$$

$$\omega_2 = \{(q_1, \ldots, q_n) \in \mathbb{H}^n : a < |q_1| < b,\ (q_2, \ldots, q_n) \in B\}, \qquad \omega = \omega_1 \cup \omega_2,$$

$$\widehat{\omega} = \{(q_1, \ldots, q_n) \in \mathbb{H}^n : |q_1| < b,\ (q_2, \ldots, q_n) \in B\},$$

then $\widehat{\omega}$ is a regular completion of ω with respect to $\mathcal{R}(\omega)$.

PROOF. – Suppose that $f \in \mathcal{R}(\omega)$ and $\rho \in (a, b)$. We set

$$B_\rho = \{(q_1, \ldots, q_n) \in \mathbb{H}^n : |q_1| < \rho,\ (q_2, \ldots, q_n) \in B\},$$

$$A_\rho = \{(q_1, \ldots, q_n) \in \mathbb{H}^n : |q_1| < \rho,\ (q_2, \ldots, q_n) \in A\}.$$

If $(q_1, \ldots, q_n) \in A_\rho$, we have, from (2),

$$f(q_1, \ldots, q_n) = \frac{1}{2\pi^2} \int_{|q|=\rho} G(q - q_1) Dq f(q, q_2, \ldots, q_n).$$

Setting

$$F(q_1, \ldots, q_n) = \frac{1}{2\pi^2} \int_{|q|=\rho} G(q - q_1) Dq f(q, q_2, \ldots, q_n),$$

we define a function $F: B_\rho \to \mathbb{H}$, which extends $f|_{A_\rho}$ to B_ρ. We have

$$\Delta_1 F(q_1, \ldots, q_n) = \frac{1}{2\pi^2} \int_{|q|=\rho} \Delta_1 G(q - q_1) Dq f(q, q_2, \ldots, q_n) = 0,$$

because G is regular.

Since Δ_h is a real operator, we also have, for $h = 2, \ldots, n$,

$$\Delta_h F(q_1, \ldots, q_n) = \frac{1}{2\pi^2} \int_{|q|=\rho} G(q - q_1) Dq \Delta_h f(q, q_2, \ldots, q_n) = 0,$$

because f is regular. Hence F is harmonic in B_ρ and regular in A_ρ. Then F is necessarily regular in B_ρ. Since $\omega \cap B_\rho$ is a connected open set, which contains A_ρ, from the Principle of Identity (cf. [P1], p. 67), we obtain $F = f$ in $\omega \cap B_\rho$. Finally, if we set

$$\widehat{f} = \begin{cases} F & \text{in } B_\rho, \\ f & \text{in } \widehat{\omega} - B_\rho, \end{cases}$$

we obtain that $\widehat{f}$ is a regular extension of f in $\widehat{\omega}$. q.e.d.

3. – A first geometrical property of the domains of regularity is a consequence of the following

THEOREM 2. – *Let K be a compact connected subset of $\mathbb{H}^{n-1}$ $(n > 1)$ containing the origin O' of $\mathbb{H}^{n-1}$, and set $D_r = \{q_1 \in \mathbb{H}: |q_1| \leq r\}$. Then let $f: \mathbb{H}^n \to \mathbb{H}^n$ be an invertible affine right linear map. If A is a domain of regularity of $\mathbb{H}^n$ and if $f(D_r \times \{O'\}) \cup f(\partial D_r \times K)$ is contained in A, then $f(D_r \times K)$ is also contained in A.*

PROOF. – We set, for $m \in \mathbb{N}$,

$$V_m = \left\{ q \in \mathbb{H}^{n-1} : |q| < \frac{1}{m} \right\},$$

$$B_m = \left\{ q \in \mathbb{H}^{n-1} : d(q, K) < \frac{1}{m} \right\},$$

where d denotes the euclidean distance in $\mathbb{H}^{n-1}$.

Obviously $V_m \subsetneq B_m$, $\forall m \in \mathbb{N}$.

We define now, for $m \in \mathbb{N}$,

$$\widetilde{D}_m = \left\{ (q_1, q) \in \mathbb{H} \times \mathbb{H}^{n-1} : |q_1| < r + \frac{1}{m}, \; q \in V_m \right\} \cup$$

$$\left\{ (q_1, q) \in \mathbb{H} \times \mathbb{H}^{n-1} : r - \frac{1}{m} < |q_1| < r + \frac{1}{m}, \; q \in B_m \right\}.$$

$\widetilde{D}_m$ is an open neighbourhood of $(D_r \times \{O'\}) \cup (\partial D_r \times K)$; moreover if W is any neighbourhood of $(D_r \times \{O'\}) \cup (\partial D_r \times K)$, there exists $m \in \mathbb{N}$, such that $\widetilde{D}_m \subsetneq W$.

Then we can fix $m \in \mathbb{N}$ such that $f(\widetilde{D}_m) \subsetneq A$.

From Theorem 1,

$$\hat{D}_m = \left\{ (q_1, q) \in \mathbb{H} \times \mathbb{H}^{n-1} : |q_1| < r + \frac{1}{m}, \; q \in B_m \right\}$$

is a regular completion of $\widetilde{D}_m$ with respect to $\mathcal{R}(\widetilde{D}_m)$. Then from Proposition 1, $f(\hat{D}_m)$ is a regular completion of $f(\widetilde{D}_m)$ with respect to $\mathcal{R}(f(\widetilde{D}_m))$. Since A is a domain of regularity, it follows $f(\hat{D}_m) \subsetneq A$ and hence $f(D_r \times K) \subsetneq A$. q.e.d.

4. – In the first section we have identified $\mathbb{C}^{2n}$ and $\mathbb{H}^n$. With this identification, it is easy to prove the following

PROPOSITION 5 ([P1], p. 69). – *A function* f: $\mathbb{H}^n = \mathbb{C}^{2n} \to \mathbb{C} = \{q = z_1 + jz_2 \in \mathbb{H} : z_2 = 0\}$ *is regular if and only if it is holomorphic.*

From this proposition it follows easily

PROPOSITION 6 ([P1], p. 128). – *Let A be an open subset of $\mathbb{H}^n = \mathbb{C}^{2n}$. If A is a domain of holomorphy of $\mathbb{C}^{2n}$, then it is also a domain of regularity of $\mathbb{H}^n$. In particular, every convex open subset of $\mathbb{H}^n$ is a domain of regularity.*

3. - Domains of regularity and regular convexity.

1. - In this section we will prove a theorem of Cartan-Thullen type.

We begin with some definitions. Let us consider an open set A of $\mathbb{H}^n$ and a compact subset K of A. We define the $\mathcal{R}(A)$-*hull* $\hat{K}_A$ *of* K by

$$\hat{K}_A = \{q \in A \colon |f(q)| \leq \max_K |f|, \ \forall f \in \mathcal{R}(A)\}.$$

It is clear that $K \subseteq \hat{K}_A$ and $(\hat{\hat{K}}_A)_A = \hat{K}_A$. Moreover $\hat{K}_A$ is bounded and closed in A ([P1], p. 124). The open set A is said to be *regularly convex*, if, for every compact subset K of A, the $\mathcal{R}(A)$-hull $\hat{K}_A$ is also compact.

If $n = 1$, $\hat{K}_A$ is the union of K and all connected components of $A - K$, which are relatively compact in A (cf. [P1], p. 54). Hence every open subset A of $\mathbb{H}$ is regularly convex.

2. - Before giving some inequalities on the derivatives of a regular function, we fix the notations. If $\alpha = (\alpha_0^{(1)}, \alpha_1^{(1)}, \alpha_2^{(1)}, \alpha_3^{(1)}, \alpha_0^{(2)}, \dots, \alpha_3^{(n)}) \in \mathbb{N}^{4n}$ is a multi-index we set

$$\alpha! = \prod_{\substack{\lambda = 0, \dots, 3 \\ h = 1, \dots, n}} \alpha_\lambda^{(h)}! \quad \text{and} \quad |\alpha| = \sum_{\substack{\lambda = 0, \dots, 3 \\ h = 1, \dots, n}} \alpha_\lambda^{(h)}.$$

Moreover if $q = (q_1 \dots, q_n) \in \mathbb{H}^n$, with $q_h = x_0^{(h)} + ix_1^{(h)} + jx_2^{(h)} + kx_3^{(h)}$, for $h = 1, \dots, n$, and $\alpha = (\alpha_0^{(1)}, \dots, \alpha_3^{(n)}) \in \mathbb{N}^{4n}$ is a multi-index, we define a real number q^α by

$$q^\alpha = \prod_{\substack{\lambda = 0, \dots, 3 \\ h = 1, \dots, n}} (x_\lambda^{(h)})^{\alpha_\lambda^{(h)}}.$$

Then, if f is a function and $\alpha \in \mathbb{N}^{4n}$, we set

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_0^{(1)\alpha_0^{(1)}} \partial x_1^{(1)\alpha_1^{(1)}} \dots \partial x_3^{(n)\alpha_3^{(n)}}}.$$

Thus, if q, q_0, $\xi \in \mathbb{H}$, $\xi \neq q$, q_0, and $|q - q_0| < |\xi - q_0|$, since $G(\xi - q) = G((\xi - q_0) - (q - q_0))$, we obtain, from Taylor's formula,

$$G(\xi - q) = \sum_{\alpha \in \mathbb{N}^4} (-1)^{|\alpha|} \frac{D^\alpha G(\xi - q_0)}{\alpha!} (q - q_0)^\alpha. \tag{4}$$

3. - Consider now a regular function $f \colon A \subseteq \mathbb{H}^n \to \mathbb{H}$ and let $q_0 = (q_1^0, \dots, q_n^0) \in A$.

Let $\varepsilon_1, \dots, \varepsilon_n$ be positive real numbers such that

$$\{q = (q_1, \dots, q_n) \in \mathbb{H}^n \colon |q_h - q_h^0| \leq \varepsilon_h,\ h = 1, \dots, n\} \subseteq A.$$

If $q = (q_1, \dots, q_n) \in A$ and $|q_h - q_h^0| < \varepsilon_h$, for $h = 1, \dots, n$, from (3) and (4), we obtain

$$(5) \quad f(q_1, \dots, q_n) = \frac{1}{2^n \pi^{2n}} \sum_{\alpha = (\alpha_1, \dots, \alpha_n)} \frac{(-1)^{|\alpha|}}{\alpha!} (q - q_0)^{\alpha} \cdot$$

$$\left\{ \int\limits_{|\xi_1 - q_1^0| = \varepsilon_1} \dots \int\limits_{|\xi_n - q_n^0| = \varepsilon_n} D^{\alpha_1} G(\xi_1 - q_1^0) D\xi_1 \dots D^{\alpha_n} G(\xi_n - q_n^0) D\xi_n f(\xi_1, \dots, \xi_n) \right\}.$$

From this formula we deduce that

$$(6) \quad D^{\alpha} f(q_1^0, \dots, q_n^0) = \frac{(-1)^{|\alpha|}}{2^n \pi^{2n}} \int\limits_{|\xi_1 - q_1^0| = \varepsilon_1} \dots \int\limits_{|\xi_n - q_n^0| = \varepsilon_n} D^{\alpha_1} G(\xi_1 - q_1^0) D\xi_1 \dots$$
$$D^{\alpha_n} G(\xi_n - q_n^0) D\xi_n f(\xi_1, \dots, \xi_n), \qquad \forall \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^{4n},$$

and hence (cf. [P1], p. 19)

$$(7) \quad |D^{\alpha} f(q_1^0, \dots, q_n^0)| \leq \frac{1}{2^n \pi^{2n}} \int\limits_{|\xi_1 - q_1^0| = \varepsilon_1} \dots \int\limits_{|\xi_n - q_n^0| = \varepsilon_n} |D^{\alpha_1} G(\xi_1 - q_1^0)| \dots$$
$$|D^{\alpha_n} G(\xi_n - q_n^0)|\, |f(\xi_1, \dots, \xi_n)|, \qquad \forall \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^{4n}.$$

Now we must extimate the derivatives of G.

PROPOSITION 7. – *If $\alpha \in \mathbb{N}^4$ we have*

$$|D^{\alpha} G(q)| \leq \frac{125}{e^4} \frac{(20e)^{|\alpha|} \alpha!}{|q|^{|\alpha| + 3}}, \qquad \textit{for all } q \in \mathbb{H} - \{0\}.$$

PROOF. – Suppose $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, where $\alpha_i \in \mathbb{N}$. Since G is harmonic we have (cf. [Pu], p. 197)

$$|D^{\alpha} G(q)| \leq \frac{4^{\alpha_0} e^{\alpha_0 - 1} \alpha_0!}{(|q|/5)^{\alpha_0}} \max_{|\xi - q| \leq (|q|)/5} |D^{(0, \alpha_1, \alpha_2, \alpha_3)} G(\xi)| \leq$$

$$\frac{4^{\alpha_0 + \alpha_1} e^{\alpha_0 + \alpha_1 - 2} \alpha_0! \alpha_1!}{(|q|/5)^{\alpha_0 + \alpha_1}} \max_{|\xi - q| \leq (2|q|)/5} |D^{(0, 0, \alpha_2, \alpha_3)} G(\xi)| \leq$$

$$\frac{4^{\alpha_0 + \alpha_1 + \alpha_2} e^{\alpha_0 + \alpha_1 + \alpha_2 - 3} \alpha_0! \alpha_1! \alpha_2!}{(|q|/5)^{\alpha_0 + \alpha_1 + \alpha_2}} \max_{|\xi - q| \leq (3|q|)/5} |D^{(0, 0, 0, \alpha_3)} G(\xi)| \leq$$

$$\frac{(4e)^{|\alpha|}e^{-4}\alpha!}{(|q|/5)^{|\alpha|}}\max_{|\xi-q|\leq(4|q|)/5}|G(\xi)| = \frac{(4e)^{|\alpha|}e^{-4}\alpha!}{(|q|/5)^{|\alpha|+3}} =$$

$$\frac{125}{e^4}\frac{(20e)^{|\alpha|}\alpha!}{|q|^{|\alpha|+3}}. \qquad \text{q.e.d.}$$

Now we suppose that $\varepsilon_1 = \ldots = \varepsilon_n = \varepsilon$ and we set

$$\Gamma(q_0, \varepsilon) = \{\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{H}^n \colon |\xi_h - q_h^0| = \varepsilon, \text{ for } h = 1, \ldots, n\}$$

From (7) and Proposition 7 we obtain easily

PROPOSITION 8. – *If f is regular in A and $q_0 \in A$, $\varepsilon > 0$ are as above, then*

$$\frac{|D^\alpha f(q_0)|}{\alpha!} \leq \left(\frac{125}{e^4}\right)^n \left(\frac{(20e)}{\varepsilon}\right)^{|\alpha|} \max_{\Gamma(q_0,\varepsilon)} |f|, \qquad \forall \alpha \in \mathbb{N}^{4n}.$$

4. – Now we can prove a theorem of Cartan-Thullen type (cf. also [N]).

THEOREM 3. – *Every domain of regularity of $\mathbb{H}^n$ is regularly convex.*

PROOF. – Let K be a compact subset of a domain of regularity A of $\mathbb{H}^n$ and let $\hat{K}$ denote its $\mathcal{R}(A)$-hull.

If $p = (p_1, \ldots, p_n)$, $q = (q_1, \ldots, q_n) \in \mathbb{H}^n$, with $p_h = y_0^{(h)} + iy_1^{(h)} + jy_2^{(h)} + ky_3^{(h)}$, $q_h = x_0^{(h)} + ix_1^{(h)} + jx_2^{(h)} + kx_3^{(h)}$, for $h = 1, \ldots, n$, we define

$$\delta(p,q) = \max_{1\leq i\leq n} |p_i - q_i|, \qquad \Delta(p,q) = \max_{\substack{1\leq h\leq n\\ 0\leq\lambda\leq 3}} |y_\lambda^{(h)} - x_\lambda^{(h)}|,$$

$$\delta(q,K) = \inf_{p\in K} \delta(q,p), \qquad \delta = \inf_{q\in\partial A} \delta(q,K),$$

$$\Delta(q,\hat{K}) = \inf_{p\in\hat{K}} \Delta(q,p), \qquad \Delta = \inf_{q\in\partial A} \Delta(q,\hat{K}).$$

Since K is compact we have $\delta > 0$; then we can fix $r \in (0, \delta)$. We set $K_r = \{q \in \mathbb{H}^n \colon \delta(q,K) \leq r\}$. Obviously K_r is a compact subset of A. Let us fix $q_0 \in K$. We have $\{q \in \mathbb{H}^n \colon \delta(q,q_0) \leq r\} \subseteq K_r \subseteq A$.

Now consider $f \in \mathcal{R}(A)$. If $\delta(q,q_0) < r$ we have

$$f(q) = \sum_{\alpha\in\mathbb{N}^{4n}} \frac{D^\alpha f(q_0)}{\alpha!}(q - q_0)^\alpha$$

and, from Proposition 8,

$$\frac{|D^\alpha f(q_0)|}{\alpha!} \leq \left(\frac{125}{e^4}\right)^n \left(\frac{(20e)}{r}\right)^{|\alpha|} \max_{K_r} |f|, \qquad \forall \alpha \in \mathbb{N}^{4n}.$$

Hence

$$(8)\qquad \max_K \frac{|D^\alpha f|}{\alpha!} \leqslant \left(\frac{125}{e^4}\right)^n \left(\frac{(20e)}{r}\right)^{|\alpha|} \max_{K_r} |f|, \qquad \forall \alpha \in \mathbb{N}^{4n}.$$

We fix now $q_1 \in \hat{K}$. If $\Delta(q, q_1) < r/20e$, we define

$$g(q) = \sum_{\alpha \in \mathbb{N}^{4n}} \frac{D^\alpha f(q_1)}{\alpha!} (q - q_1)^\alpha.$$

Since $D^\alpha f$ is regular in A and $q_1 \in \hat{K}$ we have, from (8),

$$\frac{|D^\alpha f(q_1)|}{\alpha!} \leqslant \max_K \frac{|D^\alpha f|}{\alpha!} \leqslant \left(\frac{125}{e^4}\right)^n \left(\frac{(20e)}{r}\right)^{|\alpha|} \max_{K_r} |f|.$$

Thus if $\Delta(q, q_1) < r/20e$ we obtain, $\forall \alpha \in \mathbb{N}^{4n}$,

$$\left| \frac{D^\alpha f(q_1)}{\alpha!} (q - q_1)^\alpha \right| \leqslant \left(\frac{125}{e^4}\right)^n b^{|\alpha|} \max_{K_r} |f|,$$

where $b = \Delta(q, q_1)(20e/r) < 1$.

Hence g is a real analytic function on

$$P = \left\{ q \in \mathbb{H}^n \colon \Delta(q, q_1) < \frac{r}{20e} \right\}.$$

Because g coincides with the regular function f in a neighbourhood of q_1, it follows that g is regular in P. If we denote by C the connected component of $A \cap P$ which contains q_1, we have $g|_C = f|_C$. Hence P is a regular completion of C with respect to $\mathcal{R}(A)|_C$, and then P is contained in A. Thus $\Delta \geqslant r/20e > 0$.

Hence $\hat{K}$ is closed in $\mathbb{H}^n$; since it is also bounded, we deduce that $\hat{K}$ is compact. q.e.d.

REMARK. – We think that, as in the complex case, the converse of Theorem 3 is also true, but we have not been able to prove it.

4. – Another geometrical property of domains of regularity.

1. – In this section we prove that the boundary of a domain of regularity must satisfy a differential geometric condition. Suppose that A is an open subset of $\mathbb{H}^n$, with boundary of class C^2. Let ν be the inner normal versor of ∂A: it determines an orientation on ∂A. We denote by h the second fundamental form of ∂A with respect to this orientation, and we say that h is the *second fundamental form of ∂A relatively to A*. It can be noted that, if Φ

is a real function of class C^2 such that $A = \{\Phi < 0\}$, $\partial A = \{\Phi = 0\}$, with grad $\Phi \neq 0$ on ∂A, we have $\nu = -\operatorname{grad}\Phi / |\operatorname{grad}\Phi|$.

Moreover, if $0 \in \partial A$ and $T_0(\partial A) = \{x_3^{(n)} = 0\}$, we get

$$(9) \quad h_0((u_0^{(1)}, \dots, u_2^{(n)}, 0), (v_0^{(1)}, \dots, v_2^{(n)}, 0)) = \frac{1}{|\operatorname{grad}\Phi(0)|} \sum \frac{\partial^2 \Phi}{\partial x_\alpha^{(h)} \partial x_\beta^{(s)}} (0)\, u_\alpha^{(h)} v_\beta^{(s)}.$$

2. – Now we can prove the following

THEOREM 4. – *Let A be any domain of regularity of $\mathbb{H}^n$ $(n > 1)$ and let $q_0 \in \partial A$. Suppose that the boundary ∂A is of class C^2, in a neighbourhood of q_0. Then there exist no one-dimensional right linear subspaces of $\mathbb{H}^n$, contained in $T_{q_0}(\partial A)$, on which the second fundamental form of ∂A relatively to A is negative definite.*

PROOF. – We use reductio ad absurdum. Let π be any one-dimensional right linear subspace of $\mathbb{H}^n$, contained in $T_{q_0}(\partial A)$, on which h_{q_0} is negative definite.

By Proposition 1, we can suppose

$$q_0 = 0, \qquad T_0(\partial A) = \{x_3^{(n)} = 0\}, \qquad \pi = \{(q_1, 0, \dots, 0) \in \mathbb{H}^n : q_1 \in \mathbb{H}\}.$$

Let Φ be a C^2-function defined on a neighbourhood U of 0 such that $\partial A \cap U = \{\Phi = 0\}$, $A \cap U = \{\Phi < 0\}$, with grad $\Phi \neq 0$ on ∂A. Then

$$\Phi(q_1, \dots, q_n) = \frac{\partial \Phi}{\partial x_3^{(n)}}(0)\, x_3^{(n)} + \frac{1}{2} \sum \frac{\partial^2 \Phi}{\partial x_\alpha^{(h)} \partial x_\beta^{(s)}}(0)\, x_\alpha^{(h)} x_\beta^{(s)} + o(|q|^2),$$

where $q = (q_1, \dots, q_n)$, $q_h = x_0^{(h)} + i x_1^{(h)} + j x_2^{(h)} + k x_3^{(h)}$, for $h = 1, \dots, n$. Thus, from (9) we get

$$\Phi(q_1, 0, \dots, 0) = \frac{|(\partial \Phi / \partial x_3^{(n)})(0)|}{2} h_0((q_1, 0, \dots, 0), (q_1, 0, \dots, 0)) + o(|q_1|^2).$$

Since h_0 is negative definite on π, we have

$$\frac{h_0((q_1, 0, \dots, 0), (q_1, 0, \dots, 0))}{|q_1|^2} \leqslant -C < 0, \qquad \text{for } q_1 \neq 0.$$

Then

$$\frac{\Phi(q_1,0,\dots,0)}{|q_1|^2} \leqslant -\frac{C}{2}\left|\frac{\partial\Phi}{\partial x_3^{(n)}}(0)\right| + \chi(q_1),$$

where $\lim_{q_1\to 0}\chi(q_1)=0$.

Then there exist δ, $\varepsilon>0$ such that, if we set

$$D_\delta=\{q_1\in\mathbb{H}\colon |q_1|\leqslant\delta\},\qquad K=\{(q_2,\dots,q_n)\in\mathbb{H}^{n-1}\colon |q_i|\leqslant\varepsilon,\ \text{for } i=2,\dots,n\},$$

we have $\Phi<0$ on $\partial D_\delta\times K$, and hence $\partial D_\delta\times K\subseteq A$. Moreover there exists $t\in[-\varepsilon,\varepsilon]$ such that, if we set $P=(0,\dots,0,tk)\in\mathbb{H}^{n-1}$, we have $D_\delta\times\{P\}\subseteq A$, with $P\in K$. By Theorem 2, then we obtain $D_\delta\times K\subseteq A$. In particular $0\in A$. Contradiction. q.e.d.

5. – An extension theorem of Hans Lewy type.

1. – Let us consider any real hypersurface S of $\mathbb{H}^n$ is class C^4; let Φ be a real function of class C^4, defined on a neighbourhood of S, such that $S=\{\Phi=0\}$, and grad $\Phi\neq 0$ on S. A function $f\colon S\to\mathbb{H}$ of class C^4 is said to be *admissible*, if it can be extended, on a neighbourhood of S, to a function F of class C^4, such that $\partial F/\partial\bar{q}_h=\Phi^2\psi_h$, where the functions ψ_h are of class C^3, for $h=1,\dots,n$.

The definition is independent of the function Φ, which defines S.

Obviously all traces of regular functions are admissible. Moreover, it is easy to check that admissible functions verify all trace conditions, expressed by differential equations of both first and second order, that we have introduced in [P4]. Conversely in [P4] (pp. 475-477) it is shown that if S is of class C^6, every function of class C^6 which verifies these differential conditions is admissible. (In [P3] and [P4], by mistake, we wrote C^7 instead of C^6.) Thus, for functions of class C^6, admissibility is equivalent to the differential conditions of [P4].

2. – It is easy to show the following

PROPOSITION 9. – *Let $f\colon S\to\mathbb{H}$ be admissible. If $g\colon\mathbb{H}^n\to\mathbb{H}^n$ is an invertible affine right linear map, then $f\circ g\colon g^{-1}(S)\to\mathbb{H}$ is also admissible.*

In [P4] (pp. 475-476) it is also proved the following

PROPOSITION 10. – *Let $f: S \to \mathbb{H}$ be admissible. Then f can be extended, on a neighbourhood of S, to a function F of class C^3 such that $\partial F / \partial \bar{q}_h = \Phi^3 \psi_h$, where the functions ψ_h are of class C^2, for $h = 1, \ldots, n$, and Φ is a function which defines S.*

Theorems 4, 5 and 6 of [P4] can be summarized in the following form:

THEOREM 5. – *Let U be a bounded open set of $\mathbb{H}^n$ ($n > 1$), with boundary of class C^4, such that $\mathbb{H}^n - \overline{U}$ is connected, and let $f: \partial U \to \mathbb{H}$ be an admissible C^4-function. Then there exists $F \in C^2(\overline{U})$, regular in U, which extends f.*

3. – By the same methods, we prove now an extension theorem of Hans Lewy type.

THEOREM 6. – *Let S be a real hypersurface of $\mathbb{H}^n$ ($n > 1$) of class C^4, and let $q_0 \in S$. Let Φ be a C^4-function, defined on a neighbourhood U of q_0, such that $U \cap S = \{\Phi = 0\}$, and $\operatorname{grad} \Phi \neq 0$ on $U \cap S$. Suppose that there exists a one-dimensional right linear subspace π of $\mathbb{H}^n$, contained in $T_{q_0}(S)$, on which the second fundamental form of S, relatively to $\{\Phi < 0\}$, is positive definite. Then there exists a neighbourhood ω of q_0 such that, every admissible C^4-function $f: \omega \cap S \to \mathbb{H}$, can be extended to a function $F: \omega \to \mathbb{H}$ of class C^2, which is regular in $\omega^- = \omega \cap \{\Phi \leqslant 0\}$.*

PROOF. – By Proposition 1 and 9, we can suppose $q_0 = 0$, $T_0(S) = \{x_3^{(n)} = 0\}$, $\pi = \{(q_1, 0, \ldots, 0) \in \mathbb{H}^n \colon q_1 \in \mathbb{H}\}$.

As in the proof of Theorem 4, we have

$$\Phi(q_1, 0, \ldots, 0) =$$

$$\frac{|(\partial \Phi / \partial x_3^{(n)})(0)|}{2} h_0((q_1, 0, \ldots, 0), (q_1, 0, \ldots, 0)) + o(|q_1|^2),$$

and

$$\frac{h_0((q_1, 0, \ldots, 0), (q_1, 0, \ldots, 0))}{|q_1|^2} \geqslant C > 0, \qquad \text{for } q_1 \neq 0.$$

Then

$$\frac{\Phi(q_1, 0, \ldots, 0)}{|q_1|^2} \geqslant \frac{C}{2} \left| \frac{\partial \Phi}{\partial x_3^{(n)}}(0) \right| + \chi(q_1), \qquad \textit{where} \qquad \lim_{q_1 \to 0} \chi(q_1) = 0.$$

Then there exist δ, $\varepsilon > 0$ such that, setting $\tilde{\omega} = \{(q_1, \ldots, q_n) \in \mathbb{H}^n$: $|q_1| < \delta$, $|q_2|, \ldots, |q_n| < \varepsilon\}$, we have $\partial^2 \Phi / (\partial x_0^{(1)} \partial x_0^{(1)}) > 0$ on $\tilde{\omega}$ (cf. (9)), and $\Phi(q_1, \ldots, q_n) > 0$ if $|q_1| = \delta$ and $|q_2|, \ldots, |q_n| < \varepsilon$. Then, if we fix $q_2^0, \ldots, q_n^0$ such that $|q_2^0|, \ldots, |q_n^0| < \varepsilon$,

(10) the set $\{(q_1, q_2^0 \ldots, q_n^0) \in \tilde{\omega}: \Phi(q_1, q_2^0 \ldots, q_n^0) > 0\}$ is connected.

Indeed, since $\partial^2 \Phi / (\partial x_0^{(1)} \partial x_0^{(1)}) > 0$, Φ cannot have local maxima.

Moreover there exists $t \in (-\varepsilon, \varepsilon)$ such that $\Phi(q_1, 0, \ldots, 0, tk) > 0$, for all q_1 which satisfies $|q_1| \leq \delta$. Then there exist $\sigma \in (0, \varepsilon)$ and a neighbourhood V of tk in $\mathbb{H}$, contained in $\{q \in \mathbb{H}: |q| < \varepsilon\}$, such that

(11) $\Phi > 0$ on $W =$

$$\{(q_1, \ldots, q_n) \in \mathbb{H}^n: |q_1| < \delta, \ |q_2|, \ldots, |q_{n-1}| < \sigma, \ q_n \in V\}.$$

We set now

$$\omega = \{(q_1, \ldots, q_n) \in \mathbb{H}^n: |q_1| < \delta, \ |q_2|, \ldots, |q_{n-1}| < \sigma, \ |q_n| < \varepsilon\}.$$

From (10) we obtain that

(12) $$\omega^+ = \{q \in \omega: \Phi(q) > 0\} \quad \text{is connected},$$

and from (11)

(13) $$W \subseteq \omega^+.$$

Before completing the proof, we must show the following (cf. Theorem 1 of [P4])

PROPOSITION 11. – *Let $f = (f_1, \ldots, f_n) \in C^k(\omega, \mathbb{H}^n)$ ($n > 1$, $k \geq 2$) be a function such that $f|_{\omega^+} = 0$. There exists $u \in C^k(\omega)$ such that $\partial u / \partial \bar{q}_h = f_h$, for $h = 1, \ldots, n$, and $u|_{\omega^+} = 0$, if and only if*

(14) $$\Delta_s f_h = \frac{\partial}{\partial \bar{q}_h} \frac{\partial f_s}{\partial q_s}, \quad \textit{for } s, h = 1, \ldots, n.$$

PROOF. – Obviously (14) is a necessary condition (cf. [P4]). Conversely suppose that (14) holds true, and consider the function $u \in C^k(\omega)$ defined by

$$u(q_1, \ldots, q_n) = -\frac{1}{2\pi^2} \int_{|\xi| < \delta} G(\xi - q_1) f_1(\xi, q_2, \ldots, q_n).$$

We have $\partial u/\partial\bar{q}_1 = f_1$ (cf. [P2], p. 51). Moreover, from (12),

$$\Delta_n u(q_1,\dots,q_n) = -\frac{1}{2\pi^2}\int_{|\xi|<\delta} G(\xi - q_1)\,\Delta_n\, f_1(\xi, q_2,\dots,q_n) =$$

$$-\frac{1}{2\pi^2}\int_{|\xi|<\delta} G(\xi - q_1)\,\frac{\partial}{\partial\bar{q}_1}\,\frac{\partial f_n}{\partial q_n}\,(\xi, q_2,\dots,q_n) = \frac{\partial f_n}{\partial q_n}\,(q_1,\dots,q_n).$$

The last equality follows from Theorem 1 of [P2].

Thus, we have obtained,

$$\frac{\partial}{\partial q_n}\left(\frac{\partial u}{\partial\bar{q}_n} - f_n\right) = 0.$$

If we fix $q_1^0,\dots,q_{n-1}^0$, such that $|q_1^0| < \delta$, $|q_2^0|,\dots,|q_{n-1}^0| < \sigma$, the function g defined by

$$g(q) = \left(\frac{\partial u}{\partial\bar{q}_n} - f_n\right)(q_1^0, q_2^0,\dots,q_{n-1}^0, q), \qquad \text{for } |q| < \varepsilon,$$

is anti-regular, and from (13), it vanishes on V. It follows that this function vanishes identically; hence

$$\frac{\partial u}{\partial\bar{q}_n} = f_n \quad \text{on } \omega. \tag{15}$$

Then, if $h = 2,\dots,n-1$, we have, from (14) and (15),

$$\Delta_n\,\frac{\partial u}{\partial\bar{q}_h} = \frac{\partial}{\partial\bar{q}_h}\Delta_n u = \frac{\partial}{\partial\bar{q}_h}\,\frac{\partial f_n}{\partial q_n} = \Delta_n f_h,$$

and hence

$$\Delta_n\left(\frac{\partial u}{\partial\bar{q}_h} - f_h\right) = 0.$$

By the same arguments, we deduce again that $\partial u/\partial\bar{q}_h = f_h$ on ω.

In particular u is regular on ω^+. Since $u|_W = 0$, from (12) and from the Principle of Identity, we obtain $u|_{\omega^+} = 0$.

This concludes the proof of Proposition 11.

Now we can return to the proof of Theorem 6.

If f is an admissible C^4-function on $\omega \cap S$, by Proposition 10, it can be extended to a C^3-function F_1 such that $\partial F_1/\partial\bar{q}_h = \Phi^3\psi_h$, for $h = 1,\dots,n$, where the functions ψ_h are of class C^2. As in [P4] we define now

$$v_h = \begin{cases} \Phi^3\psi_h & \text{on } \omega^-, \\ 0 & \text{on } \omega^+. \end{cases}$$

The functions v_h verify (14); then, from Proposition 11, there exists $u \in C^2(\omega)$ such that $\partial u/\partial\bar{q}_h = v_h$, for $h = 1,\dots,n$, and $u|_{\omega^+} = 0$.

Then $F = F_1 - u$ is an extension of f, which is regular in ω^-. q.e.d.

4. – As in the complex case, we can prove the following

PROPOSITION 12. – *Let S be a real hypersurface of class C^1, which is contained in an open subset A of $\mathbb{H}^n$, and let $f: A \to \mathbb{H}$ be a continuous function, which is regular in $A - S$. Then f is regular in A.*

Then, from this Proposition and from Theorem 6, we obtain

THEOREM 7. – *Let S be a real hypersurface of $\mathbb{H}^n$ $(n > 2)$ of class C^4, and let $q_0 \in S$. Suppose that there exist two one-dimensional right linear subspaces π_1, π_2 of $\mathbb{H}^n$, which are contained in $T_{q_0}(S)$, such that the second fundamental form of S is negative definite on π_1, and positive definite on π_2. Then there exists a neighbourhood ω of q_0 such that, every admissible C^4-function $f: \omega \cap S \to \mathbb{H}$ can be extended to a regular function $F: \omega \to \mathbb{H}$.*

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