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Quantum field corrections of the equation of state in cosmological space-time

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Abstract. A proper quantum statistical field theory framework for the decoupling of the cosmological plasma in curved space-time implies the appearance of corrections to the classical kinetic form of the stress-energy tensor of freely streaming particles. Such quantum corrections can become relevant even at long times after the decoupling and can significantly modify the relation between energy density and pressure, that is the equation of state.

1. Introduction

In cosmology we are usually interested in studying the evolution in the expanding universe of different distributions of matter. The key ingredient is the equation of state, that is the relation between the energy density ε of the distribution and its pressure p. Such relation is often determined using the classical expression of the stress-energy tensor and its evolution is studied by means of the Boltzmann equation [2]. However there might be overlooked quantum effects whose responsible is the interplay between quantum statistical mechanics and quantum field theory in curved space-time. The cosmological space-time is usually described by the Robertson-Walker-Friedman-Lemaitre (RWFL) metric which in comoving coordinates reads:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1-Kr^2} + r^2\mathrm{d}\Omega^2\right), \tag{1}$$

where $a(t)$ is the *scale factor* and K is the *space-curvature*, $K = 0, \pm 1$. The scale factor $a(t)$ is assumed to be a positive defined, increasing function of t, thus the metric (1) describes an homogeneous, isotropic, expanding universe. The only allowed form for a stress-energy tensor is then the *perfect fluid* form:

$$T_{\mu\nu} = (\varepsilon + p)\, u_\mu u_\nu - p g_{\mu\nu}, \quad u_\mu = (1, \mathbf{0})\,. \tag{2}$$

The equation of state of ordinary or dark matter is inferred imposing thermodynamic equilibrium relations. An important case of study regards the evolution of a distribution of matter after its decoupling with the cosmological plasma. Due to the interactions between the distribution and the plasma, thermodynamic equilibrium can be reached. However the expansion of the universe ultimately will overcome the interactions and, at a time t_0, called *decoupling time*, the rate of expansion of the universe will be greater than the rate of the interactions. After some time then the interactions will completely cease and the distribution of matter freezes-out and starts to evolve freely. After the freeze-out, the classical phase space distribution function $f(x,k)$ is the so-called *free-streaming* solution of the Boltzmann equation and the relation between stress-energy tensor and distribution function reads [1]:

$$T^{\mu\nu} = \int \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3 \frac{k^\mu k^\nu}{k^0 \sqrt{-g}} f(x,k) \,.$$

In the approximation of sudden freeze-out, that is of an instantaneous transition from local thermodynamic equilibrium to a non interacting system, the energy density and pressure of freely streaming neutral particles are given by:

$$\varepsilon(t) = \frac{1}{(2\pi)^3 a^4(t)} \int \mathrm{d}k^3 \sqrt{\mathrm{k}^2 + m^2 a^2(t)} f(t_0, k) \,, \tag{3}$$

$$p(t) = \frac{1}{(2\pi)^3 a^4(t)} \int \mathrm{d}k^3 \frac{\mathrm{k}^2}{3\sqrt{\mathrm{k}^2 + m^2 a^2(t)}} f(t_0, k) \tag{4}$$

where $f(t_0,k)$ is the local thermodynamic equilibrium distribution function calculated at the decoupling time t_0, and $\mathrm{k}^2 = k_x^2 + k_y^2 + k_z^2$.

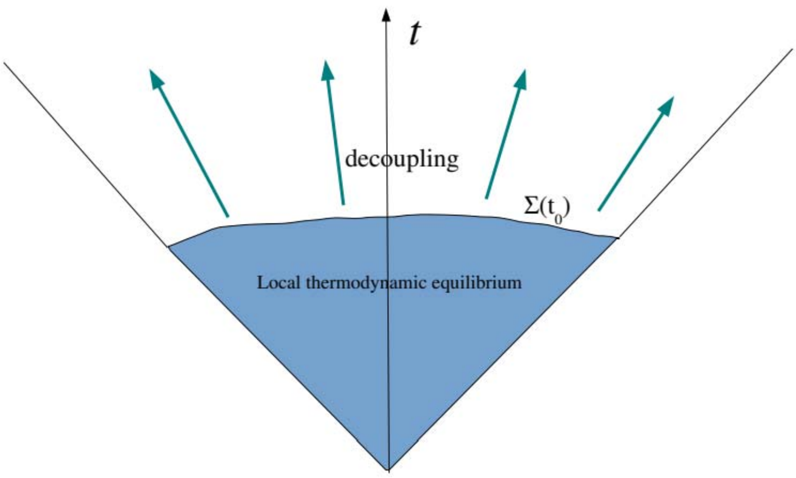

Figure 1. Schematic illustration of the evolution of a matter field as a function of cosmological time in RWFL space-time. Until some time t_0 and the corresponding space-like hypersurface $\Sigma(t_0)$, the field is coupled to the cosmological plasma, thereafter it decouples and evolves freely.

The equation of state can be readily find from the expressions (3) and (4) and it is

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usually expressed in terms of the parameter W defined by:

$$W \doteq \frac{p(t)}{\varepsilon(t)} = \frac{\int \mathrm{dk}^3 \frac{\mathrm{k}^2}{\sqrt{\mathrm{k}^2+m^2a^2(t)}} f_0\left(t_0, k\right)}{3\int \mathrm{dk}^3 \sqrt{\mathrm{k}^2+m^2a^2(t)} f_0\left(t_0, k\right)}.$$

The classical theory for freely streaming massive particles predicts a positive defined W which vanishes in the the far future $a(t) \to \infty$. This implies that, no matter how energetic where the particles when they decoupled, they will ultimately become non relativistic at late times. If the particles are massless instead the equation of state is the one associated to the radiation: $W = 1/3$.

We will show that a proper quantum statistical handling will bring to the appearance of corrections to the expressions of the energy density and the pressure and thus to the relation between the two. Such corrections cannot be found without using a quantum field description for the distribution of matter and thus their origin stems from the interplay between the quantum statistical mechanics and the quantum field theory in curved space-time.

2. Quantum statistical operator of local equilibrium

The renormalized expectation value of the stress-energy tensor in curved space-time is a key ingredient for the solution of the semi-classical Einstein equation:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \left\langle \widehat{T}_{\mu\nu} \right\rangle_{ren}. \tag{5}$$

We aim to calculate the renormalized expectation value originated from matter at local thermodynamic equilibrium which thereafter decouples at some time t_0. In a quantum statistical framework the state of local thermodynamic equilibrium is described by the proper *density operator* $\widehat{\rho}_{LE}$ obtained maximizing the entropy $S = -\operatorname{Tr}\left[\widehat{\rho}_{LE} \ln \widehat{\rho}_{LE}\right]$ with the constraints of given energy and momentum densities [3], [5]:

$$\widehat{\rho}_{LE} = \frac{1}{Z_{LE}} \exp\left[-\int_\Sigma \mathrm{d}\Sigma_\mu \widehat{T}^{\mu\nu} \beta_\nu\right], \tag{6}$$

where Σ is a space-like hypersurface and $\beta_\nu = u_\nu/T$ is the four-temperature which for a fluid at rest is $\beta_\nu = (1/T, \mathbf{0})$. Confining to the case of flat RWFL space-time, $K = 0$, then $\mathrm{d}\Sigma = a^3(t)\mathrm{d}x^3$ and the local operator turns out to be

$$\widehat{\rho}_{LE}(t) = \frac{1}{Z_{LE}} \exp\left[-\widehat{H}(t)/T(t)\right], \quad \widehat{H}(t) \doteq a^3(t) \int \mathrm{d}x^3 \widehat{T}^{00}\left(t, \mathbf{x}\right). \tag{7}$$

The operator $\widehat{H}(t)$ plays the role of an effective hamiltonian and it is not conserved i.e it depends on time t while the finite temperature T quantifies the degrees of excitation of the system respect to the vacuum. From this point of view a natural procedure for the definition of the renormalized expectation value implies the subtraction of the vacuum contribution [8], [9] obtained in the limit $T \to 0$:

$$\left\langle \widehat{T}^{\mu\nu} \right\rangle_{ren} \doteq Tr\left[\widehat{\rho}_{LE}\left(t_0\right) \widehat{T}^{\mu\nu}\right] - \left\langle 0_{t_0}\right| \widehat{T}^{\mu\nu} \left|0_{t_0}\right\rangle, \tag{8}$$

where the state $|0_{t_0}\rangle$ is the vacuum of the hamiltonian at a fixed time t_0. This formula implies that we take the density operator defining the state of the Universe as the fixed (as it should be in the Heisenberg representation) local thermodynamic equilibrium state at the decoupling/freeze-out time t_0. In the limit $T(t_0) \to 0$ the RHS of the above expression goes to zero as expected. Morevore being the operator $\widehat{\rho} = \widehat{\rho}_{LE}(t_0)$ and the state $|0_{t_0}\rangle$ fixed in time the renormalized expectation value is covariantly conserved as the corresponding quantum operator thus being a proper RHS for the Einstein equations:

$$\nabla_\mu \widehat{T}^{\mu\nu} = 0 \Rightarrow \nabla_\mu \left\langle \widehat{T}^{\mu\nu} \right\rangle_{ren} = 0. \tag{9}$$

From the (2) we can obtain the expressions for the energy density ε and the pressure p for a distribution of matter in local thermodynamic equilibrium at the time t_0:

$$\varepsilon(t)_{ren} = \mathrm{Tr}\left[\widehat{\rho}_{LE}(t_0)\,\widehat{T}^{00}\right] - \langle 0_{t_0}|\,\widehat{T}^{00}\,|0_{t_0}\rangle \tag{10}$$

$$p(t)_{ren} = a^{-2}(t)\,\mathrm{Tr}\left[\widehat{\rho}_{LE}(t_0)\,\widehat{T}^{jj}\right] - a^{-2}(t)\,\langle 0_{t_0}|\,\widehat{T}^{jj}\,|0_{t_0}\rangle\,, \tag{11}$$

where we have taken advantage of the isotropy defining:

$$\mathrm{Tr}\left[\widehat{\rho}_{LE}(t_0)\,\widehat{T}^{jj}\right] \doteq \mathrm{Tr}\left[\widehat{\rho}_{LE}(t_0)\,\frac{1}{3}\sum_{i=1}^{3}\widehat{T}^{ii}\right]$$

To explicit the above expressions we must calculate the expectation values of the components of the quantum stress-energy operator built in terms of the fundamental fields operators of the underlying theory.

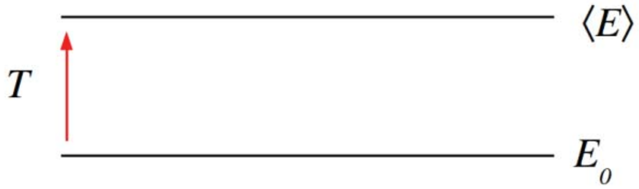

Figure 2. The subtraction of the istantaneous vacuum implies that the LHS of the (8) vanishes in the limit $T(0) \to 0$ thus all the vacuum terms are neglected respect to the excitations induced by the thermal bath.

It is important to stress that, being the hamiltonian $\widehat{H}(t)$ time dependent, the state $|0_{t_0}\rangle$ chosen as the vacuum at the time t_0 is no longer the vacuum at a later time $t > t_0$. However, as we will show in the next section, this is an expected result for a quantum field theory in curved space-time where the choice of the vacuum is, at large extent, arbitrary.

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3. Quantum scalar field in cosmological space-time

The stress-energy tensor form is fixed by the request that it must be the RHS of the classical Einstein equations:

$$\widehat{T}^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S\left[\widehat{\phi}, g\right]}{\delta g_{\mu\nu}},$$

where $S\left[\widehat{\phi}, g\right]$ is the action of the system and $\widehat{\phi}$ are the fundamentals fields of the theory. We consider the simplest case of a free real scalar field for which

$$S\left[\widehat{\phi}, g\right] = \frac{1}{2}\int \mathrm{d}x^4 \sqrt{-g}\left(\nabla_\mu \widehat{\phi} \nabla^\mu \widehat{\phi} - m^2 \widehat{\phi}^2 + \xi R \widehat{\phi}^2\right), \tag{12}$$

where m^2 is the square-mass, $R = R^\mu_\mu$ is the Ricci scalar, ξ is a real number and ∇_μ is the covariant derivative. The non-minimal coupling with the curvature ξR must be included being renormalizble in four-dimensions with ξ free parameter of the theory. However two values of ξ are of particular physical interest, $\xi = 0$ which corresponds to the *minimal coupling* and $\xi = 1/6$ which corresponds to the *conformal coupling*. In the latter case the scalar field turns out to be conformally coupled with the metric in the massless limit and the stress-energy tensor results to be trace-less

$$m \to 0: \; \widehat{T}^\mu_\mu = 0, \quad (\xi = 1/6)\,.$$

Given the action (12) the stress-energy tensor is:

$$\begin{aligned}\widehat{T}_{\mu\nu} = &\nabla_\mu \widehat{\phi} \nabla_\nu \widehat{\phi} - \frac{g_{\mu\nu}}{2}\left(\nabla^\sigma \widehat{\phi} \nabla_\sigma \widehat{\phi} - m^2 \widehat{\phi}^2\right) \\ &+ \xi\left(G_{\mu\nu} + g_{\mu\nu} \nabla_\sigma \nabla^\sigma - \nabla_\mu \nabla_\nu\right) \widehat{\phi}^2,\end{aligned} \tag{13}$$

while the equation of motions are:

$$\left(\nabla_\sigma \nabla^\sigma + m^2 - \xi R\right) \widehat{\phi} = 0. \tag{14}$$

The main goal is to calculate the explicit expressions for the energy density (10) and the pressure (11) for a scalar field which is coupled with the cosmological plasma, and thus in local thermodynamic equilibrium, until the time t_0 after which evolves freely in the expanding universe. Being free, from t_0 onward, the field $\widehat{\phi}$ satisfies the equations (14) with initial conditions fixed by the request that the state of the field is the described by the operator $\widehat{\rho}_{LE}(t_0)$. To obtain ε and p then we must solve (14), express the stress-energy tensor in terms of its solutions and finally calculate the expectation values.

The solution of the Klein-Gordon equation in the FRWL metric has been extensively studied in literature [4]. This is easier to obtain in conformal coordinates, where the t coordinate is replaced by:

$$\eta(t) = \int_{t_0}^{t} \frac{\mathrm{d}t'}{a(t')}, \quad \eta(t_0) = 0, \tag{15}$$

and the flat RWFL metric became

$$\mathrm{d}s^2 = a^2(\eta)\left(\mathrm{d}\eta^2 - \mathrm{d}x^2\right).$$

Like for the Minkowski space-time the field can be expandend in plane waves:

$$\widehat{\chi} = a(\eta)\widehat{\phi} = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k} \left(v_k\left(\eta\right) e^{i\mathbf{k}\cdot\mathbf{x}} \widehat{a}_k + \mathrm{c.c} \right), \tag{16}$$

where $\widehat{a}_k$ are the annihilation operators and $v_k(\eta)$ are the so-called *mode functions*. Plugging the field expansion (16) in the equation of motion (14) gives us the following equation for the mode functions:

$$v_k''(\eta) + \Omega_k^2(\eta) v_k(\eta) = 0, \quad \Omega_k^2(\eta) \doteq \mathrm{k}^2 + m^2 a^2(\eta) - (1-6\xi)\frac{a''(\eta)}{a(\eta)}, \tag{17}$$

where the prime denotes a derivative respect to η. The orthonormality condition for different solutions of the (14) respect to the Klein-Gordon inner product implies a conditions for the modes v_k which in turn implies that the commuation relation for the operators $\widehat{a}_k$ are the usual ones:

$$W\left[v_k\right] \doteq v_k v_k^{\prime *} - v_k^* v_k' = i \iff \left[\widehat{a}_k, \widehat{a}_{k'}^\dagger\right] = \delta^3\left(\mathbf{k} - \mathbf{k}'\right). \tag{18}$$

The mode expansion (16) and the initial conditions for the equation (17) are fixed by the definition of the vacuum of the theory which is the state such that:

$$\widehat{a}_k \left|0\right\rangle = 0, \ \forall k.$$

From this state any multiparticle state can be created by the repeated action of the creation operator $\widehat{a}_k^\dagger$.

3.1. The choice of the vacuum

The quantization procedure in curved space-time follows the one usually implemented in flat space-time. The main difference respect to the flat case is that the choice of the vacuum, and thus the choice of the field expansion (16), is ambiguous. In Minkwoski only one vacuum state invariant under the isometries of the metric, the Poincaré group, exists. Such state is the only state which is the same vacuum for every inertial observer and its defined as the state associated with positive frequency plane-waves mode functions:

$$v_k(t) = \frac{1}{2\sqrt{\omega_k}} e^{-i\omega_k t}, \quad \omega_k \doteq \sqrt{\mathrm{k}^2 + m^2}. \tag{19}$$

This modes are the eigenfunctions of the global time-like Killing vector $\partial/\partial t$ and they define the state which is the vacuum at every time t. In a curved space-time we cannot invoke the Poincaré symmetry to fix the vacuum, moreover, in general, no global time-like Killing vector exists. Thus, given a set of modes v_k associated with the operaotrs $\widehat{a}_k$, another set of modes w_k associated to the operators $\widehat{c}_k$ can be defined by means of a *Bogolubov transformation*:

$$w_k = A_k v_k + B_k v_k^*, \quad \widehat{c}_k = A_k^* \widehat{a}_k - B_k^* \widehat{a}_{-k}^\dagger. \tag{20}$$

If the new modes w_k satisfies the orthonormality condition:

$$W[w_k] = w_k w_k^{\prime *} - w_k^* w_k' = i \iff \left[\widehat{c}_k, \widehat{c}_{k'}^\dagger\right] = \delta^3\left(\mathbf{k} - \mathbf{k}'\right),$$

then the set $\{w_k;\widehat{c}_k\}$ is an equally valid set of modes and operators in which expand the field. It's a straightforward consequence of the (20) that the vacuum defined by the operators $\widehat{a}_k$ is not the vacuum defined by the operators $\widehat{c}_k$. That is the two states

$$\widehat{a}_k\,|0_a\rangle = 0, \quad \widehat{c}_k\,|0_c\rangle = 0,$$

have a non vanishing expectation value for the operator $\widehat{c}_k^\dagger\widehat{c}_k$ and $\widehat{a}_k^\dagger\widehat{a}_k$ respectively. This ambiguity in the vacuum leads to an ambiguity in the interpretation of the concept of particles in curved space-time and its at the base of the gravitational particle production mechanism [4], [7]. Following the arguments of the previous section our goal is to find the set of modes v_k and their associated annihilation operators having as vacuum the instantaneous lowest lying state of the hamiltonian (7). The zero-zero component of the stress-energy tensor is:

$$\widehat{T}_{00} = \frac{1}{2a^4}\left[\widehat{\chi}'^2 + \vec{\nabla}\widehat{\chi}\cdot\vec{\nabla}\widehat{\chi} + m^2a^2\widehat{\chi}^2 + (1-6\xi)\left(\frac{a'^2}{a^2}\widehat{\chi}^2 - 2\frac{a'}{a}\widehat{\chi}'\widehat{\chi}\right)\right]. \quad (21)$$

Plugging the field expansion (16) inside the above expression gives us:

$$\widehat{H}(\eta) = \frac{1}{2a(\eta)}\int \mathrm{dk}^3\omega_k(\eta)\left[K_k(\eta)\left(\widehat{a}_k\widehat{a}_k^\dagger + \widehat{a}_k^\dagger\widehat{a}_k\right) + \Lambda_k(\eta)\widehat{a}_k\widehat{a}_{-k} + \Lambda_k^*(\eta)\widehat{a}_k^\dagger\widehat{a}_{-k}^\dagger\right] \quad (22)$$

with:

$$\omega_k^2(\eta) = \mathrm{k}^2 + m^2a^2(\eta) + (1-6\xi)\,\frac{a'^2(\eta)}{a^2(\eta)}, \quad (23)$$

$$K_k(\eta) = \frac{1}{\omega_k}\left[|v_k'|^2 + \omega_k^2\,|v_k|^2 - 2\,(1-6\xi)\,\frac{a'}{a}\mathrm{Re}\,(v_k'v_k^*)\right], \quad (24)$$

$$\Lambda_k(\eta) = \frac{1}{\omega_k}\left[(v_k')^2 + \omega_k^2\,(v_k)^2 - (1-6\xi)\,\frac{a'}{a}v_k'v_k\right] \quad (25)$$

The vacuum state $|0_{t_0}\rangle$ is thus defined as the state such that the energy at the decoupling $\eta(t_0)=0$, $\langle 0_{t_0}|\,\widehat{H}\,(0)\,|0_{t_0}\rangle = E_0$, is minimum. Minimizing E_0 corresponds to minimize the function $K_k(0)$ which in turn implies the following conditions for the mode functions:

$$v_k(0) = \frac{1}{\sqrt{2\omega_\xi\,(k,0)}}, \quad v_k'\,(0) = -\frac{i}{2v_k(0)} + a'(0)\,(1-6\xi)\,v_k(0), \quad (26)$$

where $\omega_\xi\,(k,0) = \left(\omega_k(0) - (1-6\xi)^2\,a'^2(0)\right)^{1/2}$ is the effective hamiltonian eigenvalue at the decoupling‡. The defined vacuum state is called *istantaneous diagonalization vacuum*, being the state that, at fixed time, diagonalize the effective hamiltonian:

$$\widehat{H}(0) = \frac{1}{2}\int \mathrm{dk}^3\omega_\xi\,(k,0)\left(\widehat{a}_k^\dagger\widehat{a}_k + \frac{1}{2}\right), \quad (27)$$

where we imposed $a\,(t_0) = a\,(\eta=0) = 1$. In the flat space-time limit the vacuum defined by the above prescription becomes time independent and results to be the eigenstate

‡ For a general coupling ξ the eigenvalue $\omega_\xi\,(k,0)$ could become negative. In this case the hamiltonian is unbounded from below and the theory is unstable. However for $\xi=0$ and $\xi=1/6$, which are the most interesting cases, the resulting eigenvalue is always positive defined.

of minimum eigenvalue of the stationary hamiltonian. Thus our renormalization prescription reduces to the normal ordering prescription which is the one usually implemented for a non-interactive quantum field theory at finite temperature in Minkwoski space-time.

4. The expectation value of the stress-energy tensor

From the expression (27) the density operator $\widehat{\rho}_{LE}(0)$ is given and we are now able to calculate the expectation value of the stress-energy operator:

$$\left\langle \widehat{T}^{\mu\nu} \right\rangle_{ren} = \frac{1}{Z} \operatorname{Tr}\left[\widehat{\rho}_{LE}(0)\widehat{T}^{\mu\nu}\right] - \langle 0_0 | \widehat{T}^{\mu\nu} | 0_0 \rangle .$$

For this purpose we shall need the expectation values of quadratic combinations of creation and annihilation operators such as:

$$\left\langle \widehat{a}^\dagger_k \widehat{a}_{k'} \right\rangle = \frac{1}{Z} \operatorname{Tr}\left[\widehat{\rho}_{LE}(0)\widehat{a}^\dagger_k \widehat{a}_{k'}\right] .$$

Such forms are straightforward to work out, as the Hamiltonian is diagonal in the creation and annihilation operators (27). By using the traditional methods of thermal field theory, one readily finds:

$$\begin{aligned}
\langle \widehat{a}_k \widehat{a}_{k'} \rangle &= \left\langle \widehat{a}^\dagger_k \widehat{a}^\dagger_{k'} \right\rangle = 0, \\
\left\langle \widehat{a}_k \widehat{a}^\dagger_{k'} \right\rangle &= n_B\left(\frac{\omega_\xi(k,0)}{T(0)}\right) \delta^3(\mathbf{k}-\mathbf{k}'), \\
\left\langle \widehat{a}^\dagger_k \widehat{a}_{k'} \right\rangle &= \left[n_B\left(\frac{\omega_\xi(k,0)}{T(0)}\right)+1\right] \delta^3(\mathbf{k}-\mathbf{k}'),,
\end{aligned} \tag{28}$$

where $n_B(x) = 1/(e^x - 1)$ is the Bose-Einstein distribution and $T(0)$ is the temperature at the decoupling. To calculate the searched expectation value we plug the field expansion (16) inside the (13) and then we calculate the resulting trace finding [6]:

$$\operatorname{Tr}\left[\widehat{\rho}_{LE}(0)\widehat{T}_{00}\right] = \frac{1}{(2\pi)^3 a^4} \int \mathrm{dk}^3 \omega_k(\eta) K_k(\eta) \left[n_B\left(\frac{\omega_\xi(k,0)}{T(0)}\right) + \frac{1}{2}\right], \tag{29}$$

$$\operatorname{Tr}\left[\widehat{\rho}_{LE}(0)\widehat{T}_{jj}\right] = \frac{1}{(2\pi)^3 a^4} \int \mathrm{dk}^3 \omega_k(\eta) \Gamma_k(\eta) \left[n_B\left(\frac{\omega_\xi(k,0)}{T(0)}\right) + \frac{1}{2}\right], \tag{30}$$

where the function Γ_k is defined as

$$\Gamma_k(\eta) = \frac{1}{\omega_k}\left[(1-4\xi)\left|v'_k\right|^2 + \gamma_k \left|v_k\right|^2 - 2(1-6\xi)\operatorname{Re}\left(v'_k v^*_k\right)\right], \tag{31}$$

with γ_k given by:

$$\gamma_k(\eta) = (12\xi - 1)\,\mathrm{k}^2 + 3(4\xi-1)\,m^2 a^2 + 3(1-6\xi)\frac{a'^2}{a^2} - 12\xi(1-6\xi)\frac{a''}{a} \tag{32}$$

The factor 1/2 stems from the commutation relation of the operators $\widehat{a}_k, \widehat{a}^\dagger_k$ and its presence implies a divergence in the expectation value. To remove it we subtract to the energy density and pressure their limit for $T(0) \to 0$, leading to the elimination of the factor 1/2. In conclusion the energy density and the pressure are given by:

$$a^4(\eta)\varepsilon_{ren}(\eta) = \frac{1}{(2\pi)^3}\int \mathrm{dk}^3 \omega_k(\eta) K_k(\eta) n_B \left[\frac{\omega_\xi(k,0)}{T(0)}\right], \tag{33}$$

$$a^4(\eta)p_{ren}(\eta) = \frac{1}{(2\pi)^3}\int \mathrm{dk}^3 \omega_k(\eta) \Gamma_k(\eta) n_B \left[\frac{\omega_\xi(k,0)}{T(0)}\right]. \tag{34}$$

The above expressions are the main result of this work and must be compared with the one obtained using the classical Boltzmann equation (3), (4). Both the energy density and the pressure are different respect to the classical expressions and depends on the full solution of the Klein-Gordon field equation (17). In particular correction terms can be extracted. For the energy density:

$$a^4(\eta)\Delta\varepsilon(\eta) = \frac{1}{(2\pi)^3}\int \mathrm{dk}^3 \omega_k(\eta) \left[K_k(\eta) - 1\right] n_B \left[\frac{\omega_\xi(k,0)}{T(0)}\right], \tag{35}$$

while for the pressure:

$$a^4(\eta)\Delta p(\eta) = \frac{1}{(2\pi)^3}\int \mathrm{dk}^3 \omega_k(\eta) \left[\Gamma_k(\eta) - \frac{\mathrm{k}^2}{3\omega_k^2(\eta)}\right] n_B \left[\frac{\omega_\xi(k,0)}{T(0)}\right]. \tag{36}$$

Such corrections are vanishing in the flat space-time limit and thus depends on the coupling of the scalar field with the metric. Moreover, in the massless limit, for a conformally coupled field ($\xi = 1/6$) they are exactly 0. To know the impact of these correction and their weight respect to the classical term one must solve the equation of motion (17) for a given scale factor and study the behaviour at late times.

5. Conclusions and outlook

Using a quantum field approach in calculating the energy density and the pressure of a distribution of matter from its decoupling from the cosmological plasma we found corrections respect to the classical expressions for the energy density and the pressure. These corrections depends on the solution of the classical equation of motion and are vanishing in the flat space-time limit. To estimate how important are and if they survive even at late times, one must solve the (17) for different scale factors $a(\eta)$, and then study the behaviour of the functions $K_k(\eta)$, $\Gamma_k(\eta)$. For instance the sign of the function $\Gamma_k(k)$ is not positive defined thus corrections can bring to negative pressures which results in $W < 0$. The next step then will regard the study of different solutions to the equation of motion (17) for different scale factors. In particular the radiation dominated universe $a(t) \sim t^{1/2}$ and the de Sitter universe $a(t) \sim \exp(t)$ will be analyzed being the first decelerated in its expansion while the latter is accelerated resulting in very different behaviours for the solutions of the equation of motion.

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