UNIQUENESS FOR RICCATI EQUATIONS WITH APPLICATION TO THE OPTIMAL BOUNDARY CONTROL OF COMPOSITE SYSTEMS OF EVOLUTIONARY PARTIAL DIFFERENTIAL EQUATIONS

PAOLO ACQUISTAPACE AND FRANCESCA BUCCI

Abstract. In this article we address the issue of uniqueness for differential and algebraic operator Riccati equations, under a distinctive set of assumptions on their unbounded coefficients. The class of boundary control systems characterized by these assumptions encompasses diverse significant physical interactions, all modeled by systems of coupled hyperbolic-parabolic partial differential equations. The proofs of uniqueness provided tackle and overcome the obstacles raised by the peculiar regularity properties of the composite dynamics. These results supplement the theories of the finite and infinite time horizon linear-quadratic problem devised by the authors jointly with I. Lasiecka, as the unique solution to the Riccati equation enters the closed-loop form of the optimal control.

1. Introduction

Well-posedness of Riccati equations is a fundamental question within control theory of Partial Differential Equations (PDE). While the issues of existence and uniqueness for the corresponding solutions are both natural to be addressed and significant in themselves, when seen in the context of linear-quadratic optimal control uniqueness proves particularly relevant, not exclusively from a theoretical perspective. This is because it brings about *in a univocal manner* the (optimal cost, or Riccati) operator which occurs in the feedback representation of the optimal control, thereby allowing its synthesis.

In the present work focus is on the differential and algebraic Riccati equations arising from optimal control problems with quadratic functionals for the class of infinite dimensional abstract control systems dealt with in our earlier works [2] and [4], joint with Lasiecka. The basic characteristics of these linear systems – which read as $y'(t) = Ay(t) + Bu(t)$, $t \in [0, T)$, according to a standard notation – are the following: the free dynamics operator A is the infinitesimal generator of a C_0-semigroup $\{e^{At}\}_{t\ge 0}$ on the state space Y, while the control operator B is *unbounded*, meaning that B maps continuously the control space U into a larger functional space than Y, that is the extrapolation space $[\mathcal{D}(A^*)]'$; see the Assumptions 2.1. It is well known that the latter is an intrinsic feature of differential systems describing evolutionary PDE with boundary (and also point) control; see, e.g., [8] and [21].

We require more specifically that several assumptions on the dynamics operator coefficients (A and B) are fulfilled, recorded as Assumptions 2.3 in Section 2. These are regularity properties that pertain to the adjoint of the kernel $e^{At}B$, with respective PDE counterparts. It is worth emphasizing here that not only the aforesaid control-theoretic properties do not prescribe analyticity of the semigroup e^{At}, in accordance with the fact that the class of systems under consideration – introduced by these authors with Lasiecka in [2] – is inspired by and tailored on systems of coupled hyperbolic-parabolic PDE, subjected to boundary/interface control. They are also more general than the full *singular estimates* (SE)[1] for $e^{At}B$ which are known to be equally effective – even in the absence of analiticity of e^{At} – for the study of the quadratic optimal control problem on both a finite and infinite time horizon, as proved in [16], [17], [19, 20].

Another distinguishing feature of the coefficients of the Riccati equations under study, whose algebraic form is

$$(Px, Az)_Y + (Ax, Pz)_Y - (B^*Px, B^*Pz)_U + (Rx, Rz)_Z = 0\,, \quad x, z \in \mathcal{D}(A)\,,$$

is that the requirement on the *observation* operator R – that is iiib) of the Assumptions 2.3 – may allow R to be the identity, thereby including the integral of the full quadratic energy of the physical system among the viable cost functionals; see the Remarks 2.4.

Working within the functional-analytic framework described above, a theory for both the finite and infinite time horizon LQ-problem has been devised in [2] and [4], the latter under the Assumptions 2.8 (replacing Assumptions 2.3). The strenght of these theories is confirmed by the boundary regularity results – of independent value – that have been established for the solutions to significant PDE systems comprising hyperbolic and parabolic components. Indeed, the novel class introduced in [2] has proven successful in attaining solvability of the quadratic optimal control problems associated with a diverse range of physical interactions such as mechanical-thermal, acoustic-structure, fluid-elasticity ones; see [3], [10], [13], [14] (time interval of finite lenght), [4], [11] and the recent [1] (infinite time horizon).

We recall that since in [2] and [4] we followed an established variational approach, by using the optimality conditions a bounded operator – to wit, $P(t)$ or P, depending on $T < +\infty$ or $T = +\infty$ – is constructed in terms of the optimal state and only subsequently shown to satisfy the corresponding Riccati equations. For this reason the works [2] and [4] provide existence for Riccati equations, but not uniqueness.

The goal of this work is to prove *uniqueness* results for both the differential and algebraic Riccati equations (in short, DRE and ARE, from now on), thereby completing the complex of findings of [2] and [4], respectively. These results – our main results – are stated as Theorems 2.6 and 2.10. Uniqueness holds in appropriate (respective) classes of linear bounded operators, which are consistent with the distinctive property of the *gain operator* ($B^*P(t)$ or B^*P in the quadratic term of the DRE or ARE) emerged in [2] and [4]; see the statements S4. and A4. in Theorem 2.5 and Theorem 2.9, respectively.

We feel it is important to emphasize that even when following avenues previously pursued in the past literature, our proofs of uniqueness need to face novel challenges. This is due in the first place to the specific assumptions on the dynamics operator coefficients (A, B). These are notably weaker than the ones which characterize the parabolic class (as well as its generalization), more elaborate than

[1] We recall that in the present context with SE we mean that $e^{At}B \in \mathcal{L}(U,Y)$ in a right neighbourhood I of $t = 0$, and in particular that $\|e^{At}B\|_{\mathcal{L}(U,Y)} = \mathcal{O}(t^{-\gamma})$ holds true for some $\gamma \in (0,1)$ and any $t \in I$. This explains the adjective "singular". In the PDE realm the membership alone $e^{At}B \in \mathcal{L}(U,Y)$ amounts to an enhanced interior regularity of the solutions to the IBVP with homogeneous boundary data and 'rough' initial data.

the *admissibility* condition (also termed "abstract trace regularity") which is satisfied in the case of significant hyperbolic PDE with boundary control, while the hypothesis on R essentially demands that the operator R^*R *preserves regularity.* And yet, a careful use of the hypotheses allows us to provide a clean and direct (first) proof of Theorem 2.5; see Section 3.

However, since the finite lenght of the interval $[0,T]$ is central to the aforementioned proof, in order to attain uniqueness for the ARE as stated in Theorem 2.9 we pursue an alternative line of argument which is borrowed from classical control theory, but it is used in a nonstandard fashion in the present work; see Section 4. This path turns out to be effective for both the DRE and ARE, thereby bringing about a second proof of Theorem 2.5. An outline of the paper is found in Section 1.2.

1.1. **A glimpse of composite systems of PDE relevant to the context.** Before introducing the optimal control problem formulation under the Assumptions 2.3 (or Assumptions 2.8, for the infinite time horizon case), we revisit two PDE illustrations of initial-boundary value problems (IBVP) which fall within the functional-analytic framework. Our objective here is to give a flavour of the boundary control problems that the Assumptions 2.3 may mirror, as well as of the considerable task that checking their validity entails. We keep the focus on the main facts – from a PDE perspective – and refer the reader to earlier work for all details. (The reading of this section can be postponed, if one aims at focusing on the core topic of the present study.)

1.1.1. *A PDE model of acoustic-structure interaction.* We consider a PDE system which describes the interaction between the acoustic waves in a three-dimensional domain Ω (the acoustic chamber) and the vibrations of a flexible flat portion of the chamber's boundary, say Γ_0; Γ_1 is the so called *hard* wall. Thus, let $\Omega\subset\mathbb{R}^3$ be a bounded domain, with smooth boundary $\partial\Omega =: \Gamma=\overline{\Gamma_0\cup\Gamma_1}$, where $\Gamma_i\subset\mathbb{R}^2$, $i=0,1$, are open, simply connected and disjoint. As thermal effects are also taken into consideration, the PDE system comprises a wave equation for the acoustic velocity potential $z=z(t,x)$, $x\in\Omega$, and a thermoelastic system for the pair of the plate's vertical displacement and the temperature $(w(t,x),\theta(t,x))$, $t\in(0,T)$, $x\in\Gamma_0$.

The wave equation is supplemented with Neumann boundary condition (BC), while the thermoelastic system (is supplemented) with clamped BC and is subject to Dirichlet boundary control. Thus, the IBVP is as follows:

$$\begin{cases} z_{tt}=\Delta z & \text{in } (0,T)\times\Omega \\ \frac{\partial z}{\partial\tilde\nu}+d_1 z=0 & \text{on } (0,T)\times\Gamma_1=:\Sigma_1 \\ \frac{\partial z}{\partial\tilde\nu}=w_t & \text{on } (0,T)\times\Gamma_0=:\Sigma_0 \\ w_{tt}-\rho\Delta w_{tt}+\Delta^2 w+\Delta\theta+z_t=0 & \text{in } \Sigma_0 \\ \theta_t-\Delta\theta-\Delta w_t=0 & \text{in } \Sigma_0 \\ w=\frac{\partial w}{\partial\nu}=0\,,\ \theta=g & \text{on } (0,T)\times\partial\Gamma_0 \\ z(0,\cdot)=z^0,\ z_t(0,\cdot)=z^1 & \text{in } \Omega \\ w(0,\cdot)=w^0,\ w_t(0,\cdot)=w^1;\ \theta(0,\cdot)=\theta^0 & \text{in } \Gamma_0; \end{cases} \tag{1.1}$$

T may be finite or infinite. The symbols $\tilde{\nu}$ and ν above denote the outward unit normals to Γ and to the curve $\partial\Gamma_0$, respectively; d_1 and ρ are positive constants (ρ is proportional to the thickness of the plate). We note that since $\rho > 0$, the (uncoupled) fourth order plate equation is a Khirchhoff equation, which is known to be of hyperbolic type. Therefore, the overall system comprises *three* evolutionary PDE, two hyperbolic equations and a parabolic one.

The natural (finite energy) state space for the PDE problem (1.1) is

$$Y = H^1(\Omega) \times L^2(\Omega) \times H^2_0(\Gamma_0) \times H^1_0(\Gamma_0) \times L^2(\Gamma_0)\,.$$

Take $U = L^2(\partial\Gamma_0)$ as the control space, $\mathcal{U} := L^2(0,\infty;U)$ as the space of admissible controls.

That the IBVP (1.1) can be reformulated as the Cauchy problem (2.1) (having set $y(t) := y(t,\cdot) \equiv (z(t,\cdot), w(t,\cdot), w_t(t,\cdot), \theta(t,\cdot))$, $u(t) = u(t,\cdot)$, $y_0 := (z^0, z^1, w^0, w^1, \theta^0)$, and with the operators A and B explicitly identified, is the outcome of the analysis carried out formerly in [25, Theorem 1.1] and later in [9, 10]. In particular, [25] contains two fundamental results regarding the *uncontrolled* problem, namely, the semigroup generation and exponential stability, whereas the works [9] and [10] tackle the IBVP with nonhomogeneous boundary data (i.e. control actions, in that context as well as here).

Optimal control problems arise naturally, motivated by the goal of reducing the noise within the chamber and /or the vibrations of the elastic wall. Unlike

- the archetypical PDE model of acoustic-structure interactions studied in [5] – where the flexible wall is modeled via a (damped) Euler-Bernoulli equation, in the absence of thermal effects, and the control action affects the mechanical component –, as well as
- the same PDE system in (1.1) under *hinged* – rather than clamped – mechanical BC, or subject to Neumann (thermal) boundary control,

the operator $e^{At}B$ does *not* yield a singular estimate. Indeed, the analysis carried out in the two works [9, 10] revealed that this coupled dynamics inherits the very same control-theoretic properties possessed by the thermoelastic system alone, proved earlier in the works [12, 3].

The decomposition of the operator $B^*e^{A^*\cdot}$ demanded by the Assumptions 2.3, along with the complex of regularity (in time and space) estimates for either component, together with the key hypothesis iiib) on $B^*e^{A^*\cdot}A^{*\epsilon}$, amount to respective properties of the boundary traces $\frac{\partial\theta}{\partial\nu}$ and $\frac{\partial\theta_t}{\partial\nu}$ on $\partial\Gamma_0$, for the solutions (z, z_t, w, w_t, θ) of the uncontrolled problem (i.e. the IBVP with homogeneous boundary data). The full statement of the obtained results is found in [10, Theorem 2.3] (for the finite time horizon problem); the infinite time horizon problem has been discussed briefly (and resolved) in the recent [1, Section 3.2].

Remark 1.1. We remark that a key finding for the proof of Theorem 2.3 in [10] is the exceptional regularity of certain boundary traces of the mechanical component: to wit, the estimate

$$\exists C_T > 0\colon \qquad \int_0^T \|\Delta w\|_{L^2(\partial\Gamma_0)}\, dt \le C_T \|y_0\|_Y^2\,, \qquad y_0 \in Y\,, \tag{1.2}$$

holds true; see [10, Proposition 3.2]. We note that

- the regularity result (1.2), i.e. $\Delta w|_{\partial\Gamma_0} \in L^2(0,T;L^2(\partial\Gamma_0))$, plainly does not follow from the interior regularity $w \in H^2(\Gamma_0)$ of finite energy solutions, via well-known trace theory;
- its proof, carried out by using energy /multiplier methods, owes much to [6] while it additionally exploits the regularity theory of hyperbolic equations with nonhomogeneous Neumann boundary data; see the estimate (3.9) in [10], and the proper bibliographical references therein (with [29] providing the most general result).
- Lastly, the theory of interpolation spaces is the tool which eventually enables to attain iiib) of the Assumptions 2.3.

1.1.2. *A PDE model of fluid-ealsticity interaction.* For fluid-structure interaction (FSI) it is meant the coupling of physical phenomena /evolutions that appear in the distinct fields of fluidodynamics and structural mechanics. Thus the mathematical modeling of FSI gives rise to composite systems of PDE which exhibit dynamics of different type – with respective typical characteristics –, most often parabolic and hyperbolic (indeed, relative to fluid and structure, respectively).

As a second illustration we introduce the linearization of a recognized PDE model for the interaction of an elastic body fully immersed in a fluid. For context, we mention that the PDE problem we consider here goes at least as far back as J.-L. Lions' work; the reader is referred to the study [7] of 2007 for some background, literature review (till then), and an insight into the mathematical challenges overcome therein to prove the well-posedness for the true nonlinear problem (in a natural functional setting).

The open, bounded and smooth domain representing the fluid-solid region is denoted by $\Omega \subset \mathbb{R}^d$, $d = 2,3$, with Ω_f and Ω_s the (open and smooth) domains occupied by the fluid and the solid, respectively; then Ω is the interior of $\overline{\Omega}_f \cup \overline{\Omega}_s$. The interaction occurrs at the solid boundary $\partial\Omega_s =: \Gamma_s$ – the *interface* –, which is assumed *fixed.* (This feature finds a physical justification in the fact that the motion of the solid is considered as entirely due to infinitesimal displacements – fast, though.) Finally, Γ_f is the outer boundary of Ω_f, namely, $\Gamma_f = \partial\Omega_f \setminus \Gamma_s$.

The dynamics of the fluid and the solid are described by the equations of Stokes flow in the vector-valued variable u (the fluid velocity field) and scalar variable p (the pressure), plus the Lamé system of dynamic elasticity in the variable w, respectively. The mathematical formulation of the boundary control problem is as follows:

$$
\begin{cases}
u_t - \operatorname{div}\epsilon(u) + \nabla p = 0 & \text{in } \; Q_f := (0,T)\times\Omega_f \\
\operatorname{div} u = 0 & \text{in } \; Q_f \\
w_{tt} - \operatorname{div}\sigma(w) = 0 & \text{in } \; Q_s := (0,T)\times\Omega_s \\
u = 0 & \text{on } \; \Sigma_f := (0,T)\times\Gamma_f \\
\epsilon(u)\cdot\nu = \sigma(w)\cdot\nu + p\nu + g & \text{on } \; \Sigma_s := (0,T)\times\Gamma_s \\
w_t = u & \text{on } \; \Sigma_s \\
u(0,\cdot) = u_0 & \text{in } \; \Omega_f \\
w(0,\cdot) = w_0\,, \quad w_t(0,\cdot) = w_1 & \text{in } \; \Omega_s\,,
\end{cases}
\tag{1.3}
$$

where σ and ϵ denote the elastic stress tensor and the strain tensor, respectively, that are

$$\epsilon_{ij}(w)=\frac{1}{2}\Big(\frac{\partial w_i}{\partial x_j}+\frac{\partial w_j}{\partial x_i}\Big)\,, \qquad \sigma_{ij}(w)=\lambda\sum_{k=1}^{3}\epsilon_{kk}(w)\delta_{ij}+2\mu\epsilon_{ij}(w)\,,$$

λ,μ are the Lamé constants and δ_{ij} is the Kronecker symbol. We note that $\nu=\nu(x)$ is the outward unit normal for the fluid region Ω_f; accordingly, it is pointing towards the interior of the solid region Ω_s. The control function acting on the interface is denoted by the letter g in place of u (in order to avoid abuse of notation); the class of admissible controls $\mathcal{U}=L^2(0,T;L^2(\Gamma_s))$.

As the study of the IBVP (1.3) – geared towards attaining the complex of regularity results for the solutions to the uncontrolled problem that allow to infer solvability of the associated quadratic optimal control problems – is long and technical (and presented in full elsewhere), we refer the reader to the analysis in [13], [14] and [11]. We point out that once the IBVP is shown to correspond to the Cauchy problem (2.1) (with u replaced by g) in a natural energy space Y, with the operator A and B explicitly identified, then the decomposition of the operator $B^*e^{A^*\cdot}$ demanded by the Assumptions 2.3, along with the complex of regularity (in time and space) estimates for either component, as well as the key hypothesis iiib) for $B^*e^{A^*\cdot}A^{*\epsilon}$, amount to respective properties of the normal traces $\frac{\partial u}{\partial\nu}$ and $\frac{\partial u_t}{\partial\nu}$ of the thermal variable on $\partial\Gamma_0$, for solutions (u,w,w_t) of the IBVP with homogeneous boundary data.

Then, we remark that

- the sought boundary regularity results for the fluid velocity field are established combining carefully the parabolic regularity of the fluid variable with the enhanced regularity of the normal component $\sigma(w)\cdot\nu$ of the stress tensor on the interface, valid for semigroup solutions;
- however, unlike the case of the previous acoustic-structure interaction, the proof of the (exceptional) boundary regularity of the hyperbolic component – which owes much to the analysis performed in the works [7] and [23], yet it requires the development of two novel lemmas (*cf.* [14, Lemmas 2.3 and 2.4]) that are problem-specific – cannot dispense with the tools of microlocal analysis;
- once again, interpolation is utilized to eventually conclude that iiib) of the Assumptions 2.3 holds true, with the interpolation spaces between fractional Sobolev spaces and dual spaces (of fractional Sobolev spaces) computed thanks to [26, Theorem 12.5].

The study of the infinite time horizon quadratic optimal control problem has been carried out in [11], on a recognized variant of (1.3) whose free dynamics is exponentially stable. In that case, multiplier /energy methods (plus semigroup theory and interpolation) suffice, instead.

1.2. **An outline of the paper.** In the next Section 2 we state our uniqueness results, that are Theorem 2.6 and Theorem 2.10, after having recalled the underlying framework (shaped by the Assumptions 2.3 and 2.8) along with the core findings of the theories of the finite and infinite time horizon LQ-problem devised in [2] and [4], respectively. The subsection 2.3 contains a mini-guide to the mathematical proofs.

In Section 3 we present a first proof of Theorem 2.6. This proof follows a line of argument more akin to those used in the past to establish uniqueness for the DRE in the parabolic and hyperbolic cases. It is preceded by Lemma 3.1, which establishes two integral forms of the differential Riccati equation. An integral form of the algebraic Riccati equation is derived as well in Lemma 3.2, for subsequent use.

In Section 4 we develop a different line of argument culminating with the proof of Theorem 2.10 on uniqueness for the ARE. As the very same approach can be pursued for the DRE as well, we provide a second distinct proof of Theorem 2.6. A preliminary major step that is carried out is the derivation of two independent results, one of which establishes a fundamental identity while the other one discusses a built closed loop equation, for *any* solution to the ARE (or DRE) within the proper class $\mathcal{Q}$ ($\mathcal{Q}_T$, respectively). These are Lemma 4.1 and Lemma 4.2 for the DRE, Lemma 4.3 and Lemma 4.4 for the ARE.

In Appendix A we gather several regularity results (some old, some new) which are used throughout the paper.

2. Abstract framework, main results

2.1. **The LQ problem: abstract dynamics and setting.** Let Y and U be two separable Hilbert spaces, the *state* and *control* space, respectively. We consider the abstract (linear) control system $y' = Ay + Bu$ and the corresponding Cauchy problems

$$\begin{cases} y'(t) = Ay(t) + Bu(t)\,, & 0 \le t < T \\ y(0) = y_0 \in Y\,, \end{cases} \tag{2.1}$$

under the following basic Assumptions.

Assumptions 2.1 (Basic Assumptions). *Let Y, U be separable complex Hilbert spaces.*

- *The closed linear operator $A : \mathcal{D}(A) \subset Y \to Y$ is the infinitesimal generator of a strongly continuous semigroup $\{e^{At}\}_{t\ge 0}$ on Y;*
- *$B \in \mathcal{L}(U, [\mathcal{D}(A^*)]')$.*

The basic assumptions on the operators A and B which characterize the abstract dynamics in (2.1) reflect two intrinsic features stemming from coupled systems of hyperbolic-parabolic PDEs with nonhomogeneous boundary data, to wit,

i. the control operator B will *not* be bounded from the control space U into the state space Y, and
ii. the semigroup e^{At} typically will be neither analytic nor a group.

We recall that the first property is shared by PDE systems subject to point control in the interior of the domain; a wealth of illustrations are found in [18], [8] and [21].

Thus, given $y_0 \in Y$, the Cauchy problem (2.1) possesses a unique *mild* solution given by

$$y(t) = e^{At}y_0 + \int_0^t e^{A(t-s)}Bu(s)\,ds\,, \qquad t \in [0,T)\,, \tag{2.2}$$

where

$$L\colon u(\cdot) \longrightarrow (Lu)(t) := \int_0^t e^{A(t-s)}Bu(s)\,ds$$

is the *input-to-state mapping*, that is the operator which associates to any control function $u(\cdot)$ the solution to the Cauchy problem (2.1) with $y_0 = 0$, and (2.2) makes sense at least in the extrapolation space $[\mathcal{D}(A^*)]'$; see [21, § 0.3, p. 6, and Remark 7.1.2, p. 646].
We will use the notation L throughout the paper. We point out here the definition (A.1) of the operator L_s, which will occur later; the symbol L_0, in place of L, is avoided for the sake of simplicity.

To the state equation (2.1) we associate the quadratic functional

$$J(u) = \int_0^T \left(\|Ry(t)\|_Z^2 + \|u(t)\|_U^2\right) dt\,, \tag{2.3}$$

where Z is a third separable Hilbert space – the so called observation space (possibly, $Z \equiv Y$) – and at the outset the *observation* operator R simply satisfies

$$R \in \mathcal{L}(Y, Z)\,. \tag{2.4}$$

The formulation of the optimal control problem under study is classical. The adjectives finite or infinite time horizon problem refer to the cases $T < +\infty$ or $T = +\infty$, respectively.

Problem 2.2 (The optimal control problem). *Given $y_0 \in Y$, seek a control function $u \in L^2(0, T; U)$ which minimizes the cost functional* (2.3)*, where $y(\cdot) = y(\cdot\,; y_0, u)$ is the solution to* (2.1) *corresponding to the control function $u(\cdot)$ (and with initial state y_0) given by* (2.2)*.*

It is well known that aiming at solving Problem 2.2, certain principal facts need to be ascertained, beside the existence of a unique optimal pair $(\hat{u}(\cdot, s; y_0), \hat{y}(\cdot, s; y_0))$ (which can be easily inferred via convex optimization arguments). Namely,

- that the optimal control $\hat{u}(t)$ admits a (pointwise in time) *feedback* representation, in terms of the optimal state $\hat{y}(t)$;
- that the optimal cost operator $P(t)$ (P, when $T = +\infty$) solves the corresponding Differential (Algebraic) Riccati equation; thus, the issue of well-posedness of the DRE (ARE) arises, requiring
- that a meaning is given to the gain operator $B^*P(t)$ (B^*P) on the state space Y (by means of extensions, or – and this will be the case here –, as a *bounded* operator on a dense subset of Y).

2.2. **Theoretical results: finite and infinite time horizon problems.** We begin by recalling the theory of the LQ-problem on a finite time interval developed in [2]. This theory pertains to the class of control systems – introduced in the very same [2] – whose dynamics, control and observation operators are subject to the following assumptions.

Assumptions 2.3 (Finite time horizon case). *Let Y, U and Z be separable complex Hilbert spaces, and let $T > 0$ be given. The pair (A, B) (which describes the state equation* (2.1)*) fulfils Assumptions 2.1, with the additional property $A^{-1} \in \mathcal{L}(Y)$, while the observation operator R (which occurs in the cost functional* (2.3)*) satisfies the basic condition* (2.4)*.*

The operator $B^*e^{A^*t}$ can be decomposed as

$$B^*e^{A^*t}x = F(t)x + G(t)x\,, \qquad 0 \le t \le T\,,\ x \in \mathcal{D}(A^*)\,, \tag{2.5}$$

where $F(t)\colon Y \longrightarrow U$ *and* $G(t)\colon \mathcal{D}(A^*) \longrightarrow U$, $t>0$, *are bounded linear operators satisfying the following assumptions:*

i) *there exist constants* $\gamma \in (0,1)$ *and* $N>0$ *such that*

$$\|F(t)\|_{\mathcal{L}(Y,U)} \le N\, t^{-\gamma}\,, \qquad 0<t\le T\,; \tag{2.6}$$

ii) *the operator* $G(\cdot)$ *belongs to* $\mathcal{L}(Y, L^p(0,T;U))$ *for all* $p\in[1,\infty)$;

iii) *there exists* $\epsilon>0$ *such that:*

a) *the operator* $G(\cdot){A^*}^{-\epsilon}$ *belongs to* $\mathcal{L}(Y, C([0,T];U))$, *with*

$$\sup_{t\in[0,T]} \|G(t){A^*}^{-\epsilon}\|_{\mathcal{L}(Y,U)} < \infty\,;$$

b) *the operator* R^*R *belongs to* $\mathcal{L}(\mathcal{D}(A^\epsilon), \mathcal{D}(A^{*\epsilon}))$, *i.e.*

$$\|A^{*\epsilon}R^*RA^{-\epsilon}\|_{\mathcal{L}(Y)} \le c < \infty\,;$$

c) *there exists* $q\in(1,2)$ *(depending, in general, on* ϵ*) such that the map* $x\longmapsto B^*e^{A^*t}A^{*\epsilon}x$ *has an extension which belongs to* $\mathcal{L}(Y, L^q(0,T;U))$.

Remarks 2.4. 1. We note that it is assumed at the very outset that $0\in\rho(A)$, i.e. the dynamics operator A is boundedly invertible on Y. This property happens to hold true for an ample variety of coupled systems of hyperbolic-parabolic PDE, such as thermoelastic systems as well as models of acoustic- and fluid-structure interaction. This allows in particular to define the fractional powers $(-A)^\alpha$, $\alpha\in(0,1)$; see [30, § 1.15.1-2], [28], [27]. (We note that in order to make the notation lighter, we wrote – and shall write throughout – A^α in place of $(-A)^\alpha$.)

2. We remark that the class of control systems shaped by the Assumptions 2.3 is *more general* than the one characterized by singular estimates (SE, recalled in the Introduction) for the operator $e^{At}B$. This unless $G(t)\equiv 0$ in the decomposition (2.5) of $B^*e^{tA^*}$, in which case $B^*e^{tA^*}$ reduces to the component $F(t)$ that indeed yields the estimate (2.6).

3. We may assert that the term $G(t)x$ embodies both the parabolic and the hyperbolic components, via the coupling. Its enhanced regularity mirrors the combination of parabolic regularity with the peculiar (exceptional) regularity of the hyperbolic boundary traces. An enlightening comparison with earlier classes studied in the literature is found in [2, Section 2.1]; see also [4, pp. 1827-1828].

4. The requirement iiib) essentially demands that the operator R^*R somewhat maintains regularity. We note, in particular, that when the fractional powers $\mathcal{D}(A^\epsilon)$ can be computed as interpolation spaces $[\mathcal{D}(A),Y]_{1-\epsilon}$ – which happens e.g. if e^{tA} is a s.c. *contraction* semigroup, precisely the case of the PDE problems discussed in Section 1.1 –, then $\mathcal{D}(A)\equiv\mathcal{D}(A^*)$ implies $\mathcal{D}(A^\epsilon)\equiv\mathcal{D}(A^{*\epsilon})$, at least for sufficiently small ϵ. In this circumstance, $R=I$ is a viable observation operator.

Under the listed Assumptions 2.3, a full solution to the optimal control Problem 2.2, as detailed by the complex of statements S1.–S6. collected in Theorem 2.5 below, was obtained in [2]. Despite the lack of continuity (in time) of the optimal control $\hat u(\cdot)$ (see S1. below) – a novelty over the parabolic or hyperbolic theories –, and the fact that the gain operator $B^*P(t)$ is bounded only on a suitable dense subset of Y (contained in $\mathcal{D}(A)$, though; *cf.* S4.), all the sought properties (the aforementioned "principal facts") are achieved.

Theorem 2.5 (Finite time horizon theory; cf. [2], Theorem 2.3)**.** *With reference to the control problem* (2.1)–(2.3)*, under the Assumptions 2.3, the following statements are valid for each* $s \in [0,T)$.

S1. *For each* $x \in Y$ *the optimal pair* $(\hat{u}(\cdot,s;x), \hat{y}(\cdot,s;x))$ *satisfies*

$$\hat{y}(\cdot,s;x) \in C([s,T];Y), \quad \hat{u}(\cdot,s;x) \in \bigcap_{1\le p<\infty} L^p(s,T;U).$$

S2. *The linear bounded (on* Y*) operator* $\Phi(t,s)$*, defined by*

$$\Phi(t,s)x = \hat{y}(t,s;x) = e^{A(t-s)}x + [L_s\hat{u}(\cdot,s;x)](t)\,, \quad s \le t \le T\,,\; x \in Y\,, \tag{2.7}$$

is an evolution operator, i.e.

$$\Phi(t,t) = I_Y\,, \qquad \Phi(t,s) = \Phi(t,\sigma)\Phi(\sigma,s) \quad \text{for } s \le \sigma \le t \le T.$$

S3. *For each* $t \in [0,T]$ *the operator* $P(t) \in \mathcal{L}(Y)$*, defined by*

$$P(t)x = \int_t^T e^{A^*(\tau-t)} R^*R\Phi(\tau,t)x\; d\tau\,, \qquad x \in Y, \tag{2.8}$$

is self-adjoint and positive; it belongs to $\mathcal{L}(Y, C([0,T];Y))$ *and is such that*

$$(P(s)x,x)_Y = J_s(\hat{u}(\cdot,s;x), \hat{y}(\cdot,s;x)) \qquad \forall s \in [0,T]\,.$$

S4. *The gain operator* $B^*P(\cdot)$ *belongs to* $\mathcal{L}(\mathcal{D}(A^\varepsilon), C([0,T];U))$ *and the optimal pair satisfies for* $s \le t \le T$

$$\hat{u}(t,s;x) = -B^*P(t)\hat{y}(t,s;x) \qquad \forall x \in Y. \tag{2.9}$$

S5. *The operator* $\Phi(t,s)$ *defined in* (2.7) *satisfies for* $s < t \le T$*:*

$$\frac{\partial \Phi}{\partial s}(t,s)x = -\Phi(t,s)(A - BB^*P(s))x \in L^{1/\gamma}(s,T;[\mathcal{D}(A^{*\varepsilon})]')$$

for all $x \in \mathcal{D}(A)$*, and*

$$\frac{\partial \Phi}{\partial t}(t,s)x = (A - BB^*P(t))\Phi(t,s)x \in C([s,T],[\mathcal{D}(A^*)]')$$

for all $x \in \mathcal{D}(A^\varepsilon)$.

S6. *The operator* $P(t)$ *defined by* (2.8) *satisfies the following (differential) Riccati equation in* $[0,T)$*:*

$$\begin{cases} \frac{d}{dt}(P(t)x,y)_Y + (P(t)x,Ay)_Y + (Ax,P(t)y)_Y + (Rx,Ry)_Z \\ \qquad\qquad\qquad -(B^*P(t)x,B^*P(t)y)_U = 0 \qquad \forall\, x,y \in \mathcal{D}(A) \\ P(T) = 0\,. \end{cases} \tag{2.10}$$

The assertion S6. in Theorem 2.5 shows the existence of at least one solution to the DRE (2.10) corresponding to problem (2.1)-(2.3). The question as to whether the optimal cost operator $P(\cdot)$ (defined in (2.8)) is actually the *unique* solution to the DRE – at least within an appropriate class of operators –, is an issue which was not dealt with in the paper [2]. Thus, in order to render the finite time horizon theory devised in [2] complete, we complement assertion S6. of Theorem 2.5 with the (novel) achievement of *uniqueness*, thereby concluding the proof of well-posedness of the DRE.

As we will see, uniqueness is meant within a suitable class – that is class $\mathcal{Q}_T$ in (2.11) below – of linear, bounded, self-adjoint operators also meeting an additional requirement, which is consistent with the regularity property displayed by the gain operator in assertion S4. above.

Theorem 2.6 (**Uniqueness for the DRE**)**.** *With reference to the control problem* (2.1)–(2.3)*, let the Assumptions 2.3 hold. Then, the differential Riccati equation* (2.10) *has a unique solution within the class*

$$\begin{aligned}\mathcal{Q}_T = \big\{Q \in C([0,T];\mathcal{L}(Y))\colon\; Q(t) = Q(t)^* \ge 0\,,\; Q(T) = 0\,,\\ B^*Q(\cdot) \in \mathcal{L}(\mathcal{D}(A^\epsilon), C([0,T];U))\big\}\,.\end{aligned} \tag{2.11}$$

The optimal cost operator $P(\cdot)$ defined by (2.8) *is consequently that solution.*

Remark 2.7 (*A technical point*)**.** The findings of the work [2], summarized as Theorem 2.5 above, were actually established under the *weaker* regularity assumption

iiic)' there exists $q \in (1,2)$ such that the map $x \longmapsto B^*e^{A^*t}R^*RA^\epsilon x$ has an extension which belongs to $\mathcal{L}(Y, L^q(0,T;U))$,

rather than iiic). Indeed, iiic)' of the Assumptions 2.3, combined with iiib), implies iiic), as already pointed out in [2, p. 1401] (with a reversed notation, though).

However, on one side the present iiic) – more precisely, the boundary regularity result that (case by case) the control-theoretic condition iiic) translates to – has been shown over the years to hold true in the case of distinct PDE systems studied in the aforementioned references [3], [10], [13, 14], [11]. On the other side, uniqueness of solutions to the Riccati equations appears to be in need of it: both within the first proof of Theorem 2.5 given in the next section (specifically to perform the estimates which bring about (3.7)), and also to show Lemma 4.2, instrumental to the distinct proof of the same result proposed in Section 4. Furthermore, the stronger (A.3) – which is central to the proof of Lemma 4.4 relevant to the infinite time horizon case – is based upon iiic) of Assumptions 2.3.

In the infinite time horizon case, i.e. when $T = +\infty$ in (2.3), the hypotheses on both the semigroup e^{At} and the component $F(t)$ (in the decomposition of the operator $B^*e^{A^*t}$) are strenghtened, assuming their exponential decay as $t \to +\infty$; see (2.12) and (2.13) below, respectively. For the sake of completeness and the reader's convenience, the hypotheses pertaining to the infinite time horizon are wholly recorded below.

Assumptions 2.8 (**Infinite time horizon case**)**.** *Let Y, U and Z be separable complex Hilbert spaces, and let the basic Assumptions 2.1 be valid, with the additional property that the C_0-semigroup e^{At} is exponentially stable on Y, $t \ge 0$; namely, there exist constants $M \ge 1$ and $\omega > 0$ such that*

$$\|e^{At}\|_{\mathcal{L}(Y)} \le M\, e^{-\omega t} \qquad \forall t \ge 0\,. \tag{2.12}$$

Then in particular, $A^{-1} \in \mathcal{L}(Y)$.

The operator $B^*e^{A^*t}$ admits the decomposition (2.5)*, where $F(t)\colon Y \longrightarrow U$, $t \ge 0$, is a bounded linear operator such that*

i)' *there exist constants $\gamma \in (0,1)$ and $N, \eta > 0$ such that*

$$\|F(t)\|_{\mathcal{L}(Y,U)} \le N\, t^{-\gamma}\, e^{-\eta t} \qquad \forall t > 0\,, \tag{2.13}$$

while ii)-iiia)-iiib)-iiic) of the Assumptions 2.3 on the (linear, bounded) component $G(t)\colon \mathcal{D}(A^*) \longrightarrow U$, $t \ge 0$, *hold true for some* $T > 0$.

We note that the functional (2.3) with $T = +\infty$ makes sense at least for $u \equiv 0$. This again in view of the exponential stability of the semigroup e^{At} ((2.12) of Assumptions 2.8), which combined with (2.4) ensures $Ry(\cdot, y_0; 0) \in L^2(0, \infty; Y)$. (The analysis carried out in the present paper easily extends to more general quadratic functionals, like

$$J(u) = \int_0^\infty \big(\|Ry(t)\|_Z^2 + \|\tilde{R}u(t)\|_U^2\big)\,dt\,,$$

provided $\tilde{R}$ is a coercive operator in U. We take $\tilde{R} = I$ just for the sake of simplicity and yet without loss of generality.)

Theorem 2.9 (Infinite time horizon theory; cf. [4], Theorem 1.5)**.** *Under the Assumptions 2.8, the following statements are valid.*

A1. *For any* $y_0 \in Y$ *there exists a unique optimal pair* $(\hat{u}(\cdot), \hat{y}(\cdot))$ *for Problem* (2.1)-(2.3)*, which satisfies the following regularity properties*

$$\begin{aligned} &\hat{u} \in \bigcap_{2\le p<\infty} L^p(0,\infty;U)\,,\\ &\hat{y} \in C_b([0,\infty);Y) \cap \Big[\bigcap_{2\le p<\infty} L^p(0,\infty;Y)\Big]\,. \end{aligned}$$

A2. *The family of operators* $\Phi(t)$, $t \ge 0$, *defined by*

$$\Phi(t)y_0 := \hat{y}(t) = y(t, y_0; \hat{u}) \tag{2.14}$$

is a C_0*-semigroup on* Y, $t \ge 0$, *which is exponentially stable.*

A3. *The operator* $P \in \mathcal{L}(Y)$ *defined by*

$$Px := \int_0^\infty e^{A^*t} R^* R\Phi(t)x\,dt, \qquad x \in Y, \tag{2.15}$$

is the optimal cost operator*;* P *is (self-adjoint and) non-negative.*

A4. *The following (*pointwise in time*) feedback representation of the optimal control is valid for any initial state* $y_0 \in Y$*:*

$$\hat{u}(t) = -B^*P\hat{y}(t) \qquad \text{for a.e. } t \in (0,\infty),$$

where the gain operator satisfies $B^*P \in \mathcal{L}(\mathcal{D}(A^\epsilon), U)$ *(that is, it is just densely defined on* Y *and yet it is bounded on* $\mathcal{D}(A^\epsilon)$*).*

A5. *The infinitesimal generator* A_P *of the (optimal state) semigroup* $\Phi(t)$ *defined in* (2.14) *coincides with the operator* $A(I - A^{-1}BB^*P)$*; more precisely,*

$$\begin{aligned} &A_P \equiv A(I - A^{-1}BB^*P)\,,\\ &\mathcal{D}(A_P) \subset \big\{x \in Y:\ x - A^{-1}B\,B^*Px \in \mathcal{D}(A)\big\}\,. \end{aligned}$$

A6. *The operator* $e^{At}B$*, defined in* U *and a priori with values in* $[\mathcal{D}(A^*)]'$*, is such that*

$$e^{\delta\cdot}e^{A\cdot}B \in \mathcal{L}(U, L^p(0,\infty;[\mathcal{D}(A^{*\epsilon})]') \quad \forall p \in [1, 1/\gamma) \tag{2.16}$$

for all $\delta \in [0, \omega \wedge \eta)$*; almost the very same regularity is inherited by the operator* $\Phi(t)B$*:*

$$e^{\delta\cdot}\Phi(\cdot)B \in \mathcal{L}(U, L^p(0,\infty;[\mathcal{D}(A^{*\epsilon})]') \quad \forall p \in [1, 1/\gamma)\,,$$

with $\delta > 0$ sufficiently small.

A7. *The optimal cost operator P defined in* (2.15) *is a solution to the algebraic Riccati equation (ARE) corresponding to Problem* (2.1)-(2.3)*, that is*

$$\begin{aligned}(Px, Az)_Y + (Ax, Pz)_Y - (B^*Px, B^*Pz)_U \\ + (Rx, Rz)_Z = 0\,, \quad x, z \in \mathcal{D}(A)\,.\end{aligned} \tag{2.17}$$

The ARE can be rewritten as

$$(A^*Px, z)_Y + (x, A^*Pz)_Y - (B^*Px, B^*Pz)_U + (Rx, Rz)_Z = 0\,,$$

when $x, z \in \mathcal{D}(A_P)$.

In order to render the infinite time horizon theory devised in [4] complete, we complement assertion A7. of Theorem 2.9 about existence of solutions to the ARE (2.17) corresponding to Problem (2.1)-(2.3), with the achievement of *uniqueness*, thereby concluding the proof of well-posedness of the ARE. Just as in the finite time horizon case, the (linear, bounded, self-adjoint) operators that belong to the class $\mathcal{Q}$ in (2.18) are characterized by a requirement that is consistent with the regularity property displayed by the gain operator in assertion A4. of Theorem 2.9.

Theorem 2.10 (**Uniqueness for the ARE).** *Consider the optimal control problem* (2.1)-(2.3)*, with $T = +\infty$, under the Assumptions 2.8. Then, the algebraic Riccati equation* (2.17) *has a unique solution P within the class $\mathcal{Q}$ defined as follows:*

$$\mathcal{Q} := \big\{Q \in \mathcal{L}(Y) \colon Q = Q^* \geq 0\,,\ B^*Q \in \mathcal{L}(\mathcal{D}(A^\epsilon), U)\big\}\,. \tag{2.18}$$

The optimal cost operator P defined by (2.15) *is consequently that solution.*

2.3. **An insight into the mathematical proofs.** We provide two proofs of Theorem 2.6, that establishes uniqueness for the DRE (2.10). This result is relevant for the optimal control problem on a finite time horizon (i.e. Problem 2.2 with $T < +\infty$), under the Assumptions 2.3.

The first proof, given in Section 3, follows the method employed in [21, Vol. I, Theorem 1.5.3.3; Vol. II, Theorems 8.3.7.1], up to a certain point. The basic rationale is standard: one proceeds by contradiction, assuming there exists another solution $P_1(t)$ to the DRE, besides the optimal cost operator $P(t)$ which is known to solve the DRE, as from S6. of Theorem 2.5. On the basis of the integral form of the DRE – in the present case two forms, derived in Lemma 3.1 –, one finds that the difference $Q(t) = P_1(t) - P(t)$ solves a suitable integral equation. It is in the estimates performed afterwards, that the paths diverge, with iiic) of the Assumptions 2.3 playing a major role here, together with the class of operators $P(t)$ and $P_1(t)$ belong to. The proofs carried out in the aforementioned results instead take advantage of either the enhanced regularity of the analytic semigroup e^{At} that describes the free dynamics, or the additional regularity assumed e.g. on the operator $R^*Re^{At}B$ (or on other combinations of R with e^{At} and/or B).

It is unlikely that the method of proof described above could be adjusted in order to establish uniqueness for the ARE, relevant for Problem 2.2 with $T = +\infty$. This owing to the argument employed when $T < +\infty$: $Q(t)$ is shown to be zero on some subinterval of $[0, T]$, with the soughtafter goal attained in a finite number of steps.

Other methods of proof are certainly worth to be explored. One might attempt to proceed along the lines of the proof of [21, Vol. I, Theorem 2.4.5], despite the absence of analyticity of the semigroup e^{At}, as well as of the optimal state semigroup

$\Phi(t)$. If this were the case, a preliminary analysis which appears unavoidable would pertain to issues connected to a given solution P_1 to the ARE (a priori, distinct from the optimal cost operator P): in particular, the possible generation of a C_0-semigroup on Y by the operator $A - BB^*P_1$, in turn to be suitably defined. We choose a different path, instead.

In order to prove Theorem 2.10, we borrow from the dynamic programming approach to the LQ-problem a key element in attaining that the optimal control admits a (pointwise in time) feedback representation. This element is fulfilled by the so called *fundamental identity*. In a direct approach, the fundamental identity builds a bridge between the nonlinear Riccati equation – whose well-posedness is studied in a first step, independently from the minimization problem, as recalled above – and the actual closed loop form of the optimal control. The latter goal (i.e. the feedback representation of the optimal control) was already attained in [2] and [4]; see the statements S4. of in Theorem 2.5 and A4. of Theorem 2.9, respectively. And yet, the identities we establish in Lemma 4.1 and Lemma 4.3 constitute a major (and technically nontrivial) step in our analysis, allowing to achieve uniqueness for both differential and algebraic Riccati equations, respectively. Theorem 2.6 and Theorem 2.10 are thus established, via methods of proof which are akin to each other.

3. A first proof of uniqueness for the DRE

In this Section we derive integral forms of both the differential and algebraic Riccati equations, and present a first proof of Theorem 2.6.

3.1. **Finite time interval, differential Riccati equations.** In this subsection we make reference to the optimal control problem (2.1)–(2.3), with $T < +\infty$. We address the issue of uniqueness of solutions to the Cauchy problem (2.10) for the Riccati equation corresponding to problem (2.1)–(2.3), under the Assumptions 2.3.

We begin by relating the differential form (2.10) of the Riccati equation to an integral form of it, which in turn can be further interpreted.

Lemma 3.1 (Integral forms of the Riccati equation)**.** *Let $\mathcal{Q}_T$ be the class defined in* (2.11)*, and let $Q(\cdot) \in \mathcal{Q}_T$ be a solution to the DRE* (2.10)*. Then the following assertions hold true.*

1. *$Q(\cdot)$ solves the integral Riccati equation (in short, IRE), that is*

$$\begin{aligned}\big(Q(t)e^{A(t-s)}x, e^{A(t-s)}y\big)_Y &= (Q(s)x,y)_Y - \int_s^t \big(Re^{A(r-s)}x, Re^{A(r-s)}y\big)_Z\,dr \\ &\quad + \int_s^t \big(B^*Q(r)e^{A(r-s)}x, B^*Q(r)e^{A(r-s)}y\big)_U\,dr\,,\end{aligned} \tag{3.1}$$

with $0 \le s \le t \le T$ and $x, y \in \mathcal{D}(A^\epsilon)$.

2. $B^*Q(\cdot)e^{A(\cdot-s)} \in \mathcal{L}(Y, L^2(s,T;U))$.

3. *The IRE* (3.1) *can be rewritten in the form*

$$\big(e^{A^*(t-s)}Q(t)e^{A(t-s)}x,y\big)_Y = (Q(s)x,y)_Y - \int_s^t \big(e^{A^*(r-s)}R^*Re^{A(r-s)}x,y\big)_Y\,dr + \int_s^t \big(e^{A^*(r-s)}Q(r)BB^*Q(r)e^{A(r-s)}x,y\big)_Y\,dr\,, \tag{3.2}$$

valid for any $x,y\in Y$ *and with* $0\le s\le t\le T$.

Proof. **1.** Let $x,y\in\mathcal{D}(A)$: then $e^{A\cdot}x$, $e^{A\cdot}y$ are differentiable, and therefore, using (2.10), there exists

$$\begin{aligned}
&\frac{d}{dr}\big(Q(r)e^{A(r-s)}x,e^{A(r-s)}y\big)_Y\\
&\quad=-\big(Q(r)e^{A(r-s)}x,Ae^{A(r-s)}y\big)_Y-\big(Ae^{A(r-s)}x,Q(r)e^{A(r-s)}y\big)_Y\\
&\qquad-\big(Re^{A(r-s)}x,Re^{A(r-s)}y\big)_Z+\big(B^*Q(r)e^{A(r-s)}x,B^*Q(r)e^{A(r-s)}y\big)_U\\
&\qquad+\big(Q(r)Ae^{A(r-s)}x,e^{A(r-s)}y\big)_Y+\big(Q(r)e^{A(r-s)}x,Ae^{A(r-s)}y\big)_Y\\
&\quad=-\big(Re^{A(r-s)}x,Re^{A(r-s)}y\big)_Z+\big(B^*Q(r)e^{A(r-s)}x,B^*Q(r)e^{A(r-s)}y\big)_U\,.
\end{aligned}$$

Integrating the above identity in $r\in[s,t]$, one readily obtains the IRE (3.1), valid for $x,y\in\mathcal{D}(A)$. In view of Lemma A.5, the validity of the IRE is extended to all $x,y\in\mathcal{D}(A^\epsilon)$ by density.

2. By taking now in (3.1) $t=T$, $x=y\in\mathcal{D}(A^\epsilon)$, since $P(T)=0$ we establish

$$\int_s^T \big\|B^*P(r)e^{A(r-s)}x\big\|_U^2\,dr \le \int_s^T \big\|Re^{A(r-s)}x\big\|_Z^2\,dr \le C\|x\|_Y^2$$

by density.

3. The equivalent form (3.2) of the IRE follows in view of **2.** and by density.

□

A first proof of Theorem 2.6. We follow the proof of Theorem 1.5.3.3 in [21], up to a point. The subsequent arguments and estimates are driven by the distinctive assumptions on the adjoint of the kernel $e^{At}B$, as well as by the different class of regularity the solutions to the DRE are sought.

We know already that the optimal cost operator $P(\cdot)$ defined by (2.8) solves (the Cauchy problem (2.10) for) the differential Riccati equation, as well as that $P\in\mathcal{Q}_T$. Assume there exists another operator in $\mathcal{Q}_T$, say $P_1(\cdot)$, which solves (2.10), and set $Q(t):=P_1(t)-P(t)$, $t\in[0,T]$; we aim to prove that $Q(t)\equiv 0$. By construction $Q(\cdot)\in\mathcal{Q}_T$. By Lemma 3.1, both $P_1(\cdot)$ and $P(\cdot)$ satisfy the IRE (3.1). Then, taking in particular $t=T$, we find that $Q(s)$ satisfies

$$(Q(s)x,y)_Y = -\int_s^T (B^*Q(r)e^{A(r-s)}x,B^*P_1(r)e^{A(r-s)}y)_U\,dr - \int_s^T (B^*P(r)e^{A(r-s)}x,B^*Q(r)e^{A(r-s)}y)_U\,dr\,, \tag{3.3}$$

for any $x, y \in D(A^\epsilon)$. To render the computations cleaner, set $V(r) := B^*Q(r)$ (that r belongs to $[s,T]$ is omitted here and below, as clear from the context). Because $Q(\cdot) \in \mathcal{Q}_T$, it holds $V(r)^* \in \mathcal{L}(U, [\mathcal{D}(A^\epsilon)]')$, along with

$$\|V(r)^*\|_{\mathcal{L}(U,[\mathcal{D}(A^\epsilon)]')} = \|V(r)\|_{\mathcal{L}(\mathcal{D}(A^\epsilon),U)} \le \|V(\cdot)\|_{\mathcal{L}(\mathcal{D}(A^\epsilon),C([s,T];U)} =: c\,.$$

We see that

$$\big|\langle V(r)^* w, y\rangle_{[\mathcal{D}(A^\epsilon)]',\mathcal{D}(A^\epsilon)}\big| \le c\,\|w\|_U\|y\|_{D(A^\epsilon)}$$

consequently, as well as that $A^{*-\epsilon}V(r)^* \in \mathcal{L}(U,Y)$, with

$$\big|\big(A^{*-\epsilon}V(r)^*w, x\big)_Y\big| = \big|\langle V(r)^*w, A^{-\epsilon}x\rangle_{[\mathcal{D}(A^\epsilon)]',\mathcal{D}(A^\epsilon)}\big| \le c\,\|w\|_U\|x\|_Y\,.$$

The same observations apply to $[B^*P_1(r)]^*$ and $[B^*P(r)]^*$, bringing about analogous estimates.

We may now rewrite (3.3) as

$$\begin{aligned}(Q(s)x,y)_Y = &-\int_s^T (e^{A^*(r-s)}A^{*-\epsilon}[B^*P_1(r)]^*V(r)e^{A(r-s)}x, A^\epsilon y)_Y\,dr \\ &-\int_s^T (e^{A^*(r-s)}A^{*-\epsilon}V(r)^*B^*P(r)e^{A(r-s)}x, A^\epsilon y)_Y\,dr\,,\end{aligned}$$

which tells us that

$$\begin{aligned}A^{*\epsilon}\int_s^T \Big[&e^{A^*(r-s)}A^{*-\epsilon}[B^*P_1(r)]^*V(r)e^{A(r-s)} \\ &+ e^{A^*(r-s)}A^{*-\epsilon}V(r)^*B^*P(r)e^{A(r-s)}\Big]x\,dr\,,\end{aligned}$$

a priori an element of $[\mathcal{D}(A^\epsilon)]'$, in fact coincides with $-Q(s)x \in Y$ by the very definition of adjoint operator. We deduce

$$\begin{aligned}Q(s)x = -A^{*\epsilon}\int_s^T \Big[&e^{A^*(r-s)}A^{*-\epsilon}[B^*P_1(r)]^*V(r)e^{A(r-s)} \\ &+ e^{A^*(r-s)}A^{*-\epsilon}V(r)^*B^*P(r)e^{A(r-s)}\Big]x\,dr\,,\end{aligned} \tag{3.4}$$

valid for every $x \in D(A^\epsilon)$, where, as pointed out above, the right hand side is an element of Y. As $x \in \mathcal{D}(A^\epsilon)$, $B^*Q(s)x$ is meaningful, and we are allowed to apply B^* to both sides of (3.4), thus obtaining

$$\begin{aligned}V(s)x = -B^*A^{*\epsilon}\int_s^T \Big[&e^{A^*(r-s)}A^{*-\epsilon}[B^*P_1(r)]^*V(r)e^{A(r-s)} \\ &+ e^{A^*(r-s)}A^{*-\epsilon}V(r)^*B^*P(r)e^{A(r-s)}\Big]x\,dr\,.\end{aligned} \tag{3.5}$$

It is here where iiic) of Assumptions 2.3, that is

$$\exists\, q \in (1,2)\,,\ C = C(T) > 0\colon \quad \|B^*e^{A^*(\cdot-s)}A^{*\epsilon}x\|_{L^q(s,T;U)} \le C\,\|x\|_Y \qquad \forall x \in Y\,,$$

becomes crucially important: indeed, it yields as well

$$\|[B^*e^{A^*(\cdot-s)}A^{*\epsilon}]^*g(\cdot)\|_Y \le C\,\|g\|_{L^{q'}(s,T;U)}$$

(q' denotes the conjugate exponent of q), so that in particular

$$\|[B^*e^{A^*(\cdot-s)}A^{*\epsilon}]^*w\|_Y \le C\,(T-s)^{1/q'}\|w\|_U \qquad \forall w \in U\,. \tag{3.6}$$

We return to (3.5), and highlight a few blocks within its right hand side, as follows:

$$V(s)x = -\int_s^T \left[B^* e^{A^*(r-s)} A^{*\epsilon}\right] A^{*-\epsilon} [B^* P_1(r)]^* V(r) e^{A(r-s)} x\, dr$$
$$- \int_s^T \left[B^* e^{A^*(r-s)} A^{*\epsilon}\right] A^{*-\epsilon} V(r)^* B^* P(r) e^{A(r-s)} x\, dr\,;$$

multiply next both members by $w \in U$, to find

$$(V(s)x, w)_U = -\int_s^T \big(V(r) e^{A(r-s)} x, [B^* P_1(r) A^{-\epsilon}] \left[B^* e^{A^*(r-s)} A^{*\epsilon}\right]^* w\big)_U\, dr$$
$$- \int_s^T \big(B^* P(r) e^{A(r-s)} x, [V(r) A^{-\epsilon}] \left[B^* e^{A^*(r-s)} A^{*\epsilon}\right]^* w\big)_U\, dr\,.$$

We now proceed to estimate either summand in the right hand side, making use of (3.6); this leads to

$$\begin{aligned}
&|(V(s)x, w)_U| \\
&\le M\, \|V(\cdot)\|_{\mathcal{L}(\mathcal{D}(A^\epsilon), C([s,T];U))} \|x\|_{\mathcal{D}(A^\epsilon)} \|B^* P_1(\cdot)\|_{\mathcal{L}(\mathcal{D}(A^\epsilon), C([s,T];U))} \|w\|_U (T-s)^{1/q'} \\
&\quad + M\, \|B^* P(\cdot)\|_{\mathcal{L}(\mathcal{D}(A^\epsilon), C([s,T];U))} \|x\|_{\mathcal{D}(A^\epsilon)} \|V(\cdot)\|_{\mathcal{L}(\mathcal{D}(A^\epsilon), C([s,T];U))} \|w\|_U (T-s)^{1/q'}\,.
\end{aligned}$$

Therefore, there exists a positive constant C (depending on P and P_1) such that

$$\big|(V(s)x, w)_U\big| \le C\, \|V(\cdot)\|_{\mathcal{L}(\mathcal{D}(A^\epsilon), C([s,T];U))} (T-s)^{1/q'} \|w\|_U \|x\|_{\mathcal{D}(A^\epsilon)}\,,$$

which establishes

$$\|V(s)\|_{\mathcal{L}(\mathcal{D}(A^\epsilon), U)} \le C\, \|V(\cdot)\|_{\mathcal{L}(\mathcal{D}(A^\epsilon), C([s,T];U))} (T-s)^{1/q'}\,, \tag{3.7}$$

for any $s \in [0, T)$.

The argument is now pretty standard: set s_0 such that $(T - s_0)^{1/q'} < 1/C$; since the estimate (3.7) holds true in particular for any $s \in [s_0, T)$, we have

$$\|V(\cdot)\|_{\mathcal{L}(\mathcal{D}(A^\epsilon), C([s_0,T];U))} \le C\, (T - s_0)^{1/q'}\, \|V(\cdot)\|_{\mathcal{L}(\mathcal{D}(A^\epsilon), C([s_0,T];U))}$$

which is impossible unless $V(\cdot) \equiv 0$ on $[s_0, T]$. Iterating the same argument, in a finite number of steps we obtain $V(s) \equiv 0$ on $[0, T]$. This in turn implies, by (3.3),

$$(Q(s)x, y)_Y = 0 \qquad \forall s \in [0, T]\,,\ \forall x, y \in \mathcal{D}(A^\epsilon)\,;$$

by density we obtain $(Q(s)x = 0$ for any $x \in Y$ first, and then) $Q(\cdot) \equiv 0$, that is $P_1(\cdot) \equiv P(\cdot)$, as desired.

□

3.2. **Infinite time interval. Preparatory material.** We turn now our attention to the optimal control problem (2.1)–(2.3), with $T = +\infty$, and to the ARE. The following Lemma contributes to the preparatory material for the forthcoming analysis in Section 4, that follows a different line of argument than the one utilized to prove uniqueness for the DRE. Since an integral form of the ARE will prove more effective (than its algebraic form) to accomplish our goal – just like the integral forms of the DRE in Lemma 3.1 provide fundamental tools for both proofs of Theorem 2.6 –, we derive the said integral form of the ARE here. Its proof is not difficult, yet it is explicitly given for the reader's convenience.

Lemma 3.2 (Integral form of the ARE)**.** *Let $\mathcal{Q}$ be the class defined in* (2.18)*, and let $P_1 \in \mathcal{Q}$ be a solution to the algebraic Riccati equation* (2.17)*. Then, P_1 solves the following integral form of the ARE valid for all $x, y \in \mathcal{D}(A^\epsilon)$:*

$$\begin{aligned}\big(P_1 e^{A(t-s)}x, e^{A(t-s)}y\big)_Y = (P_1 x, y)_Y &+ \int_s^t \big(B^* P_1 e^{A(r-s)}x, B^* P_1 e^{A(r-s)}y\big)_U \, dr \\ &- \int_s^t \big(R e^{A(r-s)}x, R e^{A(r-s)}y\big)_Z \, dr\,,\end{aligned} \tag{3.8}$$

with $0 \le s \le t$.

Proof. Let $P_1 \in \mathcal{Q}$ be a solution to the ARE (2.17), that we record for the reader's convenience:

$$(P_1 x, Ay)_Y + (Ax, P_1 y)_Y - (B^* P_1 x, B^* P_1 y)_U + (Rx, Ry)_Z = 0\,, \quad x, y \in \mathcal{D}(A)\,.$$

With $e^{A(t-s)}x$, $e^{A(t-s)}y \in \mathcal{D}(A)$ in place of x, y, and with $0 \le s \le t$, the equation becomes

$$\begin{aligned}&(P_1 e^{A(t-s)}x, Ae^{A(t-s)}y)_Y + (Ae^{A(t-s)}x, P_1 e^{A(t-s)}y)_Y \\ &\quad - (B^* P_1 e^{A(t-s)}x, B^* P_1 e^{A(t-s)}y)_U + (Re^{A(t-s)}x, Re^{A(t-s)}y)_Z = 0\,,\end{aligned}$$

that is nothing but

$$\begin{aligned}\frac{d}{dt}\big(P_1 e^{A(t-s)}x, e^{A(t-s)}y\big)_Y &= \big(B^* P_1 e^{A(t-s)}x, B^* P_1 e^{A(t-s)}y\big)_U \\ &\quad - \big(Re^{A(t-s)}x, Re^{A(t-s)}y\big)_Z\,, \quad x, y \in \mathcal{D}(A)\,.\end{aligned} \tag{3.9}$$

Integrating both sides of (3.9) between s and t we attain (3.8), initially for any $x, y \in \mathcal{D}(A)$. Its validity is then extended to all $x, y \in \mathcal{D}(A^\epsilon)$ by density, since $P_1 \in \mathcal{Q}$. □

While the integral form (3.8) of the ARE will constitute the starting point for the proof of Theorem 2.10, it is important to emphasize the central role of the distinguishing (and improved) regularity properties of the operator $B^* e^{A^* \cdot} A^{*\epsilon}$. We refer the reader to Appendix A, where we collected and highlighted several instrumental results, with the aim of displaying their statements in a clear sequence and framework. See, more specifically, Proposition A.6 therein.

4. A unified method of proof of uniqueness for both DRE and ARE

In this Section we provide a second proof of Theorem 2.6 and then show Theorem 2.10, thereby settling the question of uniqueness for the differential and algebraic Riccati equations corresponding to the optimal control problem (2.2)–(2.3). We recall from Section 2.3 that a crucial intermediate step to achieve either goal is an identity which is classical in control theory.

4.1. **Finite time interval, differential Riccati equations.** In this subsection we focus on the optimal control problem (2.1)–(2.3), with $T < +\infty$, along with the corresponding Riccati equation. In approaching the second proof of Theorem 2.6, we start by showing the above-mentioned fundamental identity. Despite being a standard element in classical optimal control theory, the identity should not be taken for granted in the absence of evident beneficial regularity properties of the kernel $e^{At}B$ – such as analiticity of the semigroup or more generally singular estimates. Achieving the said equality requires that the Assumptions 2.3 are fully exploited. The delicate, careful computations are carried out in the following Lemma.

Lemma 4.1 (Fundamental identity)**.** *Let $Q \in \mathcal{Q}_T$ be a solution to the integral Riccati equation* (3.1)*. With $u \in L^2(s,T;U)$ and $x \in \mathcal{D}(A^\epsilon)$, let $y(\cdot)$ be the semigroup solution to the state equation in* (2.1) *corresponding to $u(\cdot)$, with $y(s) = x$, that is*

$$y(t) = e^{A(t-s)}x + \int_s^t e^{A(t-r)}Bu(r)\,dr = e^{A(t-s)}x + L_s u(t)\,, \quad t \in [s,T]\,.$$

Then, the following identity is valid: for $t \in [s,T]$

$$\begin{aligned}(Q(t)y(t),y(t))_Y - (Q(s)x,x)_Y = &-\int_s^t \left[\|Ry(r)\|_Z^2 + \|u(r)\|_U^2\right]\,dr \\ &+ \int_s^t \|u(r) + B^*Q(r)y(r)\|_U^2\,dr\,.\end{aligned} \tag{4.1}$$

Proof. Assume initially that $u \in L^\infty(s,T;U)$. We examine the right hand side of the identity (4.1). For the first term we have

$$\begin{aligned}-\int_s^t \|Ry(r)\|_Z^2\,dr = &-\int_s^t \left\|Re^{A(r-s)}x\right\|_Z^2 dr - \int_s^t \left\|RL_s u(r)\right\|_Z^2 dr \\ &- 2\mathrm{Re}\int_s^t \left(Re^{A(r-s)}x, RL_s u(r)\right)_Z dr =: \sum_{j=1}^3 R_j\,.\end{aligned}$$

We note that each summand R_j makes sense, just considering the space regularity originally singled out in [2] and here recalled in Proposition A.1; more specifically, $u \in L^\infty(s,T;U)$ implies $L_s u \in C([s,T];Y)$ by its fourth assertion. We consider next the remainder

$$-\int_s^t \|u(r)\|_U^2\,dr + \int_s^t \|u(r) + B^*Q(r)y(r)\|_U^2\,dr\,.$$

Computing the square in the second integral, discarding additive inverses and replacing again the expression of $y(r)$, we get

$$\begin{aligned}
&-\int_s^t \|u(r)\|_U^2\,dr + \int_s^t \|u(r)+B^*Q(r)y(r)\|_U^2\,dr \\
&\quad = 2\mathrm{Re}\int_s^t \big(B^*Q(r)e^{A(r-s)}x, u(r)\big)_U\,dr + 2\mathrm{Re}\int_s^t \big(B^*Q(r)L_s u(r), u(r)\big)_U\,dr \\
&\qquad + \int_s^t \big\|B^*Q(r)e^{A(r-s)}x\big\|_U^2\,dr + 2\mathrm{Re}\int_s^t \big(B^*Q(r)e^{A(r-s)}x, B^*Q(r)L_s u(r)\big)_U\,dr \\
&\qquad + \int_s^t \big\|B^*Q(r)L_s u(r)\big\|_U^2\,dr =: \sum_{j=1}^5 C_j\,.
\end{aligned}$$

That each summand C_j makes sense as well is justified by the following observations: $B^*Q(\cdot)e^{A(\cdot-s)}x \in L^2(s,T;U)$ because of item 2. of Lemma 3.1; in addition, since $L^\infty(s,T;U)\subset L^{q'}(s,T;U)$, Lemma A.2 yields the improved regularity $L_s u\in C([s,T];\mathcal{D}(A^\epsilon))$, which in turn implies $B^*Q(\cdot)L_s u(\cdot)\in C([s,T];U)$, as shown in Lemma A.5.

By using the original form (3.1) of the integral Riccati equation (IRE), with $x=y$, we find that

$$\begin{aligned}
R_1 + C_3 &= -\int_s^t \|Re^{A(r-s)}x\|_Z^2\,dr + \int_s^t \big\|B^*Q(r)e^{A(r-s)}x\big\|_U^2\,dr \\
&= \big(Q(t)e^{A(t-s)}x, e^{A(t-s)}x\big)_Y - (Q(s)x,x)_Y\,.
\end{aligned} \tag{4.2}$$

Next,

$$\begin{aligned}
R_3 + C_4 + C_1 &= -2\mathrm{Re}\int_s^t \big(Re^{A(r-s)}x, RL_s u(r)\big)_Z\,dr \\
&\qquad + 2\mathrm{Re}\int_s^t \big(B^*Q(r)e^{A(r-s)}x, B^*Q(r)L_s u(r)\big)_U\,dr \\
&\qquad + 2\mathrm{Re}\int_s^t \big(B^*Q(r)e^{A(r-s)}x, u(r)\big)_U\,dr \\
&= -2\mathrm{Re}\int_s^t \Big(R^*Re^{A(r-s)}x, \int_s^r e^{A(r-\sigma)}Bu(\sigma)\,d\sigma\Big)_Y\,dr \\
&\qquad + 2\mathrm{Re}\int_s^t \Big\langle Q(r)BB^*Q(r)e^{A(r-s)}x, \int_s^r e^{A(r-\sigma)}Bu(\sigma)\,d\sigma\Big\rangle_{[\mathcal{D}(A^\epsilon)]',\mathcal{D}(A^\epsilon)}\,dr \\
&\qquad + 2\mathrm{Re}\int_s^t \big(B^*Q(r)e^{A(r-s)}x, u(r)\big)_U\,dr\,,
\end{aligned}$$

where the duality in the penultimate term is based on the membership $Q(\cdot)\in\mathcal{Q}_T$ (along with the estimate (A.2)) which yields $Q(r)B\in\mathcal{L}(U,[\mathcal{D}(A^\epsilon)]')$, combined as before with $L_s u \in C([s,T];\mathcal{D}(A^\epsilon))$. The above leads to

$$\begin{aligned}
R_3 + C_4 + C_1 = -2\mathrm{Re} & \int_s^t \int_s^r \big(B^* e^{A^*(r-\sigma)} R^* R e^{A(r-s)} x, u(\sigma)\big)_U \, d\sigma \, dr \\
& + 2\mathrm{Re} \int_s^t \int_s^r \big(B^* e^{A^*(r-\sigma)} Q(r) B B^* Q(r) e^{A(r-s)} x, u(\sigma)\big)_U \, d\sigma dr \\
& + 2\mathrm{Re} \int_s^t (B^* Q(\sigma) e^{A(\sigma-s)} x, u(\sigma))_U \, d\sigma \,,
\end{aligned}$$

which can be rewritten, exchanging the order of integration, as follows:

$$\begin{aligned}
R_3 + C_4 + C_1 = -\, 2\mathrm{Re} \int_s^t \Big(B^* \Big\{ & \int_\sigma^t e^{A^*(r-\sigma)} R^* R \big[e^{A(r-s)} x\big] \, dr \\
& - \int_\sigma^t e^{A^*(r-\sigma)} Q(r) B B^* Q(r) e^{A(r-\sigma)} \big[e^{A(\sigma-s)} x\big] \, dr \\
& - Q(\sigma) \big[e^{A(\sigma-s)} x\big] \Big\}, u(\sigma)\Big)_U \, d\sigma \,.
\end{aligned} \tag{4.3}$$

Let us focus on the expression inside the curly bracket. Because $Q(\cdot)$ solves the IRE (3.1), as well as its second form (3.2) valid for any pair $x, y \in Y$, then the following identity – a *strong* form of the IRE, when $Q(\cdot)$ is unknown – holds true:

$$\begin{aligned}
e^{A^*(t-\sigma)} Q(t) e^{A(t-\sigma)} z = Q(\sigma) z - & \int_\sigma^t e^{A^*(r-\sigma)} R^* R z \, dr \\
& + \int_\sigma^t e^{A^*(r-\sigma)} Q(r) B B^* Q(r) e^{A(r-\sigma)} z \, dr \,, \quad 0 \le \sigma \le t \le T \,, \ z \in Y \,.
\end{aligned}$$

Thus, returning to (4.3) with this information and setting in particular $z = e^{A(\sigma-s)} x$, we find that $R_3 + C_4 + C_1$ simply reads as follows:

$$\begin{aligned}
R_3 + C_4 + C_1 &= -2\mathrm{Re} \int_s^t \big(B^* e^{A^*(t-\sigma)} Q(t) e^{A(t-s)} x, u(\sigma)\big)_U \, d\sigma \\
&= -2\mathrm{Re}\big(Q(t) e^{A(t-s)} x, L_s u(t)\big)_Y \,.
\end{aligned} \tag{4.4}$$

We examine next the sum

$$R_2 + C_5 = -\int_s^t \|R L_s u(r)\|_Z^2 \, dr + \int_s^t \big\|B^* Q(r) L_s u(r)\big\|_U^2 \, dr \,,$$

where, again, since $u \in L^\infty(s,T;U) \subset L^{q'}(s,T;U)$, we know from Lemma A.2 that $L_s u \in C([s,T];\mathcal{D}(A^\epsilon))$. Consequently, one gets

$$\begin{aligned}
R_2 + C_5 &= -\int_s^t \big\langle \big[R^* R - Q(r) B B^* Q(r)\big] L_s u(r), L_s u(r)\big\rangle_{[\mathcal{D}(A^\epsilon)]', \mathcal{D}(A^\epsilon)} \, dr \\
&= -\mathrm{Re} \int_s^t \Big(A^{*-\epsilon} \big[R^* R - Q(r) B B^* Q(r)\big] A^{-\epsilon} \int_s^r A^\epsilon e^{A(r-\lambda)} B u(\lambda) \, d\lambda, \\
&\qquad\qquad \int_s^r A^\epsilon e^{A(r-\mu)} B u(\mu) \, d\mu\Big)_Y \, dr \,.
\end{aligned} \tag{4.5}$$

It is important to emphasize that in going from the duality to the inner product in (4.5), two facts have been crucially used, besides $Q(\cdot) \in \mathcal{Q}_T$: the hypothesis (2.3) on the observation operator R (that is iiib) of the Assumptions 2.3), and once again, Lemma A.2. Further handling of the right hand side of (4.5) leads to the triple integral

$$R_2 + C_5 = -\mathrm{Re} \int_s^t I(r,s)\, dr\,,$$

having set

$$I(r,s) = \int_s^r \int_s^r \big(B^* e^{A^*(r-\mu)} \big[R^*R - Q(r)BB^*Q(r)\big] e^{A(r-\lambda)} Bu(\lambda), u(\mu)\big)_U \, d\lambda\, d\mu\,.$$

Let us focus on the inner double integral $I(r,s)$. We note that this integral pertains to a symmetric function of (λ,μ), and hence the integral over the square $[s,r]\times[s,r]$ can be replaced by twice the integral over the triangle

$$\{(\lambda,\mu)\colon\, s\le \mu\le\lambda\le r\}\,.$$

It follows that

$$\begin{aligned}
I(r,s) &= 2\int_s^r d\lambda \int_s^\lambda d\mu \,\big(B^* e^{A^*(r-\mu)} \big[R^*R - Q(r)BB^*Q(r)\big] e^{A(r-\lambda)} Bu(\lambda), u(\mu)\big)_U \\
&= 2\int_s^r \Big[\int_s^\lambda \big(B^* e^{A^*(\lambda-\mu)}\, e^{A^*(r-\lambda)} \big[R^*R - Q(r)BB^*Q(r)\big] e^{A(r-\lambda)} Bu(\lambda), u(\mu)\big)_U \, d\mu \Big]\, d\lambda \\
&= 2\int_s^r \big(e^{A^*(r-\lambda)} \big[R^*R - Q(r)BB^*Q(r)\big] e^{A(r-\lambda)} Bu(\lambda), \int_s^\lambda e^{A(\lambda-\mu)} Bu(\mu)\, d\mu\big)_Y \, d\lambda \\
&= 2\int_s^r \big(e^{A^*(r-\lambda)} \big[R^*R - Q(r)BB^*Q(r)\big] e^{A(r-\lambda)} Bu(\lambda), L_s u(\lambda)\big)_Y \, d\lambda\,.
\end{aligned}$$

Inserting the expression of $I(r,s)$ obtained above in the outer integral yields

$$\begin{aligned}
&R_2 + C_5 \\
&\quad = -2\mathrm{Re} \int_s^t \int_s^r \big(e^{A^*(r-\lambda)} \big[R^*R - Q(r)BB^*Q(r)\big] e^{A(r-\lambda)} Bu(\lambda), L_s u(\lambda)\big)_Y d\lambda\, dr\,;
\end{aligned}$$

next we exchange the order of integration and also move the first argument of the inner product, to achieve

$$\begin{aligned}
&R_2 + C_5 \\
&\quad = -2\mathrm{Re} \int_s^t \int_\lambda^t \big(e^{A^*(r-\lambda)} \big[R^*R - Q(r)BB^*Q(r)\big] e^{A(r-\lambda)} Bu(\lambda), L_s u(\lambda)\big)_Y dr\, d\lambda \\
&\quad = -2\mathrm{Re} \int_s^t \int_\lambda^t \big(u(\lambda), B^* e^{A^*(r-\lambda)} \big[R^*R - Q(r)BB^*Q(r)\big] e^{A(r-\lambda)} L_s u(\lambda)\big)_Y dr\, d\lambda \\
&\quad = -2\mathrm{Re} \int_s^t \big(u(\lambda), B^* \int_\lambda^t e^{A^*(r-\lambda)} \big[R^*R - Q(r)BB^*Q(r)\big] e^{A(r-\lambda)}\, L_s u(\lambda)\, dr\big)_U \, d\lambda\,.
\end{aligned} \tag{4.6}$$

It is apparent that the second form (3.2) of the IRE (with λ in place of s) – in fact, a strong form of it – provides once more the tool, just like in deriving (4.4) from (4.3). With $z = L_s u(\lambda)$, replace the integral

$$\int_\lambda^t e^{A^*(r-\lambda)} \big[R^*R - Q(r)BB^*Q(r)\big] e^{A(r-\lambda)} z\, dr$$

by $[Q(\lambda) - e^{A^*(t-\lambda)}Q(t)e^{A(t-\lambda)}]z$, to find

$$
\begin{aligned}
R_2 + C_5 &= -2\mathrm{Re} \int_s^t \big(u(\lambda), B^*\big[Q(\lambda) - e^{A^*(t-\lambda)}Q(t)e^{A(t-\lambda)}\big]L_s u(\lambda)\big)_U \, d\lambda \\
&= -2\mathrm{Re} \int_s^t \big(B^*\big[Q(\lambda) - e^{A^*(t-\lambda)}Q(t)e^{A(t-\lambda)}\big]L_s u(\lambda), u(\lambda)\big)_U \, d\lambda \,.
\end{aligned}
\tag{4.7}
$$

Thus, adding C_2 to (4.7), we see that a useful simplification occurs, as detailed below:

$$
\begin{aligned}
R_2 + C_5 + C_2 = -2\mathrm{Re} &\int_s^t \big(B^*Q(\lambda)L_s u(\lambda), u(\lambda)\big)_U \, d\lambda \\
&+ 2\mathrm{Re} \int_s^t \big(B^* e^{A^*(t-\lambda)}Q(t)e^{A(t-\lambda)}L_s u(\lambda), u(\lambda)\big)_U \, d\lambda \\
&+ 2\mathrm{Re} \int_s^t \big(B^*Q(r)L_s u(r), u(r)\big)_U \, dr \\
= 2\mathrm{Re} &\int_s^t \int_s^\lambda \big(Q(t)e^{A(t-\sigma)}Bu(\sigma), e^{A(t-\lambda)}Bu(\lambda)\big)_Y \, d\sigma \, d\lambda \,.
\end{aligned}
$$

Owing to the simmetry of the latter integrand in (σ, λ), we may replace twice the integral over the triangle $\{(\lambda, \sigma) \colon s \le \lambda \le \sigma \le t\}$ by the integral over the square $[s,t] \times [s,t]$, and finally get

$$
\begin{aligned}
R_2 + C_5 + C_2 &= \mathrm{Re} \int_s^t \int_s^t \big(Q(t)e^{A(t-\sigma)}Bu(\sigma), e^{A(t-\lambda)}Bu(\lambda)\big)_Y \, d\lambda \, d\sigma \\
&= \mathrm{Re}\big(Q(t)L_s u(t), L_s u(t)\big)_Y = \big(Q(t)L_s u(t), L_s u(t)\big)_Y \,.
\end{aligned}
\tag{4.8}
$$

Combining (4.8) with (4.2) and (4.4), we finally obtain

$$
\begin{aligned}
\sum_{i=1}^3 R_i + \sum_{j=1}^5 C_j &= \big(Q(t)e^{A(t-s)}x, e^{A(t-s)}x\big)_Y - (Q(s)x, x)_Y \\
&\quad + 2\mathrm{Re}(Q(t)e^{A(t-s)}x, L_s u(t))_Y + (Q(t)L_s u(t), L_s u(t))_Y \\
&= (Q(t)y(t), y(t))_Y - (Q(s)x, x)_Y \,,
\end{aligned}
$$

which establishes the fundamental identity (4.1) in the case $u \in L^\infty(s, T; U)$. Finally, the identity extends to $u \in L^2(s, T; U)$ by density, which concludes the proof of Lemma 4.1.

□

We next introduce an integral equation that involves a given operator solution $Q(t)$ to the Riccati equation corresponding to optimal control problem (2.1)–(2.3). Once uniqueness for the DRE (2.10) is established, so that $Q(t)$ must coincide with the Riccati operator $P(t)$, then it will be clear that the said integral equation ((4.9) below) is nothing but the well known *closed-loop equation*, of central importance for the synthesis of the optimal control. (This justifies the use of the term "closed-loop equation" for (4.9)).

As we shall see, the following Lemma 4.2 and (the independent) Lemma 4.1 constitute the core elements for the proof of Theorem 2.6.

Lemma 4.2. *Let ϵ be as in iii) of Assumptions 2.3. Let $Q \in \mathcal{Q}_T$, where $\mathcal{Q}_T$ is the class defined by (2.11). Then, for every $x \in \mathcal{D}(A^\epsilon)$, the closed loop equation*

$$y(t) = e^{At}x - \int_0^t e^{A(t-s)}BB^*Q(\sigma)y(\sigma)\,d\sigma\,, \qquad t \in [0,T]\,, \tag{4.9}$$

has a unique solution in the space

$$X = \Big\{ y \in C([0,T];\mathcal{D}(A^\epsilon))\colon \sup_{t\in[0,T]} \big(e^{-rt}\|y(t)\|_{\mathcal{D}(A^\epsilon)}\big) < \infty \Big\} \tag{4.10}$$

endowed with the norm

$$\|y\|_{X,r} = \sup_{t\in[0,T]} e^{-rt}\|y(t)\|_{\mathcal{D}(A^\epsilon)}\,, \qquad y \in X\,,$$

provided $r > 0$ is chosen sufficiently large.

Proof. With $x \in \mathcal{D}(A^\epsilon)$, we set $E(t) = e^{At}x$. By semigroup theory we know that $E(\cdot) \in C([0,T];\mathcal{D}(A^\epsilon))$; even more, since e^{At} is exponentially stable, it holds $E(\cdot) \in X$ provided r is sufficiently large. As the integral equation (4.9) has the clear form

$$y(t) + \big[LB^*Q(\cdot)y(\cdot)\big](t) = E(t)\,, \quad t \in [0,T]\,,$$

we appeal to a classical argument of functional analysis: we will prove that $LB^*Q(\cdot)$ is a contraction mapping in X, having chosen r sufficiently large. This will in turn imply that $I+LB^*Q(\cdot)$ is invertible in X, thus providing the sought unique solution to (4.9).

For each $y \in X$, $z \in \mathcal{D}(A^{*\epsilon})$, $t \in [0,T]$, we have by Lemma A.5

$$\begin{aligned}
&\big|(e^{-rt}LB^*Q(\cdot)y(\cdot), A^{*\epsilon}z)_Y\big| \\
&\quad = \Big|\int_0^t e^{-r(t-s)}\,\big(B^*Q(s)e^{-rs}y(s), B^*e^{A^*(t-s)}A^{*\epsilon}z\big)_U\,ds\Big| \\
&\quad \le \int_0^t e^{-r(t-s)}\|B^*Q(\cdot)e^{-r\cdot}y(\cdot)\|_{C([0,T];U)}\,\big\|B^*e^{(t-s)A^*}A^{*\epsilon}z\big\|_U\,ds \\
&\quad \le \|B^*Q(\cdot)\|_{\mathcal{L}(C([0,T];\mathcal{D}(A^\epsilon)),C([0,T];U))}\,\|y\|_{X,r}\int_0^t e^{-r\sigma}\|B^*e^{A^*\sigma}A^{*\epsilon}z\|_U\,d\sigma \\
&\quad \le \|B^*Q(\cdot)\|_{\mathcal{L}(C([0,T];\mathcal{D}(A^\epsilon)),C([0,T];U))}\,\|y\|_{X,r}\,\Big[\int_0^t e^{-r\sigma q'}\,d\sigma\Big]^{1/q'}\|B^*e^{\cdot A^*}A^{*\epsilon}z\|_{L^q(0,T;U)} \\
&\quad \le \|B^*Q(\cdot)\|_{\mathcal{L}(C([0,T];\mathcal{D}(A^\epsilon)),C([0,T];U))}\,\frac{1}{(rq')^{1/q'}}\,\|B^*e^{A^*\cdot}A^{*\epsilon}\|_{\mathcal{L}(Y,L^q(0,T;U))}\,\|z\|_Y\,\|y\|_{X,r}\,.
\end{aligned}$$

We note that in going from the antepenultimate to the penultimate estimate we used iiic) of the Assumptions 2.3. Therefore, there exist positive constants c, c' such that

$$\big\|e^{-rt}\big[L\,B^*Q(\cdot)y(\cdot)\big](t)\big\|_{\mathcal{D}(A^\epsilon)} \le \frac{c}{(rq')^{1/q'}}\,\|y\|_{X,r} \le \frac{c'}{r^{1/q'}}\,\|y\|_{X,r}\,,$$

so that by taking a sufficiently large r we see that $LB^*Q(\cdot)$ is a contraction mapping in X. The conclusion of the Lemma follows. $\square$

Uniqueness for the DRE is now a consequence of Lemmas 4.1 and 4.2, its proof following a somewhat familiar path.

Proof of Theorem 2.6. For the optimal pair $(\hat{y}, \hat{u})$ corresponding to the initial state $x \in Y$ it holds

$$(P(s)x, x)_Y = J(\hat{u}) = \int_s^T \Big(\|R\hat{y}(r)\|_Z^2 + \|\hat{u}(r)\|_U^2\Big)\, dr\,, \quad 0 \le s \le T\,,$$

where $P(\cdot)$ is the Riccati operator defined in (2.8), i.e.

$$P(t)x = \int_t^T e^{A^*(r-t)} R^* R\Phi(r,t)x\, dr\,, \qquad x \in Y\,,$$

while $\Phi(r,t)$ denotes the evolution operator

$$\Phi(r,t)x = e^{A(r-t)}x + L_t\hat{u}(r)\,, \qquad r \in [t,T]\,.$$

Let $Q(\cdot) \in \mathcal{Q}_T$ be another solution to the DRE (2.10): by Lemma 3.1 $Q(\cdot)$ solves the IRE (3.1) as well; then, with $u \in L^2(s,T;U)$ and $x \in \mathcal{D}(A^\epsilon)$, the identity (4.1) holds true by Lemma 4.1. With $t = T$, since $Q(T) = 0$, from (4.1) we see that

$$\begin{aligned}(Q(s)x, x)_Y &= \int_s^T \big[\|Ry(r)\|_Z^2 + \|u(r)\|_U^2\big]\, dr - \int_s^T \|u(r) + B^*Q(r)y(r)\|_U^2\, dr \\ &\le \int_s^T \big[\|Ry(r)\|_Z^2 + \|u(r)\|_U^2\big]\, dr = J(u)\,.\end{aligned}$$

In particular, when $u = \hat{u}$, we establish

$$(Q(s)x, x)_Y \le J(\hat{u}) = (P(s)x, x)_Y \qquad \forall s \in [0,T]\,,\ \forall x \in \mathcal{D}(A^\epsilon)\,. \tag{4.11}$$

Conversely, let $y(\cdot)$ be the solution to the closed-loop equation (4.9) corresponding to $x \in \mathcal{D}(A^\epsilon)$, guaranteed by Lemma 4.2, and let $u(\cdot) = -B^*Q(\cdot)y(\cdot)$. By construction $u \in L^2(s,T;U)$, and the fundamental identity becomes

$$(Q(s)x, x)_Y = \int_s^t \Big(\|Ry(r)\|_Z^2 + \|u(r)\|_U^2\Big)\, dr + (Q(t)y(t), y(t))_Y\,,$$

which in turn gives, for $t = T$,

$$(Q(s)x, x)_Y = J(u) \ge J(\hat{u}) = (P(s)x, x)_Y \qquad \forall s \in [0,T]\,,\ \forall x \in \mathcal{D}(A^\epsilon)\,. \tag{4.12}$$

The inequality (4.12), combined with (4.11), establishes – via the usual polarization (first) and density (next) arguments – $Q(s) \equiv P(s)$ on $[0,T]$, as desired.

□

4.2. **Infinite time interval, algebraic Riccati equations.** In this Section we prove our second main result, that is Theorem 2.10, which pertains to uniqueness for the algebraic Riccati equation (2.17), under the standing Assumptions 2.8. Instrumental results are the counterparts of Lemmas 4.1 and 4.2, along with the integral form (3.8) of the ARE, already obtained in Section 3; see Lemma 3.2 therein.

The first Lemma is the infinite time horizon version of the fundamental identity established in Lemma 4.1.

Lemma 4.3 (Fundamental identity ($T = +\infty$))**.** *Recall the class $\mathcal{Q}$ defined in* (2.18)*. Let $Q \in \mathcal{Q}$ be a solution to the integral Riccati equation* (3.8)*. With $u \in L^2_{loc}(0,\infty;U)$ and $x \in \mathcal{D}(A^\epsilon)$, let $y(\cdot)$ be the semigroup solution to the state equation*

(2.1) *corresponding to* $u(\cdot)$*, with initial state* x*, given by* (2.2)*. Then, the following identity holds true, for any* $t \geq 0$*:*

$$
\begin{aligned}
&(Qy(t), y(t))_Y - (Qx, x)_Y \\
&\qquad = -\int_0^t \Big(\|Ry(s)\|_Z^2 + \|u(s)\|_U^2\Big)\, ds + \int_0^t \big\|u(s) + B^*Qy(s)\big\|_U^2\, ds\,.
\end{aligned}
\tag{4.13}
$$

Proof. It suffices to proceed along the lines of the proof of Lemma 4.1, replacing the interval $[s,t]$ by $[0,t]$ and assuming initially $u \in L^\infty_{\text{loc}}(0,\infty;U)$; the proof is actually slightly simpler, since here Q is independent of t. The details are omitted for the sake of conciseness. □

The next Lemma is the infinite time horizon version of Lemma 4.2, dealing with an integral equation which – once uniquenss for the ARE is ascertained – will turn out to be the closed-loop equation.

Lemma 4.4. *Let* ϵ *be as in the Assumptions 2.8. Recall the class* $\mathcal{Q}$ *defined by* (2.18)*, and let* $Q \in \mathcal{Q}$*. For every* $x \in \mathcal{D}(A^\epsilon)$ *and a suitably large* $r > 0$ *there exists a unique solution* $y(\cdot)$ *to the closed loop equation*

$$
y(t) = e^{At}x - \int_0^t e^{A(t-s)}BB^*Qy(s)\, ds\,, \qquad t > 0\,, \tag{4.14}
$$

in the space

$$
X = \big\{y \in C([0,\infty);\mathcal{D}(A^\epsilon))\colon \quad \sup_{t\geq 0} e^{-rt}\|y(t)\|_{\mathcal{D}(A^\epsilon)} < \infty\big\} \tag{4.15}
$$

endowed with the norm

$$
\|y\|_{X,r} = \sup_{t>0} e^{-rt}\|y(t)\|_{D(A^\epsilon)} \quad \forall y \in X\,, \quad r > 0\,.
$$

Proof. The argument is pretty much the same employed in the proof of Lemma 4.2. A technically decisive (distinct) element here comes from the extended (and enhanced) regularity in time of the operator $B^*e^{A^*\cdot}A^{*\epsilon}$ over the half line $[0,\infty)$, which is guaranteed by [4, Proposition 3.2], recalled here as Proposition A.6. The computation is included for the reader's convenience.

Let $x \in \mathcal{D}(A^\epsilon)$ be given. By setting $E(t) = e^{At}x$, and recalling the input-to-state map L, the integral equation (4.14) reads as $\big([I + LB^*Q]y(\cdot)\big)(t) = E(t)$, in short. For any function $y(\cdot) \in X$ and any $z \in \mathcal{D}(A^{*\epsilon})$, we have

$$
\begin{aligned}
|(e^{-rt}LB^*Qy(t), A^{*\epsilon}z)_Y &= \Big|\int_0^t e^{-r(t-s)}(B^*Qy(s)e^{-rs}, B^*e^{A^*(t-s)}A^{*\epsilon}z)_Y\, ds\Big| \\
&\leq \int_0^t e^{-r(t-s)}\|B^*Q\|_{\mathcal{L}(\mathcal{D}(A^\epsilon),U)}\|y\|_{X,r}\, e^{-\delta(t-s)}\, \|e^{\delta(t-s)}B^*e^{A^*(t-s)}A^{*\epsilon}z\|_U\, ds \\
&\leq \|B^*Q\|_{\mathcal{L}(\mathcal{D}(A^\epsilon),U)}\|y\|_{X,r} \Big(\int_0^t e^{-(r+\delta)(t-s)q'}\, ds\Big)^{1/q'} \cdot \\
&\qquad\qquad \cdot \Big(\int_0^t \|e^{\delta(t-s)}B^*e^{A^*(t-s)}A^{*\epsilon}z\|_U^q\, ds\Big)^{1/q} \\
&\leq \frac{1}{[(r+\delta)q']^{1/q'}}\, \|B^*Q\|_{\mathcal{L}(\mathcal{D}(A^\epsilon),U)}\, \|e^{\delta\cdot}B^*e^{A^*\cdot}A^{*\epsilon}\|_{\mathcal{L}(Y,L^q(0,\infty;U))}\, \|y\|_{X,r}\, \|z\|_Y\,,
\end{aligned}
$$

where δ belongs to the interval $(0,\omega\wedge\eta)$ (ω and η being like in the Assumptions 2.8). The above estimate implies readily that there exists a constant $C>0$ such that

$$\|LB^*Qy\|_{X,r} \le \frac{C}{(r+\delta)^{1/q'}}\,\|y\|_{X,r}\|e^{\delta\cdot}B^*e^{A^*\cdot}A^{*\epsilon}\|_{\mathcal{L}(Y,L^q(0,\infty;U))}$$

so that

$$\|LB^*Qy\|_{X,r} \le \frac{1}{2}\|y\|_{X,r}\,,$$

provided r is sufficiently large. The conclusive argument is standard. □

Proof of Theorem 2.10. Let $y_0\in Y$, and let $(\hat y,\hat u)$ the optimal pair of the optimal control problem (2.1)-(2.3) (with $T=+\infty$), corresponding to the initial state y_0. Recall that

$$(Py_0,y_0)_Y = J(\hat u) = \int_0^\infty \|R\hat y(s)\|_Z^2\,ds + \int_0^\infty \|\hat u(s)\|_U^2\,ds\,,$$

where the (optimal cost) operator P is defined in terms of the optimal state semigroup $\Phi(t)$ via (2.15). In addition, P belongs to the class $\mathcal{Q}$ and solves the ARE (2.17); consequently, by Lemma 3.2 P solves the integral form (3.8) of the ARE.

Let now $Q\in\mathcal{Q}$ be another solution to the ARE. By Lemma 4.3, we know that for any given $y_0\in\mathcal{D}(A^\epsilon)$, and any admissible control $u(\cdot)$, the identity (4.13) holds true (with x replaced by y_0), where $y(\cdot)$ is the solution to the state equation corresponding to the control u and the initial state y_0. Consequently,

$$(Qy_0,y_0)_Y \le (Qy(t),y(t))_Y + J(u) \qquad \forall u\in L^2_{\rm loc}(0,\infty;U)\,,\ \forall t>0\,;$$

by choosing in particular the admissible pair (y_T,u_T) defined as follows,

$$u_T = \hat u\cdot\chi_{[0,T]}\,,\qquad y_T(t) = \begin{cases} \hat y(t) & \text{if } t\le T\\ e^{At}y_0 + e^{A(t-T)}L\hat u(T) & \text{if } t>T,\end{cases}$$

we find $(Qy_0,y_0)_Y \le (Qy_T(t),y_T(t))_Y + J(u_T)$, valid for arbitrary $t\ge T>0$. By letting $t\to+\infty$ in the previous inequality, one obtains readily

$$(Qy_0,y_0)_Y \le J(u_T) \qquad \forall y_0\in\mathcal{D}(A^\epsilon) \quad \forall T>0\,, \tag{4.16}$$

in view of the fact that the semigroup e^{At} decays exponentially; so $\|y_T(t)\|_Y\longrightarrow 0$, as $t\to+\infty$.

Observe now that

$$\begin{aligned} J(u_T) &= \int_0^\infty \|Ry_T(s)\|_Z^2\,ds + \int_0^\infty \|u_T(s)\|_U^2\,ds\\ &= \int_0^T \|R\hat y(s)\|_Z^2\,ds + \int_T^\infty \big\|R\big(e^{sA}y_0 + e^{(s-T)A}L\hat u(T)\big)\big\|^2\,ds + \int_0^T \|\hat u(s)\|_U^2\,ds\,,\end{aligned}$$

so that $J(u_T)\longrightarrow J(\hat u)$, as $T\to+\infty$. Keeping this in mind, return to (4.16) and let $T\to+\infty$ to find

$$(Qy_0,y_0)_Y \le J(\hat u) = (Py_0,y_0)_Y \qquad \forall y_0\in\mathcal{D}(A^\epsilon)\,. \tag{4.17}$$

On the other hand, given $y_0\in\mathcal{D}(A^\epsilon)$ (and still with $Q\in\mathcal{Q}$ another solution to the ARE), let $y(\cdot)$ be the solution to the closed loop equation guaranteed by Lemma 4.4; by construction $y\in L^2_{\rm loc}(0,\infty;Y)$. Take now the control

$u(\cdot) = -B^*Qy(\cdot)$, which belongs to $L^2_{\rm loc}(0,\infty;U)$. Then, the identity (4.13) holds true for any positive t, that is

$$\begin{aligned}(Qy_0,y_0)_Y &= (Qy(t),y(t))_Y + \int_0^t \Big(\|Ry(s)\|_Z^2 + \|u(s)\|_U^2\Big)\,ds \\ &\quad \cancel{-\int_0^t \big\|u(s)+B^*Qy(s)\big\|_U^2\,ds} \ge \int_0^t \|Ry(s)\|_Z^2\,ds + \int_0^t \|u(s)\|_U^2\,ds\,.\end{aligned} \tag{4.18}$$

As $t\to+\infty$, this shows that $Ry\in L^2(0,\infty;Z)$, $u\in L^2(0,\infty;U)$, as well as $(Qy_0,y_0)_Y \ge J(u)$. By minimality we find

$$(Qy_0,y_0)_Y \ge J(u) \ge J(\hat u) = (Py_0,y_0)_Y \qquad \forall y_0\in\mathcal{D}(A^\epsilon)\,. \tag{4.19}$$

On the basis of (4.17) and (4.19), a standard polarization (first) and density (next) argument confirms that $Q=P$, thereby concluding the proof of Theorem 2.10. □

Appendix A. Instrumental results

In this appendix we gather several results (some old, some new) which single out certain regularity properties – in time and space – of

- the input-to-state map L,
- the operator $B^*Q(\cdot)$, when $Q(t)\in\mathcal{Q}_T$,
- the operator $B^*e^{A^*t}A^{*\epsilon}$ and its adjoint.

All of them stem from the Assumptions 2.3 or 2.8 on the (dynamics and control) operators A and B. The role played by the assertions of the novel Lemma A.2 and Lemma A.5 in the proofs of our uniqueness results is absolutely critical.

Initially, it is useful to recall from [2] and [4] the basic regularity properties of the input-to-state map L. The first result pertains to the finite time horizon problem. The reader is referred to [2, Appendix B] for the details of the computations leading to the various statements in the following Proposition.

Proposition A.1 ([2], Proposition B.3)**.** *Let L_s be the operator defined by*

$$L_s\colon u(\cdot)\longrightarrow (L_su)(t) := \int_s^t e^{A(t-r}Bu(r)\,dr\,, \qquad 0\le s\le t\le T\,. \tag{A.1}$$

Under the Assumptions 2.3, the following regularity results hold true.

1. *If $p=1$, then $L_s\in\mathcal{L}(L^1(s,T;U),L^{1/\gamma}(s,T;[\mathcal{D}(A^{*\epsilon})]'))$;*
2. *if $1<p<\frac{1}{1-\gamma}$, then $L_s\in\mathcal{L}(L^p(s,T;U),L^r(s,T;Y))$, with $r=\frac{p}{1-(1-\gamma)p}$;*
3. *if $p=\frac{1}{1-\gamma}$, then $L_s\in\mathcal{L}(L^p(s,T;U),L^r(s,T;Y))$ for all $r\in[1,\infty)$;*
4. *if $p>\frac{1}{1-\gamma}$, then $L_s\in\mathcal{L}(L^p(s,T;U),C([s,T];Y))$.*

Moreover, in all cases the norm of L_s does not depend on s.

The space regularity in the last assertion can be actually enhanced. To be more precise, L_s maps control functions $u(\cdot)$ which belong to $L^{q'}(s,T;U)$ into functions which take values in $\mathcal{D}(A^\epsilon)$ (q' being the conjugate exponent of q in the Assumptions 2.3). We highlight this property – appparently left out of the work [2] – as a separate result, since it will be used throughout in the paper. The proof is omitted, as it is akin to (and somewhat simpler than) the one carried out to establish assertion (v) of the subsequent Proposition A.3.

Lemma A.2. *Let ϵ and q be as in (iii) of the Assumptions 2.3. Then, for the operator L_s defined in* (A.1) *we have*

$$L_s \in \mathcal{L}(L^{q'}(s,T;U), C([s,T];\mathcal{D}(A^\epsilon)))\,.$$

A counterpart of Proposition A.1 specific for the infinite time horizon problem was proved in [4, Proposition 3.6]. The collection of findings on the regularity of the input-to-state map L is recorded here for the reader's convenience.

Proposition A.3 ([4], Proposition 3.6)**.** *Let L be the operator defined by*

$$L\colon u(\cdot) \longrightarrow (Lu)(t) := \int_0^t e^{A(t-r}Bu(r)\,dr\,, \qquad t\ge 0\,.$$

Under the Assumptions 2.8, the following regularity results hold true.

(i) $L \in \mathcal{L}(L^1(0,\infty;U), L^r(0,\infty;[\mathcal{D}(A^{*\epsilon})]')$*, for any* $r\in[1,1/\gamma)$*;*
(ii) $L \in \mathcal{L}(L^p(0,\infty;U), L^r(0,\infty;Y))$*, for any* $p\in(1,1/(1-\gamma))$ *and any* $r\in[p,p/(1-(1-\gamma)p)]$*;*
(iii) $L \in \mathcal{L}(L^{\frac{1}{1-\gamma}}(0,\infty;U), L^r(0,\infty;Y))$*, for any* $r\in[1/(1-\gamma),\infty)$*;*
(iv) $L \in \mathcal{L}(L^p(0,\infty;U), L^r(0,\infty;Y)\cap C_b([0,\infty);Y))$*, for any* $p\in(1/(1-\gamma),\infty)$ *and any* $r\in[p,\infty)$*;*
(v) $L \in \mathcal{L}(L^r(0,\infty;U), C_b([0,\infty);\mathcal{D}(A^\epsilon))$*, for any* $r\in[q',\infty]$.

Because they occur in the present work, besides being central to the analysis of [4], we need to recall the L^p-spaces with weights. Set

$$L^p_g(0,\infty;X) := \big\{f\colon (0,\infty)\longrightarrow X\,,\ g(\cdot)f(\cdot)\in L^p(0,\infty;X)\big\}\,,$$

where $g\colon (0,\infty)\longrightarrow \mathbb{R}$ is a given (*weight*) function. We will use more specifically the exponential weights $g(t)=e^{\delta t}$, along with the following (simplified) notation:

$$L^p_\delta(0,\infty;X) := \big\{f\colon (0,\infty)\longrightarrow X\,,\ e^{\delta\cdot}f(\cdot)\in L^p(0,\infty;X)\big\}\,.$$

Remark A.4. As pointed out in [4, Remark 3.8], all the regularity results provided by the statements contained in the Propositions A.1 and A.3 extend readily to natural analogues involving L^p_δ spaces (rather than L^p ones), maintaining the respective summability exponents p.

We now move on to a result which clarifies the regularity of the operator $B^*Q(\cdot)$, $Q(t)\in\mathcal{Q}_T$, when acting upon *functions* (with values in $\mathcal{D}(A^\epsilon)$) rather than on *vectors* – namely, on elements of the space $\mathcal{D}(A^\epsilon)$.

Lemma A.5. *Let ϵ be as in (iii) of the Assumptions 2.3. If $Q(\cdot)\in\mathcal{Q}_T$ and $f\in C([0,T];\mathcal{D}(A^\epsilon))$, then*

$$B^*Q(\cdot)f(\cdot)\in C([0,T];U)\,.$$

Proof. We proceed along the lines of the proof of [2, Lemma A.3]. Let $Q(\cdot)\in\mathcal{Q}_T$ and let $t_0\in[0,T]$. By the definition of $\mathcal{Q}_T$, there exists $c_1>0$ such that

$$\|B^*Q(t)z\|_U \le c_1\|z\|_{\mathcal{D}(A^\epsilon)} \qquad \forall t\in[0,T]\,,\ \forall z\in\mathcal{D}(A^\epsilon)\,. \tag{A.2}$$

Since $f(t_0)\in\mathcal{D}(A^\epsilon)$, then $B^*Q(\cdot)f(t_0)\in C([0,T];U)$. Then

$$\begin{aligned}
&\|B^*Q(t)f(t)-B^*Q(t_0)f(t_0)\|_U\\
&\qquad\le \|B^*Q(t)[f(t)-f(t_0)]\|_U + \|B^*Q(t)f(t_0)-B^*Q(t_0)f(t_0)\|_U\\
&\qquad\le c_1\|f(t)-f(t_0)\|_{\mathcal{D}(A^\epsilon)} + \|[B^*Q(t)-B^*Q(t_0)]f(t_0)\|_U = o(1)\,,\quad t\longrightarrow t_0\,.
\end{aligned}$$

□

Essential as well in this work, and more specifically in the proof of Theorem 2.10, is a stronger property of the operator $B^*e^{A^*\cdot}A^{*\epsilon}$, namely, (A.3) below, which holds true for appropriate δ, under the Assumptions 2.8. Originally devised in [4], this result reveals that once the validity of iiic) of Assumptions 2.3 is ascertained on some bounded interval $[0,T]$, then the very same regularity estimate extends to the half line, along with an enhanced summability of the function $B^*e^{A^*\cdot}A^{*\epsilon}x$, $x \in Y$. The key to this is the exponential stability of the semigroup, i.e. (2.12); see [4, Proposition 3.2].

Proposition A.6 ([4], Proposition 3.2)**.** *Let ω, η and ϵ like in the Assumptions 2.8. For each $\delta \in (0, \omega\wedge\eta)$ the map*

$$t \longrightarrow e^{\delta t}B^*e^{A^*t}A^{*\epsilon}$$

has an extension which belongs to $\mathcal{L}(Y, L^q(0,\infty;U))$. In short,

$$B^*e^{A^*\cdot}A^{*\epsilon} \in \mathcal{L}(Y, L^q_\delta(0,\infty;U))\,. \tag{A.3}$$

We conclude providing a result that takes a more in-depth glance at the regularity of the operator $B^*e^{A^*t}A^{*\epsilon}$ and its adjoint.

Lemma A.7. *Under the Assumptions 2.8, the following regularity results are valid, for any $\delta \in (0,\omega\wedge\eta)$:*

$$\begin{aligned} &a) \quad e^{\delta\cdot}A^\epsilon e^{A\cdot}B \in \mathcal{L}(L^{q'}(0,\infty;U),Y)\,,\\ &b) \quad e^{\delta\cdot}B^*e^{A^*\cdot}A^{*-\epsilon} \in \mathcal{L}(L^r(0,\infty;Y),U) \qquad \forall r > \frac{1}{1-\gamma}\,. \end{aligned} \tag{A.4}$$

The respective actions of the operators in (A.4) *are made explicit by* (A.5) *and* (A.6)*.*

Proof. The regularity results in (A.4) are, in essence, dual properties of the regularity result in Proposition A.6 and of assertion A6. in Theorem 2.9, respectively. To infer (a), we introduce the notation S for the mapping from Y into $L^q(0,\infty;U)$ defined by

$$Y \ni z \longrightarrow [Sz](t) := e^{\delta t}B^*e^{A^*t}A^{*\epsilon}z\,, \qquad t>0\,.$$

For any $z\in Y$ and any $h \in L^{q'}(0,\infty;U)$, it must be $S^* \in \mathcal{L}(L^{q'}(0,\infty;U),Y)$ and more precisely,

$$\begin{aligned} \langle S^*h, z\rangle_Y = \langle h, Sz\rangle_{L^{q'}(0,\infty;U),L^q(0,\infty;U)} &= \int_0^\infty \big\langle h(t), e^{\delta t}B^*e^{A^*t}A^{*\epsilon}z\big\rangle_U\,dt\\ &= \Big\langle \int_0^\infty e^{\delta t}A^\epsilon e^{At}Bh(t)\,dt, z\Big\rangle_Y\,. \end{aligned}$$

Therefore,

$$S^*h = \int_0^\infty e^{\delta t}A^\epsilon e^{At}Bh(t)\,dt\,, \qquad h\in L^{q'}(0,\infty;Y)\,. \tag{A.5}$$

To achieve (b) of (A.4), we recall instead the assertion A6. in Theorem 2.9, which further tells us that

$$e^{\delta\cdot}A^{-\epsilon}e^{A\cdot}B \in \mathcal{L}(U, L^p(0,\infty;Y))\,, \quad \text{for any } p \text{ such that } 1\le p<\frac{1}{\gamma}.$$

Similarly as above, we introduce the notation T for the mapping from U into $L^p(0,\infty;Y)$ defined by

$$U \ni w \longrightarrow [Tw](t) := e^{\delta t} A^{-\epsilon} e^{At} Bw\,, \qquad t>0\,;$$

by construction, $T^* \in \mathcal{L}(L^{p'}(0,\infty;Y),U)$ for all $p'>1/(1-\gamma)$. More precisely, for any $w\in U$ and any $g\in L^{p'}(0,\infty;Y)$ we have

$$\begin{aligned}\langle T^*g,w\rangle_U &= \langle g,Tw\rangle_{L^{p'}(0,\infty;Y),L^p(0,\infty;Y)} = \int_0^\infty \big\langle g(t), e^{\delta t}A^{-\epsilon}e^{At}Bw\big\rangle_Y\,dt \\ &= \Big\langle \int_0^\infty e^{\delta t}B^* e^{A^*t}A^{*-\epsilon} g(t)\,dt, w\Big\rangle_U\,,\end{aligned}$$

which establishes

$$T^*g = \int_0^\infty e^{\delta t}B^*e^{A^*t}A^{*-\epsilon}g(t)\,dt\,. \qquad \forall g\in L^{p'}(0,\infty;Y)\,. \tag{A.6}$$

The integrals in (A.5) and (A.6) are the sought respective representations of the adjoint operators in (A.4).

$\square$

Acknowledgements

This research has been performed in the framework of the MIUR-PRIN Grant 2020F3NCPX "Mathematics for industry 4.0 (Math4I4)."

The research of F. Bucci was partially supported by the Università degli Studi di Firenze under the 2019 Project *Metodi ed Applicazioni per Equazioni Differenziali Ordinarie e a Derivate Parziali*. Bucci is a member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM), and has been participant to the GNAMPA Projects *Controllabilità di PDE in modelli fisici e in scienze della vita* (2019) and *Problemi inversi e di controllo per equazioni di evoluzione e loro applicazioni* (2020). She has been also a member of the French-German-Italian Laboratoire International Associé (LIA) COPDESC in Applied Analysis.

The research of P. Acquistapace was partially supported by the PRIN-MIUR Project 2017FKHBA8 of the Italian Education, University and Research Ministry.

References

[1] P. Acquistapace, F. Bucci, On the infinitesimal generator of an optimal state semigroup, *Semigroup Forum* **105** (2022), 46-72. https://doi.org/10.1007/s00233-022-10288-0

[2] P. Acquistapace, F. Bucci, I. Lasiecka, Optimal boundary control and Riccati theory for abstract dynamics motivated by hybrid systems of PDEs, *Adv. Differential Equations* **10** (2005), no. 12, 1389-1436.

[3] P. Acquistapace, F. Bucci, I. Lasiecka, A trace regularity result for thermoelastic equations with application to optimal boundary control, *J. Math. Anal. Appl.* **310** (2005), no. 1, 262-277.

[4] P. Acquistapace, F. Bucci, I. Lasiecka, A theory of the infinite horizon LQ-problem for composite systems of PDEs with boundary control, *SIAM J. Math. Anal.* **45** (2013), no. 3, 1825-1870.

[5] G. Avalos, I. Lasiecka, Differential Riccati equation for the active control of a problem in structural acoustics, *J. Optim. Theory Appl.* **91** (1996), no. 3, 695-728.

[6] G. Avalos, I. Lasiecka, Exponential stability of a thermoelastic system without mechanical dissipation, Rend. Istit. Mat. Univ. Trieste **28** (1996), suppl., 1-28 (1997).

[7] V. Barbu, Z. Grujić, I. Lasiecka, A. Tuffaha, Existence of the energy-level weak solutions for a nonlinear fluid-structure interaction model, *Contemp. Math.* (2007) **440**, 55-81.

[8] A. Bensoussan, G. Da Prato, M. Delfour, S. Mitter, *Representation and Control of Infinite Dimensional Systems*, 2^{nd} edition, Birkhäuser, Boston, 2007.

[9] F. Bucci, Control-theoretic properties of structural acoustic models with thermal effects, I. Singular estimates, *J. Evol. Equations* **7** (2007), 387-414.

[10] F. Bucci, Control-theoretic properties of structural acoustic models with thermal effects, II. Trace regularity results, *Appl. Math.* **35** (2008), n. 3, 305-321.

[11] F. Bucci, Improved boundary regularity for a Stokes-Lamé system, *Evol. Equ. Control Theory* **11** (2022), no. 1, 325-346.

[12] F. Bucci, I. Lasiecka, Singular estimates and Riccati theory for thermoelastic plate models with boundary thermal control. *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* **11** (2004), no. 4, 545-568.

[13] F. Bucci, I. Lasiecka, Optimal boundary control with critical penalization for a PDE model of fluid-solid interactions, *Calc. Var. Partial Differential Equations* **37** (2010), no. 1-2, 217-235.

[14] F. Bucci, I. Lasiecka, Regularity of boundary traces for a fluid-solid interaction model, *Discrete Contin. Dyn. Syst. Ser. S* **4** (2011), no. 3, 505-521.

[15] F. Bucci, I. Lasiecka, Feedback control of the acoustic pressure in ultrasonic wave propagation, *Optimization* **20** (2019), no. 10, 1811-1854 (published online: 19 Aug. 2018).

[16] I. Lasiecka, *Mathematical Control Theory of Coupled PDEs*, CBMS-NSF Regional Conf. Ser. in Appl. Math., vol. 75, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 2002.

[17] I. Lasiecka, Optimal control problems and Riccati equations for systems with unbounded controls and partially analytic generators. Applications to Boundary and Point control problems, in: *Functional analytic methods for evolution equations*, 313–369, Lecture Notes in Math., 1855, Springer-Verlag, Berlin, 2004.

[18] I. Lasiecka, R. Triggiani, Differential and algebraic Riccati equations with application to boundary/point control problems: continuous theory and approximation theory. *Lecture Notes in Control and Information Sciences*, 164, Springer-Verlag, Berlin, 1991. xii+160 pp.

[19] I. Lasiecka, R. Triggiani, Optimal Control and Algebraic Riccati Equations under Singular Estimates for $e^{At}B$ in the Absence of Analiticity. I. The stable case, in: Differential Equations and Control Theory, Athens, OH, 2000, Lecture Notes in Pure and Appl. Math., vol. 225, Dekker, New York, 2002, pp. 193-219.

[20] I. Lasiecka, R. Triggiani, Optimal Control and Differential Riccati Equations under Singular Estimates for $e^{At}B$ in the Absence of Analiticity, Advances in dynamics and control, 270-307, *Nonlinear Syst. Aviat. Aerosp. Aeronaut. Astronaut.* **2**, Chapman & Hall/CRC, Boca Raton, FL, 2004.

[21] I. Lasiecka, R. Triggiani, *Control Theory for Partial Differential Equations: Continuous and Approximation Theories.* I. Abstract Parabolic Systems; II. Abstract Hyperbolic-like Systems over a Finite Time Horizon, Encyclopedia Math. Appl., 74–75, Cambridge University Press, Cambridge, 2000.

[22] I. Lasiecka, A. Tuffaha, Riccati equations for the Bolza problem arising in boundary/point control problems governed by C_0-semigroups satisfying a singular estimate, *J. Optim. Theory Appl.* **136** (2008), no. 2, 229-246.

[23] I. Lasiecka, A. Tuffaha, Riccati theory and singular estimates for a Bolza control problem arising in linearized fluid-structure interaction, *Systems and Control Letters* **58** (2009), 499-509.

[24] I. Lasiecka, A. Tuffaha, A Bolza optimal synthesis problem for singular estimate control systems, *Control Cybernet.* **38(4B)** (2009), 1429-1460.

[25] C. Lebiedzik, Uniform stability of a coupled structural acoustic system with thermoelastic effects, *Dynam. Contin. Discrete Impuls. Systems* **7** (2000), no. 3, 369-383.

[26] J.-L. Lions, E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications, Vols. I and II*, Springer Verlag, 1972.

[27] A. Lunardi, *Interpolation Theory*, 2nd edn. Appunti, Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)] Edizioni della Normale, Pisa (2009).

[28] C. Martínez, M. Sanz, L. Marco, Fractional powers of operators, *J. Math. Soc. Japan* **40** (1988), n. 2, 331-347.

[29] D. Tataru, On the regularity of boundary traces for the wave equation, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **26** (1998), 185-206.

[30] H. Triebel, *Interpolation theory, function spaces, differential operators*, Second edition, Johann Ambrosius Barth, Heidelberg, 1995, 532 pp.

[31] A. Tuffaha, Riccati equations for generalized singular estimate control systems, *Appl. Anal.* **92** (2013), no. 8, 1559-1596.

Università di Pisa, Dipartimento di Matematica, Largo Bruno Pontecorvo 5, 56127 Pisa, ITALY

Email address: paolo.acquistapace(at)unipi.it

Francesca Bucci, Università degli Studi di Firenze, Dipartimento di Matematica e Informatica, Via S. Marta 3, 50139 Firenze, ITALY

Email address: francesca.bucci(at)unifi.it