



Effective methods for plane quartics, their theta characteristics and the Scorza map

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Abstract

This is a revised version of the lecture notes prepared for the workshop on “Plane quartics, Scorza map and related topics”, held in Catania, January 19–21, 2016. The last section contains eight Macaulay2 scripts on theta characteristics and the Scorza map, with a tutorial. The first sections give an introduction to these scripts. The tutorial contains a list of the 36 Scorza preimages of the Edge quartic.

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1 Introduction

1.1 How to write down plane quartics and their theta characteristics

Plane quartics make a relevant family of algebraic curves because their plane embedding is the canonical embedding. As a byproduct, intrinsic and projective geometry are strictly connected. A theta characteristic θ is by definition a square root of the canonical bundle K , namely $2\theta = K$ in the additive notation, so θ is a priori an intrinsic object. There are $64 = 2^6$ theta characteristic, since the Jacobian variety of the plane quartic has real dimension $2g = 6$. It is not a surprise that the 64 theta characteristics of a plane quartic show up in many projective constructions. There are 28 odd theta characteristic (such that $h^0(\theta) = 1$) and 36 even theta characteristic (such that $h^0(\theta) = 0$). The first well known fact is that the 28 odd theta characteristics correspond to the 28 bitangents of the plane quartic curve. Indeed if a line is tangent to the curve in two points P, Q , it is easy to see that $\theta := P + Q$ satisfies $2\theta = K$, since the canonical divisor K corresponds to the hyperplane divisor. Moreover $h^0(P + Q) = 1$ since $P + Q$ is an effective divisor. It is less known that the 36 even theta characteristics may be visualized as the 36 plane quartic curves that are the preimages through the Scorza map (see §5).

Definition 1.1 A set of seven bitangents $\{\theta_1, \dots, \theta_7\}$, that we identify with their odd theta characteristic, is called an Aronhold system of bitangents if for every $\{i, j, k\}$ such that $1 \leq i < j < k \leq 7$ we have

$$h^0(2K - \theta_i - \theta_j - \theta_k) = 0.$$

Replacing $2K = 2\theta_i + 2\theta_j$ this is equivalent to say that $\theta_i + \theta_j - \theta_k$ is an even theta characteristic. For more informations see [34, Defs. 4.1, 4.2].

To write down explicitly a plane quartic and its theta characteristics, we will adopt the following four different descriptions

- A homogeneous polynomial f of degree 4 in x_0, x_1, x_2 . Computationally, this is a vector with 15 homogeneous coordinates.
- A symmetric linear determinantal representation of f , namely a symmetric 4×4 matrix A with linear entries in x_0, x_1, x_2 , such that $\det(A) = f$. The exact sequence

$$0 \longrightarrow \mathcal{O}(-2)^4 \xrightarrow{A} \mathcal{O}(-1)^4 \longrightarrow \theta \longrightarrow 0$$

gives a even theta characteristic θ , as a line bundle supported on the curve $\{f = 0\}$. For a general f , there are 36 classes of matrices A such that $\det(A) = f$ is given, up to $GL(4)$ -congruence, corresponding to the 36 even theta characteristics. Computationally, we have a net of symmetric 4×4 matrices $\langle A_0, A_1, A_2 \rangle$.

- Let $C = \{f = 0\}$. A symmetric $(3, 3)$ -correspondence in $C \times C$ has the form $T_\theta = \{(x, y) \in C \times C \mid h^0(\theta + x - y) \neq 0\}$ corresponding to an even theta characteristic θ . Computationally, the divisor T_θ is cut on the Segre variety $\mathbb{P}^2 \times \mathbb{P}^2$ by six bilinear equations in x, y .

4. • Let $\{P_1, \dots, P_7\}$ be seven general points in \mathbb{P}^2 . The net of cubics through these seven points defines a 2 : 1 covering $\mathbb{P}^2(P_1, \dots, P_7) \xrightarrow{\pi} \mathbb{P}^2$, with source the blow-up of \mathbb{P}^2 at the points P_i , which ramifies over a quartic $C \subset \mathbb{P}^2$. In equivalent way, seven general lines in \mathbb{P}^2 define a unique quartic curve C such that the seven lines give an Aronhold system of bitangents of C . Computationally, these data can be encoded in a 7×3 matrix (up to reorder the rows) or also in a 2×3 matrix of the following type

$$\begin{pmatrix} l_0 & l_1 & l_2 \\ q_0 & q_1 & q_2 \end{pmatrix} \tag{1.1}$$

where $\deg l_i = 1$, $\deg q_i = 2$ and l_i are independent. The maximal minors of this matrix vanish on seven points. If $Z = \{P_1, \dots, P_7\}$, then the resolution of the ideal $I_Z(3) := I_Z \otimes \mathcal{O}(3)$ is

$$0 \longrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-2) \xrightarrow{g} \mathcal{O}^3 \longrightarrow I_Z(3) \longrightarrow 0$$

where the matrix of g is the transpose of (1.1).

The previous descriptions have an increasing amount of data, in the sense that 2. and 3. are equivalent, while for the other items we have

$$\{4.\text{seven points}\} \begin{matrix} \xrightarrow{8:1} \\ \xrightarrow{36:1} \end{matrix} \left\{ \begin{matrix} 2. \text{ symmetric determinant} \\ 3. \text{ symm. (3,3) corresp.} \end{matrix} \right\} \xrightarrow{36:1} \{1.\text{quartic polynomial}\} \tag{1.2}$$

By forgetting the theta characteristic in the intermediate step, we get the interesting correspondence

$$\{\text{seven points}\} \xrightarrow{288:1} \{\text{quartic polynomial}\} \tag{1.3}$$

We will see how to “move on the right” in the diagrams (1.2) and (1.3), from one description to another one. These moves, connecting the several descriptions, are $SL(3)$ -equivariant. Any family of quartics that is invariant by the $SL(3)$ -action of projective linear transformations can be described by invariants or covariants in the above descriptions.

From the real point of view, it is interesting to recall the following Table from [16, Prop. 5.1] for the theta characteristics of real quartics.

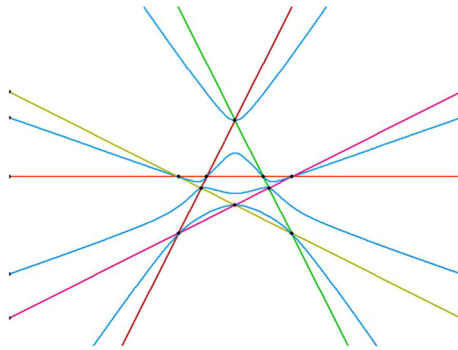
topological classification	# real odd theta	# real even theta
empty	4	12
one oval	4	4
two nested ovals	4	12
two non nested oval	8	8
three ovals	16	16
four ovals	28	36

(1.4)

1.2 Clebsch and Lüroth quartics

A famous example is given by Clebsch quartics, which are the quartics which can be written as $f = \sum_{i=1}^5 l_i^4$ (Waring decomposition), while the general quartic needs six summands consisting of 4th powers of linear forms, in contrast with the naive numerical expectation. Clebsch quartics can be detected by a determinantal invariant of degree 6 in the 15 coefficients of the quartic polynomial (description 1.), called the catalecticant invariant or the Clebsch invariant.

Description of special quartics may be quite different depending on the different descriptions we choose. The archetypal example is that of Lüroth quartics, which by definition contain the 10 vertices of a complete pentalateral, like in the picture



The pentalateral defines in a natural way a particular even theta characteristic θ , that it is called the pentalateral theta (the 10 vertices of the pentalateral make the divisor $2K + \theta$). The pair (f, θ) consisting of a Lüroth quartic f with a pentalateral theta θ can be described in a relatively easy way by looking at the determinantal representation (description 2.): if the symmetric 4×4 matrix is $\sum_{i=0}^2 x_i A_i$ then the condition of being Lüroth with a pentalateral theta is expressed by the vanishing of the 6th degree Pfaffian of the matrix

$$\begin{pmatrix} 0 & A_2 & -A_1 \\ -A_2 & 0 & A_0 \\ A_1 & -A_0 & 0 \end{pmatrix}, \quad (1.5)$$

see [24, Theor. 4.1, Prop. 6.1]. There is a second way to describe a Lüroth quartic, if we can profit of the additional data of a Aronhold system of seven bitangents (description 4.). Indeed the seven points which give a Lüroth quartic correspond surprisingly to the seven eigenvectors of a plane cubic. See §8 for details, roughly this means that there exists a cubic polynomial c such that in (1.1) it holds $\frac{\partial c}{\partial l_i} = q_i$. If we forget these additional data, it is tremendously difficult to detect a Lüroth quartic looking just at it defining polynomial (description 1.).

Lüroth quartic were studied deeply in the period 1860-1918, starting from Lüroth paper and their description in terms of net of quadrics, culminating with Morley brilliant description of the invariant in terms of the seven points, showing finally that the degree of Lüroth invariant, in terms of the fifteen coefficients of the quartic, is 54. The explicit expression of this degree 54 invariant is challenging even nowadays

with the help of a computer. Lüroth quartics became again popular in 1977 when Barth showed that the jumping curve of a stable 2-bundle on \mathbb{P}^2 with Chern classes $(c_1, c_2) = (0, 4)$ is a Lüroth quartic. LePotier and Tikhomirov showed in 2001 that the moduli space of the above bundles can be described in terms of Lüroth quartics, the degree 54 turned out to be a Donaldson invariant of \mathbb{P}^2 .

1.3 The Scorza map

The Scorza map associates to a general quartic f a pair $(S(f), \theta)$ where $S(f)$ is another quartic (the Aronhold covariant) and θ is an even theta characteristic on $S(f)$. Its precise definition needs the Aronhold invariant of plane cubics and it will be recalled in §5. Its main property is the Theorem of Scorza that the map

$$f \mapsto (S(f), \theta)$$

is dominant on the variety of pairs (g, θ) where θ is an even theta characteristic on the quartic g . In a previous paper, Scorza showed that if f is Clebsch then $S(f)$ is Lüroth with pentalateral θ . This fact is the first step in the proof of the Theorem of Scorza. We will give a computational description of the Scorza map and its inverse, especially in Algorithms 6 and 7.

1.4 Description of the content

Sections from 2 to 6 describe a few basics about theta characteristics and Scorza map for plane quartics. The goal is to introduce the terminology to understand the algorithms, we refer to the literature for most of the proofs. Some emphasis is given to Clebsch and Lüroth quartics, which correspond through the Scorza map. More emphasis is given on the construction of a quartic from seven points, which give an Aronhold system of seven bitangents. Section 7 is a brief survey about invariant theory of plane quartics. Section 8 considers the link with the seven eigenvectors of a plane cubic, as sketched in 1.2. Section 9 is the core of this paper and contains the Macaulay2[15] scripts. These scripts are available as ancillary files of the arXiv version of this paper or by contacting the author. We have tried to use the verbatim text style to indicate Macaulay2 input.

1.5 Summary of the eight M2 scripts presented in Sect. 9

1. INPUT: seven general lines l_1, \dots, l_7
OUTPUT: the quartic having l_1, \dots, l_7 as Aronhold system of bitangents
2. INPUT: a 2×3 matrix with 2-minors vanishing on $Z = \{l_1, \dots, l_7\}$
OUTPUT: the quartic having Z as Aronhold system of bitangents
3. INPUT: seven general lines l_1, \dots, l_7
OUTPUT: a 8×8 symmetric matrix (the bitangent matrix) collecting in each row the 8 Aronhold systems of bitangents for the quartic having l_1, \dots, l_7 as Aronhold

system of bitangents, equivalent to l_1, \dots, l_7 . In particular, every principal 4×4 minor of the bitangent matrix gives a symmetric determinantal representation.

4. INPUT: a 2×3 matrix with 2-minors vanishing on $Z = \{l_1, \dots, l_7\}$
OUTPUT: a symmetric determinantal 4×4 representation of the quartic having Z as Aronhold system of bitangents
5. INPUT: a quartic f
OUTPUT: the image $S(f)$ through the Scorza map
6. INPUT: a quartic f and a point $q \in S(f)$
OUTPUT: a determinantal representation of the image $(S(f), \theta)$ through the Scorza map
7. INPUT: a determinantal representation of a quartic g corresponding to (g, θ)
OUTPUT: the quartic f such that the image $(S(f), \theta)$ through the Scorza map corresponds to (g, θ)
The tutorial contains a list of the 36 Scorza preimages of the Edge quartic
8. INPUT: a plane quartic f
OUTPUT: the order of the automorphism group of linear invertible transformations which leave f invariant

The algorithms 4, 6, 7 are computationally expensive. I wonder if there are simpler solutions and shortcuts, from the computational point of view.

1.6 Acknowledgements

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2 Apolarity, Waring decompositions

Our base ring is $S^*(V) = \mathbb{C}[x_0, x_1, x_2]$. The dual ring of differential operators is $S^*(V^\vee) = \mathbb{C}[\partial_0, \partial_1, \partial_2]$ with the action satisfying $\partial_i(x_j) = \delta_{ij}$.

A differential operator $g \in S^*(V^\vee)$ such that $g \cdot f = 0$ is called apolar to f . Differential operators of degree d can be identified with plane curves of degree d in the dual plane.

Apolarity is very well implemented in M2 by the command `diff`, with the caveat that differential operators are written with the same variables x_i of the ring where they act.

Definition 2.1 Denote $P_a = \sum_i a_i \partial_i$. The polar of $f \in \text{Sym}^d V$ at a is $P_a(f) \in \text{Sym}^d V$.

If f corresponds to the symmetric multilinear form $f(x, \dots, x)$ then $P_a(f)$ corresponds to the multilinear form $f(a, x, \dots, x)$. It follows that after d iterations we get

$$P_a^d(f) = \underbrace{P_a \circ \dots \circ P_a}_{d \text{ times}}(f) = d!f(a).$$

An example important for Morley construction is the following, which is discussed in [28]. Take a cubic f with a nodal point Q . Then $P_a(f)$ is the nodal conic consisting of the two nodal lines making the tangent cone at Q .

Note that f depends essentially on ≤ 2 variables (namely it is a cone) if and only if there is a differential operator of degree 1 apolar to f , in this case we say that there is a line apolar to f .

The lines apolar to f make the kernel of the contraction map

$$C_f^1: V^\vee \rightarrow \text{Sym}^3 V$$

In the same way, the conics apolar to f make the kernel of the contraction map

$$C_f^2: \text{Sym}^2 V^\vee \rightarrow \text{Sym}^2 V$$

The map C_f^2 is called the middle catalecticant map and there are conics apolar to f if and only if the middle catalecticant of f vanishes.

In equivalent way, the Clebsch quartics f of section 1.2 can be defined by the condition $\det C_f^2 = 0$.

3 The Aronhold invariant of plane cubics

References for this section: [9, 18, 25, 36].

The Aronhold invariant is the equation of the $SL(3)$ -orbit of the Fermat cubic $x^3 + y^3 + z^3$ in $\mathbb{P}^9 = \mathbb{P}(\text{Sym}^3 \mathbb{C}^3)$. By construction, it is an $SL(3)$ -invariant in the 10 coefficients of a plane cubic. In other words, it is the equation of the 3-secant variety to the 3-Veronese embedding of \mathbb{P}^2 , which is an hypersurface of degree 4.

Theorem 3.1 [18, Ex. 1.2.1], [25, Theor. 1.2] *Let $f(x, y, z) \in \text{Sym}^3 \mathbb{C}^3$ be a homogeneous cubic polynomial. Let $C(f_x)$, $C(f_y)$, $C(f_z)$ be the three 3×3 symmetric matrices which are the Hessian of the three partial derivatives of f .*

All the 8-pfaffians of the 9×9 skew-symmetric matrix

$$\begin{pmatrix} 0 & C(f_z) & -C(f_y) \\ -C(f_z) & 0 & C(f_x) \\ C(f_y) & -C(f_x) & 0 \end{pmatrix} \quad (3.1)$$

coincide (up to scalar) with the Aronhold invariant. Note the beautiful analogy with (1.5).

Let $\text{End}_0 \mathbb{C}^3$ be the space of traceless endomorphisms of \mathbb{C}^3 . The matrix (3.1) describes [25, §2] the contraction

$$A_f : \text{End}_0 \mathbb{C}^3 \rightarrow \text{End}_0 \mathbb{C}^3,$$

which in the case $f = v^3$ satisfies

$$A_{v^3}(M)(w) = (M(v) \wedge v \wedge w) v \quad \forall M \in \text{End} \mathbb{C}^3, \quad \forall w \in \mathbb{C}^3. \quad (3.2)$$

Theorem 3.2 (Nonabelian apolarity for plane cubics) *Let $f = l_1^3 + l_2^3 + l_3^3$ with l_i not collinear linear forms. Then l_i are (simultaneous) eigenvectors of all the matrices $M \in \ker A_f \subset \text{End}_0 \mathbb{C}^3$. This allows to recover l_i from f .*

Proof When $f = v^3$ the equation (3.2) shows that $\text{Ker} A_{v^3} = \{M \in \text{End}_0(\mathbb{C}^3) | v \text{ is an eigenvector of } M\}$. Hence we have the containment

$$\text{Ker} A_f \supseteq \{M \in \text{End}_0(\mathbb{C}^3) | l_i \text{ are eigenvectors of } M\}$$

and equality holds since both spaces are 2-dimensional. \square

Remark 3.3 Let M be a general matrix of $\text{Ker} A_f$. then $\ker A_f$ is obtained as the intersection of $\langle I, M, M^2 \rangle$ with $\text{End}_0 \mathbb{C}^3$. In particular l_i can be found as eigenvectors of M . An efficient implementation is via the numerical command

```
eigenvectors(sub(M, CC))
```

of M2. An alternative way to recover l_i from f is sketched in Proposition 3.5. It is theoretically simpler but not so efficient from the computational point of view. Moreover the technique of Theorem 3.2 can be generalized to other cases, for the case of general plane quintics see [23, Algorithm 1].

The explicit expression of the Aronhold invariant Ar (sometimes called also S in the literature, that we cannot use to avoid ambiguity with the Scorza map) has 25 monomials and it can be found in [36] Prop. 4.4.7 or in [10] (5.13.1), or as output of the following M2 script

```
R=QQ[x,y,z,c_0..c_9]
x1=matrix{{x,y,z}}
x3=symmetricPower(3,x1)
```



```
f=(matrix{{c_0..c_9}}*transpose x3)_(0,0)
m=matrix{{0,z,-y},{-z,0,x},{y,-x,0}}
---following is 9*9 matrix
m9=diff(m,diff(x1,diff(transpose x1,f)))
---following is Aronhold invariant
aronhold=(mingens pfaffians(8,m9))_(0,0)
```

By a slight abuse of notation we denote by Ar also the corresponding multilinear form. The classical symbolic expression for Ar is

$$Ar(x^3, y^3, z^3, w^3) = (x \wedge y \wedge z)(x \wedge y \wedge w)(x \wedge z \wedge w)(y \wedge z \wedge w).$$

Proposition 3.4 *The closure of the orbit $SL(3) \cdot (x^3 + y^3 + z^3)$ contains the following three orbits*

	Equation	Dim	rk
Fermat	$x^3 + y^3 + z^3$	8	3
Cuspidal	$y^2z - x^3$	7	4
smooth conic+tg line	$x(y^2 - xz)$	6	5

These are all the cubics with border rank three. Of course the closure contains also the cubics with border rank ≤ 2 , which make other three orbits.

The Aronhold invariant vanishes on the above three orbits and on all the cubics depending on essentially one or two variables, they have border rank ≤ 2 and they are cones with a point as vertex.

In the classical terminology, a Fermat cubic $l_1^3 + l_2^3 + l_3^3$ has a polar 3-lateral given by $l_1l_2l_3$. A recipe to compute the polar 3-lateral in the first case of Prop. 3.4, alternative to Theorem 3.2 is given by the case (i) of the following Proposition.

- Proposition 3.5** (i) *The Hessian of $l_1^3 + l_2^3 + l_3^3$ factors as $l_1l_2l_3$.*
 (ii) *The Hessian of a cuspidal cubic splits as $l_1^2l_2$ where l_1^2 is the tangent cone at the cusp.*
 (iii) *The Hessian of {smooth conic} \cup {tg line l } is l^3 .*

Proposition 3.6 *Real Fermat cubics[2] The real cubics with complex border rank three make four $SL(3, \mathbb{R})$ -orbits, the Fermat case in Prop. 3.4 splits into the two cases*

- (i') *orbit of $x^3 + y^3 + z^3$, real Fermat*
 (i'') *orbit if $(x + \sqrt{-1}y)^3 + (x - \sqrt{-1}y)^3 + z^3 = 2x^3 - 6x^2y + z^3$, imaginary Fermat*

4 Three descriptions of an even theta characteristic

4.1 The symmetric determinantal description

Let A_0, A_1, A_2 be three symmetric 4×4 matrices. The expression

$$\det(xA_0 + yA_1 + zA_2)$$

defines a plane quartic with an even theta characteristic θ given by

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2)^4 \xrightarrow{A} \mathcal{O}_{\mathbb{P}^2}(-1)^4 \longrightarrow \theta \longrightarrow 0$$

As an example, the Edge quartic

$$f = 25(x^4 + y^4 + z^4) - 34(x^2y^2 + x^2z^2 + y^2z^2) \quad (4.1)$$

studied by Edge in [12, §14] (set $\gamma = -2$) has the symmetric determinantal representation found by Edge

$$\begin{pmatrix} 0 & x + 2y & 2x + z & y - 2z \\ x + 2y & 0 & y + 2z & -2x + z \\ 2x + z & y + 2z & 0 & x - 2y \\ y - 2z & -2x + z & x - 2y & 0 \end{pmatrix}$$

We refer to [30, 34] for the correspondence between the 28 bitangents of the quartic and the lines joining the eight base points of the net of quadrics. The seven lines joining one base point with the other seven base points make an Aronhold system of bitangents, according to Def. 1.1. The seven corresponding points in \mathbb{P}^3 are Gale dual (see [13, 30]) of the seven bitangents.

4.2 The sextic model

Let K be the canonical bundle over a smooth plane quartic. The line bundle $K + \theta$ has 4 independent sections which give a linear system of effective divisors of degree 6. It gives an embedding of the curve as a degree 6 (and genus 3) curve in $\mathbb{P}^3 = \mathbb{P}(H^0(K + \theta)^\vee)$. It has the resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-3)^3 \xrightarrow{M} \mathcal{O}_{\mathbb{P}^3}(-2)^4 \xrightarrow{g} \mathcal{O}_{\mathbb{P}^3}(1) \longrightarrow K + \theta \longrightarrow 0 \quad (4.2)$$

The symmetric determinantal description in 4.1 gives a $4 \times 4 \times 3$ -tensor that we considered as a 4×4 symmetric matrix with three linear entries. The tensor has a second flattening as the 4×3 matrix M with four linear entries. The four maximal minors of M give the cubic equations of the sextic and define the map g in (4.2). The

Edge quartic (4.1) gives the following

$$M = \begin{pmatrix} u_1 + 2u_2 & 2u_1 + u_3 & u_2 - 2u_3 \\ u_0 - 2u_3 & 2u_0 + u_2 & 2u_2 + u_3 \\ 2u_0 + u_3 & u_1 - 2u_3 & u_0 + 2u_1 \\ -2u_1 + u_2 & u_0 - 2u_2 & -2u_0 + u_1 \end{pmatrix}$$

The isomorphism between the quartic model with coordinates (x, y, z) and the sextic model with coordinates (u_0, \dots, u_3) is guaranteed by the system

$$M(u) \cdot (x, y, z)^t = 0. \tag{4.3}$$

Given u such that $\text{rk}M(u) = 2$, the system (4.3) defines a unique (x, y, z) . Conversely, given (x, y, z) on the quartic plane model, the system (4.3) defines a unique (u_0, \dots, u_3) in the sextic space model.

The M2 code to construct M from A is the following

```
R=QQ[x, y, z, u_0..u_3]
A= matrix({{0, x + 2*y, z + 2*x, -2*z + y},
{x + 2*y, 0, 2*z + y, z - 2*x},
{z + 2*x, 2*z + y, 0, x - 2*y},
{-2*z + y, z - 2*x, x - 2*y, 0}});
f=det(A)
uu=transpose matrix {{u_0..u_3}}
M=diff(x,A)*uu|diff(y,A)*uu|diff(z,A)*uu
```

An elegant and alternative way to construct the divisor $K + \theta$ is the following

Theorem 4.1 (Dixon) *Given a symmetric determinantal representation A , the four principal minors define four cubics which are contact cubics, namely they cut the quartic $\det(A)$ in a nonreduced divisor supported on a degree six divisor which is $K + \theta$.*

More generally, if A^{adj} is the adjugate matrix, defined by the signed 3-minors of A (such that $AA^{adj} = (\det A)I_4$), then $u^t A^{adj} u$ parametrizes contact cubics, making a 3-fold of degree 8 in \mathbb{P}^9 .

From the sextic model, one finds four independent sections $u_i \in H^0(K + \theta)$. Then the sections $u_i u_j$ may be lifted to cubics in \mathbb{P}^2 , with proper scaling guaranteed by the equation $(u_i + u_j)^2 = u_i^2 + u_j^2 + 2u_i u_j$. Then the adjugate of the 4×4 matrix $(u_i u_j)$ has degree 9 polynomials which contain f^2 as a factor, after dividing by f^2 we get the corresponding symmetric determinantal description.

Remark 4.2 Dixon Theorem 4.1 generalizes to $d \times d$ symmetric representation, in this case the degree $d(d - 1)/2$ divisor, which is the support of the nonreduced divisor, is $H + \theta$.

Remark 4.3 Note that a general 4×3 matrix M with linear entries in u_0, \dots, u_3 defines a sextic curve of genus 3, but the line bundle which gives the embedding has the form

$K + L$, with L a degree 2 line bundle which is not necessarily a theta characteristic. The condition to be a theta characteristic is equivalent to the fact that the 4×4 matrix with entries in x_0, \dots, x_2 obtained by flattening M may be symmetrized by row/columns operations.

4.3 The (3, 3)-correspondence

References for this section: [8, 10]. Let θ be an even theta characteristic on C of genus 3. It is defined a (3, 3) correspondence from the following divisor on $C \times C$

$$T_\theta = \{(P, Q) \in C \times C \mid h^0(\theta + P - Q) > 0\}. \quad (4.4)$$

It follows from Serre duality that the correspondence is symmetric. By Riemann-Roch, $\forall P \in C$, $\theta + P$ is linearly equivalent to a unique effective divisor of degree 3, hence we get a (3, 3) correspondence.

Remark 4.4 An analogous (g, g) correspondence may be defined starting from a general line bundle L of degree $g - 1$, but this correspondence is in general not symmetric, the symmetry is guaranteed from L being a theta characteristic (see Remark 4.3).

In the sextic model the correspondence has the following form: pick coordinates $(u_0, \dots, u_3), (v_0, \dots, v_3)$ and we have, for any $(u, v) \in C \times C$

$$(u, v) \in T_\theta \iff v \cdot M(u) = 0.$$

This works because $\ker M(u)$ is a 3-secant line of the sextic model (see [33]).

5 The Aronhold covariant of plane quartics and the Scorza map

Main references for this section are [10, 24, 28]. Recall by Definition 2.1 that $P_x f$ is the polar of f at the point x . The Scorza map is defined as

$$f \mapsto (S(f), \theta),$$

where $S(f) = \{x \in \mathbb{P}^2 \mid \text{Ar}(P_x f) = 0\}$ is the Aronhold covariant (the notation with the letter S goes back to Clebsch and has nothing to do with Scorza, see §7 in Ciani's monograph [5]) and the (3, 3)-correspondence T_θ which encodes θ as in (4.4) is defined by

$$\{(x, y) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid \text{rk}(P_x P_y f) \leq 1\}.$$

The main result regarding the Scorza map is the following Theorem, proved by Scorza in 1899[33].

Theorem 5.1 (Scorza) *The map $f \mapsto S(f)$ is dominant and is generically $36 : 1$.*

Since there are 36 even theta characteristic on a general quartic curve, Theorem 5.1 implies that the general pair (C, θ) where C is a plane quartic and θ is an even theta characteristic on C comes from a unique quartic f through the Scorza map.

Assume that $P_x f = l_1^3 + l_2^3 + l_3^3$ and denote $x_{ij} = \{l_i = l_j = 0\}$. The divisor θ is linearly equivalent to the divisor

$$x_{12} + x_{13} + x_{23} - x.$$

This description allows to compute explicitly both the Scorza map and its inverse (see the Algorithm 7). Note that the determinantal description can be obtained in the coefficient field of f if $S(f)$ contains a point x lying in the same field.

Theorem 5.2 [10, §7.3] *If f is Clebsch then $(S(f), \theta)$ is Lüroth with the pentalateral θ . Conversely the general (g, θ) , where g is a Lüroth curve with pentalateral θ , comes from a unique Clebsch curve f such that $S(f) = g$.*

Corollary 5.3 *The infinitely many decomposition of a Clebsch quartic, with $f = \sum_{i=0}^4 l_i^4$ and l_i circumscribed to the conic C , give infinitely many pentalateral inscribed in $S(f)$.*

A reflection on this Theorem allows to understand why the invariant description of Lüroth condition for the pair (f, θ) (regarding net of quadrics, see (1.5)) is much simpler than the one for f itself.

Since the Scorza map is $SL(3)$ -equivariant, we remark the following consequence

Proposition 5.4 *Let $Aut(f)$ be the automorphsim group of linear transformation of \mathbb{P}^2 leaving the quartic f invariant. Then*

$$Aut(f) \subseteq Aut(S(f))$$

is a group inclusion. In particular the order of $Aut(f)$ divides the order of $Aut(S(f))$.

As a corollary, note that both double conics (having $Aut(f) = SL(2)$) and Klein quartic (having $Aut(f)$ the simple group of order 168, the group of higher order among all irreducible quartics) both satisfy $Aut(f) = Aut(S(f))$.

Moreover, the 36-preimages $S^{-1}S(f)$ divide into $SL(3)$ -orbits O_i for $i = 1, \dots, k$, which are the same as $Aut(S(f))$. These orbits are studied in the literature regarding the action of the automorphism group on the even spin structures of the curve.

For any $f \in O_i$ the size of $Aut(f)$ is fixed and we get

$$|O_i| = \frac{|Aut(S(f))|}{|Aut(f)|},$$

which can be used jointly with the obvious identity

$$\sum_{i=1}^k |O_i| = 36.$$

See the comments and the tables before the M2 script of Algorithm 7 in Sect. 9.

6 Contact cubics and contact triangles

For any even theta characteristics θ , the effective divisors of degree six corresponding to $K + \theta$ can be computed with the following trick. Let A be a 4×4 symmetric determinantal representation corresponding to θ , as in 4.1. Any principal minor of A defines a contact cubic, which cuts the quartic in a nonreduced divisor, supported on $K + \theta$. More generally, for any $u = (u_0, \dots, u_3)$ there is a contact cubic given by

$$\det \begin{pmatrix} A & u^t \\ u & 0 \end{pmatrix}. \quad (6.1)$$

This was the classical formula which gives the entries of the adjugate matrix of A , it is quite convenient from the computational point of view when regarding matrices with symbolic entries, compare with Dixon Theorem 4.1.

When the diagonal elements of A are zero we have a further description, indeed the contact cubics are the triangles given by three bitangents, that can be found from the matrix

$$\begin{pmatrix} 0 & l_{01} & l_{02} & l_{03} \\ l_{01} & 0 & l_{12} & l_{13} \\ l_{02} & l_{12} & 0 & l_{23} \\ l_{03} & l_{13} & l_{23} & 0 \end{pmatrix}$$

where l_{ij} are bitangents. The notation l_{ij} for the 28 bitangents come from the 28 pairs joining the eight base points P_0, \dots, P_7 of a net of quadrics in \mathbb{P}^3 , as in §4.1 and it is implemented in Algorithm 3 in §9. In this case the four principal minors are contact triangles, like

$$\det \begin{pmatrix} 0 & l_{01} & l_{02} \\ l_{01} & 0 & l_{12} \\ l_{02} & l_{12} & 0 \end{pmatrix}$$

which gives $l_{01}l_{02}l_{12}$.

There are $\binom{8}{3} = 56$ contact cubics given by three bitangents in the family (6.1) of contact cubics, all together they are $56 \cdot 36 = 2016$. Indeed the three bitangents can be found by the principal 3-minors of the 8×8 bitangent matrix of Algorithm 3. All of these triples $\{\theta_i, \theta_j, \theta_k\}$ have the six contact points which do not lie on a conics, which is equivalent to $h^0(2K - \theta_i - \theta_j - \theta_k) = 0$, exactly as in Definition 1.1. The 36 families of contact cubics correspond to 36 2-Veronese 3folds in \mathbb{P}^9 (of degree 8). They do not meet the 2-secant variety of the 3-Veronese surface, namely the variety of special triangles given by three collinear lines. They meet the variety of all triangles in 120 points. According to Mukai [22], a triangle is biscribed by a curve C if it is both circumscribed and inscribed, that is, each vertex lies on C and each side is a tangent. A secant line \overline{ab} with $a, b \in C$ is a strict tangent if the divisor class $H - a - b - 2r$ is effective for some $r \in C$. A triangle is strictly biscribed if all sides are strict tangents. There are 8 strictly biscribed triangles, according to Mukai [22], each one counts with

multiplicity 8 for a total of 64, indeed note that $56 + 64 = 120 = 8 \cdot 15$ (degree of intersection of Veronese 3fold with the variety of triangles).

There are other 28 families of contact cubics, beyond the 36 families as in (6.1), that can be constructed starting from any bitangent ℓ in the following way. The contact cubics c corresponding to ℓ satisfy the equation $f = \ell c + q^2$ for some conic q . Each of these families is 3-dimensional. In each of these families there are 45 contact triangles given by three bitangents. Note that $45 \cdot 28 = 1260$ and $1260 + 2016 = 3276 = \binom{28}{3}$ so that we have counted exactly once all triples of bitangents. The 1260 triples are special since they have six contact points on a conic. They meet the 3-Veronese surface of triple lines in one point of multiplicity 6, which corresponds to the bitangent at power 3, like l^3 . In any case, the expression $ml^3 + q^2 = f$ does not hold for any line m and conic q . Label the 28 bitangents as ij where $0 \leq i < j \leq 7$, corresponding to the pairs of base points of the net of quadrics, as in 4.1. These 45 contact triangles divide in two types, 30 of them like 12.23.34, other 15 of them like 12.34.56 (see [5, §13]). In the first type there is a fourth bitangent 41 such that the 8 contact points lie on a conic, there are 210 4-ples of this kind. Also in the second type there is a fourth bitangent 07 such that the 8 contact points lie on a conic, there are 105 4-ples of this kind. Altogether, there are $210 + 105 = 315$ 4-ples of bitangents such that the 8 contact points lie on a conic.

7 The invariant ring of plane quartics

It is worth to advertise that the complete determination of invariant ring of plane quartics is a big achievement of computer algebra, the final step was presented at MEGA 2013 in Frankfurt by Andreas-Stephan Elsenhans [14], relying on previous work by Shioda and Dixmier, so solving a classical question which went back to Emmy Noether's doctoral thesis.

The following result was conjectured by Shioda in 1967, who computed the Hilbert series.

Theorem 7.1 *The invariant ring $\left[\bigoplus_d \text{Sym}^d \left(\text{Sym}^4 \mathbb{C}^3 \right) \right]^{SL(3)}$ is generated by invariants of degree 3, 6, 9, 9, 12, 12, 15, 15, 18, 18, 21, 21, 27. The relations are known.*

Dixmier found in 1987 the invariants of degree 3, 6, 9, 12, 15, 18, 27 which are algebraically independent, so that the invariant ring is an algebraic extension of the ring generated by these ones (primary invariants). The invariants up to degree 18 (together with the discriminant of degree 27) can be found in Salmon book[32], compare also with [5, §7]. So only the invariants of degree 21 were missing in the 19th century. Apparently the first who produced the complete generators of invariant ring was T. Ohno in an unpublished work in 2007. Note the elementary fact that the degree of any invariants is divisible by 3. The Clebsch invariant, defining quartics of rank 5, has degree 6. Some classical facts regarding the cubic invariant are recalled in [27].

8 The link with the seven eigentensors of a plane cubic

Let q be a nondegenerate quadratic form on V . Given a plane cubic $f \in \text{Sym}^3 V^\vee$, an eigentensor of f is $v \in V$ such that

$$f(v, v, x) = \lambda q(v, x) \quad (8.1)$$

for every $x \in V$. When q is the euclidean metric (in the real setting), which gives an identification between V and V^\vee , the previous equation can be written as $f(v^2) = \lambda v$, which is the way the eigentensor equation is commonly written in the numerical setting, and it is the natural generalization of the eigenvector condition for symmetric matrices.

In the metric setting, the generalization is more transparent. Indeed, the eigenvectors of a symmetric matrix q are the *critical points of the distance function* from q to the Veronese variety $(\mathbb{P}^2, \mathcal{O}(2))$. In the same way, the seven eigentensors are the seven *critical points of the distance function* from f to the Veronese variety $(\mathbb{P}^2, \mathcal{O}(3))$, see [11].

A dimensional count shows that seven eigenvectors of a cubic cannot be seven general point, so that it is interesting to understand their special position according to (1.3). The following is the geometric counterpart of [1, Prop. 5.3].

Theorem 8.1 (Bateman) (i) *The seven points p_1, \dots, p_7 which are eigentensors of a cubic satisfy the following property:*

the seven nodal conics C_i for $i = 1, \dots, 7$ which correspond to $P_{p_i} H_i$, where H_i is the unique cubic passing through all p_j and singular at p_i are harmonic, that is $\Delta C_i = 0$, where Δ is the Laplacian.

(ii) *Seven points v_i are eigentensors of a cubic f with respect to some nondegenerate conic q as in (8.1) if and only the seven points give a Lüroth quartic in the correspondence (1.3).*

Proof Part (i) is a reformulation of Morley differential identity, see formula (20) in §9 of [28], where $\theta = x_0^2 + x_1^2 + x_2^2$ and the fact that (with the notations of [28]), the Morley form $M(P_i, X)$ coincides with $H_i(X)$, see the paragraph after Corollary 3.3 in [28]. Part (ii) is a reformulation of Theorem 10.4 in [28] \square

9 Eight algorithms and Macaulay2 scripts, with a tutorial

Algorithm 1 • *INPUT: seven general lines l_1, \dots, l_7*

• *OUTPUT: the quartic having l_1, \dots, l_7 as Aronhold system of bitangents*

The steps of the algorithm are the following

1. Compute the 2×3 matrix

$$\begin{pmatrix} a_0(x) & a_1(x) & a_2(x) \\ q_0(x) & q_1(x) & q_2(x) \end{pmatrix}$$

with $\deg a_i = 1$, $\deg q_i = 2$ degenerating on $Z = \{l_1, \dots, l_7\}$ seen as seven points in the dual space. This matrix is computed from the resolution of the ideal vanishing on the points and it can be taken as input (as in Algorithm 2).

2. Construct the net of cubics passing through Z as

$$\text{net}(x, y) = \det \begin{pmatrix} a_0(y) & a_1(y) & a_2(y) \\ a_0(x) & a_1(x) & a_2(x) \\ q_0(x) & q_1(x) & q_2(x) \end{pmatrix}$$

3. Construct the jacobian of the net as the determinant of the 3×3 matrix

$$\text{jac}(x) = \det \left[\frac{\partial^2 \text{net}}{\partial x_i \partial y_j} \right]$$

which is a sextic (in coordinates x) nodal at the seven points.

4. The quartic in output is obtained by eliminating x from the four equations $\text{jac}(x)$ and $y_i - \frac{\partial \text{net}(x, y)}{\partial y_i}$.

As a running example, start with input given the seven points which are the rows of the following matrix

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & -2 \\ 5 & 5 & 3 \\ 5 & -3 & 5 \\ 3 & 5 & -5 \\ -1 & 1 & 1 \end{pmatrix}$$

get as output the quartic

$$f = 25y_0^4 - 34y_0^2y_1^2 + 25y_1^4 - 34y_0^2y_2^2 - 34y_1^2y_2^2 + 25y_2^4.$$

In alternative, start from

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ (3x_0^2 + 2x_0x_1 + x_2^2) & (x_0^2 + 2x_1x_2) & (x_1^2 + 2x_0x_2) \end{pmatrix}$$

where the second row is the gradient of $x_0^2x_1 + x_1^2x_2 + x_2^2x_0 + x_0^3$ and get as output the Lüroth quartic

$$f = 2y_0^3y_1 - 2y_0^2y_1^2 - y_0y_1^3 - y_0^3y_2 - 2y_0^2y_1y_2 + 4y_0y_1^2y_2 + 2y_1^3y_2 + y_0^2y_2^2 - 2y_0y_1y_2^2 + y_1^2y_2^2 - y_0y_2^3 - y_1y_2^3.$$

The M2 script is the following

```

KK=QQ
R=KK[x_0..x_2,y_0..y_2]
x1=matrix{{x_0..x_2}}
y1=matrix{{y_0..y_2}}
----the rows of p2 contain the seven points,
below are shown two samples
p2=random(R^{7:0},R^{3:0})
p2=matrix{{1,2,0},{2,0,1},{0,1,-2},{5,5,3},
{5,-3,5},{3,5,-5},{-1,1,1}}
---computation of the 2*3 matrix mat
I7=minors(2,p2^{0}||x1)
for i from 0 to 6 do I7=intersect(I7,
minors(2,p2^{i}||x1))
r7=res I7
betti r7
mat=transpose r7.dd_2
-----in alternative one can start from a
2*3 matrix as above
---we construct now the Morley form
nc=(sub((mat)^{0},apply(3,i->(x_i=>y_i))||mat)
net2=diff(matrix{{y_0..y_2}},det(nc))
---jacobian of the net, is a plane sextic nodal
at seven points
jac=ideal(det(diff(transpose x1,net2)))
-----get the quartic f eliminating x_i from the net
of cubics and the jacobian
ff=eliminate({x_0,x_1,x_2},ideal(matrix{{y_0..y_2}}
-net2)+jac)
f=((gens ff)_(0,0))
----f is our OUTPUT
---check the seven starting points are really bitangents
for i from 0 to 6 do print(i, degree ideal(f,(p2^{i}
*transpose y1)_(0,0)),
degree radical ideal(f,(p2^{i}*transpose y1)_(0,0)))

```

Algorithm 2

- *INPUT*: a 2×3 matrix with 2-minors vanishing on $Z = \{l_1, \dots, l_7\}$
- *OUTPUT*: the quartic having Z as Aronhold system of bitangents

This is just the combination of steps 2...4 of Algorithm 1.

A typical application is the matrix $\begin{pmatrix} x_1 & x_2 & x_0 \\ x_0^2 & x_1^2 & x_2^2 \end{pmatrix}$ which gives the Klein quartic $y_0 y_1^3 + y_1 y_2^3 + y_0^3 y_2$. Let $\tau = e^{2\pi\sqrt{-1}/7}$ a 7-th root of unity. In this case, the seven lines in Z

are represented by the columns of the matrix

$$(y_0 \ y_1 \ y_2) \cdot \begin{pmatrix} 1 & \tau^3 & \tau^6 & \tau^2 & \tau^5 & \tau & \tau^4 \\ 1 & \tau^5 & \tau^3 & \tau & \tau^6 & \tau^4 & \tau^2 \\ 1 & \tau^2 & \tau^4 & \tau^6 & \tau & \tau^3 & \tau^5 \end{pmatrix}$$

but the script does not need τ and works on the field where the 2×3 matrix is defined.

The M2 script is the following

```
KK=QQ
R=KK[x_0..x_2,y_0..y_2]
x1=matrix{{x_0..x_2}}
y1=matrix{{y_0..y_2}}
----with the following matrix we find Klein quartic !!
mat=matrix{{x_1,x_2,x_0},{x_0^2,x_1^2,x_2^2}}
nc=(sub((mat)^{0},apply(3,i->(x_i=>y_i)))|mat)
net2=diff(matrix{{y_0..y_2}},det(nc))
codim ideal net2, degree ideal net2
---jacobian of the net, is a plane sextic nodal
at seven points
jac=ideal(det(diff(transpose x1,net2)))
----get the quartic f eliminating x_i from the net
of cubics and the jacobian
ff=eliminate({x_0,x_1,x_2},ideal(matrix{{y_0..y_2}}
-net2)+jac)
f=((gens ff)_(0,0))
---f is our output
```

Algorithm 3 • *INPUT: seven general lines l_1, \dots, l_7*

- *OUTPUT: a 8×8 symmetric matrix (the bitangent matrix) collecting in each row the 8 Aronhold systems of bitangents for the quartic having l_1, \dots, l_7 as Aronhold system of bitangents, equivalent to l_1, \dots, l_7 . In particular, every principal 4×4 minor of the bitangent matrix gives a symmetric determinantal representation.*

These are the steps

1. Write the coordinates of the seven points $\{l_1, \dots, l_7\}$ as columns of a 3×7 matrix P and compute a Gale transform in \mathbb{P}^3 given by the seven rows $\{m_1, \dots, m_7\}$ of a 7×4 matrix M such that $PM = 0$. These are well defined modulo $SL(4)$ -action.
2. Compute the net of quadrics through $\{m_1, \dots, m_7\}$, spanned by the 4×4 symmetric matrices Q_0, Q_1, Q_2 .
3. Compute the symmetric determinantal representation of the quartic curve as $\det(\sum_i x_i Q_i)$.
4. Compute (with a saturation), the eighth base point m_0 of the net and correspondingly stack it as a first row over M so obtaining the 8×4 matrix M_0 . The theory guarantees that the 28 lines $m_i m_j$ for $0 \leq i < j \leq 7$ are the 28 bitangent of the quartic constructed in the previous step, but we have to apply a linear projective transformation as in next step.

5. Compute the linear projective transformation g from the space of lines in \mathbb{P}^3 through m_0 to our dual \mathbb{P}^2 which takes the line m_0m_i to l_i for $i = 1, \dots, 7$.
6. The bitangent matrix is $M_0(\sum_i g(x)_i Q_i)M_0^t$ (see [30, (3.4)]).

An example is given by the following seven lines in input

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_0 + x_1 + x_2 \\ x_0 + 2x_1 + 3x_2 \\ 2x_0 + 3x_1 + x_2 \\ 3x_0 + x_1 + 2x_2 \end{pmatrix} \text{ The}$$

corresponding quartic is

$$81x_0^4 + 198x_0^3x_1 - 41x_0^2x_1^2 - 198x_0x_1^3 + 81x_1^4 - 198x_0^3x_2 - 1561x_0^2x_1x_2 - 1561x_0x_1^2x_2 + 198x_1^3x_2 - 41x_0^2x_2^2 - 1561x_0x_1x_2^2 - 41x_1^2x_2^2 + 198x_0x_2^3 - 198x_1x_2^3 + 81x_2^4$$

and the bitangent matrix in output has the following first three columns (the whole matrix is too big to be printed, it can be found by running the M2 script).

$$\begin{pmatrix} 0 & 3x_0 & 3x_1 & 3x_2 & (777/143)x_0 + (1239/143)x_1 - (378/143)x_2 & (777/143)x_0 + (1239/143)x_1 - (378/143)x_2 \\ 3x_0 & 0 & 3x_1 & 3x_2 & (1239/143)x_0 - (378/143)x_1 + (777/143)x_2 & (1239/143)x_0 - (378/143)x_1 + (777/143)x_2 \\ 3x_1 & (777/143)x_0 + (1239/143)x_1 - (378/143)x_2 & 0 & 3x_1 & (1239/143)x_0 - (378/143)x_1 + (777/143)x_2 & (1239/143)x_0 - (378/143)x_1 + (777/143)x_2 \\ 3x_2 & (1239/143)x_0 - (378/143)x_1 + (777/143)x_2 & (777/143)x_0 + (1239/143)x_1 - (378/143)x_2 & 0 & (1239/143)x_0 - (378/143)x_1 + (777/143)x_2 & (1239/143)x_0 - (378/143)x_1 + (777/143)x_2 \\ 11x_0 + 11x_1 + 11x_2 & (126/13)x_0 + (287/13)x_1 + (133/13)x_2 & (126/13)x_0 + (287/13)x_1 + (133/13)x_2 & (126/13)x_0 + (287/13)x_1 + (133/13)x_2 & (126/13)x_0 + (287/13)x_1 + (133/13)x_2 & (126/13)x_0 + (287/13)x_1 + (133/13)x_2 \\ -x_0 - 2x_1 - 3x_2 & -(756/143)x_0 - (448/143)x_1 - (525/143)x_2 & -(756/143)x_0 - (448/143)x_1 - (525/143)x_2 & -(756/143)x_0 - (448/143)x_1 - (525/143)x_2 & -(756/143)x_0 - (448/143)x_1 - (525/143)x_2 & -(756/143)x_0 - (448/143)x_1 - (525/143)x_2 \\ -2x_0 - 3x_1 - x_2 & -(189/143)x_0 - (1113/143)x_1 + (119/143)x_2 & -(189/143)x_0 - (1113/143)x_1 + (119/143)x_2 & -(189/143)x_0 - (1113/143)x_1 + (119/143)x_2 & -(189/143)x_0 - (1113/143)x_1 + (119/143)x_2 & -(189/143)x_0 - (1113/143)x_1 + (119/143)x_2 \\ -3x_0 - x_1 - 2x_2 & -(84/143)x_0 - (161/143)x_1 - (392/143)x_2 & -(84/143)x_0 - (161/143)x_1 - (392/143)x_2 & -(84/143)x_0 - (161/143)x_1 - (392/143)x_2 & -(84/143)x_0 - (161/143)x_1 - (392/143)x_2 & -(84/143)x_0 - (161/143)x_1 - (392/143)x_2 \end{pmatrix}$$

The M2 script is the following

```
restart
R=QQ[x_0..x_2,y_0..y_3]
x1=matrix{{x_0..x_2}}
y2=symmetricPower(2,matrix{{y_0..y_3}})
y1=matrix{{y_0..y_3}}
p2=matrix{{1,2,0},{2,0,1},{0,1,-2},{5,5,3},
{5,-3,5},{3,5,-5},{-1,1,1}}
--- p2 contains seven points in P2
-----p3 contains the seven points in P^3 which
are the Gale transform
p3=gens kernel transpose p2
---p3 contains the seven points in P3 which are Gale
dual, defined up to PGL(4)
qp3=symmetricPower(2,p3^{0})
for i from 1 to 6 do qp3=qp3||symmetricPower
(2,p3^{i})
net3=y2*gens kernel qp3
---net3 is the net of quadrics in P3
---note that even if the net is defined up to PGL(4),
----the locus of singular quadrics is well defined
in the net
```

```

----the important fact is that the net of quadrics
is dual to the original  $P^2$ 
for i from 0 to 2 do Q_i=diff(transpose y1,diff
( y1,net3_(0,i)))
--for i from 0 to 2 do print Q_i
f=det(sum(3,i->x_i*diff(transpose y1,diff
( y1,net3_(0,i))))))
---f is the plane quartic curve, living in projective
plane of nets of quadrics,
---it is not yet in the right coordinate system
----now we use the eighth point to make a projectivity
with starting plane
pp3=minors(2,p3^{0}||y1)
for i from 1 to 6 do pp3=intersect(pp3,
minors(2,p3^{i}||y1))
codim pp3, degree pp3
---pp3 is the ideal of seven points in  $P^3$ , obtained by
Gale transform
eighth=saturate(ideal(net3),pp3)
cord8=sub(matrix{apply(4,i->(y_i%eighth))},y_3=>1)
---cord8 contains coordinates of 8th point.
cc=sub(net3,apply(4,i->(y_i=>p3_(0,i)+2*cord8_(0,i))))
for i from 1 to 6 do cc=cc||sub(net3,apply(4,j->
(y_j=>p3_(i,j)+2*
cord8_(0,j))))
----cc contains the coordinates of the seven
lines through the 8th point,
---in the coordinates of the net
---now we look for projectivity between cc and p2, in the
SAME order of points
R1=QQ[g_0..g_8]
gg=transpose genericMatrix(R1,3,3)
----following are set of points we want to
make in projection
----we impose conditions to a unknown projective
transformation gg
p21=sub(p2,R1),cc1=sub(cc,R1)
IG=minors(2,p21^{0}*gg||cc1^{0})
for i from 1 to 3 do IG=IG+minors(2,p21^{i}*gg||cc1^{i})
codim IG, degree IG
for i from 4 to 6 do IG=IG+minors(2,p21^{i}*gg||cc1^{i})
codim IG, degree IG
gc=sub(matrix{apply(9,i->(g_i)%IG)},g_8=>1)
gm=submatrix(gc,{0..2})||submatrix(gc,{3..5})||submatrix
(gc,{6..8})
----gm is the projective transformation found

```

```

det gm
gi=inverse (sub(gm,QQ))
ff=sub(f,apply(3,i->(x_i=>(gi*transpose x1)_(i,0))))
----ff is the quartic in the starting(correct)
coordinate system
---qq is the determinantal representation of the quartic
-- with given Aronhold system of bitangents
qq=sum(3,i->((gi*transpose x1)_(i,0))*Q_i)
factor(det(qq))
---we want to find now the bitangent matrix,
according to Hesse
---np3 cotains the eight base points of the net
np3=cord8||p3
btm=np3*qq*(transpose np3)*(1/20)
---btm is the bitangent matrix

---check that all 28 lines are bitangents
for i from 0 to 6 do for j from i+1 to 7 do print(i,j,
degree radical ideal(ff,btm_(i,j)))

```

Algorithm 4

- *INPUT*: a 2×3 matrix with 2-minors vanishing on $Z = \{l_1, \dots, l_7\}$
- *OUTPUT*: a symmetric determinantal 4×4 representation of the quartic having Z as Aronhold system of bitangents

Starting with the the matrix $\begin{pmatrix} x_1 & x_2 & x_0 \\ x_0^2 & x_1^2 & x_2^2 \end{pmatrix}$ which gives the Klein quartic $y_0y_1^3 + y_1y_2^3 + y_0^3y_2$ we find the following determinantal representation

$$\begin{pmatrix} 2x_2 & 0 & 0 & -x_1 \\ 0 & 2x_0 & 0 & -x_2 \\ 0 & 0 & 2x_1 & -x_0 \\ -x_1 & -x_2 & -x_0 & 0 \end{pmatrix}$$

These are the steps of the algorithm

1. Compute the net of cubics through the seven points and choose two independent cubics in the net. In the matrix description, these are given by two 2×2 minors.
2. With a saturation, compute the two additional points where the two cubics vanish and compute the 4-dimensional space of conics $\langle Q_0(x), \dots, Q_3(x) \rangle$ vanishing at these two points.
3. A Gale transform in \mathbb{P}^3 can be found by eliminating x_i from the net of cubics and from the four equations $y_i - Q_i(x)$ for $i = 0, \dots, 3$. The result is an ideal p_3 in coordinates y generated by three quadrics and one cubic, vanishing in seven points. With these conventions, the net net_3 of the three quadrics $\langle F_0(y), \dots, F_2(y) \rangle$ vanish on a eighth point which has coordinates $(y_0, y_1, y_2, y_3) = (0, 0, 0, 1)$.

4. We eliminate y_i from the ideal generated by $x_i - F_i(y)$ and from the three cubics (in y) obtained by eliminating y_3 from p_3 . The result is a net of cubics (in x_i) that we call I_7 net.
5. We find a linear 3×3 transformation in coordinates x_i from the net of plane cubics we started with, to the net I_7 net, after this linear transformation is applied to $F_i(y)$, the resulting net of quadrics gives the determinantal representation in output.

```

KK=QQ
R=KK[x_0..x_2]
x1=matrix{{x_0..x_2}}
mat=matrix{{x_1,x_2,x_0},{x_0^2,x_1^2,x_2^2}}
----we redefine I7 accordingly, so that I7
comes from mat
I7=gens minors(2,mat)
---chose two cubics in the minors of mat
and find two additional points.
twocub=ideal submatrix(gens minors(2,mat),,{0,1})
quad=super basis(2,saturate(twocub,ideal
I7)) ---four conics in x_i
R1=KK[x_0..x_2,y_0..y_3,g_0..g_8]
x1=matrix{{x_0..x_2}}
x3=symmetricPower(3,matrix{{x_0..x_2}})
y1=matrix{{y_0..y_3}}
---we find now p3, gale transform
p3=eliminate({x_0,x_1,x_2},sub(ideal I7,R1)
+ideal(y1-sub(quad,R1)))
codim p3, degree p3
betti p3
----p3 is seven points in P^3, three quadrics
and one cubic in y_i
----net3, only the three quadrics in y_i remain
net3=submatrix(gens p3,,{0..2})
codim ideal net3, degree ideal net3
i7net=eliminate({y_0,y_1,y_2,y_3},ideal(matrix
{{x_0,x_1,x_2}}-net3)+
eliminate({y_3},p3))
codim i7net, degree i7net
dnet3=diff(symmetricPower(3,matrix{{x_0..x_2}}),
transpose gens i7net)
gg=transpose genericMatrix(R1,g_0,3,3)
----we find now a 3*3 linear transformation
from I7 to I7net
----we stack to I7, considered as 3*10 matrix,
each of the cubics in I7net
--- (as a 10-dimensional vector), transformed
with a unknown 3*3 matrix gg

```

```

---and we ask that the resulting 4*10 matrix
  is degenerate
----if all the 4*4 minors are too many,
one may select only some of them
IG=ideal(0_R1)
for j from 0 to 2 do IG=IG+minors(4,diff(x3,
sub(sub((I7)_(0,j),R1),
apply(3,i->(x_i=>((gg)*transpose x1)_(i,0))))
||dnet3)
time SIG=saturate (IG,ideal(det(gg)));
codim SIG, degree SIG
----with Edge quartic we find three solutions,
--the following decompose command can be
skipped if SIG vanishes on just 1 point
dgg=decompose SIG
gc=sub(matrix{apply(9,i->(g_i)%dgg_0)},g_8=>1)
gm=submatrix(gc,{0..2})||submatrix(gc,{3..5})|
|submatrix(gc,{6..8})
----gm is the projective transformation found
nq3=net3*(transpose gm)*transpose x1
----following mrep is the determinantal
representation found
mrep=diff(transpose y1,diff( y1,nq3))

```

Algorithm 5

- *INPUT*: a quartic f
- *OUTPUT*: the image $S(f)$ through the Scorza map

The script uses the expression of the Aronhold invariant in [18], example 1.2.1. Let $H(f_x)$ be the Hessian matrix of the cubic f_x . All the 8-Pfaffians of the following 9×9 matrix (written in 3×3 block form) coincide, up to scalar, with the equation of $S(f)$.

$$\begin{bmatrix} 0 & H(f_z) & -H(f_y) \\ -H(f_z) & 0 & H(f_x) \\ H(f_y) & -H(f_x) & 0 \end{bmatrix}$$

In the M2 script we check that starting from $f = x^4 + y^4 + z^4 + 2\alpha(x^3y + y^3z + z^3x) - \alpha(x^2y^2 + y^2z^2 + x^2z^2) + (\alpha - \frac{2}{3})(xy^3 + yz^3 + zx^3) - 4(x^2yz + xy^2z + xyz^2)$, with $\alpha = \frac{-1+\sqrt{-7}}{2}$, then $S(f) = xy^3 + yz^3 + zx^3$.

```

K2=toField ((ZZ/31991)[t]/ideal(t^2+7))
R=K2[y_0..y_2]
xx=basis(1,R)
a=(-1+t)/2
---change f in following line
f=y_0^4+y_1^4+y_2^4+2*a*(y_0^3*y_1+y_1^3*y_2+y_2^3*y_0)
-a*(y_0^2*y_1^2+y_1^2*y_2^2+y_0^2*y_2^2)+

```



```
(a-2/3)*(y_0*y_1^3+y_1*y_2^3+y_2*y_0^3)-4*(y_0^2*y_1*y_2
+y_0*y_1^2*y_2+y_0*y_1*y_2^2)
m=matrix({0,y_2,-y_1},{-y_2,0,y_0},{y_1,-y_0,0})
m9= diff(m,diff(xx,diff(transpose xx,f)));
scorzamap=(mingens paffians(8,m9))_(0,0)
---output is y_0*y_1^3+y_0^3*y_2+y_1*y_2^3
```

Algorithm 6

- *INPUT: a quartic f and a point q ∈ S(f)*
- *OUTPUT: a determinantal representation of the image (S(f), θ) through the Scorza map*

The steps are the following

1. Compute the polar $P_q(f)$, which is a Fermat cubic curve since by assumption $q \in S(f)$. The Hessian determinant of $P_q(f)$ was classically called the Polihessian and it splits in three lines, its singular locus gives the three points in $S(f)$ corresponding to q in the (3, 3)-correspondence defined by the θ . Add to the divisor given by these three points a generic hyperplane divisor, call DP the resulting degree 7 divisor. We want to find all the effective divisors of degree 6 linearly equivalent to $DP - q$. To achieve this goal call $SE = R/(S(f))$ the quotient ring, call $IP/(S(f))$ and $I_q/(S(f))$ the images respectively of the ideals of DP and q in the quotient ring, pick the first generator h of $IP/(S(f))$ and compute $h \cdot I_q/(S(f)) : IP/(S(f))$. This is an ideal generated by four cubics that represent the four generators of $H^0(C, K + \theta)$, in the following sense. The common base locus of C_i is a degree six effective divisor on $S(f)$, say $K + \theta$. Each cubic of the system vanishes on $K + \theta$ and in additional six points which define a linearly equivalent degree six effective divisor. Call C_i , for $i = 0, \dots, 3$, the representatives of these cubics in the original ring R . Introduce new variables u_0, \dots, u_3 and eliminate (x, y, z) from the ideal generated by $S(f)$ and $u_i - C_i(x, y, z)$. We get the ideal of the sextic curve in \mathbb{P}^3 which is defined by the linear system $K + \theta$.
2. The resolution of the ideal of the sextic curve in \mathbb{P}^3 found at previous step gives a 4×3 matrix C with linear entries. Note that a different flattening gives a 4×4 matrix with linear entries, but we have to find a symmetric representative of it.
3. The correct relation between coordinates u_i and (x, y, z) is given by the relation $C \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$ (see 4.3). After elimination of u_i we find that the polynomials $u_i u_j$ are cubic polynomials in (x, y, z) . We construct the matrix md with entries $u_i u_j$ which are cubic polynomials in x, y, z Main problem is to correctly scale them. To scale we find separately the cubic polynomial corresponding to $(u_i + u_j)^2$ (recorded in matrix ad) and we use the identity $(u_i + u_j)^2 = u_i^2 + u_j^2 + 2u_i u_j$
4. We conclude last step with a symmetric 4×4 matrix pmd with (i, j) entry given by the cubic polynomial corresponding to $u_i u_j$. Now we use Dixon technique, as explained in [30], we adjugate pmd and we factor out $(S(f))^2$ from each entry. The output is the wished linear determinantal representation.

A nice example is to start from

$$f = x^4 + y^4 + z^4 + (-6\sqrt{2} - 6)(x^2y^2 + x^2z^2 + y^2z^2)$$

which satisfies $S(f) = x^4 + y^4 + z^4$.

Choosing the point $q = (1, 0, e^{(\pi/4)\sqrt{-1}}) \in S(f)$ we get the following symmetric determinantal representation of the Fermat quartic (set $\alpha = \sqrt{-2} + \sqrt{-1}$ and $\beta = \sqrt{-2} - \sqrt{-1}$)

$$\begin{pmatrix} (-\alpha - 1)x + (\beta - 2\sqrt{2} + 5)z & (\alpha + 1)y & (\beta - 1)x + (-\beta - 1)z & (-\beta + 1)y \\ (\alpha + 1)y & (\beta - 1)x + (2\sqrt{-2} - \sqrt{-1} - \sqrt{2} + 3)z & (-\beta + 1)y & (\alpha + 1)x + (-\alpha + 3)z \\ (\beta - 1)x + (-\beta - 1)z & (-\beta + 1)y & (\alpha + 1)x + (\alpha - 1)z & (-\alpha - 1)y \\ (-\beta + 1)y & (\alpha + 1)x + (-\alpha + 3)z & (-\alpha - 1)y & (-\beta + 1)x + (\sqrt{-1} - \sqrt{2} + 1)z \end{pmatrix}$$

Actually the determinant is $-32\sqrt{-1}(x^4 + y^4 + z^4)$. This result leads to great admiration for Edge who, without a computer, finds in [12, §12] other determinantal representations of the Fermat quartic curve, with essentially the same technique but, due to computational tricks available only in the Fermat example, his output turns out to be simpler and more elegant !

A second example is the quartic f with Waring decomposition given by $f = x^4 + y^4 + (x + y)^4 + (x + y + z)^4 + (x + 2y + 3z)^4 + (-5x + 7y - 11z)^4$. Note that the point $(0, 0, 1)$ belongs to the first three summands, so that the polar cubic f_z is Fermat and $(0, 0, 1) \in S(f)$. The output found with the algorithm is messy but can be found in less than one minute on a PC. We do not print it here.

The M2 script is the following

```
-----INPUT: f, and a point q in S(f)
-----OUTPUT: a determinantal
representation of S(f)
K2=QQ
--K2=toField(QQ[t]/ideal(t^2-2))
R=K2[x,y,z]
x2=basis(2,R)
x1=basis(1,R)
f=x^4+y^4+(x+y)^4+(x+y+z)^4+(x+2*y+3*z)^4
+(-5*x+7*y-11*z)^4---the point q=(0,0,1)
--- kills first three summands, so it
belongs to S(f)
-----
hes=det diff(transpose x1,diff(x1,diff(z,f)))
---it is the polohessian triangle
singhes=ideal(diff(x1,hes))
---computation of S(f), called scf
m=matrix{{0,z,-y},{-z,0,x},{y,-x,0}}
scf=(mingens(pfaffians(8,diff(x1,diff
(transpose x1,diff(m,f))))))_(0,0)
sub(scf,{x=>0,y=>0,z=>1})
----we have the theta given by singhes-PP
---now we add the hyperplane divisor and we
find a degree six
```

```

---effective divisor linearly equivalent

SE=R/(scf)
DN= ideal(x,y)--negative part of divisor (1 point)
DP= intersect(sub(singhes,SE),ideal(random(1,SE)))
--positive part
---of divisor (3+4=7 points)
h=(gens DP)_(0,0)
LD=((h*DN) : DP);---four cubics which
  have base points
---on the effective divisor and give the embedding
  of scf in P^3
betti LD
---from the four cubics vanishing on the six
  points on scf
---we get the sextic embedding in P^3
----by computing linear relation between (u,x)
  with Groebner basis
R2=K2[x,y,z,u_0..u_3]
ld=sub(LD,R2)

J=ideal(sub(scf,R2),u_0-(gens ld)_(0,0),
u_1-(gens ld)_(0,1),
u_2-(gens ld)_(0,2),u_3-(gens ld)_(0,3))
----we offer two alternatives to compute
  the 4*3 cs2 matrix
---the first one, commented
below is more direct
--rJ=res eliminate({x,y,z},J)
--cs2=(rJ.dd)_2;
---the second alternative turns out to be
computationally cheaper
JJ=gens gb(J);
betti JJ
cs1=diff(transpose matrix({x,y,z}),
submatrix(JJ,,{0..3}));
cs2=transpose cs1;

-----cs2 is the 4*3 matrix

----the following commands reconstruct the 4*4
  determinantal representation
----from the 4*3 matrix, which
  should be "symmetrizable"
-----we want first to construct the matrix
  pmd with entries u_i*u_j

```

```

--- which are cubic polynomials in x,y,z
-----main problem is to correctly scale them
-----to scale we use the identity  $(u_i+u_j)^2$ 
= $u_i^2+u_j^2+2*u_i*u_j$ 

----md is the matrix pmd, up to scale,
this elimination procedure is crucial
--- and works only for "symmetrizable" 4*3 matrix
md=mutableMatrix(R2,4,4)
--next loop runs in some seconds
for i from 0 to 3 do for j from i to 3 do
md_(i,j)=(gens eliminate({u_0,u_1,u_2,u_3}
,saturate(ideal(cs2*
transpose matrix{{x,y,z}} )+ideal(u_i*u_j),
ideal(u_0,u_1,u_2,u_3))))_(0,0)
for i from 0 to 3 do for j from 0
to i-1 do md_(i,j)=md_(j,i)
-----ad contains the auxiliary
polynomials  $(u_i+u_j)^2$ 
ad=mutableMatrix(R2,4,4)
for i from 0 to 3 do for j from i to 3 do
ad_(i,j)=(gens eliminate({u_0,u_1,u_2,u_3}
,saturate(ideal(
cs2*transpose matrix{{x,y,z}} )
+ideal((u_i+u_j)^2),
ideal(u_0,u_1,u_2,u_3))))_(0,0)
-----S2 is the ring with the coefficients
p_(i,j), q_(i,j) needed to scale
S2=K2[x,y,z,u_0..u_3,p_(0,0)
..p_(3,3),q_(0,0)..q_(3,3)]
md=sub(matrix md,S2)
ad=sub(matrix ad,S2)
x3=symmetricPower(3,matrix{{x,y,z}})

-----I contains the conditions
needed in order to scale
I=ideal()
for i from 0 to 2 do for j from i+1 to 3 do
I=I+ideal(diff(x3,p_(i,i)*md_(i,i)
+p_(j,j)*md_(j,j)+
2*p_(i,j)*md_(i,j)-q_(i,j)*ad_(i,j)))
codim I, degree I
-----pp contains the solution
to the system given by I,
---this works if  $q_(2,3)$ 
is different from zero,

```

```

---otherwise the solution to
I has to be extracted in alternative way
pp=mutableMatrix(S2,4,4)
for i from 0 to 3 do for j from
  i to 3 do pp_(i,j)=(sub(p_(i,j)%I,q_(2,3)=>1))
pp
-----pmd is our goal matrix
pmd=mutableMatrix(S2,4,4)
for i from 0 to 3 do for j from i to 3 do
pmd_(i,j)=pp_(i,j)*md_(i,j)
for i from 0 to 3 do for j from 0 to
i-1 do pmd_(i,j)=pmd_(j,i)
pmd=matrix(pmd)

-----now we use Dixon technique, we adjugate pmd
---and we factor out (scf)^2 from each entry
mdf=mutableMatrix(S2,4,4)
for i from 0 to 3 do for j from 0 to 3
do mdf_(i,j)=(-1)^(i+j)*
(quotientRemainder(det submatrix'(pmd,{i},{j}),
sub((scf)^2,S2)))_0
---mdf is our output
mdf=matrix mdf
factor det mdf

```

Algorithm 7

- *INPUT*: a determinantal representation of a quartic g corresponding to (g, θ)
- *OUTPUT*: the quartic f such that the image $(S(f), \theta)$ through the Scorza map corresponds to (g, θ)

The steps are the following

1. We find the sextic model of the quartic in \mathbb{P}^3 by transforming the $4 \times 4 \times 3$ representation in a 4×3 matrix $M(u)$ with linear entries in $u = (u_0, \dots, u_3)$. The maximal minors of M define the sextic model given by the linear system $K + \theta$.
2. We define the $(3, 3)$ correspondence introducing dual coordinates $v = (v_0, \dots, v_3)$ and the ideal cor33 generated by the maximal minors of $M(u)$, $M(v)$ and by the bilinear equations given by $u \cdot M(v)$ and $v \cdot M(u)$.
3. We lift the $(3, 3)$ -correspondence to the quartic model with dual coordinates $a = (a_0, \dots, a_2)$ and $b = (b_0, \dots, b_2)$. This works by eliminating u and v from the saturation over u and v of the ideal generated by cor33 and by the bilinear equations $M(u) \cdot a^t$ and $M(v) \cdot b^t$. We get 6 biquadratic equations in (a, b) that we call eli6 . This step is computationally expensive. At the end we convert the six equations in a 6×36 matrix, where the 36 columns correspond to the 36 biquadratic monomials in (a, b) , called maab .

4. Let f_c be a generic quartic given by 15 entries c_i , we compute the 6 biquadratic equations in (a, b) obtained by the condition $\text{rk}(P_a P_b f_c) \leq 1$. We convert these equations in a 6×36 matrix, called mab , with entries depending quadratically on c_i .
5. We impose that the rows of two matrices maab and mab span the same 6-dimensional space. This gives quadratical equations in c_i . *Note that Scorza map has entries which are quartic in c_i , so that this “tour de force” has an effective gain at the end.* In principle this can be obtained by stacking each row of mab to maab and computing all the (maximal) 7×7 -minors. This is computationally too expensive, so we choose only some of these minors. For general quartics, the minors which contain the first 6 columns suffice. In the M2 script it is sketched an alternative choice of minors which works for the Klein quartic.
6. The ideal of the quadratical equations in c_i obtained in previous step has to be saturated with base locus of Scorza map. The output is a unique point in c_i which gives our output quartic. If the output is bigger, we have to repeat the previous step by adding more minors.

As a running example, we may start from the Edge quartic in (4.1) with the symmetric determinantal representation already given and find as output

$$x^4 + y^4 + z^4 + 30(x^2y^2 + x^2z^2 + y^2z^2). \quad (9.1)$$

The algorithm can be iterated for other 35 times, as described in [30]. More precisely we first find the bitangent matrix of (4.1) with Algorithm 3, then we apply [30, Theorem 3.9] with all $\{0, i, j, k\}$ such that $\{i, j, k\} \subset \{1..7\}$. All 36 quartics are real, in agreement with (1.4).

1. mother case, $x^4 + 30x^2y^2 + y^4 + 30x^2z^2 + 30y^2z^2 + z^4$
2. $(0, 1, 2, 3)$, $x^4 + (10/3)x^2y^2 + y^4 + (10/3)x^2z^2 + (10/3)y^2z^2 + z^4$
3. $(0, 1, 2, 4)$, $x^4 - (84/125)x^3y + (54/25)x^2y^2 + (492/125)xy^3 + (81/625)y^4 + (28/25)x^3z - (36/5)x^2yz - (108/25)xy^2z - (492/125)y^3z + 6x^2z^2 + (36/5)xyz^2 + (54/25)y^2z^2 + (28/25)xz^3 + (84/125)yz^3 + z^4$
4. $(0, 1, 2, 5)$, $(625/81)x^4 + (700/81)x^3y + (1250/27)x^2y^2 + (700/81)xy^3 + (625/81)y^4 + (140/27)x^3z + (500/9)x^2yz - (500/9)xy^2z - (140/27)y^3z + (50/3)x^2z^2 - (100/3)xyz^2 + (50/3)y^2z^2 - (820/27)xz^3 + (820/27)yz^3 + z^4$
5. $(0, 1, 2, 6)$, $x^4 + (164/9)x^3y + 6x^2y^2 + (164/9)xy^3 + y^4 + (164/9)x^3z + 12x^2yz + 12xy^2z + (164/9)y^3z + 6x^2z^2 + 12xyz^2 + 6y^2z^2 + (164/9)xz^3 + (164/9)yz^3 + z^4$
6. $(0, 1, 2, 7)$, $(81/625)x^4 - (492/125)x^3y + (54/25)x^2y^2 + (84/125)xy^3 + y^4 + (492/125)x^3z - (108/25)x^2yz + (36/5)xy^2z + (28/25)y^3z + (54/25)x^2z^2 - (36/5)xyz^2 + 6y^2z^2 - (84/125)xz^3 + (28/25)yz^3 + z^4$
7. $(0, 1, 3, 4)$, $(81/625)x^4 - (492/125)x^3y + (54/25)x^2y^2 + (84/125)xy^3 + y^4 - (492/125)x^3z + (108/25)x^2yz - (36/5)xy^2z - (28/25)y^3z + (54/25)x^2z^2 - (36/5)xyz^2 + 6y^2z^2 + (84/125)xz^3 - (28/25)yz^3 + z^4$
8. $(0, 1, 3, 5)$, $x^4 + (164/9)x^3y + 6x^2y^2 + (164/9)xy^3 + y^4 - (164/9)x^3z - 12x^2yz - 12xy^2z - (164/9)y^3z + 6x^2z^2 + 12xyz^2 + 6y^2z^2 - (164/9)xz^3 - (164/9)yz^3 + z^4$

9. $(0, 1, 3, 6), (625/81)x^4 + (700/81)x^3y + (1250/27)x^2y^2 + (700/81)xy^3 + (625/81)y^4 - (140/27)x^3z - (500/9)x^2yz + (500/9)xy^2z + (140/27)y^3z + (50/3)x^2z^2 - (100/3)xyz^2 + (50/3)y^2z^2 + (820/27)xz^3 - (820/27)yz^3 + z^4)$
10. $(0, 1, 3, 7), x^4 - (84/125)x^3y + (54/25)x^2y^2 + (492/125)xy^3 + (81/625)y^4 - (28/25)x^3z + (36/5)x^2yz + (108/25)xy^2z + (492/125)y^3z + 6x^2z^2 + (36/5)xyz^2 + (54/25)y^2z^2 - (28/25)xz^3 - (84/125)yz^3 + z^4)$
11. $(0, 1, 4, 5), x^4 - (15/16)y^4 + (17/4)x^3z + (45/4)xy^2z + (3/2)x^2z^2 + (17/4)xz^3 + z^4)$
12. $(0, 1, 4, 6), 15x^4 + y^4 + 45x^2yz - 17y^3z + 24y^2z^2 - 17yz^3 + z^4)$
13. $(0, 1, 4, 7), (9/26)x^4 + (15/13)x^2y^2 + (9/26)y^4 - (15/13)x^2z^2 - (15/13)y^2z^2 + z^4)$
14. $(0, 1, 5, 6), -(1/16)x^4 - (15/8)x^2y^2 - (1/16)y^4 + (15/8)x^2z^2 + (15/8)y^2z^2 + z^4)$
15. $(0, 1, 5, 7), 15x^4 + y^4 - 45x^2yz + 17y^3z + 24y^2z^2 + 17yz^3 + z^4)$
16. $(0, 1, 6, 7), x^4 - (15/16)y^4 - (17/4)x^3z - (45/4)xy^2z + (3/2)x^2z^2 - (17/4)xz^3 + z^4)$
17. $(0, 2, 3, 4), x^4 - (164/9)x^3y + 6x^2y^2 - (164/9)xy^3 + y^4 + (164/9)x^3z - 12x^2yz + 12xy^2z - (164/9)y^3z + 6x^2z^2 - 12xyz^2 + 6y^2z^2 + (164/9)xz^3 - (164/9)yz^3 + z^4)$
18. $(0, 2, 3, 5), (81/625)x^4 + (492/125)x^3y + (54/25)x^2y^2 - (84/125)xy^3 + y^4 + (492/125)x^3z + (108/25)x^2yz + (36/5)xy^2z - (28/25)y^3z + (54/25)x^2z^2 + (36/5)xyz^2 + 6y^2z^2 - (84/125)xz^3 - (28/25)yz^3 + z^4)$
19. $(0, 2, 3, 6), x^4 + (84/125)x^3y + (54/25)x^2y^2 - (492/125)xy^3 + (81/625)y^4 + (28/25)x^3z + (36/5)x^2yz - (108/25)xy^2z + (492/125)y^3z + 6x^2z^2 - (36/5)xyz^2 + (54/25)y^2z^2 + (28/25)xz^3 - (84/125)yz^3 + z^4)$
20. $(0, 2, 3, 7), (625/81)x^4 - (700/81)x^3y + (1250/27)x^2y^2 - (700/81)xy^3 + (625/81)y^4 + (140/27)x^3z - (500/9)x^2yz - (500/9)xy^2z + (140/27)y^3z + (50/3)x^2z^2 + (100/3)xyz^2 + (50/3)y^2z^2 - (820/27)xz^3 - (820/27)yz^3 + z^4)$
21. $(0, 2, 4, 5), (1/15)x^4 + (17/15)x^3y + (8/5)x^2y^2 + (17/15)xy^3 + (1/15)y^4 - 3xyz^2 + z^4)$
22. $(0, 2, 4, 6), x^4 - 30x^2y^2 - 16y^4 + 30x^2z^2 - 30y^2z^2 + z^4)$
23. $(0, 2, 4, 7), -(15/16)x^4 + y^4 + (45/4)x^2yz + (17/4)y^3z + (3/2)y^2z^2 + (17/4)yz^3 + z^4)$
24. $(0, 2, 5, 6), -(15/16)x^4 + y^4 - (45/4)x^2yz - (17/4)y^3z + (3/2)y^2z^2 - (17/4)yz^3 + z^4)$
25. $(0, 2, 5, 7), x^4 - (10/3)x^2y^2 + (26/9)y^4 + (10/3)x^2z^2 - (10/3)y^2z^2 + z^4)$
26. $(0, 2, 6, 7), (1/15)x^4 - (17/15)x^3y + (8/5)x^2y^2 - (17/15)xy^3 + (1/15)y^4 + 3xyz^2 + z^4)$
27. $(0, 3, 4, 5), -16x^4 - 30x^2y^2 + y^4 - 30x^2z^2 + 30y^2z^2 + z^4)$
28. $(0, 3, 4, 6), -(16/15)x^4 - (68/15)x^3y - (8/5)x^2y^2 - (68/15)xy^3 - (16/15)y^4 - 12xyz^2 + z^4)$
29. $(0, 3, 4, 7), x^4 + 15y^4 - 17x^3z + 45xy^2z + 24x^2z^2 - 17xz^3 + z^4)$
30. $(0, 3, 5, 6), x^4 + 15y^4 + 17x^3z - 45xy^2z + 24x^2z^2 + 17xz^3 + z^4)$
31. $(0, 3, 5, 7), -(16/15)x^4 + (68/15)x^3y - (8/5)x^2y^2 + (68/15)xy^3 - (16/15)y^4 + 12xyz^2 + z^4)$
32. $(0, 3, 6, 7), (26/9)x^4 - (10/3)x^2y^2 + y^4 - (10/3)x^2z^2 + (10/3)y^2z^2 + z^4)$
33. $(0, 4, 5, 6), x^4 - (164/9)x^3y + 6x^2y^2 - (164/9)xy^3 + y^4 - (164/9)x^3z + 12x^2yz - 12xy^2z + (164/9)y^3z + 6x^2z^2 - 12xyz^2 + 6y^2z^2 - (164/9)xz^3 + (164/9)yz^3 + z^4)$

- 34. $(0, 4, 5, 7), (81/625)x^4 + (492/125)x^3y + (54/25)x^2y^2 - (84/125)xy^3 + y^4 - (492/125)x^3z - (108/25)x^2yz - (36/5)xy^2z + (28/25)y^3z + (54/25)x^2z^2 + (36/5)xyz^2 + 6y^2z^2 + (84/125)xz^3 + (28/25)yz^3 + z^4)$
- 35. $(0, 4, 6, 7), x^4 + (84/125)x^3y + (54/25)x^2y^2 - (492/125)xy^3 + (81/625)y^4 - (28/25)x^3z - (36/5)x^2yz + (108/25)xy^2z - (492/125)y^3z + 6x^2z^2 - (36/5)xyz^2 + (54/25)y^2z^2 - (28/25)xz^3 + (84/125)yz^3 + z^4)$
- 36. $(0, 5, 6, 7), (625/81)x^4 - (700/81)x^3y + (1250/27)x^2y^2 - (700/81)xy^3 + (625/81)y^4 - (140/27)x^3z + (500/9)x^2yz + (500/9)xy^2z - (140/27)y^3z + (50/3)x^2z^2 + (100/3)xyz^2 + (50/3)y^2z^2 + (820/27)xz^3 + (820/27)yz^3 + z^4)$

The automorphism group of $\det A = 25(x^4 + y^4 + z^4) - 34(x^2y^2 + x^2z^2 + y^2z^2)$ is the octahedral group S_4 of order 24. The 36 preimages can be grouped into the S_4 -orbits which are described in the following table.

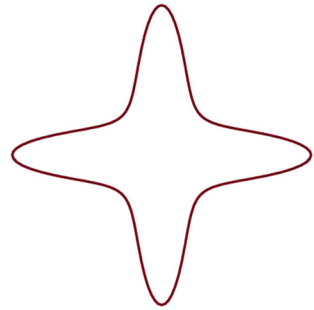
$(0, i, j, k)$	Topological classification	Automorphism group
Mother case in (9.1)	Empty	S_4 of order 24
$(0, 1, 2, 3)$	empty	S_4 of order 24
$(0, 1, 2, 4)(0, 1, 3, 7)(0, 2, 3, 6)(0, 4, 6, 7)$ $(0, 1, 2, 5)(0, 1, 3, 6)(0, 2, 3, 7)(0, 5, 6, 7)$ $(0, 1, 2, 7)(0, 1, 3, 4)(0, 2, 3, 5)(0, 4, 5, 7)$	One oval	\mathbb{Z}_2 of order 2
$(0, 1, 4, 5)(0, 1, 6, 7)(0, 2, 4, 7)$ $(0, 2, 5, 6)(0, 3, 4, 6)(0, 3, 5, 7)$	One oval	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ of order 4
$(0, 1, 4, 6)(0, 1, 5, 7)(0, 2, 4, 5)$ $(0, 2, 6, 7)(0, 3, 4, 7)(0, 3, 5, 6)$	One oval	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ of order 4
$(0, 1, 2, 6)(0, 1, 3, 5)(0, 2, 3, 4)(0, 4, 5, 6)$	One oval	S_3 of order 6
$(0, 1, 4, 7)(0, 2, 5, 7)(0, 3, 6, 7)$	Empty	D_4 of order 8
$(0, 1, 5, 6)(0, 2, 4, 6)(0, 3, 4, 5)$	One oval	D_4 of order 8

We print the figure corresponding to the star corresponding to $(0, 1, 5, 6)$, provided by Emanuele Ventura.

In the case of the Klein quartic we can follow an alternative path. Note that the 28 bitangents of the Klein quartic are printed in [35]. Only four of them are real. The Klein quartic f has the feature that

$$f = S(f). \tag{9.2}$$

Fig. 1 One of the 36 Scorza preimages of the Edge quartic



Ciani proves in [7] that the only quartics f satisfying (9.2) are the Klein quartic and the double conics (see also [8]).¹ The other 35 preimages of Klein quartic through the Scorza map (besides the Klein quartic itself) were found by Ciani in [6], although Ciani describes them intrinsically and he does not provide the equations. They divide in three $SL(3)$ -classes, two classes (conjugate to each other) containing seven quartics each one, having Aut the octahedral group of order 24, and a (self-conjugate) class containing 21 quartics, having $Aut = D_4$ the dihedral group of order 8. This class contains three real quartics by (1.4). We have found the following solution (among the possible 35), having octahedral symmetry. Let $\alpha = \tau + \tau^2 + \tau^4 = \frac{-1+\sqrt{-7}}{2}$. If $f = x^4 + y^4 + z^4 + 2\alpha(x^3y + y^3z + z^3x) - \alpha(x^2y^2 + y^2z^2 + x^2z^2) + (\alpha - \frac{2}{3})(xy^3 + yz^3 + zx^3) - 4(x^2yz + xy^2z + xyz^2)$ then $S(f) = xy^3 + yz^3 + zx^3$ is the Klein quartic. The Klein quartic belongs to the pencil generated by the double (invariant) conic $[x^2 + y^2 + z^2 + \alpha(xy + xz + yz)]^2$ and by the quadrilateral (given by four bitangents) $(x + y + z)(x + (\tau^5 + \tau^4 + 2\tau)y + (-2\tau^5 - \tau^4 + \tau^3 - \tau^2 - 2*\tau)z)((-2\tau^5 - \tau^4 + \tau^3 - \tau^2 - 2\tau)x + y + (\tau^5 + \tau^4 + 2\tau)z)((\tau^5 + \tau^4 + 2\tau)x + (-2\tau^5 - \tau^4 + \tau^3 - \tau^2 - 2\tau)y + z)$.

The M2 script is the following

```

KK=QQ
R=KK[x, y, z, u_0 . . u_3]

A = matrix({{0, x + 2*y, z + 2*x, -2*z + y},
{x + 2*y, 0, 2*z + y, z - 2*x},
{z + 2*x, 2*z + y, 0, x - 2*y},
{-2*z + y, z - 2*x, x - 2*y, 0}});
----output will be x^4+30*x^2*y^2+y^4
+30*x^2*z^2+30*y^2*z^2+z^4

det (A)

----modification M'_{1234}, according
to [Hesse]
    
```

¹ It should be interesting to compute the weight decomposition of the tangent space at the fixed points of S , in order to understand the dynamics of the iteration of the Scorza map.

```

A=matrix{{0,x-2*y,z-2*x,2*z+y},
{x-2*y,0,-2*z+y,z+2*x},
{z-2*x,-2*z+y,0,x+2*y},
{2*z+y,z+2*x,x+2*y,0}}
-----output will be
--  x^4+(10/3)*x^2*y^2+y^4+(10/3)*x^2*z^2
+(10/3)*y^2*z^2+z^4

uu=transpose matrix {{u_0..u_3}}
--following M gives the sextic model in
P^3 with coordinates u
M=diff(x,A)*uu|diff(y,A)*uu|diff(z,A)*uu

R2=KK[a_0..a_2,b_0..b_2,u_0..u_3,v_0..v_3
,MonomialOrder=>{3,3,4,4}]
aa=matrix{{a_0..a_2}}, bb=matrix{{b_0..b_2}}
cs2=sub(M,R2)
----following gives the (3,3) correspondence
on the sextic model with coord. (u,v)
cor33=minors(3,cs2)+sub(minors(3,cs2),apply
(4,i->(u_i=>v_i)))+
ideal(matrix{{v_0..v_3}}*cs2)+
ideal(matrix{{u_0..u_3}}*sub(cs2,apply(4,i->
(u_i=>v_i))))
----following lifts the (3,3) corresp. to the
quartic model with coord. (a,b)
----it is a long computation, running in about
15 minutes on a PC,
time eli=eliminate({u_0,u_1,u_2,u_3,v_0,
v_1,v_2,v_3},
saturate(ideal(cs2*transpose aa)+ideal(sub(cs2
,apply(4,i->(u_i=>v_i)))*
transpose bb)+cor33,ideal(u_0..u_3)*
ideal(v_0..v_3)))
eli6=submatrix(gens eli,,{1..6})
----the (3,3) correspondence is given by
6 biquadratic equations in a,b in eli6

R4=KK[x,y,z,a_0..a_2,b_0..b_2,c_0..c_14]
---now we compare the correspondence found
with the correspondence obtained
---starting from a generic quartic fc
---given by rank (P_aP_b(fc)) <= 1
fc=(matrix{{c_0..c_14}}*transpose
symmetricPower(4,matrix{{x,y,z}}))_(0,0)
f1=a_0*diff(x,fc)+a_1*diff(y,fc)

```

```

+a_2*diff(z,fc)
f2=b_0*diff(x,f1)+b_1*diff(y,f1)
+b_2*diff(z,f1)
xx=matrix({x,y,z})
ab=minors(2,diff(transpose xx,
diff(xx,f2)));
---previous ab is the correspondence
obtained from fc
----next aab is our correspondence
aab=ideal sub(eli6,R4);

----now to compare the two correspondences
we encode in matrices with 36 columns
---given by all biquadratic monomials
ab2=symmetricPower(2,matrix({a_0..a_2}))
**symmetricPower(2,matrix({b_0..b_2}))
maab=diff(transpose ab2, gens aab)
---following must be nonzero
det submatrix(maab,{0..5},)
det submatrix(maab,{0,1,4,7,10,14},)
mab=diff(transpose ab2, gens ab);
----we impose that each column from mab
is in the column span of maab
----in principle we have to check all
7*7 minors, but this is too expensive
----so we try with all 7*7 minors containing
the first six rows, this is enough
---in the examples found (not for
Klein quartic!)
--if at the end we get infinitely
many solutions, one has to add more conditions
---given by more minors

-----following in case of Klein quartic
I=ideal(apply(36,i->det submatrix(maab|mab_{0},
{0,1,4,7,10,14,i},)));
I=I+ideal(apply(36,i->det submatrix(maab|mab_{1},
{0,1,4,7,10,14,i},)));
I=I+ideal(apply(36,i->det submatrix(maab|mab_{2},
{0,1,4,7,10,14,i},)));
I=I+ideal(apply(36,i->det submatrix(maab|mab_{3},
{0,1,4,7,10,14,i},)));
I=I+ideal(apply(36,i->det submatrix(maab|mab_{4},
{0,1,4,7,10,14,i},)));
I=I+ideal(apply(36,i->det submatrix(maab|mab_{5},
{0,1,4,7,10,14,i},)));

```

```

I=I+ideal(apply(36,i->det submatrix(maab|mab_{6},
{0,1,4,7,10,14,i},)));
I=I+ideal(apply(36,i->det submatrix(maab|mab_{7},
{0,1,4,7,10,14,i},)));
I=I+ideal(apply(36,i->det submatrix(maab|mab_{8},
{0,1,4,7,10,14,i},)));
betti I
-----until here for Klein quartic

I=ideal(apply(30,i->det submatrix(maab|mab_{0},
{0,1,2,3,4,5,i+6},)));
I=I+ideal(apply(30,i->det submatrix(maab|mab_{1},
{0,1,2,3,4,5,i+6},)));
I=I+ideal(apply(30,i->det submatrix(maab|mab_{2},
{0,1,2,3,4,5,i+6},)));
I=I+ideal(apply(30,i->det submatrix(maab|mab_{3},
{0,1,2,3,4,5,i+6},)));
I=I+ideal(apply(30,i->det submatrix(maab|mab_{4},
{0,1,2,3,4,5,i+6},)));
I=I+ideal(apply(30,i->det submatrix(maab|mab_{5},
{0,1,2,3,4,5,i+6},)));
I=I+ideal(apply(30,i->det submatrix(maab|mab_{6},
{0,1,2,3,4,5,i+6},)));
I=I+ideal(apply(30,i->det submatrix(maab|mab_{7},
{0,1,2,3,4,5,i+6},)));
I=I+ideal(apply(30,i->det submatrix(maab|mab_{8},
{0,1,2,3,4,5,i+6},)));
betti I
sI=saturate I;
betti sI
----sI contains now the base locus of
  Scorza map plus one single point,
----we have to saturate with base
locus of Scorza, called bscorza

m=matrix({0,z,-y},{-z,0,x},{y,-x,0})
m9= diff(m,diff(xx,diff(transpose xx,fc)));
scorza=(mingens pfaffians(8,m9))_(0,0);
bscorza=ideal diff(symmetricPower(4,xx),scorza);

fiber=saturate(sI,bscorza)
----the following gives the quartic in the output
(sub(matrix {apply(15,i->(c_i)%fiber)},c_14=>1)*
transpose symmetricPower(4,matrix{x,y,z}))_(0,0)

```

Algorithm 8 • *INPUT: a plane quartic f*

- *OUTPUT: the order of the automorphism group of linear invertible transformations which leave f invariant*

This algorithm is straightforward. We add here the M2 script for the convenience of the reader. The structure of groups of small size can be found by counting the elements of a given order.

```

---work on a finite field to run quicker
R2=ZZ/31991[g_0..g_8,y_0..y_2]
gg=transpose genericMatrix(R2,3,3)
-----we impose conditions to a unknown
projective transformation gg
y4=symmetricPower(4,matrix{y_0..y_2})
y1=transpose matrix{y_0..y_2}
f=y_0^4+30*y_0^2*y_1^2+y_1^4+30*y_0^2*y_2^2
+30*y_1^2*y_2^2+y_2^4 ---order 24
IG=minors(2,diff(y4,sub(sub(f,R2),apply(3,i->(y_i=>((gg)
*y1)_i,0)))) ||
diff(y4,f));
time sig=saturate(IG,ideal(det(gg));
codim sig, degree sig
---if codim sig=8, then degree sig is
the order of automorphism group

```

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Conflict of interest The author declares that he has no Conflict of interest.

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