# Oskar Becker, On the Logic of Modalities (1930): Translation, Commentary and Analysis

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# Oskar Becker

# *On the Logic of Modalities* (1930)

# Translation, Commentary and Analysis

## {Final version}

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# Contents

# Preface

The history of twentieth-century modal logic is all too often presented as the American success story that started with the work of Clarence Irwing Lewis, while prewar modal logic research in Europe is passed off as a side-show of well-intended failures. This book is intended as a first attempt to drastically correct such a picture. It is related to the research project *Modal Logic and Austro-Polish Philosophy* awarded to the first author by the German Research Association (DFG) in Summer 2017 and started in October 2018. The preparatory work has enormously profited from the first author's one year research (2017-1018) within the research programm *Modalities and Conditionals: Systematic and Historical studies* at the University of Helsinki, whose principal investigator was Sara Negri.

The core of the project *Modal Logic and Austro-Polish Philosophy* consists in formally exploiting ideas, attempts and solutions emerging from prewar modal logic with the aim to open up unexpected, radically new paths in various areas of contemporary research in modal logic.

The starting point of the research should have been, and indeed it has been, to make the key works in modal logic appeared in prewar continental Europe accessible to a wider audience. Three works in particular come to the mind when one thinks of European research in modal logic prior to the war: Jan Łukasiewicz's *O logice trójwartościowej* (1920) [Łu20], Ernst Mally's *Grundgesetze des Sollens* (1926) [Mal26], and Oskar Becker's *Zur Logik der Modalitäten* (1930) [Bec30b].

The present book focuses on Oskar Becker's essay *On the Logic of Modalities.*

The decision to begin therewith was, in part, due to the fact that Becker seemed to have been deeply influenced by his teacher, Edmund Husserl, and an in-depth knowledge of Husserl's work appeared to be essential to correctly interpret certain choices of Becker about modalities. Such background had been elaborated in several volumes and articles that have appeared mostly by the Springer series *Synthese Library* in recent years, for instance, the monograph *Logic and Philosophy of mathematics in the Early Husserl* (*Synthese Library* 2010) [Cen10] or the volume *Essays on Husserl's Logic and Philosophy of Mathematics* (*Synthese Library* 2017) [CE17] which contains our joint work on Husserl and the algebraists of logic (*Husserl and Boole* [CM17a] and *Husserl and Schröder* [CM17b]). On the other side, two important volumes have paved the way for an adequate understanding of the philosophical significance of Oskar Becker's contribution to the history and philosophy of mathematics, namely, *Die Philosophie und die Wissenschaften. Zum Werk Oskar Beckers* edited by Annemarie Gethmann-Siefert and Jürgen Mittelstraß [GSE02] and *Oskar Becker und die Philosophie der Mathematik* edited by Volker Peckhaus [Pec05]. At a variance with the wide spectrum of topics dealt with in these volumes, our choice has been to focus exclusively on Oskar Becker's pioneering contributions to modal logic in 1930, with the aim to evaluate of Becker's scientific accomplishments on this topic and possibly to make them, so to say, bear fruit.

We first decided to give an idea of the work that had to be done on this aspect of Oskar Becker's reseach in a booklet with the title *Oskar Becker on*

*Modalities* appeared in 2019 within the series *Philosophische Hefte* edited by Axel Gelfert und Thomas Gil for the *Logos Verlag* in Berlin [CM19]. On this basis we then turned to a deeper examination and a rigorous proof of all the claims made in the booklet, but not only that.

The present work is the result of two years of intense work on Oskar Becker's *On the Logic of Modalities*. In the first place, we try to give an idea of the reasons that lead our author to deal with modalities, first of all, the birth of intuitionistic logic, two years earlier, with the publication of Arend Heyting's seminal work *The Formal Rules of Intuitionistic Logic* (1928) [Hey30]. It may perhaps turn out to be surprising that Oskar Becker is the first philosopher ever to have put forward the idea of a translation of intuitionistic logic into modal logic, even if it is Kurt Gödel that actually realized it shortly later.

This book provides the reader with the basic modal logical tools necessary to read *On the Logic of Modalities* and gives a sketch of Lewis's Logic **S3**, which is the target of Becker's essay, in the modal logical symbolism that is current nowadays. Next, two extensions of **S3** proposed by Oskar Becker are presented and all objections raised by Kurt Gödel in his *Review of Becker 1930: On the Logic of Modalities* (1931) [G3̈1] are considered with the aim of correcting some historical misunderstandings related to them.

We dwell at length on the controversy between Oskar Becker and Ernst Cassirer with respect to the meaning and the legitimacy of the use of the *ideal elements* in logic and mathematics. This results in an interesting multi-voice canon whose main actors are none other than Husserl, Weyl, Hilbert and Brouwer.

We then provide the English translation of Becker's essay, accompanying the whole translation with footnotes that explain step by step, in current symbolism, and,[*] sometimes even in words, what is going on in the text, in order to allow even a reader not versed in logic to read the text easily.

The Appendix at the end of the book contains the detailed proofs of our assertions about Becker's logical accomplishments, conjectures, and mistakes as well. It is preceded by a systematic presentation of the Kripke-style semantics for **S3**, of the completeness theorems and other related results — in a comprehensive way which, to our knowledge, is not yet to be found in the literature.

We are particularly indebted to our teacher, Ettore Casari, who aroused our interest in logic, mathematics and in Husserl's and Bolzano's writings and made us realize that restriction to one single field of research can be more of a hindrance than a help for original work. Very special thanks go to Klaus Mainzer, emeritus of excellence at the Technical University of Munich and Niklas Hebing, Head Office of the Section by Humanities and Social Science by the German Research Association (DFG), who both strongly supported this work.[*] The encouragement at a decisive moment and the friendly advice we received from Otavio Bueno, Editor in Chief of Synthese Library, were truly invaluable.



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# Introduction

Oskar Becker was involved with Nazism — he was indeed, as is written in an official file of the SS Security Service[1], “not a party member but a loyal to National Socialism, who tries to consolidate the National Socialistic ideology”.

We wish to make fully explicit that we completely distance ourselves from Becker’s political views, and that our book, which deals exclusively with Becker’s accomplishments in modal logic and its philosophy, neither endorses nor defends in any way Becker’s political and moral failings.

* * *

This book contains the first English translation of Oskar Becker’s essay *On the Logic of Modalities* (*Zur Logik der Modalitäten*)[2] that appeared in 1930 on the *Yearbook for Philosophy and Phenomenological Research*.[3] Our

[1] Cp. [LS92].

[2] [Bec30b]. Henceforth cited using the page number(s) of the original pagination and, in brackets, of the English translation contained in the present volume (the original pagination is reproduced in the margin).

[3] [Bec30b]. The *Yearbook for Philosophy and Phenomenological Research* (*Jahrbuch für Philosophie und phänomenologische Forschung*) was founded by Edmund Husserl in 1912 and served the Husserl’s circle as an important organ during Husserl’s Freiburg period (1916–1938). The first issue of the journal was published in 1913 and contains Husserl’s

commentary aims to present, to contextualize and to evaluate the pioneering contributions to modal logic contained in this work.

Oskar Becker (Leipzig 1889 – Bonn 1964) was a German philosopher, logician, mathematician and historian of mathematics. He is often remembered, together with Martin Heidegger, for being one of the most prominent students of Edmund Husserl (1859–1938). He was, together with Moritz Geiger (1880–1937), Martin Heidegger (1889–1976), Alexander Pfänder (1870–1941), Adolf Reinach (1883–1917) and Max Scheler (1874–1928), one of the members of the editorial board of the *Yearbook*.

Oskar Becker got his PhD in mathematics in 1914 with a work[4] entitled *On the Decomposition of Polygons in non-intersecting triangles on the Basis of the Axioms of Connection and Order* (*Über die Zerlegung eines Polygons in exclusive Dreiecke auf Grund der ebenen Axiome der Verknüpfung und Anordnung*). In 1922 he wrote under Husserl's supervision his *Habilitationsschrift*, *On Investigations of the Phenomenological Foundation of Geometry and their physical Application* (*Beiträge zur phänomenologischen Begründung der Geometrie und ihrer physikalischen Anwendungen*).[5] In 1927 Oskar Becker published in the *Yearbook* his masterpiece *Mathematical Existence*,[6] where he uses the Husserlian phenomenology to clarify the process of counting. In 1952 — when the study of modal logic was already well beyond its

*Ideas for a Pure Phenomenology and Phenomenological Philosophy* [Hus13]. Volume 8 includes Heidegger's masterpiece *Being and Time* (1927) [Hei27] as well as Oskar Becker's famous investigation on the logic and ontology of mathematical phenomena "*Mathematical Existence* (*Mathematische Existenz*)" [Bec27].

[4] [Bec14]. For a complete bibliography of Becker's works see [Zim69].

[5] [Bec23]. For a precise placement of Oskar Becker in the political and philosophical panorama of his time see [Wol02].

[6] [Bec27]. Hereto see at least: [Get03], [Pec05], [GSE02], [Man98]. A very important contribution for a correct understanding of Oskar Becker's position in the history of modal logic is, in our opinion, [Pec02].

pioneering era — Becker came back to the subject publishing a monograph, *Investigations on the Modal Calculus* (*Untersuchungen über den Modalkalkül*), perhaps too old-fashioned for the time.[7]

* * *

The essay *On the Logic of Modalities* represents an attempt to treat modal logical issues with a phenomenological method. This enterprise appeared from the outset not to be easy at all, for logic and phenomenology are completely different disciplines. Depending on the way in which it constructs its formal systems, formal logic can be seen as *the theory of the correct inferences*, or alternatively, as the theory *of purely formal truths*, that is, as the theory of those truths that hold without any condition. Phenomenology, instead, deals with the description of lived experiences.

Indeed, we might better say that in his investigations Becker pursued two loosely related goals. The first, more technical in character, was to find axiomatic conditions that reduced to the finite the number of logically non-equivalent combinations arising from the iterated application of the operators "not" and "it is impossible that (...)" in Lewis's modal system, as we will explain in details below. The second, more philosophically oriented and in a sense much more ambitious, was to treat the logic of modalities from a phenomenological perspective and to understand, from this perspective, the philosophical and logical-ontological problems underlying, and posed by, the intuitionism.

*On the Logic of Modalities* consists of two parts, loosely related as the above mentioned corresponding goals are. Part **I** opens with a general Introduction that shortly reviews the Aristotelian conception of modalities as well

[7] [Bec52]. It is worth noticing that Becker proposes here a Leibnizian semantics for the modal operators, which he calls the "statistical interpretation of the modal calculus", cp. [Mar69] and, in particular, [Pec02], 174 and [Cop02], 107–108.

as Hugh MacColl's pioneering modal logical investigations in his *Symbolic Logic and its Applications*[8] of 1906. It then focuses on C. I. Lewis's *Survey of Symbolic Logic*[9] of 1918. The latter work contains the first presentation of the so-called "*Survey* system", known since 1932 as "modal system **S3**."[10]

*De facto*, **S3** is the actual object of the investigations contained in Part **I** of Becker's essay. As pointed out by Emil L. Post, the system Lewis presents in 1918 *collapses* into classical logic. Lewis corrects it in a paper entitled *Strict Implication: An Emendation*[11] and published in 1920, where the system effectively becomes the logic we nowadays know as "**S3**."[12] In his essay Becker faithfully reports both that the original version of the "*Survey* system" proves the *collapse* of modalities, as well as Lewis's amendment thereof. Incidentally, "*collapse of modalities*" is a customary expression in the modal-logical jargon. It means that a modal logical system proves that *necessity* and *truth* are one and the same, or equivalently (as it is the case in the "*Survey* system") that *impossibility* and *falsity* are one and the same. Obviously, such a system is *trivial* from a modal point of view.

Becker's Introduction touches on the paradoxes of *material* and *strict implication* and sets out to establish a propositional modal logic that is *decidable* as the classical propositional logic:[13]

> The aim of the present essay is now closely related to both MacColl's and Lewis's investigations. The ultimate goal of our investigations is to develop an elementary logical calculus that takes adequately into account the *modalities* of the statement, namely in such a way *that the so-called elementary decision problem is solvable*, as in the ordinary propositional calculus.

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[8] [Mac06].

[9] [Lew18].

[10] The name appears for the first time in Appendix II of [LL32].

[11] [Lew20].

[12] Cp. [CMR16], 281 f.

[13] [Bec30b], 4 (91).

Part **I**, *On the Rank Order and the Reduction of Logical Modalities*, is specifically devoted to the problems of ranking and iteration of modalities. Becker sets out to modify **S3** by means of some additional axioms effecting the reduction of complex modalities to simple ones in order to obtain two new modal systems — he calls them "*the six modalities calculus*" (henceforth denoted here by **S3′**) and "*the ten modalities calculus*" (henceforth **S3″**) with the following properties:

(i) *the number of irreducible modalities is finite*,

(ii) *the positive* (and by consequence the negative) *modalities are arranged in a linear order with respect to logical strength.*

He believes that, since the "System of Strict Implication" has the conjunction, the negation and the impossibility as primitive logical constants, it is possible to generate within it infinitely many non equivalent nested modalities through the iteration of the negation and the impossibility operators. Such modalities, as Kurt Gödel (1906-1978) puts it in his *Review of Becker 1930*, "cannot even be linearly ordered according to their logical strength in the sense that, of any two affirming modalities, one will imply the other, and similarly for negating ones."[14] Otherwise said, there are modalities that are incomparable in Lewis's system.

That said, it is worth to be mentioned that Oskar Becker neither shows that the two systems he sets up (and others he tentatively introduces, as we will see below) really differ from one another, nor that his additional axioms cannot be derived from those of Lewis, nor that in his own systems, with *six* and, respectively, with *ten* "irreducible" modalities, such modalities cannot be further reduced.[15]

[14] [G3̈1].

[15] Hereto cp. [G3̈1].

Actually, nine years later, W. T. Parry will show, in a paper entitled *Modalities in the Survey System of Strict Implication*,[16] that, at a variance with what Becker seems to believe, **S3** has a finite number of modalities. More precisely, Parry shows, with the help of a number of suitable theses he is able to derive in the system, that it is possible to reduce all the complex modalities in **S3** to a finite number of irreducible modalities, viz. 42. He also shows that no further reduction is possible.

Part **II** of Becker's essay explores, more or less independently from Part **I**, the connection between modal and intuitionistic logic both from a formal and from a phenomenological perspective.

From a formal perspective, the particular interest of a (propositional) modal calculus with nested modalities that is *decidable* lies in the fact, so Becker, that Brouwer's idea to set up a finite logic grounded on *evidence*, or — to put it with Husserl — on *evidence levels* seems to be realizable only within the framework of a modal formal system.

Indeed, Becker is the first logician and philosopher of mathematics to put forward the idea of a *modal interpretation* of intuitionistic logic, more precisely the idea of a possible sound and faithful translation of intuitionistic logic into modal logic. However, the first actual translation is to be found in a one-page celebrated and influential paper entitled *An interpretation of the intuitionistic propositional calculus* written in 1933 by Kurt Gödel.[17] The basic idea of Gödel is similar to the one Oskar Becker outlines in *On the Logic of Modalities*.

Becker suggests to add to classical logic the predicates "(...) is provable", "(...) is such, that its negation is provable" and "(...) is undecided". Such predicates should express Brouwer's primitive logical concepts.

[16] [Par39].

[17] [G3̈3a].

Similarly, Gödel's idea is to extend the language of classical propositional logic with the unary operator "it is provable that (...)", denoted by "B", and to add to an axiomatic calculus for propositional classical logic *three* axiom-schemas and *one* rule of inference. The axiom-schemas are the modal schemas $K$, $T$ and 4 that characterize modal logics that are nowadays standard, the rule of inference is the necessitation rule that is contained in all *normal* modal systems. We will introduce both the schemas and the rule of inference in detail later on.

Notice, incidentally, that both Becker and Gödel seem to take the predicate "(...) is provable" and the operator "it is provable that (...)" as conveying the same piece of information. Actually, the predicate "(...) is provable" denotes the property of a proposition to be provable, while the operator "it is provable that (...)" takes a proposition as input and gives a different proposition as output. (Unfortunately, such practice of systematically neglecting the difference between predicate and operator is, even nowadays, quite widespread among logicians.)

Gödel writes:[18]

> One can interpret Heyting's propositional calculus by means of the notions of the ordinary propositional calculus and the notion "$p$ is provable" (written "$\mathsf{B}p$"), if one adopts for that notion the following system $\mathfrak{S}$ of axioms:
>
> 1. $\mathsf{B}p \to p$
>    if it is provable that $p$, then it is true that $p$
> 2. $\mathsf{B}p \to ((\mathsf{B}(p \to q) \to \mathsf{B}q)$
>    if it is provable that $p$ and it is provable that $p$ implies $q$, then it is provable that $q$
> 3. $\mathsf{B}p \to \mathsf{B}\mathsf{B}p$
>    if it is provable that $p$, then it is provable that it is provable that $p$
>
> In addition, [...] the new rule of inference is to be added

[18] [G33̈a], 301.

$$\frac{A}{\mathsf{B}A}$$

From $A$, it is provable that $A$ may be inferred

By substituting throughout the operator "$\mathsf{B}$" ("it is provable that (...)") by the operator "$\Box$" ("it is necessary that (...)") one obtains one of the modal logical systems that are nowadays standard, namely Lewis's system **S4**.

* * *

Becker's more logically oriented investigations, mainly contained in Part **I** of *On the Logic of Modalities*, will be carefully analyzed in Chapter 1, while his more philosophically oriented discussions, to be found in Part **II**, will be examined at length in Chapter 2. Chapter 3 contains the English translation of *On the Logic of Modalities*, together with an apparatus of explanatory footnotes. The final Chapter 4 (Appendix) provides a self-contained and comprehensive survey of the most relevant "technical" issues related to Lewis's system **S3** and other non normal modal systems in the neighbourhood.

# Chapter 1

# Part I of *On the Logic of Modalities*

## 1.1 The Conditional, or *The Crows on the Roofs*

Since Becker as well as MacColl and Lewis all refer to the old controversy about the right interpretation of conditional sentences, let us briefly dwell on this issue.

The controversy traces back to the Megarians and the Stoics. As Józef Maria Bocheński puts it in his *A History of Formal Logic*:[1]

> The definition of implication was a matter much debated among the Megarians and Stoics: All dialecticians say that a connected (proposition) is sound, when its consequent follows from its antecedent — but they dispute about when and how it follows, and propound rival criteria.
>
> Even so Callimachus, librarian at Alexandria in the 2nd century B.C., said: 'the very crows on the roofs croak about which implications are sound'.

In ancient times the quarrel was, above all, between a truth-functional and a modal interpretation of the conditional. Philo (of Megara) said that an implication is true when it is not the case that it begins with the true and ends with the false.[2] This conception of the conditional was later adopted

[1] [Boc56], 116.

[2] *Ibid.*, 117.

by Gottlob Frege (1848-1925) and by the American logician and founder of American Pragmatism Charles Sanders Peirce (1839-1914).

In his *Gedankengefüge* (1923) Frege calls conditional sentences "*hypothetische Satzgefüge*" and what is expressed by them "*hypothetische Gedankengefüge*". He writes:[3]

> [A] hypothetical compound thought is true if its consequent is true; it is also true if its antecedent is false, regardless of whether the consequent is true or false. The consequent must always be a thought. Given [...] that "A" and "B" are sentences proper, then "not (not A and B)" expresses a hypotethical compound with the sense (thought-content) of "A" as consequent and the sense of "B" as antecedent. We may also write instead: "if B, then A." But here, indeed, doubts may arise. It may perhaps be maintained that this does not square with linguistic usage. I reply, it must once again be emphasized that science has to be allowed its own terminology, that it cannot always bow to ordinary language. Just here I see the greatest difficulty for philosophy: the instrument it finds available for its work, namely ordinary language, is little suited to the purpose, for its formation was governed by requirements wholly different from those of philosophy. So also logic is first obliged to fashion a usable instrument from those already to hand. And for this purpose it initially finds but little in the way of usable instruments available. [...] The thought expressed by the compound sentence "If I own a cock which has laid eggs today, then Cologne Cathedral will collapse tomorrow morning" is [...] true. Someone will perhaps say: "But here the antecedent has no inner connection at all with the consequent." In my account, however, I required no such connection, and I ask that "if B, then A" should be understood solely in terms of what I have said and expressed in the form "not (not A and B)." It must be admitted that this conception of a hypothetical compound thought will at first be thought strange. But my account is not designed to square with ordinary linguistic usage, which is generally too vague and ambiguous for the purposes of logic.

Thus, as far as the truth-conditions of conditional propositions are concerned, Frege is, whether he knew it or not, a follower of Philo.

[3] [Fre23], 46.

In turn, Peirce is overtly a follower of Philo:[4]

> As far as I am concerned, I am a follower of Philo [...]. It is completely irrelevant, whether this conception is in accordance with ordinary language.

Diodorus of Sicily and Chrysippus of Soli raised objections against Philo's truth-functional conception of implication. While Diodorus saw the conditional as a temporal quantification — namely by regarding a conditional true if and only if, for every instant of time $t$, it is not the case that at the instant $t$ the antecedent is true and the consequent is false — Chrysippus had a virtually[5] modal conception of the conditional: according to him, a conditional is true if and only if the premise is *incompatible* with the negation of the consequence.

As we saw, Frege and Peirce shared Philo's conception. In Hugh MacColl's essay published in 1880 in the journal *Mind* and in his *Symbolic Logic* of 1906 a concept similar to that of Chrysippus can be found.[6]

Clarence Irving Lewis, on the basis of MacColl's investigations, pleaded in favor of the Chrysippean modal reading as an appropriate interpretation of the conditional, which he called "strict implication". In 1918 Lewis published his first modal system, eventually called **S3**. Later on he developed his modal systems **S1** to **S5**,[7] which are stepwise based on one another. They are conceived as alternatives to the non modal logic presented in Russell's and Whitehead's *Principia Mathematica* (1910-1913).[8]

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[4] [Pei92], 125 f.

[5] If we assume that "incompatibility" and "impossibility of the conjunction" mean the same.

[6] [Mac80] 1880, [Mac06].

[7] [LL32].

[8] [WR10].

Lewis formalized the strict implication (in symbols: "$\strictif$") in terms of negation, possibility and conjunction[9] as follows:[10]

$$p \strictif q := \neg\Diamond(p \wedge \neg q)$$

Due to the interdefinability of $\Box$ and $\Diamond$, this is logically equivalent to

$$\Box(p \to q)$$

Similarities and differences in comparison with the "material implication", as Russell called the Philonian conditional, are obvious:

$$p \to q := \neg(p \wedge \neg q)$$

Lewis's comment on this was the following:[11]

> "$p$ strictly implies $q$" is to mean "it is false that it is possbile that $p$ should be true and $q$ false" [...]
> "$p$ materially implies $q$" is to mean "it is not the case that $p$ is true and $q$ is false."

Lewis's main reason for formally introducing the Chrysippean conditional were the paradoxes of material implication, in particular:

(1) $p \to (q \to p)$ (*argumentum a fortiori*),

(2) $\neg p \to (p \to q)$ (*ex falso quodlibet*).

Lewis rephrases them as follows:[12]

---

[9] In order to comply with the now current logical notation, we prefer not to adopt Lewis's symbolic apparatus, except for the symbol "$\strictif$" of strict implication. In particular, we use "$\to$", "$\neg$", "$\wedge$", "$\vee$", "$\Box$" and "$\Diamond$" to denote, respectively, material implication, conjunction, disjunction, necessity and possibility.

[10] [LL32], 124.

[11] *Ibid.*, 124, 136.

[12] *Ibid.*, 142.

(1*) If $p$ is true, then any proposition $q$ materially implies $p$.

(2*) If $p$ is false, then $p$ materially implies any proposition $q$.

However, Lewis admitted that the strict implication also yields the following analogous paradoxical theorems:

$$(3)\ \neg\Diamond\neg p \prec (q \to p),$$

$$(4)\ \neg\Diamond p \prec (p \to q),$$

in his words:[13]

(3*) A proposition which is necessarily true is implied by any proposition [...]

(4*) A proposition which is impossible implies any proposition.

In the later *Symbolic Logic* he will carefully discuss this issue, arguing that the paradoxicality of these theorems is just illusory: they are "paradoxical only in the sense of expressing logical truths which are easily overlooked".[14]

## 1.2 The Decision Problem and *Leibniz's Dream*

Part I of Becker's essay pursues the goal, as we already said, of finding a propositional modal logic with a finite number of irreducible iterated modalities that is *decidable*. In the present section we focus on the *decision problem* and hint at the problems one encounters when searching for a decidable propositional modal logic *without an adequate semantic* (and *a completeness theorem*) for such a calculus being available.

The *Entscheidungsproblem* (German for "decision problem") is the question whether the first-order predicate logic is decidable or not. In this form the question is to be found in Hilbert & Ackermann *Outlines of Mathematical*

[13] *Ibid.*, 174.

[14] [LL32], 248 ff.

*Logic* (*Grundzüge der mathematischen Logik*) published in Berlin in 1928.[15] Is there an effective method to decide the set of of logically valid formulas of first order logic?

The conceptual issues underlying the problem have a long history and may be traced back at least to Leibniz. In his *Dissertatio de Arte Combinatoria* (1666)[16] the young Leibniz had advanced the hypothesis that all concepts could be reduced, through analysis, to a finite number of simple concepts with a procedure analogous to prime number decomposition. Such simple concepts should have been transposed in a *lingua characteristica*, i.e. in a universal sign-system capable of directly coding simple concepts and, by means of syntactical rules, complex conceptual expressions. Leibniz takes it to be possible to translate all deductive problems in the *lingua characteristica* and to decide them by means of a *calculus ratiocinator*. A typical of deductive problem has the following form: "Does the conclusion $C$ logically follow from the premises $A_1, A_2, \ldots, A_n$" In symbols:

$$A_1, A_2, \ldots, A_n \models^? C$$

A *calculus ratiocinator* may be thought of as a complex of calculation rules capable of "deciding", or of answering mechanically in a positive or negative way, any deductive problem; i.e., applied to our schematization, whether $C$ does follow from $A_1, A_2, \ldots, A_n$ or not. We interpret thus Leibniz's famous "*calculemus*". Leibniz writes:[17]

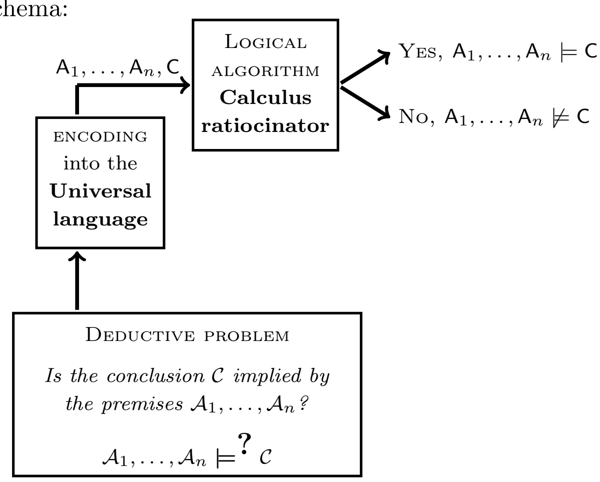
[15] [HA28], 4; 7-9.

[16] [Lei66].

[17] *De arte characteristica ad perficiendas scientias ratione nitentes*, [Lei99a]. The original Latin text reads as follows: "Sed ut redeam ad expressionem cogitationum per characteres, ita sentio nunquam temere *controversias finiri* neque *sectis* silentium imponi posse, nisi a ratiocinationibus complicatis ad *calculos* simplices, a vocabulis vagae incertaeque significationis ad *characteres* determinatos revocemur. Id scilicet efficiendum est, ut omnis *paralogismus* nihil aliud sit quam error *calculi* [. . . ] Quo facto quando orientur controversiae,

I feel that controversies can never be finished, nor silence imposed upon the sects, unless we give up complicated reasonings in favor of simple calculations, words of vague and uncertain meaning in favor of fixed symbols [*characteres*]. Thus, it will appear that every paralogism is nothing but an error of calculation [...] If controversies were to arise, there would be no more need of disputation between two philosophers than between two calculators. For it would suffice for them to take their pencils in their hands and to sit down at the abacus, and say to each other (and if they so wish also to a friend called to help): Let us calculate without further ado!

For the sake of convenience let us depict Leibniz's idea by the following schema:




To prove the decidability of a certain set, for instance of the set of all **S3**-tautologies, it suffices to exhibit a decision algorithm for it; that is, in our example, an algorithm that takes as input a formula $A$ in the language of **S3** and terminates its computation with a (conventionally fixed) output/answer **1/yes**, if $A$ is a **S3**-tautology, and **0/no**, if $A$ is not a **S3**-tautology. However, in order to prove the *undecidability* of a set, one must have a mathematical counterpart for the informal notion of *algorithm* (on which the informal no-

non magis disputatione opus erit inter duos philosophos, quam inter duos Computistas. Sufficiet enim calamos in manus sumere sedereque ad abacos, et sibi mutuo (accito si placet amico) dicere: *calculemus*".

tion of *decidable set* depends). Oskar Becker neither seems to be aware of the problem of finding an adequate formal counterpart for the concept of *decidability* nor seems to have any inkling of the necessity of making the concept of *algorithm* mathematically precise.

Thanks to the investigations of the Hilbert School, the recognition of distinct logical levels (propositional, first-order, higher-order), as it is nowadays standard, was known since the years 1917–1919, as well as the decidability of classical propositional logic. The first 'official' proof thereof is to be found in P. Bernays's *Habilitationsschrift* (1918); indeed, the result is already implicit, as an easy consequence, in the *normal form theorem* proved by Hilbert in his 1905 Lectures on *Logical Principles of Mathematical Thought*.[18]

Now, Becker seems to take **S3** as undecidable on the basis of his mistaken belief that **S3** has an infinite number of iterated modalities. His main aim in *On the Logic of Modalities* is to set up a propositional *modal* logic that is decidable, more precisely, since he seems to assume that a modal logic with a finite number of irreducible modalities has to be decidable, to find extensions of **S3** that have a finite number of irreducible modalities. To this aim, he introduces two systems, which we agreed to call **S3′** and **S3″**, that represent, so Becker, two *new* modal logics with *six* (**S3′**) and, respectively, *ten* (**S3″**) irreducible modalities.

Today we know that **S3** is decidable. Indeed, any (under very general conditions) propositional logic $L$ that is finitely axiomatizable and has the finite model property is, by Harrop's theorem, decidable.[19]

A complete semantic for **S3** (as well as for **S3′** and **S3″**) together with the finite model property would have applied to obtain the decidability of

[18] [Hil05]. Cp. [Zac99], 333-335.

[19] A logic $L$ has the finite model property if any non-theorem of $L$ is falsified by some finite model of $L$.

theoremhood for each of these logics.[20] But neither Lewis provides a semantic for **S3** nor Becker provides a semantic for **S3′** and **S3″**.

Decidability could also be proved syntactically by reducing it to terminating proof-search in an analytic (e.g. Gentzen-style) presentation of **S3**, **S3′** and **S3″**,[21] or by proving the equivalence (modulo a translation) between **S3**, **S3′** and **S3″** and some other propositional logics already known to be decidable, or by developing methods based on translations of such modal logics into suitable decidable fragments of first-order logic. None of these alternative was available at that time or had even been conjectured by Lewis or Becker.

## 1.3 Normal Modal Logics: a Quick *Resumé*

As a preliminary to the presentation and discussion of Becker's investigations on Lewis's system **S3**, to be found in the next Sections, it is convenient to review here the best known *normal* modal logics, **K**, **D**, **T**, **K4**, **B**, **S4**, **S5**, as they are usually axiomatically characterized as extensions of classical logic. More precisely, the axioms comprehend all classical tautologies (or a "sufficient" selection thereof) as well as one or more additional axioms (in schematic form) that characterize the specific logic in question; the inference rules are the *modus ponens* of classical logic and one specifically modal rule, the necessitation rule

$$\frac{A}{\Box A}$$

[20] The Kripke-style semantics for the system **S3**, together with the completeness and the finite model property theorems, is presented in full details in the Appendix, Sections 4.4, 4.5 and 4.6.

[21] Gentzen-style sequent calculi for Lewis's non normal modal systems have been introduced in [Mat60] (for **S2**) and [Ohn61] (for **S2** and **S3**). The recent paper [Tes20] provides G3-style labelled sequent calculi with good structural properties for **S2**, **S3** and some neighbour systems.

introduced, as we already said, by Gödel in his 1933 paper. It says that if a proposition $A$ is provable within the system in question, its necessitation $\Box A$ is also provable[22].

Once an axiom system for the minimal normal modal logic **K** is given, it is simple to give an axiomatization for the other above mentioned modal logics, for they amount to **K** plus a few additional axiom schemas.

Thus, we will recall the standard axiomatization for the logic **K** as well as the modal logical schemas $T$, 4, $B$, 5 or $E$, and then we will see which schemas give which logic. Doing so turns out to be useful to the aim of proving, later on, to which standard modal systems Becker's systems $\mathbf{S3}'$ and $\mathbf{S3}''$ are equivalent.

The formal modal propositional language $\mathcal{L}^{\Box}$ is defined as usual. The alphabet contains:

- denumerably many propositional variables (or 'atoms'): $p_0, p_1, p_2 \ldots$;
- the boolean connectives: $\neg\,, \vee\,, \wedge\,, \rightarrow$;
- one modal operator: $\Box$, for *necessity*;
- auxiliary symbols: parentheses.

The modal operator $\Diamond$ for *possibility* is conveniently not taken as primitive, and $\Diamond A$ is instead introduced as a metalinguistic abbreviation for $\neg\Box\neg A$.

The set of formulas of $\mathcal{L}^{\Box}$ is inductively defined as usual: atoms are formulas, if $A$ and $B$ are $\mathcal{L}^{\Box}$-formulas then also $(\neg A), (A \vee B), (A \wedge B), (A \rightarrow B), (\Box A)$ are $\mathcal{L}^{\Box}$-formulas, and nothing else is a formula.

[22] Notice that we are considering here only axiomatic systems which do not allow derivations with open assumptions. The extension of this formulation of the rule of necessitation from pure axiomatic systems to natural deduction and to axiomatic systems allowing derivability from open assumptions has lead to many misunderstandings in the literature on modal logic, as to the alleged failure of the deduction theorem — see [HN12] for a thorough discussion of this issue.

An axiomatic calculus for the basic axiom system **K** is set up as indicated in Table 1.1.

**Table 1.1** The calculus **K**

**Axioms and axiom schemas:**

— all classical tautologies

— $\Box(A \to B) \to (\Box A \to \Box B)$ (schema $K$)

**Inference rules:**

$$\frac{A \qquad A \to B}{B}\ \text{MP} \quad \text{modus ponens}$$

$$\frac{A}{\Box A}\ \text{RN} \quad \text{necessitation rule}$$

A *formal proof* in **K** is a finite list $A_1, \ldots, A_n$ of formulas such that for all $i$ $(1 \leq i \leq n)$: $A_i$ is an (instance of) an axiom (schema) of **K**, or $A_i$ follows by the *modus ponens* rule from two previous formulas $A_j, A_k$ $(j, k < i)$ in the list, or $A_i$ follows by the *necessitation rule* from a previous formula $A_j$ $(j < i)$ in the list. $A$ is a theorem of **K** (in symbols $\vdash_{\mathbf{K}} A$) iff there exists a formal proof $A_1, \ldots, A_n$ in **K** such that $A_n$ is $A$.

A modal logic which extends **K** by one ore more extra axiom schemas is called a *normal modal logic*. **K** is thus the minimal normal modal logic.

Let us now recall the modal schemas $D$, $T$, 4, $B$, $E$ and their meaning:

$T: \quad \Box A \to A$

$D: \quad \Box A \to \Diamond A$

$B: \quad A \to \Box\Diamond A$

$4: \quad \Box A \to \Box\Box A$

$E \quad : \quad \Box A \rightarrow \Box\Diamond A$

The schema $T$ claims that *if a proposition $A$ is necessary, then it is also true* ("ab necesse ad esse valet consequentia", in the Scholastics' reading). It is also known as "epistemic schema", since it is compatible with an *epistemic* interpretation of the modal operators. If we take "$\Box$" to be a placeholder for the operator "it is known that (...)" or, "the agent $x$ knows that (...)" the schema turns out to be in accordance with the Platonic conception of knowledge displayed in *Theaetetus* 201d- 210a, namely:

$x$ **knows *that*** $p$ if and only if

1. it is true *that* $p$
2. $x$ believes *that* $p$
3. $x$ is justified in believing *that* $p$

or, in other words, the schema $T$ is consistent with the platonic conception of knowledge as *true belief with an account (justified).*

The schema $D$ claims that *if a proposition $A$ is necessary, then it is also possible.* It is known as "deontic schema", since it is consistent with a *deontic* interpretation of the modal operators. If we take "$\Box$" to be a placeholder for the operator "it is obligatory that (...)" (and consequently "$\Diamond$" to be a placeholder for the operator "it is permitted that (...)" the schema says that whatever is obligatory is also permitted.

The schema $B$ is also known (after Becker[23]) as *Brouwer schema.* It claims that *if a proposition $A$ is true*, then it is *necessarily possible that it is true.*

The schemas 4 and $E$ are both consistent with an *epistemic* interpretation of modalities and are also known as "*positive*" and "*negative*" *introspection* principle, respectively. Under an epistemic reading the schema 4 says that

[23] The reason why Becker uses this name for the schema is explained in the next Section.

*if the agent x knows that p, he also knows that he knows that p*; while the schema $E$, in the equivalent reformulation:

$$E' \quad : \quad \neg\Box A \to \Box\neg\Box A$$

says that *if the agent x does not know that p, then he knows that he does not know that p.*

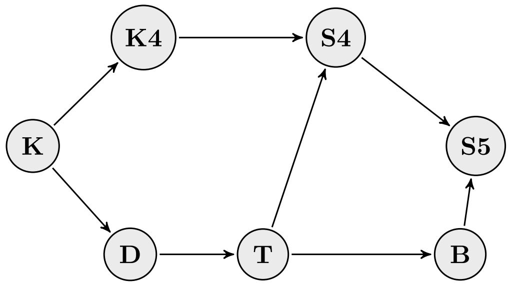
Let us now recall the most interesting axiomatic extensions of **K** which the above schemas give rise to:

- $\mathbf{D} := \mathbf{K} + D$
- $\mathbf{T} := \mathbf{K} + T$
- $\mathbf{K4} := \mathbf{K} + 4$
- $\mathbf{B} := \mathbf{K} + T + B$
- $\mathbf{S4} := \mathbf{K} + T + 4$
- $\mathbf{S5} := \mathbf{K} + T + E$

All the systems $\mathbf{K}, \mathbf{D}, \mathbf{T}, \mathbf{K4}, \mathbf{B}, \mathbf{S4}, \mathbf{S5}$ are therefore *normal* modal logics.

The "strength" relations between them can be summarized in the following diagram, where:

- an arrow leading from a system $\mathbf{L_1}$ to a system $\mathbf{L_2}$ means that $\mathbf{L_1} \subset \mathbf{L_2}$ (that is, any $\mathbf{L_1}$-theorem is also a $\mathbf{L_2}$-theorem, but not conversely),
- the *absence* of any arrow between two systems $\mathbf{L_1}$ and $\mathbf{L_2}$ means that they are **incomparable**, that is: there is at least one $\mathbf{L_1}$-theorem which is not a $\mathbf{L_2}$-theorem, and there is as well at least one $\mathbf{L_2}$-theorem which is not a $\mathbf{L_1}$-theorem.




## 1.4 Lewis's S3 and Becker's Extensions

As we already said, the formal language and the style of axiomatization employed by Lewis in his *Survey* and by Becker in his *On the Logic of Modalities* are different from the now current ones.

As primitive logical operators, they take:

- the unary operators "$-$" and "$\sim$", respectively for *negation* and *impossibility*,
- and the binary operators "$\times$" and "$=$", respectively for *conjunction* and *strict equivalence*.

Thus "$-A$", "$\sim A$", "$A \times B$", "$A = B$" correspond, respectively, to "$\neg A$", "$\neg\Diamond A$", "$A \wedge B$", "$\Box(A \leftrightarrow B)$" in our notation.

Concerning the non-primitive logical operators, both Lewis and Becker use the symbol "$\subset$", corresponding to the symbol "$\rightarrow$" in the current notation (as well as to Russell's "$\supset$"), to denote the material implication. As for the strict implication Lewis *fish-hook* symbol "$⥽$" is rendered in Becker's text (probably for typographical reasons) with the symbol "$<$", which we retain henceforth in the analysis and translation of Becker's text. The symbol for material disjunction is "$+$", corresponding to our "$\vee$". These non-primitive operators are *defined* in the expected way:

$A \subset B =_{df} -(A \times -B)$, $A < B =_{df} \sim (A \times -B)$ and $A + B =_{df} -(-A \times -B)$. To summarize:

**Table 1.2** Correspondences to current logical notation

| Logical operator | Lewis | Becker | current |
|---|---|---|---|
| – *negation* | $-$ | $-$ | $\neg$ |
| – *material conjunction* | $\times$ | $\times$ | $\wedge$ |
| – *material disjunction* | $+$ | $+$ | $\vee$ |
| – *material implication* | $\subset$ | $\subset$ | $\to$ |
| – *material equivalence* | $\equiv$ | $\equiv$ | $\leftrightarrow$ |
| – *impossibility* | $\sim$ | $\sim$ (U) | $\neg\Diamond$ |
| – *possibility* | $- \sim$ | $- \sim$ (M) | $\Diamond$ |
| – *necessity* | $\sim -$ | $\sim -$ (N) | $\Box$ |
| – *consistency* | $\circ$ | $\circ$ | $\Diamond(\ldots \wedge \ldots)$ |
| – *strict disjunction* | $\wedge$ | $\wedge$ | $\Box(\ldots \vee \ldots)$ |
| – *strict implication* | $⥽$ | $<$ | $\Box(\ldots \to \ldots)$ |
| – *strict equivalence* | $=$ | $=$ | $\Box(\ldots \leftrightarrow \ldots)$ |

In turn, Lewis's (and Becker's) axiomatization of the system **S3** is *not* given as an extension of an axiomatic calculus for classical logic by means of additional axioms and inference rules (as the axiomatizations of the normal systems reviewed in the previous Section are). Actually, it is not at all trivial to prove that all classical tautologies are theorems of the original axiomatization of **S3**.[24]

All in all, for a contemporary reader with a basic knowledge of logic it would be cumbersome to decipher and translate in the current formalism Lewis's and Becker's formulas and investigations.

[24] See [LL32], 136 ff.

In this Chapter, we therefore present the (emended) system **S3** *not* in its original formulation, but in the *equivalent* standard formulation now current in modal logic.

We then present — again, *not* in the original but in the now standard formulation — the axiomatization of Becker's **S3′** and **S3″**. Becker's claims and accomplishments will be evaluated in the next Sections.

The proofs of all our assertions (e.g. as to the equivalence or non-equivalence of the logics in question) in the present and in the next Sections are carried out in the Appendix (Chapter 4). Here we try to explain what is going on in *On the Logic of Modalities* in a way accessible to a wider audience.

## 1.5 Lewis's *Survey System* S3

An axiomatic calculus for **S3** in the style of the most known modal logics, see Section 1.3 above, is found in Table 1.3.

Thus, **S3** contains all classical tautologies, the schema $T$, the schema $K^+$, the rule of *modus ponens* and *only a restricted* version of the necessitation rule $RN$, that we indicate by "$RN^-$".

The schema $K^+$ is, like the schema $K$, a distributivity law (of the operator $\Box$ on the connective $\to$). At a a variance with $K$ it contains a further $\Box$ after the principal connective $\to$.[25]

The inference rule $RN^-$ says that if a proposition $A$ is either a classical tautology, or an instance of $T$ or $K^+$, then its necessitation $\Box A$ is a **S3**-theorem. Thus, if "$\top$" denotes any tautology, say $p \to p$, we have that $\Box\top$ is provable in **S3**. By contrast, $\Box\Box\top$ cannot be obtained from $\Box\top$ by means of RN$^-$, since it is *neither* a classical tautology nor an instance of $T$ or $K^+$. Indeed, it is possible to prove that $\Box\Box\top$ *is not* a theorem of **S3**, which im-

[25] Notice that $K$ is provable in **S3**, by using $K^+$, $T$ and the transitivity of implication.

**Table 1.3** The calculus **S3**

**Axioms and axiom schemas:**

— all classical tautologies

— $\Box A \to A$ (schema $T$)

— $\Box(A \to B) \to \Box(\Box A \to \Box B)$ (schema $K^+$)

**Inference rules:**

$$\frac{A \qquad A \to B}{B}\ \text{MP}$$ *modus ponens*

$$\frac{A}{\Box A}\ \text{RN}^-$$ provided $A$ is a tautology or an instance of $T$ or $K^+$

plies that the *unrestricted* necessitation rule $RN$ is not admissible in Lewis's system: **S3** *is not a normal modal system.*

As we already said, the original 1918 version of **S3** contained an additional axiom, which was responsible for the modal collapse (as Post pointed out to Lewis) and was therefore dropped by Lewis in his 1920 "*Emendation*". The axiom schema says: the strict implication $A \prec B$ is strictly implied by the strict implication $\neg\Diamond B \prec \neg\Diamond A$. Formally, in our notation:[26]

$$(*) \quad (\neg\Diamond B \prec \neg\Diamond A) \prec (A \prec B)$$

[26] That is, $(\sim q \prec \sim p) \prec (p \prec q)$ in Lewis's symbolism. Notice that the converse of $(*)$, $(A \prec B) \prec (\neg\Diamond B \prec \neg\Diamond A)$ is a theorem of **S3** (actually an axiom, $(p \prec q) \prec (\sim q \prec \sim p)$, in Lewis's presentation).

It is interesting to notice here how Becker accounts for the implausibility of $(*)$, by providing the following informal, yet convincing and intriguing countermodel:[27]

*Example*: One may think of a sequence of numbers generated by drawing arbitrarily many times, one after another, numbered balls from an urn, whereby, each time, the drawn ball is put back in the urn, before drawing the next ball. *It is unknown* how many balls are in the urn and how they are numbered. [. . . ]

[T]he converse

$$(\sim q <\sim p) < (p < q)$$

does not hold, as our example — suitably modified — does show. Namely, let now "$q$" and "$p$" stand for:

$q$ : "19 appears among the first 100 places."

$p$ : "19 appears among the first 200 places."

Then

$$\sim q <\sim p$$

does hold.

Actually, the *impossibility* that 19 occurs in the sequence (at a variance with its contingent, i.e., *incidental* not-occurring) can be due exclusively to the fact that no balls with the number 19 are contained in the urn. This impossibility holds for all places of the sequence, if it holds for some.

However, from $\sim q <\sim p$ it does *in no way* follow $p < q$, i.e. the implication: "If 19 appears among the first 200 places, then it necessarily appears within the first 100 places", since this sentence is trivially false.

## 1.6 Becker's *Six Modalities System* S3′

Let us now consider Becker's system **S3′**. He writes:[28]

[27] [Bec30b], 8-9 (99-101).

[28] [Bec30b], 11-12 (104-104). Recall that "$-A$" corresponds to our "$\neg A$".

Before going into the actual meaning of the reduction problem of the infinitely many nested modalities, which arise through the iteration and composition of the symbols "∼" and "−", we present a purely formal investigation, by which Lewis's system becomes a closed system, thanks to the addition of a further axiom. This can be done in several ways. [...] The assumptions introduced by Lewis are (apparently) not sufficient to obtain a closed system of irreducible modalities.
Therefore we add to Lewis's axioms the new axiom 1.9:

$$-(\sim p) < \sim (\sim p)$$

Becker is saying that — as summarized in Table 1.4 — this system (once formulated in our notation) is obtained from **S3** by the addition of one single axiom schema, namely $\Box(\Diamond A \rightarrow \Box\Diamond A)$, which is a "boxed-version" of the schema $E$.

**Table 1.4** The calculus **S3′**

**Axioms and axiom schemas:**

— all classical tautologies

| | |
|---|---|
| — $\Box A \rightarrow A$ | (schema $T$) |
| — $\Box(A \rightarrow B) \rightarrow \Box(\Box A \rightarrow \Box B)$ | (schema $K^+$) |
| — $\Box(\Diamond A \rightarrow \Box\Diamond A)$ | (schema $\Box E$) |

**Inference rules:**

| | |
|---|---|
| $\dfrac{A \qquad A \rightarrow B}{B}$ MP | *modus ponens* |
| $\dfrac{A}{\Box A}$ RN⁻ | provided $A$ is a tautology or an instance of $T$ or $K^+$ |

Becker gives a detailed proof of the fact that this system has 6 irreducible modalities:

- Positive modalities: $\Box A$, $\Diamond A$, $A$ ("factual" truth);
- Negative modalities: $\neg\Box A$, $\neg\Diamond A$, $\neg A$ ("factual" falsity).

and that they are linearly ordered, as to logical strength, as follows

- Positive modalities: $\Box A < A < \Diamond A$
- Negative modalities: $\neg\Diamond A < \neg A < \neg\Box A$.

## 1.7 Becker's *Ten Modalities System* S3″

In Brouwer's and Heyting's[29] Intuitionistic logic the *double negation principle*, $\neg\neg A \leftrightarrow A$, is not valid. More precisely, the left-to-right direction of the biconditional, $\neg\neg A \rightarrow A$, is not intuitionistically acceptable. The other direction of the biconditional,

$$A \rightarrow \neg\neg A \qquad \text{(WDN)}$$

is instead intuitionistically valid.

As we know, Becker is also trying to explore the connection between intuitionistic and modal logic. He is thus naturally led, in particular, to interpret the intuitionistic negation ("$\neg$") — which is *stronger* than classical negation — in modal terms, as *impossibility* ("$\sim$") or, as he uses to say, *absurdity* (*Absurdität*). By replacing, in the intuitionistic law (WDN), "$\neg$" with "$\sim$" and "$\rightarrow$" with "$<$" one gets

$$A < \sim\sim A$$

[29] Becker refers explicitly to [Hey30], which contains the first (complete) presentation of intuitionistic logic as a formalized calculus. The paper was published in the same year of *On the Logic of Modalities*, but was circulating since 1928.

(that is $\Box(A \to \neg\Diamond\neg\Diamond A)$, in our notation)

"Truth — as he puts it[30] — implies the absurdity of the absurdity (but not conversely!)".

Such principle is thus equivalent to the "boxed-version" of the schema $B$, $A \to \Box\Diamond A$, considered in Section 1.3. It should be now clear why Becker called it "Brouwer's *axiom*", a name still current in the literature.

According to Becker, this is a reasonable axiom to consider in order to extend Lewis's **S3**:[31]

> One can now add "Brouwer's *axiom*" to this setting (this is the weakest addition we propose):
>
> $$p = - - p \;<\; \sim\sim p \qquad (1.91)$$
>
> [...] As an [additional] axiom we choose [...]:
>
> $$\sim -p \;<\; \sim - \sim -p \qquad (1.92)$$
>
> [...] If one postulates (1.91) × (1.92) one can thus set up a ten modalities calculus.

In the current standard form we adopted, Becker's second extension of Lewis's **S3** — summarized in Table 1.5 — results from **S3** by adding *two* axiom schemas, namely $\Box(A \to \Box\Diamond A)$ (1.91), the "boxed-version" of the Brouwer's schema $B$, and $\Box(\Box A \to \Box\Box A)$ (1.92)[32] which is a "boxed-version" of the schema 4 (see Section 1.3).

Becker's claim, supported by a detailed (putative, see below, p. 40) proof, is that this system has 10 irreducible modalities:

- Positive modalities: $\Box A$, $\Diamond\Box A$, $\Diamond A$, $\Box\Diamond A$, $A$ ("factual" truth);
- Negative modalities: $\neg\Box A$, $\neg\Diamond\Box A$, $\neg\Diamond A$, $\neg\Box\Diamond A$, $\neg A$ ("factual" falsity).

[30] [Bec30b], 17 (114).

[31] [Bec30b], 17-18 (114-116).

[32] In [Bec30b], 22 f.(123 f.) Becker acutely analyzes the "concrete (phenomenological) meaning" of this schema in the light of Husserl's distinction between contingent and formal apriori.

**Table 1.5** The calculus $\mathbf{S3}''$

| | |
|---|---|
| **Axioms and axiom schemas:** | |
| — all classical tautologies | |
| — $\Box A \to A$ | (schema $T$) |
| — $\Box(A \to B) \to \Box(\Box B \to \Box B)$ | (schema $K^+$) |
| — $\Box(A \to \Box\Diamond A)$ | (schema $\Box B$) |
| — $\Box(\Box A \to \Box\Box A)$ | (schema $\Box 4$) |
| **Inference rules:** | |
| $\dfrac{A \qquad A \to B}{B}$ MP | *modus ponens* |
| $\dfrac{A}{\Box A}$ RN$^-$ | provided $A$ is a tautology or an instance of $T$ or $K^+$ |

and that they are linearly ordered, as to logical strength, as follows

- Positive modalities: $\Box A < \Diamond\Box A < A < \Box\Diamond A < \Diamond A$
- Negative modalities: $\neg\Diamond A < \neg\Box\Diamond A < \neg A < \neg\Diamond\Box A < \neg\Box A$.

## 1.8 Becker's Further "Experiments"

Becker did also tentatively consider other two possible ways to extend Lewis's **S3** in order to get a system with a finite number of irreducible modalities or, at least, a system with a possibly infinite number of irreducible modalities, yet all *pairwise comparable* with respect to logical strength. Here is a sketchy account of these two modal "experiments" (as he calls them[33]).

### *1.8.1 A Variant of* S3′

This variant, let us call it $\mathbf{S3'^*}$, is proposed in a short *Observation*[34] following the presentation and the investigation of the *six modalities calculus* $\mathbf{S3'}$. It is obtained by replacing the characteristic axiom schema $\Box E$

$$\Box(\Diamond A \rightarrow \Box\Diamond A)$$

of $\mathbf{S3'}$ (see Table 1.3) with the axiom schema $\Box E^*$

$$\Box(\Box\Diamond A \rightarrow \Box A)$$

Becker's claim is that also this new system $\mathbf{S3'^*}$ has 6 irreducible modalities, exactly the same as $\mathbf{S3'}$, and that they are ordered with respect to logical strength in the same way as they are ordered in $\mathbf{S3'}$. The only remarkable difference between the two systems, according to Becker, is that while in $\mathbf{S3'}$ $A$ is stronger than $\Box\Diamond A$ ($\sim\sim A$ in his notation), in $\mathbf{S3'^*}$ the other way around is the case: $\Box\Diamond A$ is stronger than $A$. His "formalist" conclusion is the following:[35]

> Thus, $\sim\sim p$ is stronger than $p$ (in contrast with Brouwer's conception). From a *purely formal point of view* it seems that also this approach can be carried through without contradiction; although it has perhaps no concrete meaning.

---

[33] [Bec30b], 2 (87).

[34] [Bec30b], 15-16 (111-112).

[35] [Bec30b], 16 (112).