A TEACHING INTERVENTION WITH DYNAMIC INTERACTIVE MEDIATORS TO FOSTER AN ALGEBRAIC DISCOURSE

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In this paper we analyze the developing discourse of high school students on the relations between two algebraic expressions. Using a commognitive perspective, we show in fine-grained detail how dynamic interactive mediators (DIMs) can help students with a history of low achievement find protagonists for their stories, fostering a first important step in the construction of an algebraic discourse.

INTRODUCTION AND THEORETICAL BACKGROUND

A great deal of research has documented students' difficulties in solving equations and inequalities, and in giving meaning to algebraic symbols, unknown and variables (e.g., Kieran, 2022). Common difficulties are related to a procedural view of the equal sign and to the transition to the letter-symbolic form of equations (Carpenter et al., 2005; Radford, 2022). Based on this scenario, this study investigates the effects of a didactical approach for introducing students to equations and inequalities that is based on the presentation of multiple artifacts. These are designed for representing the relationships between two expressions depending on the same variable. We used the software GeoGebra to represent expressions as moving arrows and as weights of a balance (see the next section). We investigate the effects of such an approach taking a commognitive perspective (Sfard, 2008) and analyzing students' *discourse* about the proposed *dynamic interactive mediators* (DIMs) (Antonini et al., 2020). Indeed, recent studies showed how learning to talk about well-designed DIMs can provide students with effective entry points into mathematical discourse (Baccaglini-Frank, 2021).

In line with the commognitive lens, we embrace the idea that algebra is a discourse (Caspi & Sfard, 2012) and, as such, it is characterized by its specific *words, visual mediators, narratives* and *routines* (Sfard, 2008). Learning algebra can be then described as the process of becoming able to access and express such a discourse. The framework provides many tools to capture and describe this process in a fine-grained way. We now present some of them and their use in our specific case of interest.

In this paper we investigate whether and how students accomplish a *saming process* between the two proposed DIMs, that we will call DIM_A (the one with the arrows) and DIM_B (the one with the balance), for short. Indeed, as expert mathematicians we are able to account for the fact that utterances about the reciprocal movements of two arrows realizing, for instance, 2x+1 and x+5, and corresponding utterances about the behavior of a balance with weights 2x+1 and x+5 seem to be saying "the same thing". According to Sfard (2008), we rationalize the "sameness" by conjuring *abstract objects* and speaking about arrows and balance as *realizations* of the same relation between the two expressions. It follows that the equation 2x+1=x+5 can be described as either

a *concrete object* (e.g., a balanced off balance) or as a signifier of an abstract object. Until this equation has no realizations for a student, it is just a concrete object and can only be manipulated in well-defined ways. When a student begins to speak of abstract objects and their properties, the discourse has become *objectified*. The development of this form of discourse gives us information about the students' learning process. In particular, the special property of the objectified discourse is that it *subsumes* the former independently existing discourses, in our case about DIM_A and DIM_B, making it possible to express in the new language almost everything that can be said in any of the original discourses with their own special signifiers (Sfard, 2008).

Description of the DIM designed for this study

For this study we designed a DIM realizing the relation between two expressions depending on the same variable, that we call $DIM_{(A,B)}$ because it embeds DIM_A and DIM_B (Fig.1). On the left part it shows a balance, with the plates described by algebraic expressions, dependent on the unknown weight *x* and written in a blue and a grey box. A label shows the relation between the plates: when their weight is equal, it is green; otherwise, it is yellow. On the right part there are three arrows moving like a dynagraph (Antonini et al., 2020). There are a red tick mark, realizing *x*, that is directly draggable bound to the positive x-axis, and a blue and a grey tick mark, realizing the two expressions depending on x, that move indirectly. These two tick marks correspond to the two plates of the balance, that also change simultaneously, with the dragging of *x*.
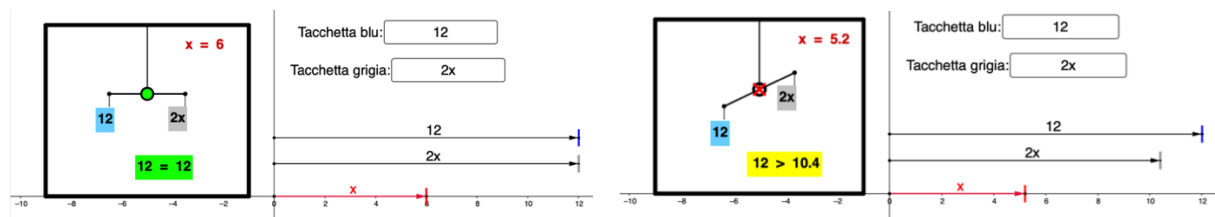

Figure 1: Two screenshots of the $DIM_{(A,B)}$ with the expressions '12' and '2x'.

We designed this DIM with the aim of promoting the process of saming between DIM_A and DIM_B. The hypothesis is that the interaction with $DIM_{(A,B)}$ can foster the emergence of new discourses that may lead to the birth of a subsuming discourse. We hypothesized this could happen once students had interacted separately with DIM_A and DIM_B and constructed an "A-discourse" and a "B-discourse". These are discourses involving the DIMs; their main features are reported in Table 1. In line with the discussion in the previous section, we thought that a possible discourse subsuming A- and B- discourses could involve algebraic symbols (see the fourth column of Table 1). Hence, in the $DIM_{(A,B)}$ we introduced labels with algebraic expressions. The protagonists of this S-discourse are objectified equations and inequalities, while the balance and dynagraph become possible realizations. In our previous examples with the two expressions, the narratives "the balance hangs to the left" in the B-discourse and "the blue arrow is to the right of the grey one" in the A-discourse find a counterpart in the S-discourse with "2x+1>x+5".

Table 1: Features of discourses about DIM_A, DIM_B and of a possible subsuming.

	A-discourse	B-discourse	S-discourse
Words	Tick mark, arrow, drag, right/left	Balance, weight, plate, up/down	Expression, equal, bigger/lower
Visual Mediators	Red, blue, and grey tick mark, ax+b and cx+d labels, gestures on DIM_A	Red, blue, and grey boxes, ax+b and cx+d labels, gestures on DIM_B	ax+b and cx+d
Narratives	"If the red tick mark is at 2, the other tick marks are aligned"	"If the unknown weight is 2, the balance is balanced off"	If x=2 then ax+b=cx+d
Routines	Drag to the right/left	Put on/take off	Algebraic manipulation

Research questions

This study is part of a greater funded research project exploring the impact of teaching interventions with second year high school students with a history of low achievement in mathematics who are asked to engage in a set of newly designed activities with DIMs in the context of algebra. In this paper, we focus on students' discourse about the relations between two expressions of the same variable. We conjecture that DIMs can play a key in students' development of this discourse, by supporting the construction of abstract objects that may become the protagonists of the subsuming discourse. To investigate such a conjecture we designed the three DIMs above and in this study asked the following specific research questions: *What characteristics (words, visual mediators, narratives and routines) of the discourses about DIM_A and DIM_B does the discourse developed by students in their interaction with the $DIM_{(A,B)}$ have? To what extent does this new emerging discourse subsume the former ones?*

METHOD

Data collection occurred in an out-of-school learning center with twelve 10th grade low achieving students volunteering from three different Italian high schools. Participants attended four 2-hour-long sessions, during which DIMs-based activities were proposed by a researcher. Data were collected in the form of video recordings, students' written productions, and screen recordings of the tablets used for the activities.

This paper focuses on two pairs of students engaged in activities with $DIM_{(A,B)}$, during the 4th session. They are given explorative tasks aimed at fostering a discourse subsuming the previously constructed A- and B- discourses and the researcher's questions seek to promote saming between DIM_A and DIM_B. We analyzed data focusing on three features that we operationalized through guiding questions and communicational indicators. We looked for such indicators to be identified in the transcripts and coded them with the typographic marks shown in Table 2.

Table 2: Analytic scheme.

Feature	Guiding questions	Examples of indicators
Type of discourse	Does it have the characteristics of A-, B- or S- discourse?	See the description of A-discourse, B-discourse and S-discourse in Table 1
Presence of abstract objects	Are there references to an expression as signifier of an abstract object?	*"x+1 is bigger than", "x+1 equals", ...*
Instances of saming	Are DIM_A and DIM_B described as realizations of the same signifier?	**"is the same", "is equal", "they make the same",** …

DATA ANALYSIS

We tell the story of Andrea (A) and Hugo (H) through the analysis of selected short episodes. Then, we present some episodes of Grazia (G) and Lucia (L) interacting with the same DIMs, for discussing similarities and differences in their emerging discourses.

Andrea and Hugo

When Andrea and Hugo first saw the $DIM_{(A,B)}$ in which the expressions 12 and 2x were defined (Fig. 1), they observed the presence of a "balance that is balanced off" and before interacting with it, they discussed with the researcher (R) as follows:

1	H:	The two plates are both at the same level
2	R:	Ok, both the plates are at the same level and what is there on the plates?
3	H:	Twelve and two x
4	R:	Mm and why are they at the same level?
5	H:	Because they both have, we can say, the same weight
6	A:	Value? The same value, let's say
7	H:	Because it also tells us that *x equals six* [He points at the red label x=6] so *two x equals twelve*
8	R:	Ok, perfect and instead on this side [She points at DIM_A]?
9	H:	**It indicates it with the tick marks** [He does not make gestures]

Through a B-discourse, Andrea and Hugo describe the equality between the two expressions for x=6. We highlight the first occurrence of the word "value" (turn 6) that can be seen as a seed of subsuming discourse, because it may refer to the "value of the expression"; however, it is used again by Andrea referring to DIM_B as the "value of the weights". At turn 3, '12' and '2x' are the objects of Hugo's discourse, to which he seems to refer not yet as abstract, but as concrete objects. Differently, at turn 7, '2x' is used without explicit references to the DIMs. The last sentence expressed by Hugo (turn 9) suggests a partial instance of saming. Indeed, an asymmetric relation emerges between the DIMs: the tick marks indicate what happens on the balance but not vice

versa. The student said that the tick marks indicate "it" referring to what has just been said for the balance, but what "it" means is missing.

The researcher then asked the students describe what happens when dragging x in $DIM_{(A,B)}$. While Andrea was dragging Hugo stated:

10   H: It depends <u>on the value of how much the red tick mark is</u> [He simulates with the finger the movement of the red tick mark on the axis], because <u>if it goes to the right</u> the balance tends to, tends to hang... to the right, while <u>if it is a number smaller than six</u>, <u>the balance tends to hang</u> to the left

This excerpt is characterized again by B-discourse, since it mainly concerns the behavior of the balance. However, there are also some A-discourse narratives and a seed of S-discourse in the utterance "if it is a number smaller than six" since it can refer to both DIMs because the subject is not made explicit by Hugo. There is another occurrence of the word "value", used as before but this time referring to a tick mark. Therefore, "value" could constitute a word in these students' emerging S-discourse. However, as evidenced by their writing at the end of the episode (Fig. 2a), the relations between the quantities are uniquely expressed in terms of balance's swing, confirming the B-discourse to be the prevailing one so far.

In the next activity, two new expressions were defined in $DIM_{(A,B)}$: x+5 and 2x+1, and students' exploration took place, prompted as before. When asked to summarize what they observed, Andrea and Hugo again produced a B-discourse to describe the relationships between the two expressions (e.g., "<u>if x equals four</u>...<u>the plates have the same weight</u>. Then, instead, <u>if x is greater than four</u> it tends to <u>dangle to the right</u>"). This summary is accompanied by an inscription that is very similar to that of the previous episode. But "the scale is even" is replaced by "they have the same value" (Fig. 2). This difference suggests the students' development of an S-discourse because in their narrative about the equality they lost the reference to the balance model by using the term "value", which had previously been used within both A- and B-discourses. The same expression could indeed refer to the position of the tick marks.
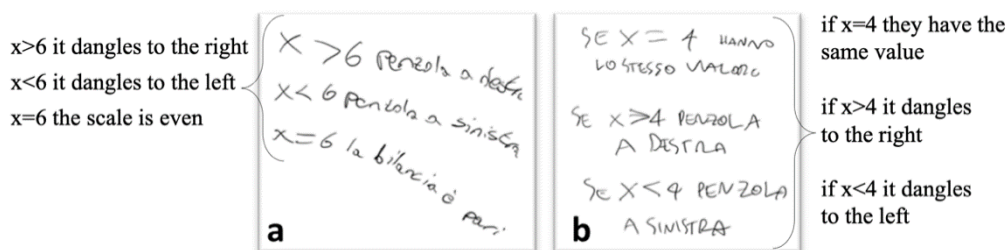

Figure 2: Andrea and Hugo's written production for the activity on the $DIM_{(A,B)}$ with the expressions (a) '12' and '2x', (b) '2x+1' and 'x+5'.

In another similar activity, involving the expressions 2x+4 and 3x, Hugo related DIM_A and DIM_B by observing that the behavior of the balance depends on that of the arrows ("<u>The balance tends to dangle</u> to the left ... because the <u>blue tick</u> let's say <u>has a greater value than the grey one</u>"). Then, about the equality, Hugo stated:

11   H: <u>All the lines are on the same number, they have the same value.</u>

12	R:	Ok, right. And <u>the balance</u>?
13	H:	Let's say, <u>it always gives the same value</u>.

The researcher's intervention promoted Hugo's move from an A-discourse (turn 11) to a B-discourse (turn 13). He used "the same value" both referring to the tick marks on the same number and to the balance plates weighing the same. This suggests the emergence of an S-discourse and a process of saming between the DIMs.

We conclude the overview of Andrea and Hugo's story with a short episode, following the previous one, in which an interweaving of A-discourse and S-discourse appeared.

14	H:	If the <u>tick mark</u>, if we want the <u>blue tick mark greater than the grey one</u>… The <u>number</u>, *<u>the x has to be smaller than four</u>*, while if we want that *three x* that the <u>grey tick mark</u> is greater it has to be a <u>number greater than four</u>
15	A:	When it is <u>at four</u>
16	H:	When it is <u>at four</u> <u>the value is equal</u>

The episode begins with an A-discourse in which Hugo mentions the tick being "greater" (turn 14), probably referring to the value taken on. Interwoven with this discourse, however, there are endorsed narratives both in A- and B- discourses. The A-discourse reappears in the next turns when they said "is at four" instead of "is four" suggesting the focus to be on the position of the red tick. This is accompanied, again, by the subsuming narrative involving "the value". Moreover, we observe that in this episode for the first time there are possible instances of abstract objects (turn 14). For example, when Hugo used "grey tick mark" as a noun for the object "three x".

Grazia and Lucia

In the first activity, after an initial exploration without having yet dragged the tick realizing x, Lucia described $DIM_{(A,B)}$ involving the expressions 12 and 2x as follows:

1	L:	Like the first 12 of the <u>balance</u> is indicated as, as it is the double of this *x* [She does not make gestures], and the same thing for the <u>grey tick mark</u>… So they correspond, that is, **it is the representation of the <u>balanced off balance</u> made <u>on the line</u>**

Lucia initially produced a B-discourse which is then followed by an A-discourse introduced by the expression "the same thing". The student seems to make explicit a saming between the two visual mediators, as also highlighted immediately afterwards by her narrative "it is the representation of the balanced off balance made on the line".

Now we share a short excerpt from the next activity, involving the expressions x+5 and 2x+1, in which we find another instance of saming in Lucia's discourse:

2	L:	That is, you also notice that <u>the tick marks are not aligned anymore</u>, as before, because <u>the balance is not balanced off anymore</u>, but they indicate exactly two different <u>values,</u> so the <u>balance</u> is not, <u>is not in balanced off anymore</u>

Lucia, by mixing A- and B-discourse, refers to a cause-and-effect relationship between the two DIMs, which bound in both directions: "the tick marks (...) because the balance (...)" and then "[the ticks] indicate (...) so the balance (...)". This highlights how DIM_A

and DIM$_B$ played a mutual role in Lucia's discourse, being both signifier and realization one for the other.

Shortly after this excerpt, the description of the behavior of the balance occurred with an interweaving of S-discourse and B-discourse, in which the former was more present:

> 3   L: If <u>*x*</u> would have been <u>smaller than 4</u> ehm we expect the opposite situation, so that <u>*2x plus one*</u> is, we say, <u>it corresponds to a value smaller than *x plus 5*</u>… And instead <u>the balance will be balanced off</u> when <u>*x* will be four</u>.

The reference to the DIM$_B$ appears only at the end of the turn. Except for this, the one produced by Lucia is a S-discourse, since it is an endorsed narrative to describe in the same terms the situation that both balance and arrows realize.

We conclude by showing what Grazia and Lucia wrote to summarize their observations about the two pairs of expressions explored in the activities in focus (Fig. 3a-b). Note how this written discourse, unlike Andrea and Hugo's, completely subsumes the A- and B-discourses constructed during the activities with the DIM$_{(A,B)}$.
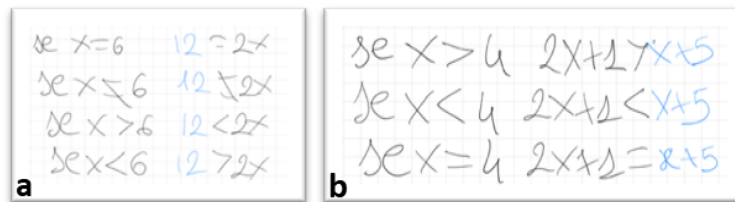

Figure 3: Grazia and Lucia's written production for the activity on the DIM$_{(A,B)}$ with the expressions (a) '12' and '2x', (b) '2x+1' and 'x+5'.

DISCUSSION AND CONCLUSIONS

The analyses presented above confirm that DIMs can foster low achieving students' participation in mathematical discourse. More specifically, activities with multiple DIMs, designed to be realizations of the same algebraic signifier, can promote the development of algebraic discourse as a form of subsuming discourse. The DIM we designed, embedding a balance (Otten et al., 2019) and a dynagraph (Antonini et al., 2020) as realizations of the relations between two algebraic expressions, accomplishes this by creating the need for a common discourse. However, significant differences between the two pairs of students' discourse emerge. On one hand, Grazia and Lucia accomplish saming between DIM$_A$ and DIM$_B$, having at their disposal the ingredients for constructing a S-discourse: algebraic expressions, and their relations, as abstract objects of which the two DIMs are realizations. Andrea and Hugo, on the other hand, do not seem to develop this degree of objectification and this results in a lack of protagonists of a possible S-discourse. We see a partially successful attempt to construct a protagonist in their use of the term "value" bridging A- and B-discourses. Even if it is only a first step of the process, this is a success for them, considering the difficulties shown in the previous sessions. A longer intervention might have made a difference, but nevertheless there is a seed of subsuming discourse.

We now discuss limitations and aspects not a priori expected. We thought that fostering saming between DIM$_A$ and DIM$_B$, through activities asking to speak in different ways

and to look for similarities between the DIMs, could support the construction of a S-discourse. However, we observed how the processes of saming and the process of constructing an S-discourse are closely intertwined, generating a kind of paradox that may be explained in discursive terms as follows: saming needs a subsuming discourse with which to talk about "the same thing" that the two DIMs realize, but, at the same time, the protagonists of subsuming discourses are the products of a saming process. The story of Andrea and Hugo shows how this circularity can be overcome through the use, within the same discourse, of words, visual mediators, narratives, or routines from different discourses; in their case, for example, the word "value".

In conclusion, in this paper we told the stories of two pairs of students, providing insights for both research and didactic reflections. Especially, the story of Andrea and Hugo can contribute to the literature on difficulties in working with the letter-symbolic form of equations and inequalities (e.g., Carpenter et al., 2005; Kieran, 2022). It also confirms the importance of designing didactical approaches aimed at fostering students' learning to talk about multiple DIMs realizing the same object, because they can open new doors into mathematical discourse for students like Andrea and Hugo.

ACKNOWLEDGMENT

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