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Memory retrieval in the demand game with a few possible splits: Unfair conventions emerge in fair settings

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ABSTRACT

Our study examines the long-run evolutionary outcome emerging in scenarios where two populations engage in a demand game with three potential splits. These populations differ in the sample sizes used when best responding to retrieved information from the past. Our findings reveal the existence of a threshold in the setting's fairness (i.e., the fairness of unfair splits) such that, below the threshold (i.e., in an unfair setting), the emerging convention is the fair one, while above the threshold (i.e., in a fair setting), the emerging convention is unfair, favoring the agents with the longer sample size. The threshold gets lower as the difference in the sample sizes increases.

1. Introduction

In this paper, we address the question how fair settings relate to fair evolutionary outcomes in the demand game. In particular, we consider settings where, beyond the fair (50/50) split, unfair outcomes exist and we let them vary in the extent of their unfairness, i.e., inequality of the distribution of resources assigned to claimants. By doing so, we address unfairness on two distinct levels: the potential unfairness of unfair splits, and the actual unfairness of the convention arising in the long run. In particular, we focus on the demand game, which is a model of bargaining introduced by Nash (1953). There are several other stylized models of simultaneous resource sharing in game theory, including the contract game (Young, 1998), which differs from the demand game because claims that sum up to less than the whole amount of resources pay nothing to the claimants, the hawk-dove (Arigapudi et al., 2021; Bilancini et al., 2022), and the battle-of-sexes (Luce and Raiffa, 1989), also in the version with an inefficient compromise option (He and Wu, 2020). If sequential resource sharing is taken into consideration, then the ultimatum game (Güth et al., 1982) can be considered as well.

We take an evolutionary perspective on the outcomes of the demand game, by following the approach of Young (1993a). At each time, two agents are selected from two distinct populations to play a demand game. The model incorporates a memory system of finite size that records the history of past plays. Each of the selected agents independently accesses this memory to review a random subset of these past interactions. The agents choose an action by best responding to the empirical distribution of their opponent's previous decisions within their subset. Each of the two populations is distinguished by the size of the memory sample it can draw upon, called sample size, meaning they have access to different numbers of past plays. Within this framework, we look at resource allocations that remain stable over time, called conventions. Given the multiplicity of possible splits, and hence possible conventions,

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we introduce noise in the dynamic process for the purpose of selecting among the possible conventions. By doing so, we thus obtain an ergodic Markov chain which allows us to find the average time spent in each convention, irrespective of the initial condition. By letting the amount of noise approach zero, we select the stochastically stable convention (Foster and Young, 1990), i.e., the convention where the dynamic system spends most of its time in the very long run in the presence of tiny perturbations.

In our model, we conceptualize the interaction among populations as global, with memory being a collective phenomenon. This scenario is plausible in contexts where interaction barriers are absent and information is readily accessible to the public. Alternatively, one could consider the scenario of local interactions, where agents are interconnected within a network structure (as in Abreu and Manea, 2012). Local interaction models are well developed in the stochastic evolutionary literature, with both theoretical analysis (Pin et al., 2017; Cui, 2014) and simulation analysis (Alós-Ferrer et al., 2021). Such a configuration lends itself well to the incorporation of individual memory models (Alós-Ferrer and Shi, 2012), creating a more nuanced and dynamic interaction landscape.

We restrict attention to demand games with only three strategies available (sometimes called mini-Nash demand games), that are demanding half of resources (M), more than half (H) and less than half (L), with the last two strategies being evenly spaced from the middle. This restriction is typical of a stake where resources are limited in number and available in indivisible units, such as in the case of bargaining over real estate and assets like cars or paintings. Beyond this and even more importantly, we believe that this restriction can capture a realistic cognitive simplification of bargaining: fractions composed of smaller numerators and denominators are inherently more salient than more complex fractions that include larger numbers (refer to "easy" and "hard" fractions in Young and Burke, 2001).

In this setting, we find that there exists a threshold in unfairness, i.e., distance of the asymmetric splits from the equal split, driving the result: if asymmetric splits are sufficiently unfair, i.e., their distance from the equal split exceeds the threshold, then the stochastically stable convention is the fair division. Conversely, if asymmetric splits are not very unfair, i.e., their distance from the equal split falls below the threshold, then the stochastically stable convention is the unfair division where the population with larger sample size takes most of the resources. We stress that, when the two populations are homogeneous in terms of sample sizes, the equal split is stochastically stable for any degree of unfairness of asymmetric outcomes (as already known from Young, 1993b). In the last part of the paper, we extend our analysis to a setting with more than three strategies by relying on the assumption of local mistakes (in the spirit of Young, 1993b), and we find that the degree of unfairness of the least unfair splits (the ones closest to the middle) determines, similarly to what obtained with three strategies only, whether the stochastically stable convention is the equal split or one of the unfair splits in favor of the population with larger sample size.

The take-home message of this paper is that more fairness in the possible divisions of resources may in fact lead to more unfair outcomes. More precisely, if unequal divisions of the resources are far from the 50/50 division, then the convention emerging in the long run is the fair one, while if unequal divisions are close to the 50/50 one, then the convention emerging in the long-run is unfair.

The main reference for our paper is Young (1993b), where the stochastically stable convention is found to be different from the fair split whenever there is heterogeneity between the two populations. The main difference in our setting concerns the number of possible splits: while Young (1993b) considers a fine grid of possible splits, we focus on the other extreme with only three possible splits, and we find results that depend on the extent of unfairness of the splits. The different setting allows us to study how stochastic stability results are affected by changes in the fairness of the setting, for a given level of heterogeneity between populations, in particular differences in the sample size. Our assumption better fits situations where the stake of bargaining has a low degree of divisibility. Also, to isolate the effect of heterogeneity between the two populations only in the sampling size, and differently from Young (1993b), we do not assume heterogeneity in the utility function. Other possible forms of heterogeneity include differences in the mistake rates, which capture different propensities to error-making, possibly as a function of expected or experienced payoffs (Mäs and Nax, 2016; Lim and Neary, 2016; Bilancini and Boncinelli, 2020; Bilancini et al., 2021).

Many other papers in the literature have studied modified versions of the demand game from an evolutionary perspective. Tröger (2002) and Ellingsen and Robles (2002) study the efficiency of the stochastically stable division in a setting where the value of the resource is determined by one party's investment decision in a pre-stage of the game. Sawa (2021) studies the influence of reference-dependent preferences on the stochastically stable division in a two-stage Nash bargaining game where players can exercise an outside option in the first stage, whose value is considered as a reference point (Kahneman and Tversky, 1979). Along this line, in Khan (2022) the reference point is determined endogenously and individuals do not have the outside option. Sáez-Martı and Weibull (1999) study stochastic stability by introducing asymmetries in the degree of sophistication of the two populations, with the agents in the more sophisticated population who best reply to the best reply of the others. A close body of literature is about the multilateral Nash demand game (Sawa, 2019; Newton, 2012; Nax, 2015; Agastya, 1999; Rozen, 2013), for which we refer to the detailed discussion in the review of Newton (2018). As concerns simultaneous bargaining, but departing from the demand game, there is a stream of literature on the evolution of conventions in the contract game (for recent contributions, see Naidu et al., 2010; Hwang and Newton, 2017; Hwang et al., 2018, and again Newton, 2018, for a detailed review).

There are a few contributions in the literature that focus on the demand game with a small number of available strategies (corresponding to different claims on the overall resources). Skyrms (1996) studies the evolution under the replicator dynamics in a population with three different groups, each programmed to play one of the possible strategies; in this setting, two possible outcomes can occur in the long run: one is the monomorphic state in which all players demand for half of the resource and second, the other is the polymorphic state in which two groups/strategies coexist, one demanding for more than half and the other one demanding for less than half. Skyrms and Zollman (2010) notes that the polymorphic state is more likely to occur when only interactions between agents of different groups are allowed. Axtell et al. (2007) show that the unfair division is favored by the endogenous formation of different classes based on an observable characteristic, even if unrelated to the payoff structure of the interaction. Such results have

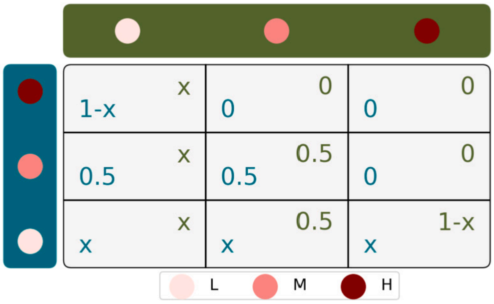

Fig. 1. Payoff table of the demand game with $x \in (0, 0.5)$ and three strategies: Low (L), Medium (M), and High (H).

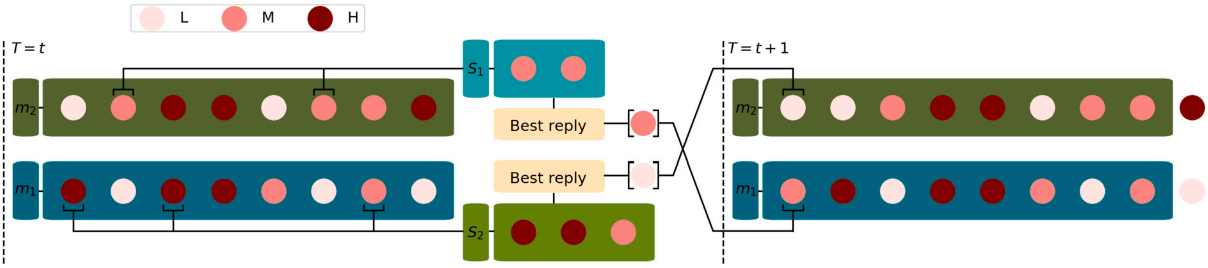

Fig. 2. Representation of an example of transition from period $T = t$ to period $T = t + 1$ with $m = 8$, $k_1 = 2$, and $k_2 = 3$. The last m plays of player 1 and player 2 are denoted with, respectively, m_1 and m_2, while S_1 and S_2 indicate the sample extracted by, respectively, player 1 and player 2.

been extended by Poza et al. (2011) considering also a spatial structure. In both contributions the analysis is carried out through agent-based simulations. When bargaining happens in a setting with local interactions and agents who imitate their best performing neighbor, the evolutionary process almost always converges to the fair division (Alexander and Skyrms, 1999). When the population is divided in classes, the relative dimension of classes matters, as showed by Bruner (2019) and O'Connor and Bruner (2019), where minorities are disadvantaged. The evolutionary implications of fundamentalism in social struggle are investigated by Arce M and Sandler (2003) using a demand game with only three strategies.

The rest of the paper is organized as follows. Section 2 introduces the model and provide results on the recurrent classes of the unperturbed dynamics. Section 3 considers the perturbed dynamics and gives the main results on stochastic stability. Section 4 provides data from simulations in the attempt to extend our results to a setting with more than three possible splits. Section 5 summarizes the contribution, discusses the results and sketches lines for future research.

2. Model

We consider two populations of finite size. Time is discrete, denoted with $t = 1, 2, \ldots$, and in each period one agent is drawn at random from each population to interact in a demand game. The agent drawn from population 1 acts as player 1 and the agent drawn from population 2 acts as player 2. Each player chooses a pure strategy s_i from a strategy set $S_i = \{L, M, H\}$, with $i \in \{1, 2\}$. Play at time t is defined as $s(t) = (s_1(t), s_2(t))$ and the amount of resources obtained by player i is $\pi_i(s(t))$, according to the matrix in Fig. 1. We assume that $0.5 > x > 0$.

We note that as x decreases the unfairness of the asymmetric divisions increases, ranging from almost fairness (for x close to 0.5) to extreme unfairness (for x close to 0). Therefore, we consider $x \in (0, 0.5)$ as a measure of the *fairness* of the setting. Every agent has the same utility function u, defined on the amount of resources obtained, which is assumed to be (weakly) concave and such that $u(0) = 0$. Agents recall the last m periods of play between both populations, hence m can be interpreted as the (collective) long-term *memory length*. A history of play encompassing the last m periods is described by $h(t) = (s(t), s(t-1), \ldots, s(t-m+1))$, with t denoting the current period. Furthermore, agents adjust their choices over time according to the adaptive learning assumptions in Young (1993a). In general, agents select the best response to a randomly drawn sample of k opponents' plays in their memory, see Fig. 2 for a graphical representation. In case of multiple best responses, all of them have positive probability to be selected. As is standard in the literature, we refer to k as *sample size*, and we interpret it as working memory.

The dynamic system under consideration is a Markov chain (S, T) (see Young, 2001, for an overview of Markov chain theory), where S is the state space composed of all possible histories, i.e., sequences of m plays of the game $(s^m, \ldots, s^\ell, \ldots, s^1)$, with $s^\ell \in \{(S_1^\ell, S_2^\ell)\}$ for all $\ell = 1, \ldots, m$. Transition between states is defined by transition matrix T, with $T_{hh'}$ being the probability of moving from history h to history h' in one period of time according to the above adjustment dynamics. It must hold that $T_{hh'} > 0$ only if h' can be obtained from h by deleting the rightmost play of the game and adding a new play of the game to the left of the sequence.

Any state h consisting of m repetitions of a strict Nash equilibrium constitutes a *convention*, that is inescapable given the defined dynamics. The Bargaining game proposed has three strict Nash equilibria: (H, L), (M, M), and (L, H). The corresponding conventions are defined by

- $h_{H,L} = (s^m, \dots, s^\ell, \dots, s^1)$ such that $s_1^\ell = H$ and $s_2^\ell = L$ for all $\ell = 1, \dots, m$
- $h_{M,M} = (s^m, \dots, s^\ell, \dots, s^1)$ such that $s_1^\ell = M$ and $s_2^\ell = M$ for all $\ell = 1, \dots, m$
- $h_{L,H} = (s^m, \dots, s^\ell, \dots, s^1)$ such that $s_1^\ell = L$ and $s_2^\ell = H$ for all $\ell = 1, \dots, m$

Lemma 1. *If $k_1 \le \frac{1}{2}m$ and $k_2 \le \frac{1}{2}m$ then $h_{H,L}$, $h_{M,M}$, and $h_{L,H}$ are the only recurrent classes.*

A recurrent class is a set of states, possibly a singleton, that cannot be exited, where each state has probability 1 to be visited again once left. It is immediate to recognize that $\{h_{H,L}\}$, $\{h_{M,M}\}$, and $\{h_{L,H}\}$ are recurrent classes, with no need of proof. In the proof in Appendix A we show that, starting from any other state, there is a positive probability to reach one of $h_{H,L}$, $h_{M,M}$, and $h_{L,H}$ in a finite number of periods.

3. Perturbed dynamics

With the aim of selecting a convention that is stabler than others, we introduce errors (in the spirit of Foster and Young, 1990). We suppose that a player does not always choose a strategy that is a best response to the sample; indeed, with a small probability ε close to zero, the player chooses one of the three strategies (H, M, L) at random. We refer to these *ε-probability* events as errors. We observe that, thanks to errors, the system can move with positive probability from any history of play h at time t to any other history h' at time $t + m$. The perturbed process, P^ε, is an irreducible and aperiodic Markov chain, thus ergodic. We use the technique of rooted trees to identify the stochastically stable conventions (Young, 1993a). A rooted tree is a directed graph where conventions are the vertices, and specifically a tree rooted at a convention is a set of two directed edges, such that there exists one and only one path connecting the two other conventions to the root of the tree. There are three rooted trees for each of the three conventions (see Fig. 3). The resistance of a transition between two conventions is defined as the minimum number of errors necessary to obtain the transition. A transition is triggered when one population (call it "the triggers") makes enough errors to make it possible for the other population (call it "the observers") to extract a sample from the memory to which the observers best respond with a different strategy. This allows the transition to another convention with no further error. Indeed, to successfully complete the transition the observers need to keep extracting for long enough samples from the memory containing the errors of the triggers. This is feasible because the memory length is at least twice the size of the sample, which is possible because the memory length at least double the sample size. After many enough plays of the new strategy by the observers have accumulated in memory, it becomes possible for the triggers to extract a sample to which they best respond with the strategy played at the beginning by error, so that a Nash equilibrium is played in the last period. If the new Nash equilibrium is played for other $m - 1$ periods in a row, which can happen with positive probability, then the transition to the new convention gets completed.

Importantly, the minimum number of errors required to move from the initial convention to the final one changes depending on the role of populations, i.e., who are the triggers and who the observers. The resistance of such transition is the minimum between the two numbers. Denote the resistance of the transition from convention i to convention j as $r_{i,j}$, with i and $j \in \{HL, MM, LH\}$ and $i \neq j$.

The resistance of a rooted tree is the sum of the resistances of the transitions along the edges composing the tree. The stochastic potential of a convention, denoted with γ_i, for $i \in \{HL, MM, LH\}$, is the minimum resistance over all trees rooted at such convention. The convention with minimal stochastic potential is stochastically stable (Young, 1993a).

Proposition 1. *For given k_1 and k_2, with $k_1 < k_2$, there exists a threshold $x^T(k_1, k_2) \in (0, 0.5)$ such that:*

- *if $x < x^T(k_1, k_2)$ then MM is stochastically stable;*
- *if $x > x^T(k_1, k_2)$ then LH is stochastically stable.*

Notice that, if $k_1 = k_2$, we already know from Young (1993a) that MM is stochastically stable. We stress that we may not have a unique stochastically stable convention, due to the ceiling effect yielding the same integer numbers. The proof of Proposition 1 rests on a tree-surgery argument, which counts the number of errors required for transitions along trees with the recurrent classes as nodes (see Appendix A).

Proposition 2. *$x^T(k_1, k_2)$ is weakly decreasing in k_2 and weakly increasing in k_1.*

In words, the threshold value x^T decreases as the difference $k_2 - k_1$ grows larger. This reduction in the threshold leads to an increase in the level of unfairness necessary to achieve an equal split as the stochastically stable outcome.

To better understand the novelty of our contribution with respect to the previous literature, we note that our result cannot be explained through the maximization of the function $F(x) = u(x)^{k_1} u(1 - x)^{k_2}$, which identifies the asymmetric Nash bargaining

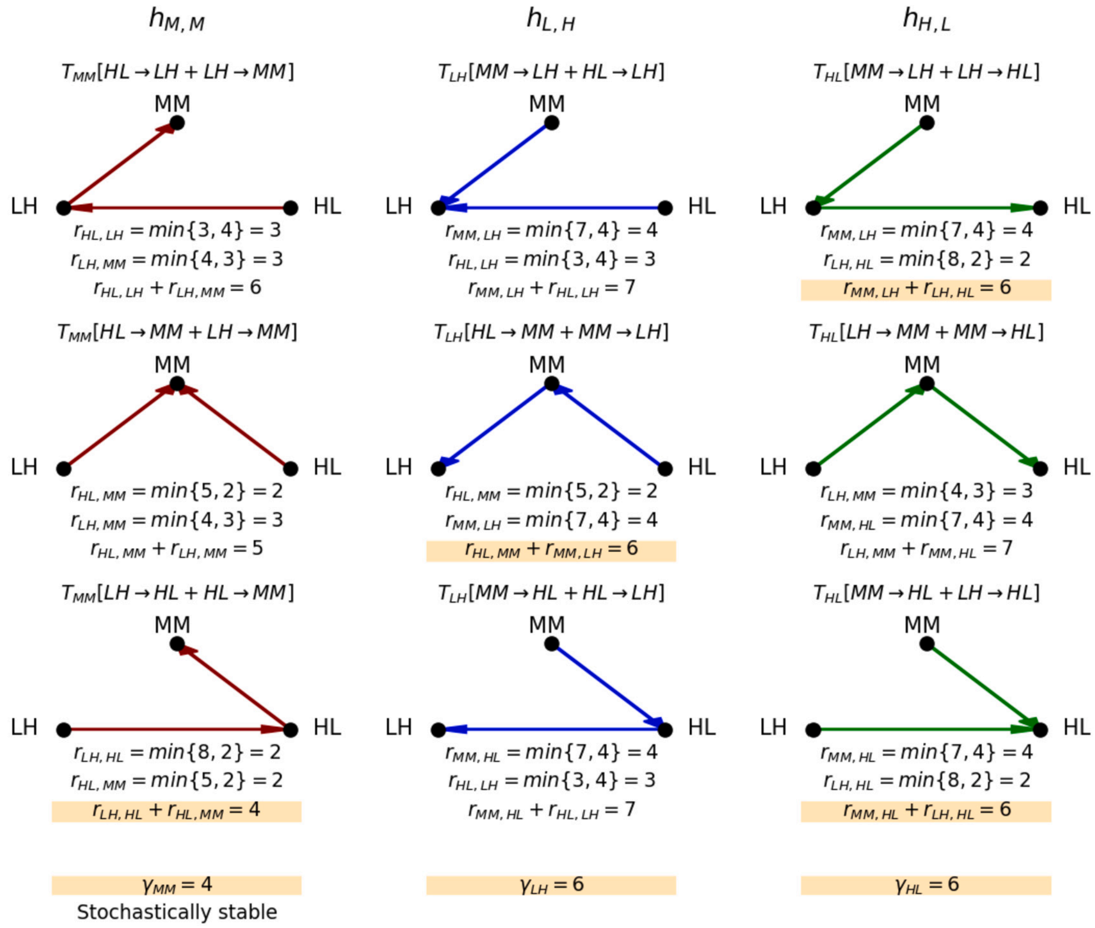

Fig. 3. Each convention has three rooted trees: on the left, the trees rooted in h_{MM} in red; on the center, the trees rooted in h_{LH} in blue; on the right, the trees rooted in h_{HL} in green. Below each tree, there are indicated the resistances of the transitions of its edges, along with their sum, which represents the resistance of the tree. The stochastic potential of each convention is highlighted in yellow. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

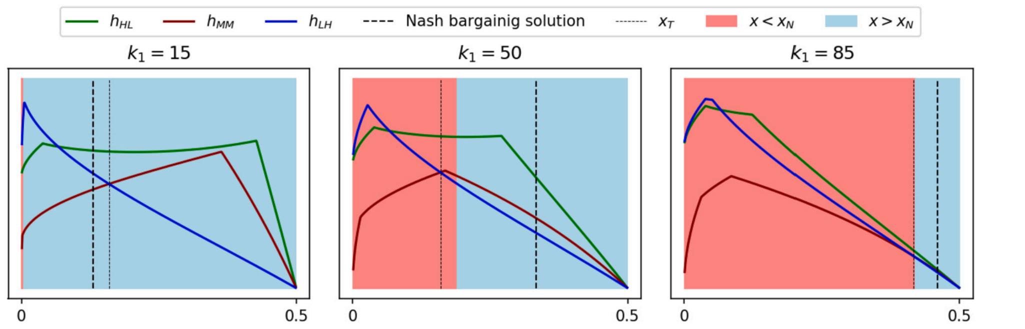

Fig. 4. The above plots depict functions from which the stochastic potentials of the conventions can be computed by taking the ceiling. The plots are drawn for $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ with $\gamma = 0.5$ and, from left to right, $k_1 = 15$, $k_1 = 50$, and $k_1 = 85$, while k_2 is kept fixed at 100. The red color in the background represents the area in which $F(x) < F(0.5)$, the blue color in the background represents the area in which $F(x) > F(0.5)$.

solution that emerges in the evolutionary model of Young (1993b). The function $F(x)$ exhibits a threshold, denoted with x_N, such that $F(x) < F(0.5)$ when $x < x_N$, and conversely $F(x) > F(0.5)$ when $x > x_N$. We observe that the x_N threshold is not the same as the x_T threshold identified in Proposition 1. We show in Fig. 4 three numerical examples in which the utility function is the root square, $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ with $\gamma = 0.5$, $k_2 = 100$, and k_1 is equal to 15, 50, and 85. In the subplot on the right, with $k_1 = 85$, x_T and x_N are close and the Nash bargaining solution is in the area in which the unfair convention is stochastically stable. In the subplot in the center, with $k_1 = 50$, x_T is significantly lower than x_N, i.e., there is a large area in which the unfair convention is stochastically

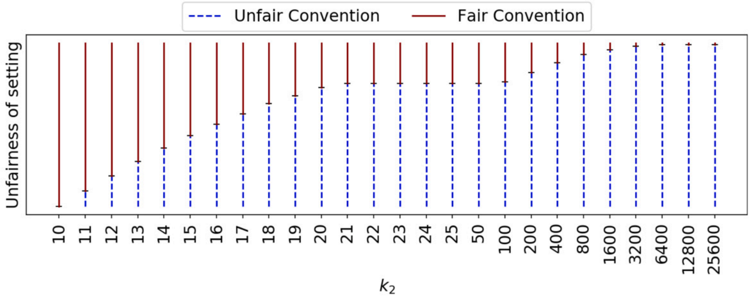

Fig. 5. k_1 is set at 10 while k_2, which is represented in the x-axis, ranges from 10 to 25600. y-axis represents the unfairness of the setting, with minimum unfairness occurring when all possible splits are permitted, and maximum unfairness arising when only the fair split and the most extreme unfair splits are allowed. The red solid line indicates that the fair convention is stochastically stable; the blue dashed line indicated that an unfair convention is stochastically stable.

stable while instead the fair convention maximizes the function $F(x)$. In the subplots on the left, with $k_1 = 15$, x_T is significantly larger than x_N, i.e., there is a large area in which the fair convention is stochastically stable while the unfair convention maximizes the function $F(x)$. In addition, the Nash bargaining solution falls within the region where the fair condition is stochastically stable. This indicates that even if one of the three conventions coincides with the Nash bargaining solution, such convention would not be stochastically stable.

In analyzing transitions with minimum resistance, it is important to highlight a key distinction between Young (1993b)'s model and our own. In Young's model, where the grid for potential splits is sufficiently refined, the transitions with minimum resistance are always triggered by a player demanding slightly more than the quantity demanded in the existing convention. In contrast, our model allows for transitions with minimum resistance that are triggered by a player who, by mistake, demands less. We also note that this latter type of transitions are those with minimum resistance in evolutionary models of bargaining based on contract games (Young, 1998), unless specific error models are considered (Naidu et al., 2010).

4. Extension

In this section we explore the robustness of our results relaxing the assumption of only three possible splits. We adopt Young (1993b)'s model with a large but finite grid of possible splits. We know that the recurrent classes are all possible pairs of demands where their sum amounts exactly to the whole resource. Starting from it, we gradually consider the cases in which we remove all splits that are close enough to the fair split. By doing so, we are increasing the degree of unfairness of the setting, in that we are eliminating potential conventions that are only moderately unfair while maintaining more unfair splits. Given the very large number of possible conventions, for simplicity we restrict attention to the case with local mistakes (Young, 2001), i.e., mistakes can lead only to choose a split that is adjacent to the current one, in that it occupies an adjacent position in the grid. We note that, in our model, moving from the fair split to an adjacent one requires a larger deviation as the unfairness of the setting increases. We say that two conventions are adjacent if their splits are adjacent. Under the assumption of local mistakes, there exists a unique tree rooted at each convention, with all conventions that are linked to the adjacent conventions.

We tackle the question of which convention is stochastically stable as we gradually remove splits around the fair one by numerical calculation. As the setting becomes more and more unfair, i.e., splits around the fair one are gradually removed, the stochastically stable convention remains unfair for a while, with a larger split for the agents with a longer sample size. After a certain threshold is passed, i.e., many enough splits around the fair one have been removed, the fair convention becomes stochastically stable. We note that the threshold value depends on the degree of heterogeneity of sample sizes between the two populations: the larger the distance in the sample sizes, the farther the threshold to the fair one. Fig. 5 illustrates these findings.

5. Conclusion

In this paper we have considered an evolutionary model of the demand game, in which we study heterogeneities in the cognitive abilities between the two populations in relation with the degree of unfairness of the setting. In particular, we have assumed that the agents in one population have a larger working memory, meaning that they draw a larger sample when retrieving information from past plays of the game. Also, we define unfairness of the setting as unfairness of the unfair divisions, i.e., the distance between the fair (50/50) division and the unfair divisions. For our results we focus on the long-run convention emerging in the model, as captured by stochastic stability analysis.

We have found that, starting from extremely low unfairness of the setting, the long-run convention is unfair (in favor of the population with larger sampling size) and exhibits an increasing degree of unfairness as the unfairness of the setting increases, and then it jumps to the 50/50 division when the unfairness of the setting exceeds a certain threshold.

When the unfairness of the setting is high and the convention is unfair, the agents receiving fewer resources are very reactive to mistakes made by the other population. Indeed, such mistakes create past plays in memory allowing for an increase in the resources obtained, which is confronted with a potential loss that is very low due to the initially extremely unfair allocation of resources. Conversely, starting from the fair convention there is a higher potential loss in the case that total demands become excessive, which makes a change in the best response harder to occur. Overall, the transition from the unfair to the fair convention turns out to be relatively easier, when the unfairness of the setting is high enough. This result shows the role played by the grid of possible allocations in determining the long-run convention of a demand game.

The non-monotonicity that results in our model between the unfairness of the setting and the unfairness of the long-run convention calls our attention on the long-run effects of interventions that reduce the unfairness of the setting, and that may not reduce the unfairness of the emerging convention. Furthermore, we confirm previous findings from the literature that reducing the gap in the working memory between the two populations either decreases or leaves unaffected the unfairness of the long-run convention.

Broadly speaking, understanding how heterogeneities between groups can lead to unequal allocation of resources helps explaining, and possibly contrast, the high degree of inequality characterizing many societies.

Declaration of generative AI and AI-assisted technologies in the writing process

During the preparation of this work the authors used ChatGPT-3.5 for grammar checking. After using this tool/service, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

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Appendix A. Proofs

In this appendix we provide the proofs of the formal results stated in the main text (Lemma 1, Proposition 1 and Proposition 2). To favor the presentation of the proofs, we also state and prove other intermediate results (Lemma 2, Lemma 3, Lemma 4, Lemma 5, and Lemma 6).

Proof of Lemma 1. The proof follows the proof of Theorem 1 in Young (1993b). Consider a generic state h. If there is a positive probability that extracted samples of previous plays are such that (L, H), or (H, L), or (M, M) are best responses, then the two players can play those strategies at the current time, which makes the same strategies be playable with positive probability the next time and all following times, thus reaching one of the three recurrent classes within m periods.

Let an agent from population 1 selects the k_1 most recent plays of population 2, call that sample α, and let S_1 be the best reply to α. Analogously, let an agent from population 2 selects the k_2 most recent plays of population 1, call that sample β, and let S_2 be the best reply to β. Since both k_1 and k_2 are smaller or equal to $m/2$, there is a positive probability that both α and β are selected for $m/2$ periods. Thus there is a positive probability that after $m/2$ periods the first half of the memory is made of $m/2$ repetitions of (S_1, S_2). At this stage there is a positive probability that player 1 selects a sample of k_1 repetitions of S_2, call that sample α', and thus plays $1 - S_2$ as best reply, where 1 is the value of the resource. Analogously, there is a positive probability that player 2 selects a sample of k_2 repetitions of S_1, call that sample β' and thus plays $1 - S_1$ as best reply. Since k_1 and k_2 are smaller or equal to $m/2$, there is a positive probability that the two samples, α' and β' are selected for $m/2$ consecutive periods. Thus, after m periods from the beginning there is a positive probability that the memory is made of $m/2$ consecutive repetitions of $(1 - S_2, 1 - S_1)$ followed by $m/2$ consecutive repetitions of (S_1, S_2). At this stage there is a positive probability that player 1 selects a sample of k_1 repetitions of $1 - S_1$ and player 2 selects a sample of k_2 repetitions of S_1. In this case the best reply is $(S_1, 1 - S_1)$, that is a Nash equilibrium. Nash equilibria are (L, H), (H, L), and (M, M). Thus, from any state, in m periods, there is a positive probability to extract a sample of previous plays such that the best reply is (L, H), or (H, L), or (M, M). □

Lemma 2. *The resistance of transitions are:*

- $r_{HL,MM} = \min\left\{ \left\lceil \frac{u(x)}{u(0.5)} k_2 \right\rceil, \left\lceil \frac{u(1-x)-u(0.5)}{u(1-x)} k_1 \right\rceil \right\}$
- $r_{MM,HL} = \min\left\{ \left\lceil \frac{u(0.5)-u(x)}{u(0.5)} k_2 \right\rceil, \left\lceil \frac{u(0.5)}{u(1-x)} k_1 \right\rceil \right\}$
- $r_{LH,MM} = \min\left\{ \left\lceil \frac{u(1-x)-u(0.5)}{u(1-x)} k_2 \right\rceil, \left\lceil \frac{u(x)}{u(0.5)} k_1 \right\rceil \right\}$
- $r_{HL,LH} = \min\left\{ \left\lceil \frac{u(x)}{u(1-x)} k_2 \right\rceil, \left\lceil \frac{u(1-x)-u(x)}{u(1-x)} k_1 \right\rceil \right\}$
- $r_{MM,LH} = \min\left\{ \left\lceil \frac{u(0.5)}{u(1-x)} k_2 \right\rceil, \left\lceil \frac{u(0.5)-u(x)}{u(0.5)} k_1 \right\rceil \right\}$
- $r_{LH,HL} = \min\left\{ \left\lceil \frac{u(1-x)-u(x)}{u(1-x)} k_2 \right\rceil, \left\lceil \frac{u(x)}{u(1-x)} k_1 \right\rceil \right\}$

Proof of Lemma 2. We show how to obtain $r_{HL,MM}$, i.e., the resistance of the transition from h_{HL} to h_{MM}. There are two possible paths, the first one triggered by mistakes made by player 1, the second by mistakes of player 2.

In the first path, player 1 plays by mistake M instead of H. We have to identify the minimal number of mistakes that, once sampled by player 2, makes M best reply for player 2. Let α be the fraction of M sampled by player 2, and, as a consequence, $1-\alpha$ is the fraction of H sampled by player 2. Then the expected utility of player 2 of choosing L and M against such a sample is:

- $E_2[L] = u(x)$
- $E_2[M] = u(0.5)\alpha$

Then the fraction of M sampled that makes player 2 indifferent between M and L is:

$$\alpha^* = \frac{u(x)}{u(0.5)}$$

Thus the minimal number of mistakes by player 1 that are necessary to trigger the transition are:

$$r^1_{HL,MM} = \left\lceil \frac{u(x)}{u(0.5)} k_2 \right\rceil \tag{1}$$

In the second path, player 2 plays by mistake M instead of L. We have to identify the minimal number of mistakes that, once sampled by player 1, makes M best reply for player 1. Let β be the fraction of M sampled by player 1, and, as a consequence, $1-\beta$ is the fraction of L sampled by player 1. Then the expected utility of player 1 of choosing H and M against such a sample is:

- $E_1[H] = u(1-x)(1-\beta)$
- $E_1[M] = u(0.5)$

Then the fraction of M sampled that makes player 1 indifferent between M and H is:

$$\beta^* = \frac{u(1-x) - u(0.5)}{u(1-x)}$$

Thus the minimal number of mistakes by player 2 that are necessary to trigger the transition are:

$$r^2_{HL,MM} = \left\lceil \frac{u(1-x) - u(0.5)}{u(1-x)} k_1 \right\rceil \tag{2}$$

In conclusion, the resistance of the transition from h_{HL} to h_{MM}, i.e., $r_{HL,MM}$, is the minimum between the values in equations (1) and (2):

$$r_{HL,MM} = \min\left\{ \left\lceil \frac{u(x)}{u(0.5)} k_2 \right\rceil, \left\lceil \frac{u(1-x) - u(0.5)}{u(1-x)} k_1 \right\rceil \right\}$$

The same process can be applied to obtain the resistance of all the other transitions. □

Lemma 3. *When $k_2 \geq k_1$, $\gamma_{HL} \geq \gamma_{MM}$.*

Proof of Lemma 3. We show that the resistance of each tree rooted at h_{HL} is always greater or equal than the resistance of at least a tree rooted at h_{MM}.

- First step: $r_{LH,HL} + r_{MM,HL} \geq r_{LH,HL} + r_{HL,MM}$, therefore we show that $r_{MM,HL} \geq r_{HL,MM}$.
  - $r_{MM,HL} = \min\left\{ \left\lceil \frac{u(0.5)-u(x)}{u(0.5)} k_2 \right\rceil, \left\lceil \frac{u(0.5)}{u(1-x)} k_1 \right\rceil \right\}$
  - $r_{HL,MM} = \min\left\{ \left\lceil \frac{u(x)}{u(0.5)} k_2 \right\rceil, \left\lceil \frac{u(1-x)-u(0.5)}{u(1-x)} k_1 \right\rceil \right\}$

  We notice that $\frac{u(1-x)-u(0.5)}{u(1-x)} k_1 < \frac{u(0.5)}{u(1-x)} k_1$ is always true, while instead $\frac{u(1-x)-u(0.5)}{u(1-x)} k_1 < \frac{u(0.5)-u(x)}{u(0.5)} k_2$ when $k_1 < k_2$. Therefore, $r_{MM,HL} \geq r_{HL,MM}$.
- Second step: $r_{MM,LH} + r_{LH,HL} \geq r_{LH,HL} + r_{HL,MM}$, therefore we show that $r_{MM,LH} \geq r_{HL,MM}$.
  - $r_{MM,LH} = \min\left\{ \left\lceil \frac{u(0.5)}{u(1-x)} k_2 \right\rceil, \left\lceil \frac{u(0.5)-u(x)}{u(0.5)} k_1 \right\rceil \right\}$
  - $r_{HL,MM} = \min\left\{ \left\lceil \frac{u(x)}{u(0.5)} k_2 \right\rceil, \left\lceil \frac{u(1-x)-u(0.5)}{u(1-x)} k_1 \right\rceil \right\}$

  By the fact that the utility function is concave, the following inequality is always true: $\frac{u(0.5)}{u(1-x)} k_2 > \frac{u(1-x)-u(0.5)}{u(1-x)} k_1$. Indeed, we have $k_2 > k_1$ by assumption and $u(1-x) < u(1) \leq 2u(0.5)$ since u is strictly increasing and concave. Moreover, this other inequality is always true: $\frac{u(0.5)-u(x)}{u(0.5)} k_1 > \frac{u(1-x)-u(0.5)}{u(1-x)} k_1$. Indeed, $u(1-x) > u(0.5)$ because u is strictly increasing and $u(0.5) - u(x) \geq u(1-x) - u(0.5)$ because u is concave.
- Third step: $r_{LH,MM} + r_{MM,HL} \geq r_{HL,MM} + r_{LH,MM}$, therefore we show that $r_{MM,HL} \geq r_{HL,MM}$. The proof of the third step is analogous to the proof of the first step. □

Lemma 4. *There exists unique* $\tilde{x} \in (0, 0.5)$ *such that:*

$$\begin{cases} r_{LH,HL} < r_{LH,MM} & if \quad x < \tilde{x} \\ r_{LH,HL} = r_{LH,MM} & if \quad x = \tilde{x} \\ r_{LH,HL} > r_{LH,MM} & if \quad x > \tilde{x} \end{cases}$$

Proof of Lemma 4. From Lemma 2:

- $r_{LH,MM} = \min\left\{ \left\lceil \frac{u(1-x)-u(0.5)}{u(1-x)} k_2 \right\rceil, \left\lceil \frac{u(x)}{u(0.5)} k_1 \right\rceil \right\}$
- $r_{LH,HL} = \min\left\{ \left\lceil \frac{u(1-x)-u(x)}{u(1-x)} k_2 \right\rceil, \left\lceil \frac{u(x)}{u(1-x)} k_1 \right\rceil \right\}$

We observe that:

$$\frac{u(1-x)-u(0.5)}{u(1-x)} k_2 < \frac{u(1-x)-u(x)}{u(1-x)} k_2 \tag{3}$$

and

$$\frac{u(x)}{u(1-x)} k_1 < \frac{u(x)}{u(0.5)} k_1 \tag{4}$$

then

$$r_{LH,MM} \gtrless r_{LH,HL} \quad \Leftrightarrow \quad \frac{u(1-x)-u(0.5)}{u(1-x)} k_2 \gtrless \frac{u(x)}{u(1-x)} k_1$$

Let $\tilde{H}(x; k_1, k_2) = [u(1-x) - u(0.5)]k_2 - u(x)k_1$. We observe that

$$\begin{aligned} \tilde{H}(0; k_1, k_2) &> 0 \\ \tilde{H}(0.5; k_1, k_2) &< 0 \\ \tilde{H}'(x; k_1, k_2) &< 0 \end{aligned}$$

and then $\exists! \tilde{x}$ s.t. $\tilde{H}(\tilde{x}; k_1, k_2) = 0$. □

Lemma 5. *There exists unique* $x^* \in (0, 0.5)$ *such that:*

$$\begin{cases} r_{LH,HL} < r_{MM,LH} & if \quad x < x^* \\ r_{LH,HL} = r_{MM,LH} & if \quad x = x^* \\ r_{LH,HL} > r_{MM,LH} & if \quad x > x^* \end{cases}$$

Proof of Lemma 5. From Lemma 2:

- $r_{MM,LH} = \min\left\{ \left\lceil \frac{u(0.5)}{u(1-x)} k_2 \right\rceil, \left\lceil \frac{u(0.5)-u(x)}{u(0.5)} k_1 \right\rceil \right\}$
- $r_{LH,HL} = \min\left\{ \left\lceil \frac{u(1-x)-u(x)}{u(1-x)} k_2 \right\rceil, \left\lceil \frac{u(x)}{u(1-x)} k_1 \right\rceil \right\}$

We observe that:

$$\frac{u(0.5)-u(x)}{u(0.5)} k_1 < \frac{u(1-x)-u(x)}{u(1-x)} k_2 \tag{5}$$

and

$$\frac{u(x)}{u(1-x)} k_1 < \frac{u(0.5)}{u(1-x)} k_2 \tag{6}$$

then

$$r_{LH,HL} \gtrless r_{MM,LH} \quad \Leftrightarrow \quad \frac{u(0.5)-u(x)}{u(0.5)} \gtrless \frac{u(x)}{u(1-x)}$$

Let $H^*(x) = \frac{u(x)}{u(1-x)} + \frac{u(x)}{u(0.5)} - 1$. We observe that

$$\begin{aligned} H^*(0) &< 0 \\ H^*(0.5) &> 0 \\ H^{*'}(x) &> 0 \end{aligned}$$

and then $\exists! x^* \in (0, 1)$ s.t. $H^*(x^*) = 0$. □

Lemma 6. *If $r_{MM,LH} \le r_{LH,MM}$ for $x = \tilde{x}$, then*

- $r_{MM,LH} < r_{LH,MM}$ *for* $x \in [\tilde{x}, 0.5)$

If $r_{MM,LH} > r_{LH,MM}$ for $x = \tilde{x}$, then there exists $\overline{x} \in (0, 0.5)$ such that:

- $r_{MM,LH} > r_{LH,MM}$ *for* $x \in [\tilde{x}, \overline{x})$
- $r_{MM,LH} = r_{LH,MM}$ *for* $x = \overline{x}$
- $r_{MM,LH} < r_{LH,MM}$ *for* $x \in [\overline{x}, 0.5)$

Proof of Lemma 6. From Lemma 2 we have:

- $r_{LH,MM} = \min\left\{ \left\lceil \frac{u(1-x)-u(0.5)}{u(1-x)} k_2 \right\rceil, \left\lceil \frac{u(x)}{u(0.5)} k_1 \right\rceil \right\}$
- $r_{MM,LH} = \min\left\{ \left\lceil \frac{u(0.5)}{u(1-x)} k_2 \right\rceil, \left\lceil \frac{u(0.5)-u(x)}{u(0.5)} k_1 \right\rceil \right\}$

When $x > \tilde{x}$, for Lemma 4, we have $r_{LH,HL} > r_{LH,MM}$ and thus:

$$[u(1-x) - u(0.5)]k_2 \le u(x)k_1 \tag{7}$$

from this follows that $\frac{u(1-x)-u(0.5)}{u(1-x)} k_2 < \frac{u(x)}{u(0.5)} k_1$. Moreover, from the previous condition we also have:

$$[u(1-x) - u(0.5)]k_2 \le u(x)k_1 < u(0.5)k_2 \tag{8}$$

from this follows that $\frac{u(1-x)-u(0.5)}{u(1-x)} k_2 < \frac{u(0.5)}{u(1-x)} k_2$. Thus when $x > \tilde{x}$ we have that $r_{LH,MM} \gtrless r_{MM,LH}$ for $\left\lceil \frac{u(1-x)-u(0.5)}{u(1-x)} k_2 \right\rceil \gtrless \left\lceil \frac{u(0.5)-u(x)}{u(0.5)} k_1 \right\rceil$.

We now make two observations. The first one is that $\frac{u(1-x)-u(0.5)}{u(1-x)} k_2 = \frac{u(0.5)-u(x)}{u(0.5)} k_1$ when $x = 0.5$. The second one is that the derivative of $\frac{u(1-x)-u(0.5)}{u(1-x)} k_2$ is negative and increasing in absolute value, while the derivative of $\frac{u(0.5)-u(x)}{u(0.5)} k_1$ is negative and non-increasing in absolute value:

$$D_x \left[\frac{u(1-x) - u(0.5)}{u(1-x)} k_2 \right] = -\frac{u(0.5)u'(1-x)k_2}{u^2(1-x)}$$

and

$$D_x \left[\frac{u(0.5) - u(x)}{u(0.5)} k_1 \right] = -\frac{u'(x)k_1}{u(0.5)}$$

From the two previous points we argue that, when $r_{MM,LH} \le r_{LH,MM}$ for $x = \tilde{x}$, then

$$\frac{u(0.5) - u(x)}{u(0.5)} k_1 \le \frac{u(1-x) - u(0.5)}{u(1-x)} k_2 \qquad \text{for} \quad x \in (\tilde{x}, 0.5)$$

and thus

$$r_{MM,LH} \le r_{LH,MM} \qquad \text{for} \quad x \in (\tilde{x}, 0.5) \tag{9}$$

On the contrary when $r_{MM,LH} > r_{LH,MM}$ for $x = \tilde{x}$, then there exists unique $\overline{x} \in (\tilde{x}, 0.5)$ such that $\frac{u(0.5)-u(x)}{u(0.5)} k_1 = \frac{u(1-x)-u(0.5)}{u(1-x)} k_2$. Moreover, $\frac{u(0.5)-u(x)}{u(0.5)} k_1 > \frac{u(1-x)-u(0.5)}{u(1-x)} k_2$ for $x \in (\tilde{x}, \overline{x})$ and $\frac{u(0.5)-u(x)}{u(0.5)} k_1 < \frac{u(1-x)-u(0.5)}{u(1-x)} k_2$ for $x \in (\overline{x}, 0.5)$. In conclusion

$$r_{MM,LH} \ge r_{LH,MM} \qquad \text{for} \quad x \in (\tilde{x}, \overline{x}) \tag{10}$$

and

$$r_{MM,LH} \le r_{LH,MM} \qquad \text{for} \quad x \in (\overline{x}, 0.5) \quad \square \tag{11}$$

Proof of Proposition 1. Define $\underline{H}(x)$ as:

$$\underline{H}(x) = u(1-x) - u(0.5) - u(x)$$

Let $\underline{x}$ be the value of x such that $\underline{H}(\underline{x}) = 0$. To prove the existence and uniqueness of $\underline{x}$ we notice that $\underline{H}(0) > 0$, $\underline{H}(0.5) < 0$, and $\underline{H}'(x) < 0$.

The proof is made of two steps:

- In the first step we study the case in which $x < \underline{x}$, call it *case I*.

Every tree rooted at h_{LH} has resistance greater or equal than at least a tree rooted at h_{HL}.

(1) $r_{MM,HL} + r_{HL,LH} \geq r_{MM,HL} + r_{LH,HL}$, that is true when $r_{HL,LH} \geq r_{LH,HL}$, where:
- $r_{HL,LH} = \min\left\{ \left\lceil \frac{u(x)}{u(1-x)} k_2 \right\rceil, \left\lceil \frac{u(1-x)-u(x)}{u(1-x)} k_1 \right\rceil \right\}$
- $r_{LH,HL} = \min\left\{ \left\lceil \frac{u(1-x)-u(x)}{u(1-x)} k_2 \right\rceil, \left\lceil \frac{u(x)}{u(1-x)} k_1 \right\rceil \right\}$

We observe that, since $k_2 \geq k_1$, $\frac{u(x)}{u(1-x)} k_2 \geq \frac{u(x)}{u(1-x)} k_1$ and $\frac{u(1-x)-u(x)}{u(1-x)} k_2 \geq \frac{u(1-x)-u(x)}{u(1-x)} k_1$, moreover we observe that $\frac{u(1-x)-u(x)}{u(1-x)} k_1 \geq \frac{u(x)}{u(1-x)} k_1$ in *case I*, indeed, $u(1-x) - u(x) \geq u(x)$ when $u(1-x) > u(0.5) + u(x)$. Thus, $r_{HL,LH} \geq r_{LH,HL}$.

(2) $r_{MM,LH} + r_{HL,LH} \geq r_{MM,LH} + r_{LH,HL}$, that is true when $r_{HL,LH} \geq r_{LH,HL}$. The condition is satisfied in *case I* as showed in the previous bullet.

(3) $r_{HL,MM} + r_{MM,LH} \geq r_{MM,LH} + r_{LH,HL}$, that is true when $r_{HL,MM} \geq r_{LH,HL}$, where:
- $r_{HL,MM} = \min\left\{ \left\lceil \frac{u(x)}{u(0.5)} k_2 \right\rceil, \left\lceil \frac{u(1-x)-u(0.5)}{u(1-x)} k_1 \right\rceil \right\}$
- $r_{LH,HL} = \min\left\{ \left\lceil \frac{u(1-x)-u(x)}{u(1-x)} k_2 \right\rceil, \left\lceil \frac{u(x)}{u(1-x)} k_1 \right\rceil \right\}$

We observe that $\frac{u(x)}{u(0.5)} k_2 > \frac{u(x)}{u(1-x)} k_1$ because $k_2 \geq k_1$ and $u(0.5) < u(1-x)$, moreover $\frac{u(1-x)-u(x)}{u(1-x)} k_2 > \frac{u(1-x)-u(0.5)}{u(1-x)} k_1$ because $k_2 \geq k_1$ and $u(x) < u(0.5)$, furthermore $\frac{u(1-x)-u(0.5)}{u(1-x)} k_1 \geq \frac{u(x)}{u(1-x)} k_1$ in *case I*, where $u(1-x) > u(0.5) + u(x)$. Thus $r_{HL,MM} \geq r_{LH,HL}$.

Therefore $\gamma_{LH} \geq \gamma_{HL}$. Moreover, by Lemma 3, $\gamma_{HL} \geq \gamma_{MM}$ and thus h_{MM} is stochastically stable under the condition $u(1-x) > u(0.5) + u(x)$.

- In the second step we study the case in which $x > \underline{x}$, call it *case II*. In this step we need some preliminary result. First, $T_{LH}[HL \to MM + MM \to LH]$ has minimal resistance between the trees rooted at h_{LH}:

(a) $r_{HL,MM} + r_{MM,LH} \leq r_{HL,LH} + r_{MM,LH}$, that is true when $r_{HL,MM} \leq r_{HL,LH}$, where:
- $r_{HL,MM} = \min\left\{ \left\lceil \frac{u(x)}{u(0.5)} k_2 \right\rceil, \left\lceil \frac{u(1-x)-u(0.5)}{u(1-x)} k_1 \right\rceil \right\}$
- $r_{HL,LH} = \min\left\{ \left\lceil \frac{u(x)}{u(1-x)} k_2 \right\rceil, \left\lceil \frac{u(1-x)-u(x)}{u(1-x)} k_1 \right\rceil \right\}$

We observe that $\frac{u(1-x)-u(0.5)}{u(1-x)} k_1 < \frac{u(1-x)-u(x)}{u(1-x)} k_1$, by the fact that u is an increasing function. Moreover $\frac{u(1-x)-u(0.5)}{u(1-x)} k_1 < \frac{u(x)}{u(1-x)} k_2$ because $u(1-x) < u(0.5) + u(x)$ in *case II* and also $k_2 > k_1$. Thus $r_{HL,MM} \leq r_{HL,LH}$.

(b) $r_{HL,LH} + r_{MM,LH} \leq r_{MM,HL} + r_{HL,LH}$, that is true when $r_{MM,LH} \leq r_{MM,HL}$, where:
- $r_{MM,LH} = \min\left\{ \left\lceil \frac{u(0.5)}{u(1-x)} k_2 \right\rceil, \left\lceil \frac{u(0.5)-u(x)}{u(0.5)} k_1 \right\rceil \right\}$
- $r_{MM,HL} = \min\left\{ \left\lceil \frac{u(0.5)-u(x)}{u(0.5)} k_2 \right\rceil, \left\lceil \frac{u(0.5)}{u(1-x)} k_1 \right\rceil \right\}$

We observe that $\frac{u(0.5)}{u(1-x)} k_2 \geq \frac{u(0.5)}{u(1-x)} k_1$ and $\frac{u(0.5)-u(x)}{u(0.5)} k_2 \geq \frac{u(0.5)-u(x)}{u(0.5)} k_1$, by the fact that $k_2 \geq k_1$. To show that $\frac{u(0.5)}{u(1-x)} > \frac{u(0.5)-u(x)}{u(0.5)}$ we initially observe that in *case II*, $\frac{u(0.5)}{u(1-x)} > \frac{u(0.5)}{u(x)+u(0.5)}$. Thus we show that $\frac{u(0.5)}{u(x)+u(0.5)} > \frac{u(0.5)-u(x)}{u(0.5)}$ which implies $\frac{u(0.5)}{u(1-x)} > \frac{u(0.5)-u(x)}{u(0.5)}$. The result follows from the fact that $u^2(0.5) > u^2(0.5) - u^2(x)$ is always true. Thus $r_{MM,LH} \leq r_{MM,HL}$.

Given (a) and (b), we conclude that $T_{LH}[HL \to MM + MM \to LH]$ has minimal resistance between the trees rooted at LH.
Second, the tree $T_{MM}[HL \to LH + LH \to MM]$ has always resistance greater or equal than at least another tree rooted at MM: $r_{HL,MM} + r_{LH,MM} \leq r_{HL,LH} + r_{LH,MM}$, that is true when $r_{HL,MM} \leq r_{HL,LH}$, where:
- $r_{HL,MM} = \min\left\{ \left\lceil \frac{u(x)}{u(0.5)} k_2 \right\rceil, \left\lceil \frac{u(1-x)-u(0.5)}{u(1-x)} k_1 \right\rceil \right\}$
- $r_{HL,LH} = \min\left\{ \left\lceil \frac{u(x)}{u(1-x)} k_2 \right\rceil, \left\lceil \frac{u(1-x)-u(x)}{u(1-x)} k_1 \right\rceil \right\}$

We already show for condition (a), that $r_{HL,MM} \leq r_{HL,LH}$ in *case II*.
Third, given Lemma 4, if $x < \tilde{x}$ then $\gamma_{MM} = r_{LH,HL} + r_{HL,MM}$, and vice versa $\gamma_{MM} = r_{HL,MM} + r_{LH,MM}$ when $x > \tilde{x}$.
We summarize the preliminary results: in the case studied in the second step, *case II*, $u(1-x) < u(0.5) + u(x)$, the stochastic potential of the convention LH is always given by the resistance of the tree $T_{LH}[HL \to MM + MM \to LH]$, while instead the stochastic potential of the convention MM is given by the resistance of the tree $T_{MM}[LH \to HL + HL \to MM]$ when $x < \tilde{x}$, and by the resistance of the tree $T_{MM}[HL \to MM + LH \to MM]$ when $x > \tilde{x}$.
Depending on whether x^* (Lemma 5) is smaller or greater than $\tilde{x}$ (Lemma 4) we obtain different results. See Fig. 6 for a graphical representation. Firstly, we analyze the case in which $x^* \leq \tilde{x}$.

- When $\underline{x} < x < \tilde{x}$.

$$\gamma_{MM} < \gamma_{LH} \qquad \text{if} \quad \underline{x} < x < x^*$$

and

$$\gamma_{MM} > \gamma_{LH} \qquad \text{if} \quad x^* < x < \tilde{x}$$

The result follows from Lemma 5.

- When $\tilde{x} < x < 0.5$.

$$\gamma_{MM} > \gamma_{LH}$$

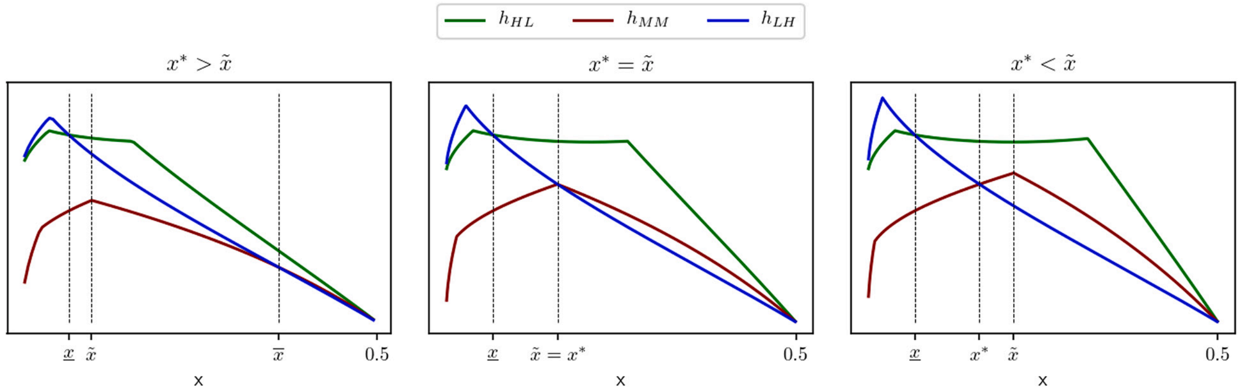

Fig. 6. The above plots depict functions from which the stochastic potentials of the conventions can be computed by taking the ceiling.

The result follows from Lemma 6.

Secondly, we analyze the case in which $x^* > \tilde{x}$.

- When $\underline{x} < x < \tilde{x} < x^*$.

$$\gamma_{MM} < \gamma_{LH}$$

The result follows from Lemma 5.

- When $\tilde{x} < x < 0.5$.

$$\gamma_{MM} < \gamma_{LH} \quad \text{if} \quad \tilde{x} < x < \overline{x}$$

and

$$\gamma_{MM} > \gamma_{LH} \quad \text{if} \quad \overline{x} < x < 0.5$$

In conclusion, when $x^* \leq \tilde{x}$,

$$x^T(k_1, k_2) = x^*$$

instead, when $x^* > \tilde{x}$,

$$x^T(k_1, k_2) = \overline{x} \quad \square$$

Proof of Proposition 2. In the conclusion of the proof of Proposition 1 we have that when $x^* \leq \tilde{x}$,

$$x^T(k_1, k_2) = x^*$$

instead, when $x^* > \tilde{x}$,

$$x^T(k_1, k_2) = \overline{x}$$

Where x^* is the value of x at which $\frac{u(x)}{u(1-x)} + \frac{u(x)}{u(0.5)} - 1 = 0$, $\tilde{x}$ is the value of x at which $[u(1-x) - u(0.5)]k_2 - u(x)k_1 = 0$, and $\overline{x}$ is the value of x at which $\frac{u(1-x)-u(0.5)}{u(1-x)}k_2 - \frac{u(0.5)-u(x)}{u(0.5)}k_1 = 0$. We notice that x^* does not depend on k_1 and k_2. Instead, $\tilde{x}$ is decreasing in k_1 and increasing in k_2. Conversely, $\overline{x}$ is increasing in k_1 and decreasing in k_2. Indeed, $\overline{x} \in [\tilde{x}, 0.5)$ is the value of x for which $r_{MM,LH} = r_{LH,MM}$ given that $x^* > \tilde{x}$. As showed in the proof of Lemma 6, $r_{MM,LH}$ and $r_{LH,MM}$ are both decreasing in x, and they are equal when $x = 0.5$. Moreover, $r_{LH,MM}$ becomes relatively steeper than $r_{MM,LH}$ as x increases. Since, $r_{LH,MM} = \left\lceil \frac{u(1-x)-u(0.5)}{u(1-x)}k_2 \right\rceil$ and $r_{MM,LH} = \left\lceil \frac{u(0.5)-u(x)}{u(0.5)}k_1 \right\rceil$, an increase in k_2 moves $\overline{x}$ to the left as a decrease in k_1, vice versa an increase in k_1 or a decrease in k_2 move $\overline{x}$ to the right. $\square$

Appendix B. Shape of the utility function

In this appendix, we further explore the role of concavity of the utility function, which is known to play a fundamental role for the selection of stochastically stable conventions, at least in 2x2 coordination games (Sawa and Wu, 2018 and Nax and Newton, 2019). In order to obtain the results of Proposition 1 and Proposition 2, it is necessary to assume that the utility function is (weakly) concave. We emphasize that the results are therefore guaranteed even in the case of linearity of the utility function. In Fig. 7, we show the numerical results obtained by computing the functions from which the stochastic potentials of the conventions can be

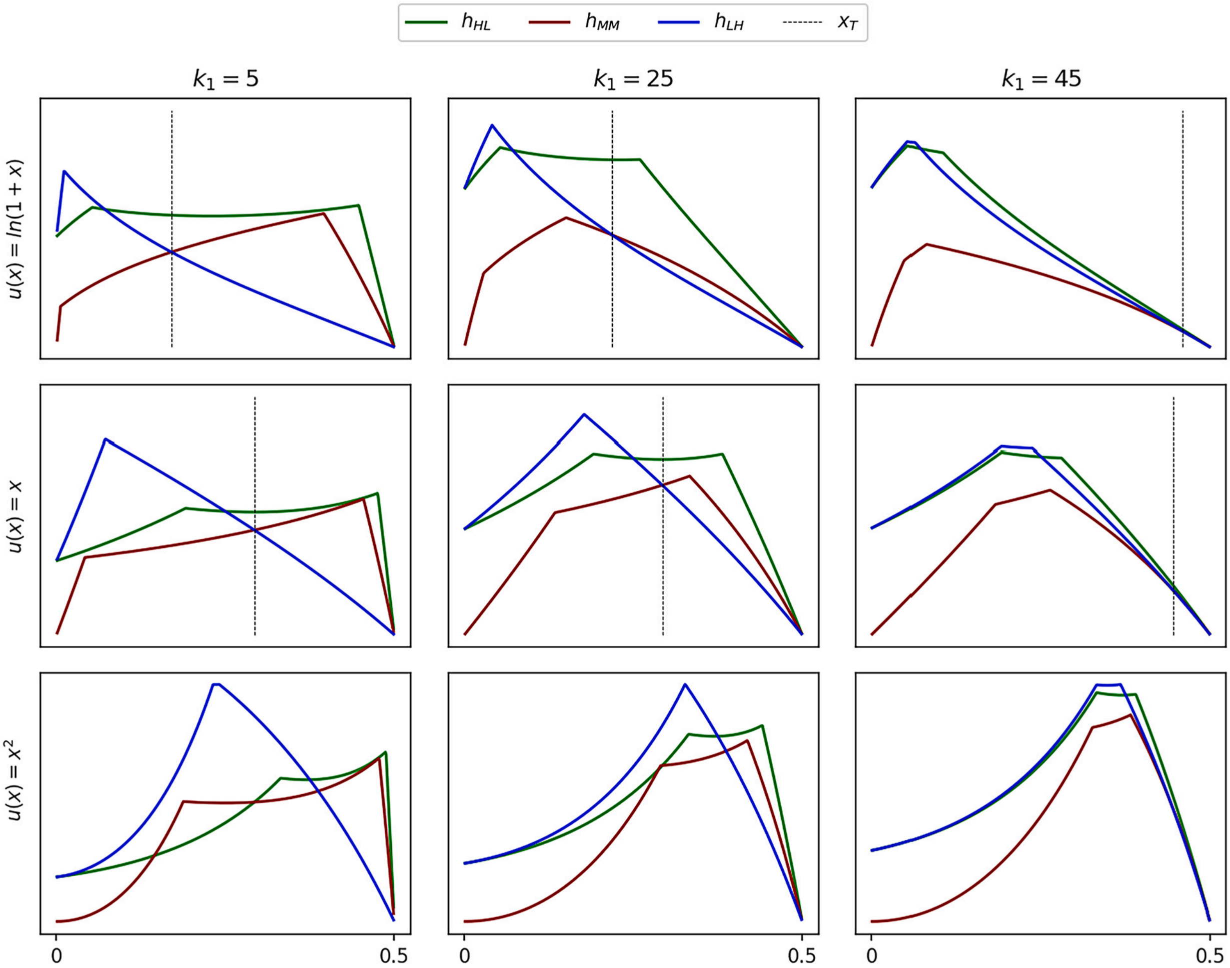

Fig. 7. The above plots depict the functions from which the stochastic potentials of the conventions can be computed by taking the ceiling. The plots are drawn, from top to bottom, for $u(x) = ln(1 + x)$, $u(x) = x$, and $u(x) = x^2$ and, from left to right, $k_1 = 5$, $k_1 = 25$, and $k_1 = 45$, while k_2 is kept fixed at 50. The threshold x_T, as it results from Proposition 1, is not identified for $u(x) = x^2$, due to the strict convexity of the utility function.

computed by taking the ceiling. In the first row, a case with a strictly concave function where $u(x) = ln(1 + x)$, and in the second row, the linear case with $u(x) = x$. Observing the subplots in the first two rows, from left to right, $k_1 = 5$, $k_1 = 25$, and $k_1 = 45$, while k_2 is kept fixed at 50, one can notice that there exists a value x_T for which if $x < x_T$ then h_{MM} is stochastically stable, whereas if $x > x_T$ then h_{LH} is stochastically stable. Furthermore, we observe that in the case where $k_1 = 5$, the Nash bargaining solution falls within a region where h_{MM} is stochastically stable. Observing the third row of Fig. 7, where the utility function is convex, with the functional form $u(x) = x^2$, it can be noted that, in the case where $k_1 = 5$, for an intermediate range of x values, the green line lies below the other lines, showing that convexity is indeed essential for Proposition 1.

However, without the assumption of (weak) concavity we can still obtain the result in Proposition 3, where we show that, if the setting is unfair enough, then the fair convention is stochastically stable.

Proposition 3. *Let $u(x)$ be such that $u'(x) > 0$ and $u(0) = 0$. When x is sufficiently close to 0, h_{MM} is stochastically stable.*

Proof of Proposition 3. We consider the resistance of transitions indicated in Lemma 2. By assumption, when x approaches 0, $u(x)$ also approaches 0. This implies that, when x is sufficiently close to 0, some of the resistances indicated in Lemma 2 are equal to 1, due to the ceiling operator. In particular, we note that $r_{LH,MM} = r_{HL,MM} = 1$. In turn, this means that $\gamma_{MM} = 2$. Because there are only three recurrent classes, and hence two links in rooted trees with each link having a resistance of at least 1, the sum of resistances over any rooted tree cannot be lower than 2. This allows us to conclude that, when x is sufficiently close to 0, h_{MM} has minimum stochastic potential and, hence, is stochastically stable. □

Appendix C. Example of transitions

In this appendix, we illustrate how transitions between conventions occur using two simple examples with a linear utility function: $u(x) = x$. In particular, we consider $x = 0.2$, $k_1 = 2$, $k_2 = 3$, and $m = 6$. The first example, depicted in Fig. 8, explores the transition

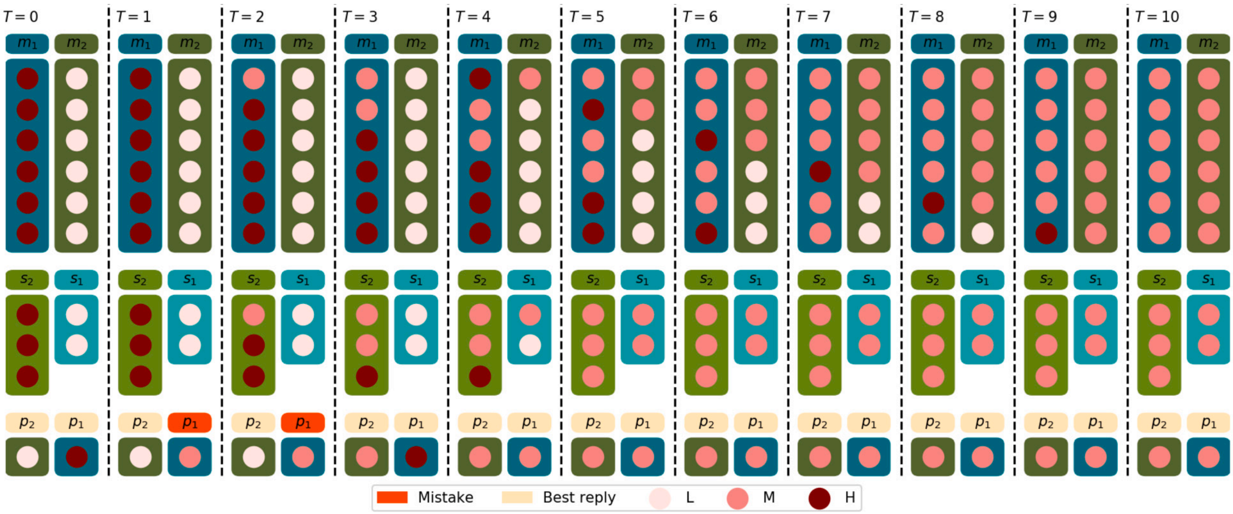

Fig. 8. Example of a transition from convention h_{HL} to convention h_{MM} triggered by the minimal number of mistakes made by player 1, assuming $u(x) = x$, $x = 0.2$, $k_1 = 2$, $k_2 = 3$, and $m = 6$.

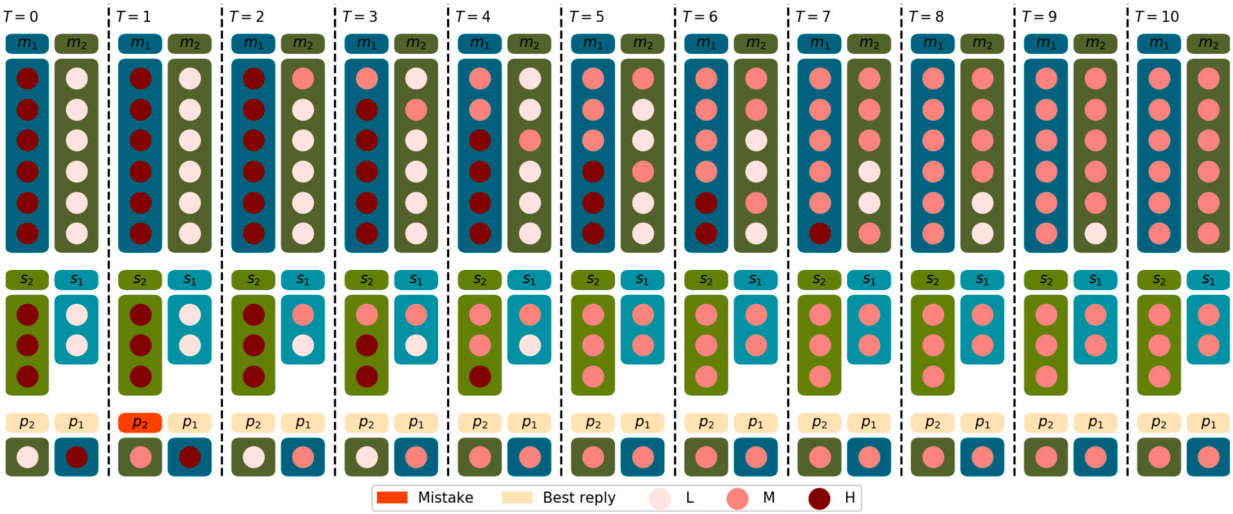

Fig. 9. Example of a transition from convention h_{HL} to convention h_{MM} triggered by the minimal number of mistakes made by player 2, assuming $u(x) = x$, $x = 0.2$, $k_1 = 2$, $k_2 = 3$, and $m = 6$.

from the convention h_{HL} to h_{MM}, triggered by mistakes made by player 1. At period 0 ($T = 0$ in Fig. 8), the system is at rest in the h_{HL} convention, with no change occurring in the best reply dynamics. In period 1, player 1 mistakenly plays M instead of H. By period 2, this error is incorporated into m_1, representing the memory of player 1's last m moves. Despite this mistake being included in player 2's sample, it is not enough to trigger a change in the best reply of player 2 under the given utility function, value of x, and sample size. Consequently, another error by player 1 is necessary for the transition to proceed. In period 3, with two accumulated mistakes in m_1, if both of them are selected in the sample of player 2, player 2's best reply shifts from L to M. In period 4, there is a positive probability that the two players select samples such that the best reply is M for both players. In this period the two players coordinate on the Nash equilibrium (M, M). From period 5 to period 9, there is a positive probability that the two players select samples such that the best reply is M for both payers. By period 10, the transition to the h_{MM} convention is complete.

In Fig. 9, we describe the same transition from h_{HL} to h_{MM} triggered in this case by mistakes of player 2. As shown in the figure, one mistake by player 2 is enough to trigger the transition.

From Lemma 2 we know that the resistance of the transition from h_{HL} to h_{MM} is:

$$r_{HL,MM} = \min \left\{ \left\lceil \frac{u(x)}{u(0.5)} k_2 \right\rceil, \left\lceil \frac{u(1-x) - u(0.5)}{u(1-x)} k_1 \right\rceil \right\}$$

Substituting the values chosen for the example, $u(x) = x$, $x = 0.2$, $k_1 = 2$, $k_2 = 3$, and $m = 6$, we obtain:

$$r_{HL,MM} = \min \{ \lceil 1.2 \rceil, \lceil 0.75 \rceil \} = \min \{2, 1\} = 1$$

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