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Bijective Construction of Equivalent Eco-systems

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ABSTRACT: *First, we explicit an infinite family of equivalent succession rules parametrized by a positive integer α, for which two specializations lead to the equivalence of rules defining the Catalan and Schröder numbers. Then, from an ECO-system for Dyck paths, we easily derive an ECO-system for complete binary trees, by using a widely known bijection between these objects. We also give a similar construction in the less easy case of Schröder paths and Schröder trees which generalizes the previous one.*

1 Introduction

The concept of succession rule was introduced in [4] by Chung et al. in the study of Baxter permutations. Later West [12], Gire [6] and Guibert [7] used succession rules for the enumeration of permutations with forbidden sequences. More recently, this concept was deepened by Barcucci et al. [2] as a fundamental tool for ECO method, which is a method for constructing and enumerating combinatorial objects. In particular, let $\mathcal{O}$ be a class of combinatorial objects and p a parameter on $\mathcal{O}$ such that $\mathcal{O}_n = \{O \in \mathcal{O} \,|\, p(O) = n\}$ is finite. ECO method provides, by means of an operator ϑ, a construction for the class $\mathcal{O}$ with respect to the parameter p. If ϑ is an operator on $\mathcal{O}$ satisfying the following conditions:

$$\begin{array}{ll} \text{(i)} & \textit{for each } O' \in \mathcal{O}_{n+1}, \textit{ there exists } O \in \mathcal{O}_n \textit{ such that } O' \in \vartheta(O), \\ \text{(ii)} & \textit{for each } O, O' \in \mathcal{O}_n \textit{ such that } O \neq O', \textit{ then } \vartheta(O) \cap \vartheta(O') = \emptyset, \end{array} \qquad \text{(E)}$$

then the family of sets $\mathcal{F}_{n+1} = \{\vartheta(O) : O \in \mathcal{O}_n\}$ is a partition of $\mathcal{O}_{n+1}$.

Note that many different operators may exist on a class $\mathcal{O}$. Consequently, when an operator ϑ is fixed on $\mathcal{O}$, we will denote it by $\vartheta_{\mathcal{O}}$, and the ECO-pair by $(\mathcal{O}, \vartheta_{\mathcal{O}})$. The subscript will be omitted when no confusion arises. The conditions (E) above state that the construction of each object $O' \in \mathcal{O}_{n+1}$ is obtained from one and only one object $O \in \mathcal{O}_n$. This construction can be described by a *generating tree* [2, 4], a rooted tree whose vertices are objects of $\mathcal{O}$. The objects having the same value of p lie at the same level, and the sons of an object are the objects produced from it by using ϑ. A generating tree can be sometimes described by means of a succession rule of the form:

$$\Lambda = \left\{ \begin{array}{l} (a) \\ (k) \rightsquigarrow (e_1)(e_2)\dots(e_k), \end{array} \right. \qquad (1)$$

where $a, k, e_i \in \mathbb{N}$, meaning that the root object has a sons, and the k objects $O'_1, \dots, O'_k$, produced by an object O are such that $|\vartheta(O'_i)| = e_i$, $1 \le i \le k$. A succession rule Ω of type (1) defines a sequence $\{f_n\}_n$ of positive integers, where f_n is the number of nodes at level n of the generating tree of Ω.

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Given an ECO-system $(\mathcal{O}, p, \vartheta_{\mathcal{O}}, \Lambda)$ and a bijection $\Phi : \mathcal{O} \longrightarrow \mathcal{V}$ between two classes of combinatorial objects, it is always possible to map formally $\vartheta_{\mathcal{O}}$ on the class $\mathcal{V}$ along the bijection Φ. Indeed, let $O \in \mathcal{O}$ then we define the ECO-system $(\mathcal{V}, p', \vartheta_{\mathcal{V}}, \Lambda)$ by

$$\Phi(O') \in \vartheta_{\mathcal{V}}(\Phi(O)) \iff O' \in \vartheta_{\mathcal{O}}(O).$$

This means that the generating tree $T_{\mathcal{O}}$ for the class $\mathcal{O}$ is mapped on the generating tree $T_{\mathcal{V}}$, where each node of $T_{\mathcal{V}}$ contains the image of the corresponding node of $T_{\mathcal{O}}$. It is clear that in this case the same succession rule is obtained, but the problem of describing the operator $\vartheta_{\mathcal{V}}$, independently from Φ, remains and is not easy in general. In Section 4, we describe explicitly this construction on two examples. Firstly, we carry out the description in the easy case of the bijection between Dyck paths and complete binary trees, and, secondly, in the less easy case of the bijection between Schröder paths and Schröder trees.

Two succession rules Λ and Λ' are *equivalent* (written $\Lambda \sim \Lambda'$) if they define the same number sequence [9]. The problem of determining classes of equivalent succession rules, is still open. In section 5, by using both a combinatorial and a generating function approach, we prove that $\Omega_\alpha \sim \Omega'_\alpha$, where Ω_α and Ω'_α are defined as follows. Let $\alpha \in \mathbb{N}^+$,

$$\Omega_\alpha = \left\{ \begin{array}{l} (\alpha) \\ (\alpha) \rightsquigarrow (\alpha+1)^\alpha \\ (k) \rightsquigarrow (\alpha+1)(\alpha+2)\dots(k-1)(k)(k+1)^\alpha, \end{array} \right.$$

and

$$\Omega'_\alpha = \left\{ \begin{array}{l} (\alpha) \\ (\alpha) \rightsquigarrow (\alpha)^{\alpha-1}(2\alpha) \\ (2k\alpha) \rightsquigarrow (\alpha)^{k\alpha}(2\alpha)^{\alpha-1}(4\alpha)^\alpha(6\alpha)^\alpha\dots(2(k-1)\alpha)^\alpha(2k\alpha)^\alpha(2(k+1)\alpha). \end{array} \right.$$

These succession rules are related to the well known classical rules for Catalan and Schröder numbers.

2 Some classical combinatorial structures

In the plane $\mathbb{Z} \times \mathbb{Z}$, we consider lattice paths using three step types: *rise steps* $(1,1)$, *fall steps* $(1,-1)$ and *k-length horizontal steps* $(k,0)$ (briefly, k-horizontal steps).

Definition 2.1. A *generalized Motzkin path* is a sequence of rise, fall and k-horizontal steps, running from $(0,0)$ to $(n,0)$, and remaining weakly above the x-axis.

These paths have been extensively studied, an account of which can be found in [11] for instance. They include many classical lattice paths, and, among others, Dyck, Motzkin and Schröder paths correspond respectively to the cases $k=0$, $k=1$ and $k=2$. A path remaining strictly above the x-axis except for $(0,0)$ and $(n,0)$ is called *elevated.* A *coloured generalized Motzkin path* is a generalized Motzkin path for which the horizontal steps can have more than one colour. We give now the classical ECO construction for Dyck and Schröder paths.

Let $\mathcal{D}$ be the class of Dyck paths, and let $D \in \mathcal{D}$. Then, $\vartheta(D)$ is the set of Dyck paths obtained by adding a peak on each point of the last sequence of D's fall steps. The rule associated to this construction is the classical rule for Catalan numbers:

$$\Omega_1 = \begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (k) \rightsquigarrow (2) \dots (k)(k+1) \end{cases}$$

Let $\mathcal{S}$ be the class of Schröder paths and let $S \in \mathcal{S}$. The set $\vartheta(S)$ contains the Schröder paths obtained from S by inserting a horizontal step at the end of S, or by inserting both a rise step in each point of the last sequence of fall and horizontal steps, and a fall one at the end of S. The rule

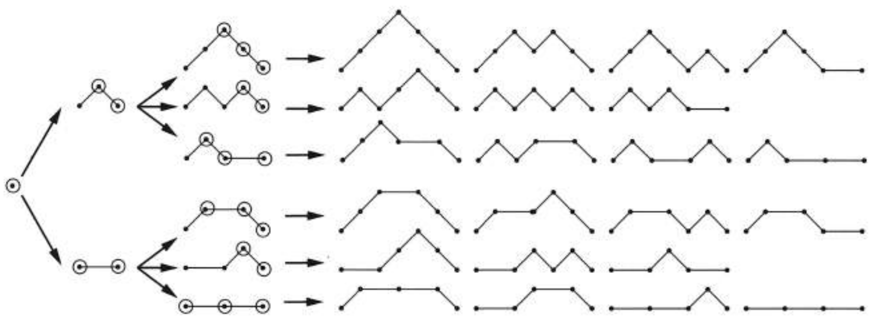

Figure 1: Classical ECO construction for Schröder paths.

$$\Omega_2 = \begin{cases} (2) \\ (2) \rightsquigarrow (3)(3) \\ (k) \rightsquigarrow (3) \dots (k)(k+1)^2 \end{cases}$$

associated to this construction is the classical rule for Schröder numbers (Fig. 1).

Definition 2.2. A *Schröder tree* is either a leaf or a list $(r, A_1, \dots, A_m)$, where $m \geq 2$, and such that each A_i is a Schröder tree.

The class $\mathcal{T}$ of Schröder trees contains planar trees whose internal nodes have degree at least two, and are enumerated by little Schröder numbers (i.e. the half of Schröder numbers) according to the number of their leaves [8]. As a particular case, the class $\mathcal{B}$ of *complete binary trees*, i. e. binary trees whose nodes have degree 0 or 2, is a subclass of $\mathcal{T}$.

3 A construction for Dyck and Schröder paths

The specialisations $\alpha = 1$ and $\alpha = 2$ of Ω'_α yield two new succession rules defining, respectively, Catalan and Schröder numbers,

$$\Omega'_1 = \begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (2k) \rightsquigarrow (1)^k(4)(6)\dots(2(k-1))(2k)(2(k+1)) \end{cases}$$

$$\Omega'_2 = \begin{cases} (2) \\ (2) \rightsquigarrow (2)(4) \\ (4k) \rightsquigarrow (2)^{2k}(4)(8)^2(12)^2\dots(4(k-1))^2(4k)^2(4(k+1)), \end{cases}$$

for which we are able to describe the corresponding constructions.

3.1 A construction for Dyck paths corresponding to Ω'_1

Each Dyck path D factors uniquely in blocks of elevated Dyck paths,

$$D = D_1 D_2 \dots D_k,$$

and, D is said of *even type* (respectively *odd type*) if $k = 2j$ for some j (resp. $k = 2j+1$). The last sequence of fall steps, or *last descent*, of D is denoted $\ell_d(D)$ and satisfies

$$\ell_d(D) = \ell_d(D_k).$$

Let $\mathbf{P}(D)$ be the set of points of $\ell_d(D)$, excepting the point at level 0. The set of Dyck paths having length $2n$ is denoted by $\mathcal{D}_n$, and the operator

$$\vartheta_{\mathcal{D}} : \mathcal{D}_n \longrightarrow 2^{\mathcal{D}_{n+1}}$$

is defined as follows:

D1. If D is of even type, then $\vartheta_{\mathcal{D}}(D)$ contains a single Dyck path, obtained by glueing a peak of height 1 at the end of D (see Fig. 2(D1)).

D2. If D is of odd type, then $\vartheta_{\mathcal{D}}(D)$ is the set of Dyck paths obtained from D by performing on each $A \in \mathbf{P}(D)$ the following actions:

   (a) insert a peak;

   (b) let A' be the leftmost point such that $A'A$ is a Dyck path; remove the subpath $A'A$ from D, elevate it by 1, and glue it at the end of D (see Fig. 2(D2)).

This construction yields the succession rule Ω'_1.

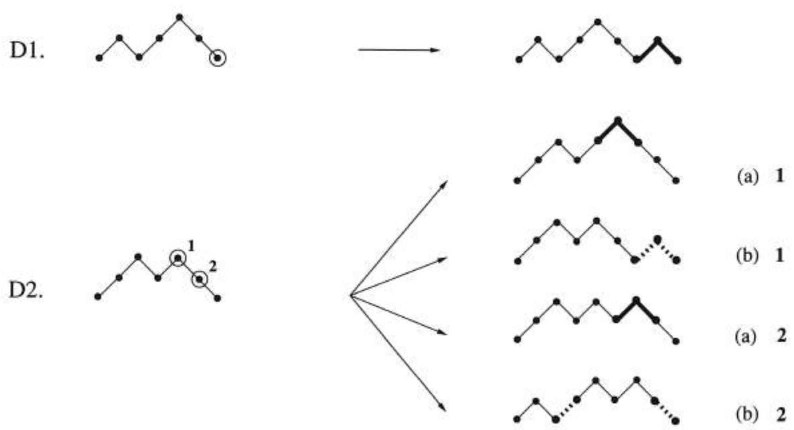

Figure 2: The construction for Dyck paths according to the rule Ω'_1.

3.2 A construction for Schröder paths corresponding to Ω'_2

We give now a similar construction for Schröder paths. Each Schröder path S factors uniquely,

$$S = S_1 S_2 \dots S_k,$$

where $S_i, 1 \leq i \leq k$, is either elevated or a horizontal step on the x-axis. The path S is said of *even type* (respectively *odd type*) if the number of elevated factors following the rightmost horizontal step is even (resp. odd). The *last descent* $\ell_d(S)$ of S is the last run of fall steps, and $\mathbf{P}(S)$ is the set of its points, excepted the last point on the x-axis.

The set of Schröder paths having length $2n$ is denoted $\mathcal{S}_n$, and the operator

$$\vartheta_{\mathcal{S}} : \mathcal{S}_n \longrightarrow 2^{\mathcal{S}_{n+1}}$$

is defined by the following rules:

S1. If S is of even type, then $\vartheta_{\mathcal{S}}(S)$ contains two Schröder paths, obtained respectively by glueing at the end of S, either a peak of height 1, resulting in an odd type path, or a horizontal step, resulting in an even type path (Fig. 3(S1)).

S2. If S is of odd type, then $\vartheta_{\mathcal{S}}(S)$ is the set obtained by performing the following actions on every point $A \in \mathbf{P}(S)$ (Fig. 3(S2)):

(a) insert a peak of height 1 or a horizontal step;

(b) let A' be the leftmost point such that $A'A$ is a Schröder path. Then cut $A'A$, elevate it by 1, and glue it at the end of S;

(c) let A'' be the first left point such that $A''A$ is a Schröder path; if $A''A$ is not empty, then replace it by a horizontal step and glue $A''A$ at the end of S; if $A''A$ is empty then glue a horizontal step at the end of S. In this way we obtain an even type path.

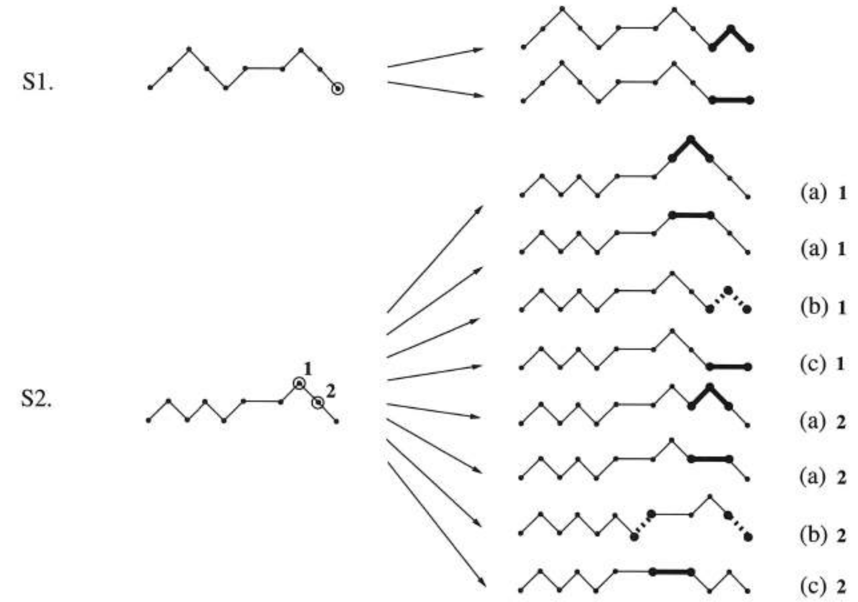

Figure 3: The construction for Schröder paths corresponding to the rule Ω'_2.

The previous construction for Schröder paths, can be easily extended to Schröder α-coloured paths by using α-coloured horizontal steps. It leads to the succession rule $\Omega'_{\alpha+1}$, with $\alpha \geq 2$. For instance, when horizontal steps of two colours are used, we obtain Schröder bi-coloured paths associated to the succession rule Ω'_3. Moreover, if we use α–coloured horizontal steps in the classical ECO construction for Schröder paths we obtain α–coloured Schröder paths to which the rule $\Omega_{\alpha+1}$, $\alpha \geq 2$, is associated. So we have proved the equivalence between Ω_α and Ω'_α in a combinatorial way.

4 A new construction for the classes $\mathcal{B}$ and $\mathcal{T}$

In this section we show how to transport an operator ϑ along a bijection, and we provide a description that is independent from the bijection in two classes of trees. The nodes of a planar tree T can be totally ordered by means of the prefix traversal, and indexed increasingly by the integers, so that, given two nodes x_i and x_j,

$$x_i < x_j \iff i < j.$$

Accordingly, the maximum of two nodes is defined by

$$\max(x_i, x_j) = x_j \iff i < j.$$

Also, the total order allows to define notions like *first, last, successor, predecessor, etc.*, consequently, for every node p of T, we denote by (see Fig. 5 and 7):

- $\ell_i(T), \ell_l(T), \ell_s(T)$ the last, respectively, internal node, leaf, internal sibling;
- $f(p)$ the set of leaves following p;
- father(p) the father of p;
- succ(p) the successor of p;

A common abuse of notation identifies a tree with the name of its root, and, consequently subtrees as nodes. The total order extends to the the class $\mathcal{F}$ of *forests*, whose objects are lists of trees, in the obvious way, making all the above definitions relevant for forests as well.

For convenience we denote the tree consisting of a single point by "$\bullet$", and define the "tree" and "raise" constructors

$$\text{tree}, \text{raise} : \mathcal{F} \longrightarrow \mathcal{T}$$

respectively, by

$$\text{tree}(T_1, T_2, \ldots, T_k) = (\bullet, T_1, T_2, \ldots, T_k),$$

and (see Fig. 4),

$$\text{raise}(T_1, T_2, \ldots, T_k) = \text{tree}(T_1, T_2, \ldots, T_k, \bullet).$$

A useful operation on trees is the substitution. Given two trees $T_1, T_2 \in \mathcal{T}$, the

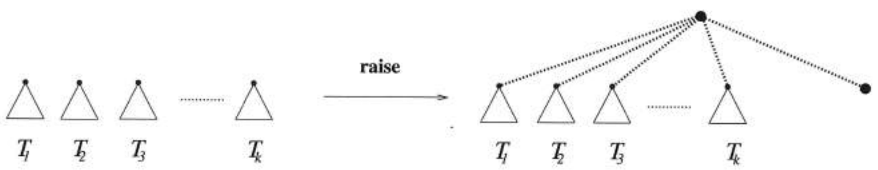

Figure 4: The raise constructor.

substituting of T_2 by T_1 ($T_2 \leftarrow T_1$) is denoted

$$\text{subs}(T_1, T_2).$$

Moreover, we say that T is of *even type* (resp. *odd type*) if the length of its rightmost branch is even (resp. odd).

>From here on, we consider this total order on two subclasses of planar trees, namely, the class $\mathcal{B}$ of complete binary trees and the class $\mathcal{T}$ of Schröder trees. The parameter p considered on these two classes of combinatorial objects is the number of leaves.

There is a well-known bijection between Dyck paths and complete binary trees,

$$\Psi : \mathcal{D} \longrightarrow \mathcal{B}$$

(for instance, see [10] and Fig. 5). For $D \in \mathcal{D}$ and $B = \Psi(D)$, define

$$\mathbf{P}(B) = f(\ell_i(B)) \setminus \{\ell_l(B)\},$$

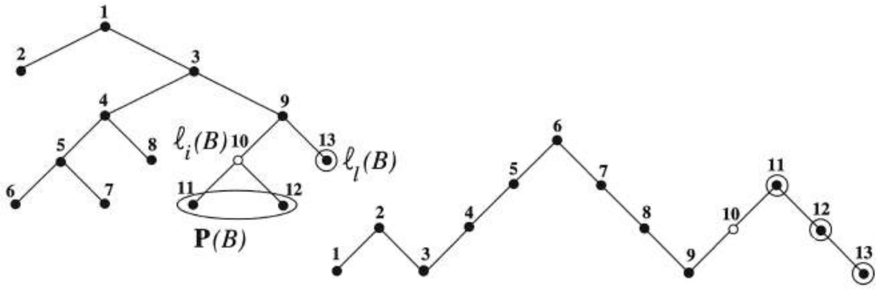

Figure 5: A complete binary tree B in $\mathcal{B}_7$, and the corresponding Dyck path.

and observe that the number of elevated Dyck paths in D corresponds to the length of the right branch of B. Moreover, we have the underlying set bijection on nodes

$$\begin{aligned} f(\ell_i(B)) &= \Psi(\ell_d(D)); \\ \mathbf{P}(B) &= \Psi(\mathbf{P}(D)). \end{aligned}$$

These observations lead to an almost direct translation of the operator $\vartheta_{\mathcal{D}}$. Indeed, let $\mathcal{B}_n$ be the set of binary trees having n leaves, and let $B \in \mathcal{B}_n$, then the operator

$$\vartheta_{\mathcal{B}} : \mathcal{B}_n \longrightarrow 2^{\mathcal{B}_{n+1}}$$

is defined as follows (see Fig. 6):

B1. if B is of even type then add two sons to $\ell_l(B)$, i.e.
$\vartheta_{\mathcal{B}}(B) = \text{subs}(\text{raise}(\bullet), \ell_l(B))$;

B2. if B is of odd type then $\vartheta_{\mathcal{B}}(B)$ is the set of complete binary trees obtained by performing on each leaf $A \in \mathbf{P}(B)$ the following actions:

(a) $\text{subs}(\text{raise}(\bullet), A)$;

(b) let A' be the largest complete binary subtree of B such that $A = \ell_l(A')$; then, do
$\text{subs}(\text{raise}(A'), \ell_l(B))$ and $\text{subs}(\bullet, A')$.

Clearly, $\vartheta_{\mathcal{D}}$ and $\vartheta_{\mathcal{B}}$ share the same succession rule Ω_1'.

4.1 A construction for Schröder trees

Let $\mathcal{S}'$ be the class of Schröder paths, without horizontal steps at level 0, and let $\vartheta_{\mathcal{S}'}$ be the restriction of $\vartheta_{\mathcal{S}}$ to $\mathcal{S}'$. That is

$$\vartheta_{\mathcal{S}'}(\mathcal{S}'_n) = \vartheta_{\mathcal{S}}(\mathcal{S}_n) \cap \mathcal{S}'_{n+1}\ , \ \forall n \geq 1.$$

As for Dyck paths, we show how to transport the operator $\vartheta_{\mathcal{S}'}$ along the bijection [8](see Fig.7)

$$\Psi' : \mathcal{S}' \longrightarrow \mathcal{T}.$$

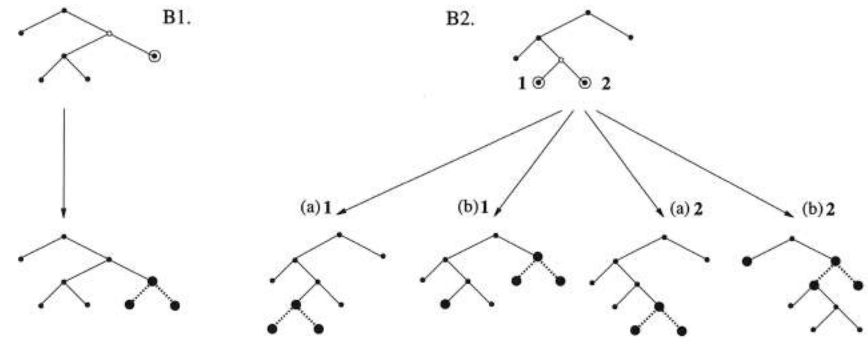

Figure 6: The construction for complete binary trees.

This bijection provides a simple interpretation of the required parameters. Indeed, a rise (resp. fall) step of S corresponds to a leftmost (resp. rightmost) sibling of T, and the horizontal steps of S correspond to the internal siblings of T, that is, those siblings strictly between the leftmost one and the rightmost one. The last run of fall steps $\ell_d(S)$ corresponds to, either the leaves following the last internal node $\ell_i(T)$, or, the last internal sibling $\ell_s(T)$ and its successors, whichever occurs the last. Therefore, define

$$z = \max(\text{succ}(\ell_i(T)), \ell_s(T)),$$

($z = 14$ in Fig. 7), and set

$$\mathbf{P}(T) = \Psi'(\mathbf{P}(S)) = \{z\} \cup f(z) \setminus \{\ell_l(T)\}.$$

Observe that this generalizes the corresponding definition in the class $\mathcal{B}$.

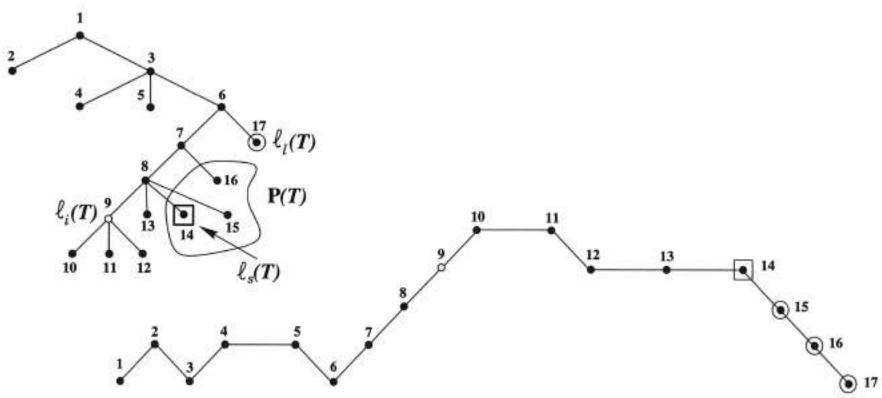

Figure 7: A Schröder tree and its corresponding path.

Let $\mathcal{T}_n$ be the set of Schröder trees having n-leaves. The operator

$$\vartheta_{\mathcal{T}} : \mathcal{T}_n \longrightarrow 2^{\mathcal{T}_{n+1}}$$

is defined as follows (see Fig. 8):

ST1. If T is of even type, then $\vartheta_{\mathcal{T}}(T) = \text{subs}(\text{raise}(\bullet), \ell_l(T))$ (see Fig. 8(ST1)).

ST2. If T is of odd type, then $\vartheta_{\mathcal{T}}(T)$ is obtained by performing on each point $A \in \mathbf{P}(T)$ the following actions (see Fig. 8(ST2)):

(a) $\text{subs}(\text{raise}(\bullet), A)$, or add a left brother to $succ(A)$;

(b) let A' be the largest Schröder subforest of T, such that $A = \ell_l(A')$; then, do
$\text{subs}(\text{raise}(A'), \ell_l(T))$ and $\text{subs}(\bullet, A')$;

(c) if $A \neq z$, let A'' be the tree having father(A) for root; then, do
$\text{subs}(A'', \ell_l(T))$, $\text{subs}(\bullet, A'')$, and add a right brother to A'' .

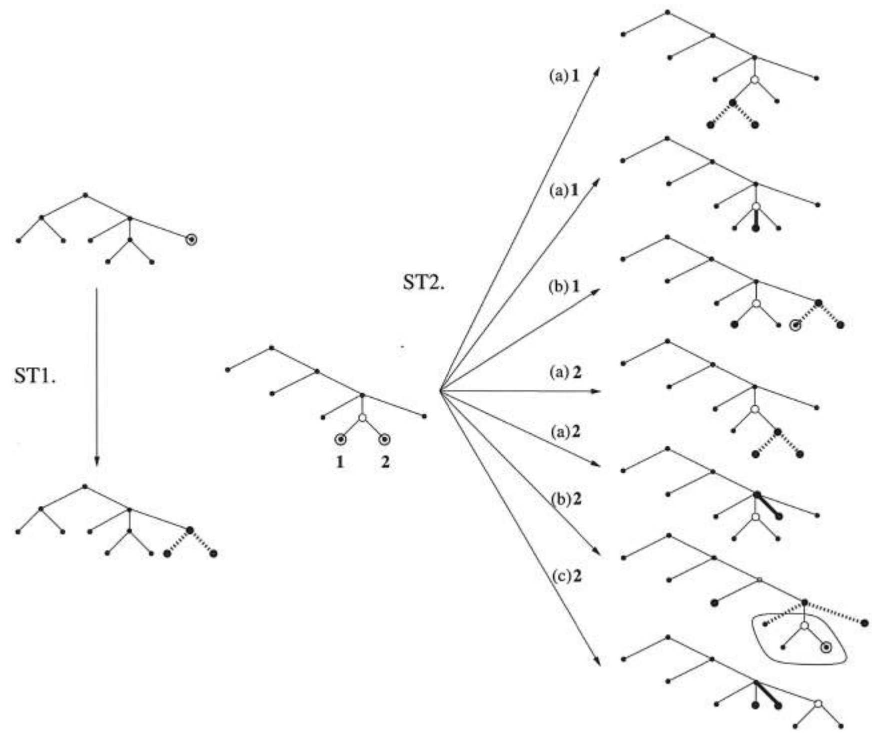

Figure 8: The construction for Schröder trees.

A careful comparison between the constructions associated to the operators $\vartheta_{\mathcal{T}}$ and $\vartheta_{\mathcal{S}}$ shows some differences. Indeed, since we are concerned with the restriction $\vartheta_{\mathcal{S}'}$, it was necessary to avoid the cases that generate a Schröder path with a horizontal step at level 0. This occurs precisely when the node z is treated.

5 Equivalence between two succession rules

We show now that the two succession rules Ω_α and Ω'_α defined in Section 1 have the same generating function. The computation is based on the *kernel method*, which was succesfully used for similar computations in [1, 3].

The bivariate generating function $F(x,y)$ counts the structures which satisfy Ω_α according to their size and the value of the associated label. Obviously, we suppose the size of the structure represented by the root of the generating tree being equal to 0. Therefore, we have:

$$F(x,y)\left(1+\frac{xy}{1-y}-xy\alpha\right)=y^\alpha+\frac{xy^{\alpha+1}}{1-y}F(x,1).$$

If

$$1+\frac{xy}{1-y}-xy\alpha=0, \tag{2}$$

then

$$y^\alpha+\frac{xy^{\alpha+1}}{1-y}F(x,1)=0.$$

The solution of the equation (2) is:

$$y_0(x)=\frac{x(\alpha-1)+1-\sqrt{(x(1-\alpha)-1)^2-4x\alpha}}{2x\alpha}$$

so, the generating function for Ω_α is:

$$\begin{aligned}F(x,1) &= \frac{y_0(x)-1}{xy_0(x)}\\ &= \frac{x(1-\alpha)+1-\sqrt{(x(1-\alpha)-1)^2-4x\alpha}}{2x}.\end{aligned}$$

In an analogous way we determine the generating function $G(x,y)$ arising from Ω'_α. After some computations we get:

$$G(x,y)=B_1(x,y)+B_2(x,y), \tag{3}$$

where

$$B_1(x,y)=y^\alpha+x(\alpha-1)y^\alpha B_1(x,1)+xy^\alpha\alpha\frac{\partial}{\partial y}B_2(x,y)\Big|_{y=1}$$

and

$$B_2(x,y)=xyB_1(x,1)+xy(\alpha-1)B_2(x,1)+\frac{xy^2\alpha}{1-y}B_2(x,1)-\frac{xy\alpha}{1-y}B_2(x,y)+xyB_2(x,y),$$

which simplify into:

$$B_1(x,1)=\frac{1+B_2(x,1)}{1-x\alpha+2x}$$

and

$$B_2(x,1) = \frac{y_0(x) - 1}{\alpha - x\alpha^2 + 3x\alpha - 2x - xy_0(x)\alpha + 2xy_0(x)}$$

where

$$y_0(x) = \frac{1 - x(\alpha - 1) - \sqrt{(x(\alpha-1)-1)^2 - 4x}}{2x}.$$

Substituting these values in (3), we have $F(x,1) = G(x,1)$, that is Ω_α and Ω'_α are equivalent.

6 Concluding remarks

The constructions we provided in this paper are natural because, in a sense, they commute. Indeed let $\pi_{\mathcal{D}}$ and $\pi_{\mathcal{B}}$ be the projections

$$\pi_{\mathcal{D}} : \mathcal{S} \longrightarrow \mathcal{D}; \quad \text{and} \quad \pi_{\mathcal{B}} : \mathcal{T} \longrightarrow \mathcal{B};$$

which erase, respectively, the horizontal steps and the internal siblings. The following diagram

$$\begin{array}{ccc} \mathcal{S}' & \xrightarrow{\Psi'} & \mathcal{T} \\ \pi_{\mathcal{D}} \downarrow & & \downarrow \pi_{\mathcal{B}} \\ \mathcal{D} & \xrightarrow{\Phi} & \mathcal{B} \end{array}$$

commutes, and the ECO-operators also commute. We believe that the problem of characterizing the natural bijections between objects (allowing the translation of ECO-operators) is a problem that is worth investigating.

References

[1] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, and D. Gouyou-Beauchamps, (2002) *Generating functions for generating trees*, Discrete Mathematics, 246 (1-3) 29–55.

[2] E. Barcucci, A. Del Lungo, E. Pergola, R. Pinzani, (1999) *ECO: a methodology for the Enumeration of Combinatorial Objects,* Journal of Difference Equations and Applications, Vol.5, 435–490.

[3] M. Bousquet-Mélou, M. Petkovšek, (2000) *Linear Recurrences with Constant Coefficients: the Multivariate Case*, Discrete Mathematics 225 (1-3)51–75.

[4] F. R. K. Chung, R. L. Graham, V. E. Hoggatt, M. Kleimann, (1978) *The number of Baxter permutations*, Journal of Combinatorial Theory Ser. A, 24, 382–394.

[5] S. Corteel, (2000) Séries génératrices exponentielles pour les eco-systèmes signés, in D. Krob, A.A. Mikhalev, A.V. Mikhalev (Eds.), *Proceedings of the 12-th International Conference on Formal Power Series and Algebraic Combinatorics, Moscow, Russia*, Springer, 655–666.

[6] S. Gire, (1993) *Arbres, permutations à motifs exclus et cartes planaires: quelques problèmes algorithmiques et combinatoires*, Thèse de l'université de Bordeaux I.

[7] O. Guibert, (1995) *Combinatoire des permutations à motifs exclus en liaison avec mots, cartes planaires et tableaux de Young*, Thèse de l'université de Bordeaux I.

[8] J.G. Penaud, E. Pergola, R. Pinzani, O. Roques, (2001) *Chemins de Schröder et hiérarchies aléatoires*, Theoretical Computer Science, 255, 345–361.

[9] E. Pergola, R. Pinzani, S. Rinaldi, (2000) *A set of well-defined operations on succession rules,* Proceedings of MCS, Versailles,**1**, 141–152.

[10] R. P. Stanley, (1986) *Enumerative Combinatorics, Vol. I,* Wadworth & Brooks/Cole, Monterey, Cal.

[11] R. A. Sulanke, (2000) *Moments of Generalized Motzkin Paths*, J. of Integer Sequences, Vol. 3, article 00.1.1.

[12] J. West, (1995) *Generating trees and the Catalan and Schröder numbers,* Discrete Mathematics, 146, 247–262.

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