Branches of forced oscillations for a class of implicit equations Involving the \Phi-Laplacian

BRANCHES OF FORCED OSCILLATIONS FOR A CLASS OF IMPLICIT EQUATIONS INVOLVING THE Φ-LAPLACIAN

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Abstract. We consider a class of parametric, implicit ordinary differential equations with a generalized Φ-Laplacian type term and we study the structure of the set of forced oscillations in presence of a periodic forcing term. Under suitable assumptions we obtain global bifurcation results whose statements require only conditions involving the well-known Brouwer degree in Euclidean spaces.

1. Introduction

In this paper we study, by topological methods, the forced oscillations of some nonlinear, implicit ordinary differential equations (ODEs) governed by a generalized Φ-Laplacian type differential operator.

Implicit ODEs arise naturally in applications and have been studied by many authors. For this kind of equations, usually the existence of homoclinic/heteroclinic orbits has been investigated with different methods, for example with a dynamical system approach (see, e.g., the recent [3] and the survey [20]). As far as we know, instead, not many papers have been devoted to the study of periodic solutions.

A typical example, in the scalar case, of equations for which it is not possible, or convenient, to write them explicitly, is provided by those involving the classical p-Laplacian operator $\Phi(x) := x|x|^{p-2}$, with $p > 1$, or, more generally, the so-called Φ-*Laplacian*: meaning a strictly increasing homeomorphism $\Phi : \mathbb{R} \to \mathbb{R}$. That is, scalar equations of the form

$$(\Phi(x'))' = f(t, x, x').$$

These equations are well-studied since they arise in some applicative models: for example, in non-Newtonian fluid theory, nonlinear elasticity, diffusion of flows in porous media, theory of capillary surfaces and, more recently, the modeling of glaciology (see, e.g., [4, 9, 32]). From a mathematical viewpoint, different kinds of boundary conditions can be associated to these equations; for example, the existence of heteroclinic solutions can be established using the method of upper/lower solutions (see, e.g., [7, 11, 13, 16, 18]). The study of periodic solutions has also been pursued in $\mathbb{R}^n$ via degree theory: we cite, for instance, [5, 6, 10, 19, 21, 22, 33].

Here we study periodic solutions of some implicit ODEs in $\mathbb{R}^n$ with generalized Φ-Laplacian type terms. We follow a topological approach based on the Brouwer degree. We consider two different problems which have to be handled separately. Namely,

$$[\phi_\lambda\big(x(t), x'(t)\big)]' = \lambda f\big(t, x(t), x'(t)\big), \quad \lambda \geq 0, \tag{1}$$

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and

$$[\phi_\lambda\big(x(t),x'(t)\big)]' = g\big(x(t),x'(t)\big) + \lambda f\big(t,x(t),x'(t)\big), \quad \lambda \geq 0, \tag{2}$$

where λ is a real parameter and, for a given open set $U \subseteq \mathbb{R}^n$, the maps $f\colon \mathbb{R}\times U \times \mathbb{R}^n \to \mathbb{R}^n$ and $g\colon U\times\mathbb{R}^n \to \mathbb{R}^n$ are continuous. We assume that f is T-periodic in the first variable, $T>0$ being fixed, and we look for periodic solutions of (1), resp. (2), of the same period T.

Concerning the left-hand side in (1) and (2), we assume that for each $\lambda \in [0,\infty)$ and $x\in U$, the map

$$\phi_\lambda(x,\cdot) : \mathbb{R}^n \to \mathbb{R}^n$$

is one-to-one and onto. Therefore, with the introduction of an auxiliary variable $y(t) = \phi_\lambda\big(x(t),x'(t)\big)$, problems (1) and (2) can be equivalently written as explicit systems in $\mathbb{R}^{2n}$. As a consequence, the properties of the solution sets of (1) and (2) can be investigated by means of earlier results, obtained by the authors also in collaboration with Massimo Furi, about periodically perturbed ODEs on differentiable manifolds (see [24, 26, 35]). We observe that the mentioned results rely upon the topological notion of degree (also called *Euler characteristic*) of a tangent vector field, see the classical books [28, 29, 30]. In spite of this fact, no advanced tools from differential topology are needed in the present paper. In fact, the technicalities are hidden in the proofs and our main results require only the well-known Brouwer degree in Euclidean spaces.

In our main results, Theorems 4.4 and 4.7, we investigate qualitative properties of the set of T-periodic solutions of (1) and (2). Roughly speaking, we obtain the existence of an unbounded connected set – a "branch" – made up of "nontrivial" T-periodic solution pairs (λ, x) of (1) and (2), that emanates from the set of the "trivial" ones – see Section 2 for more precise definitions. Thus, we may describe our main theorems as *global bifurcation results.* To prove the existence of such a branch we need only suitable assumptions on the Brouwer degree of some maps in $\mathbb{R}^n$, related with the terms f and g in the right-hand side of (1) and (2). Observe that the Brouwer degree can be computed explicitly and so, in principle, it is possible, and often not so difficult, to verify the validity of those assumptions.

Our results are somehow related to similar ones, concerning the set of T-periodic solutions of periodically perturbed differential-algebraic equations, recently obtained by the authors [8, 12, 15, 36]. In fact, under suitable conditions, also in that case it is possible to relate such equations to ODEs on manifolds, and to obtain the desired properties only in terms of the Brouwer degree in $\mathbb{R}^n$.

Another context in which the above mentioned results on periodically perturbed ODEs on manifolds have been successfully exploited is that of delay differential equations (see, e.g., [14]). We think that the results obtained here could be generalized to equations in which a dependence on delayed terms is introduced (see also [1]). However we do not pursue here this line: see Section 5 below for a brief discussion.

Observe finally that our results are not directly deducible from the standard implicit function theorem. And this for different reasons. Firstly, the implicit function theorem provides information on local properties, while the results we obtain here are of global nature. Next, the application of the implicit function theorem requires more regularity than that we assume here (i.e., the involved maps need to be of class C^1 and not merely continuous). In fact, apart from a technical assumption on the partial differentiability with respect to the parameter λ, we only require continuity. We believe that our global bifurcation results could be even generalized to a Carathéodory-type setting. This will be a further line of future investigation.

2. Setting of the problem

Assume throughout the paper that $\phi\colon [0,\infty)\times U\times\mathbb{R}^n\to\mathbb{R}^n$, U being open in $\mathbb{R}^n$, is a continuous map with the property that for each $\lambda\in[0,\infty)$ and $u\in U$, the map

$$\phi(\lambda,u,\cdot):\mathbb{R}^n\to\mathbb{R}^n$$

is one-to-one and onto; in other words, $\phi(\lambda,u,\cdot)$ is invertible with respect to the third variable in the following sense: for each $w\in\mathbb{R}^n$ there exist a unique $v\in\mathbb{R}^n$, depending on (λ,u), such that

$$w=\phi(\lambda,u,v).$$

We will denote this "partial inverse" as $\psi(\lambda,u,\cdot)$, so that $v=\psi(\lambda,u,w)$. We will also assume that, for each $w\in\mathbb{R}^n$ and $u\in U$, the map $\psi(\cdot,u,w)$ is differentiable and

$$\lambda\mapsto\partial_1\psi(\lambda,u,w)\quad\text{is continuous.}\tag{3}$$

Here and in what follows, the symbol "∂_i" denotes the partial differentiation with respect to the i-th variable.

We explicitly observe that, throughout the paper, very weak regularity assumptions are needed on ϕ, which is merely assumed to be continuous and such that (3) holds. In fact, the map ϕ will be assumed to be a perturbation of a homeomorphism; however, in the more regular case, the following remark is in order.

Remark 2.1. If the map ϕ is of class C^1, by applying the implicit function theorem, one gets that the above condition (3) is satisfied if, for any $(\lambda,u,v)\in[0,\infty)\times U\times\mathbb{R}^n$, the partial derivative

$$\partial_3\phi(\lambda,u,v)$$

is invertible.

It is sometimes convenient to write $\phi_\lambda(u,v)$ instead of $\phi(\lambda,u,v)$. Finally, we suppose that $\phi(0,\cdot,\cdot)$ depends only on the third variable, that is, by abuse of notation, $\phi(0,u,v)=:\phi_0(v)$ thus we may write $\psi(0,u,w)=\phi_0^{-1}(w)$. The latter assumption means that we regard ϕ_λ as a perturbation of some nonlinearity depending only on the third variable. With this in mind, we view $\phi(\lambda,u,v)$ as a sum $\phi_0(v)+\lambda\delta(\lambda,u,v)$, where ϕ_0 is a homeomorphism and $\delta(\lambda,u,v)$ is a perturbation such that the above assumptions are satisfied.

Example 2.2. A simple example of a map ϕ as above, in the case $n=1$, is the following:

$$\phi(\lambda,u,v)=(v+\lambda)|v+\lambda|^{p-2},\quad p=2,3,\ldots$$

which is a perturbation of the classical p-Laplacian, namely $\phi_0(v)=v|v|^{p-2}$. In this case one can explicitly compute, for $p=2$, $\psi(\lambda,u,w)=w-\lambda$, while for $p>2$,

$$\psi(\lambda,u,w)=\operatorname{sign}(w)|w|^{1/(p-1)}-\lambda,$$

in any case we get $\partial_1\psi(\lambda,u,w)=-1$ so that the above assumptions are satisfied.

Example 2.3. As another example in the case $n=1$, consider

$$\phi(\lambda,u,v)=\Phi(v)+\lambda(v-u),$$

where $\Phi:\mathbb{R}\to\mathbb{R}$, a Φ-Laplacian type operator, is a diffeomorphism such that $\Phi'(v)>0$ for any $v\in\mathbb{R}$. Even in this case one can check that the above assumptions are satisfied.

We wish to investigate the two following problems that, although somewhat similar, lead to substantially different results:

$$[\phi_\lambda\big(x(t),x'(t)\big)]' = \lambda f\big(t,x(t),x'(t)\big), \quad \lambda \geq 0, \tag{4}$$

and

$$[\phi_\lambda\big(x(t),x'(t)\big)]' = g\big(x(t),x'(t)\big) + \lambda f\big(t,x(t),x'(t)\big), \quad \lambda \geq 0. \tag{5}$$

where $f\colon \mathbb{R}\times U\times\mathbb{R}^n \to \mathbb{R}^n$ and $g\colon U\times\mathbb{R}^n \to \mathbb{R}^n$ are continuous maps and f is assumed T-periodic in the first variable for a $T>0$ given.

By a T-periodic solution of (4), corresponding to a given $\lambda \geq 0$, we mean a C^1 and T-periodic function $x\colon \mathbb{R}\to\mathbb{R}^n$ such that there exists another C^1 and T-periodic function $y\colon \mathbb{R}\to\mathbb{R}^n$ such that the system

$$\begin{cases} x'(t) = \psi\big(\lambda, x(t), y(t)\big), \\ y'(t) = \lambda f\Big(t, x(t), \psi\big(\lambda, x(t), y(t)\big)\Big), \end{cases} \tag{6}$$

is satisfied identically. The T-periodic solutions of (5), corresponding to $\lambda \geq 0$, are defined analogously as the T-periodic solutions of the system

$$\begin{cases} x'(t) = \psi\big(\lambda, x(t), y(t)\big), \\ y'(t) = g(x(t), \psi\big(\lambda, x(t), y(t)\big)\Big) + \lambda f\Big(t, x(t), \psi\big(\lambda, x(t), y(t)\big)\Big), \end{cases} \tag{7}$$

Observe that from the first equation in (6) we get $\phi_\lambda\big(x(t),x'(t)\big) = y(t)$, hence the map $t\mapsto \phi_\lambda\big(x(t),x'(t)\big)$ is C^1. Finally, notice that when the partial derivatives of ϕ_λ exist continuous along the curve $t\mapsto \big(x(t),x'(t)\big)\in\mathbb{R}^{2n}$ with $\partial_2\phi_\lambda\big(x(t),x'(t)\big)$ invertible, one has that x is actually C^2 since, in this case, we have

$$x''(t) = \partial_2\phi_\lambda\big(x(t),x'(t)\big)^{-1}\Big[\lambda f\big(t,x(t),x'(t)\big) - \partial_1\phi_\lambda\big(x(t),x'(t)\big)x'(t)\Big].$$

We denote by $C^r_T(U)$ the set of the U-valued, T-periodic, C^r functions with topology induced by the C^r norm. This is a subset of the Banach space $C^r_T(\mathbb{R}^n)$. For simplicity, when $r=0$, we write $C_T(U)$ instead of $C^0_T(U)$.

A pair $(\lambda,x)\in[0,\infty)\times C^1_T(U)$ with x taking values in U, such that x is a T-periodic solution of (4) (resp. (5)), is said to be a *T-forced pair* for (4) (resp. (5)). A T-forced pair (λ,x) is called *trivial* if x is constant and $\lambda=0$. In this paper we investigate the set of nontrivial T-forced pairs of (4) and (5).

3. Preliminaries

In order to pursue our investigation on the set of T-forced pairs of (4) and (5) we need slight generalizations of previous results, obtained in [24, 26, 35]. These results deal with periodic perturbations of autonomous ODEs on manifolds and require, in the statements, the notion of degree (also called *Euler characteristic*) of a tangent vector field.

In this preliminary section we first recall this notion of degree and its relationship with the Brouwer degree. As already pointed out, in this paper the degree is only a tool, whose use is hidden in the proofs. In spite of this, we decided to add the definition and some properties of the degree on manifolds in order to make the paper self-contained. The reader which is already familiar with it can skip Section 3.1.

3.1. The degree of a tangent vector field.

Let M be a smooth boundaryless submanifold of $\mathbb{R}^k$, and $w\colon M\to\mathbb{R}^k$ a continuous tangent vector field on M: meaning that, for all $p\in M$, we have $w(p)\in T_pM$, where $T_pM\subseteq\mathbb{R}^k$ is the tangent space of M at p. Particularly relevant in this paper is the special case when M is an open subset of $\mathbb{R}^k$. When this happens, one has $T_pM=\mathbb{R}^k$ for all $p\in\mathbb{R}^k$.

Let $U \subseteq M$ be open. The pair (w, U) is said to be *admissible* if $w^{-1}(0) \cap U$ is compact; we also say that w is admissible for the degree in U. It is known (see, e.g., [28, 29, 30]) that one can associate to any admissible pair (w, U) an integer, $\deg(w, U)$, called the *degree* of w in U. The degree, roughly speaking, counts algebraically the number of zeros of w in U.

When w is (Fréchet) differentiable at $p \in M$ and $w(p) = 0$, then its derivative at p, $w'(p)\colon T_pM \to \mathbb{R}^k$, actually maps T_pM into itself, and, therefore, its determinant $\det w'(p)$ is well-defined. If, moreover, p is a nondegenerate zero (i.e., $w'(p)\colon T_pM \to \mathbb{R}^k$ is injective) then p is an isolated zero and $\det w'(p) \neq 0$.

In the regular case, that is, when w is admissible for the degree in U and the zeros of w are all nondegenerate, then the set $w^{-1}(0) \cap U$ is finite and the following equality holds:

$$\deg(w, U) = \sum_{p \in w^{-1}(0) \cap U} \operatorname{sign} \det w'(p). \tag{8}$$

Moreover, we stress that, when $M = \mathbb{R}^k$, $\deg(w, U)$ coincides with the Brouwer degree, $\deg_B(w, V, 0)$, of the triple $(w, V, 0)$: where $V \subseteq U$ is any bounded open set, whose closure is contained in U, containing the compact set $w^{-1}(0) \cap U$, i.e.,

$$\deg_B(w, V, 0) = \deg(w, U). \tag{9}$$

All the standard properties of the classical Brouwer degree on open and bounded subsets of Euclidean spaces, such as homotopy invariance, excision, additivity, existence, are still valid in the more general context of differentiable manifolds. Indeed, it can be shown (see [25]) that the degree of a tangent vector field is uniquely determined by some of these properties used as axioms.

The classical Brouwer degree can be extended to the case when the open set U is possibly unbounded; see e.g. [17, §1.2.4] or, in the context of differentiable manifolds, [31, §1.3]. It is not difficult to show, using the excision property, that (9) is valid also in this extended context. Indeed, one could use (9) as basis for the extended notion of Brouwer degree.

In this paper, we will always omit the "target" point it in the notation for the degree since we are only interested in the case when this is 0. By abuse of notation justified by (9), we will write deg to denote both *the extended Brouwer degree* and *the degree of a tangent vector field*.

3.2. On the set of T-periodic solutions of perturbed ODEs on manifolds.

In this section we will be concerned with differential equations on manifolds. We need to introduce some further notation. Let $M \subseteq \mathbb{R}^k$ be a boundaryless differentiable manifold. By $C_T(M)$ we denote the set of the M-valued, T-periodic, continuous functions with the topology induced by the Banach space $C_T(\mathbb{R}^k)$.

Consider the following differential equations on $M \subseteq \mathbb{R}^k$ depending on a parameter $\lambda \in [0, \infty)$:

$$\dot{\xi}(t) = \mathfrak{g}\big(\xi(t)\big) + \lambda \mathfrak{f}\big(t, \xi(t), \lambda\big), \tag{10}$$

and

$$\dot{\xi}(t) = \lambda \mathfrak{f}\big(t, \xi(t), \lambda\big), \tag{11}$$

where $\mathfrak{f}\colon \mathbb{R} \times M \times [0, \infty) \to \mathbb{R}^k$ and $\mathfrak{g}\colon M \to \mathbb{R}^k$ are continuous vector fields tangent to M, $\mathfrak{f}$ being T-periodic in the first variable.

A pair $(\lambda, \xi) \in [0, \infty) \times C_T(M)$, such that (10) (resp. (11)) holds identically is a *T-pair* for (10) (resp. (11)). A T-pair (λ, ξ) is called *trivial* if ξ is constant and $\lambda = 0$. In this paper we investigate the set of nontrivial T-pairs of (10) and (11).

Given any point $p \in M$ it is convenient to introduce the notation $\overline{p}$ to denote the function constantly equal to p. Accordingly, a T-pair for (10) is trivial if and only if it is of the form $(0,\overline{p})$ for some $p \in \mathfrak{g}^{-1}(0)$. Similarly, for all $p \in M$, all pairs $(0,\overline{p})$ are trivial T-pairs for (11).

The qualitative properties of the set of T-pairs of (10) and (11) can be deduced by suitable assumptions on the degree of some tangent vector fields, see [26].

In fact, by inspection of Theorem 3.3 of [26] one immediately sees that the following result holds:

Theorem 3.1. *Let $\mathfrak{f}$ and $\mathfrak{g}$ be as in equation* (10) *and let Ω be an open subset of $[0,\infty) \times C_T(M)$. Put $\Omega_M = \big\{p \in M : (0,\overline{p}) \in \Omega\big\}$. Assume that* $\deg(\mathfrak{g}, \Omega_M)$ *is well defined and nonzero. Then there exists a connected set Γ of nontrivial T-pairs in Ω whose closure in $[0,\infty) \times C_T(M)$ intersects the set $\big\{(0,\overline{p}) \in [0,\infty) \times C_T(M) : p \in \mathfrak{g}^{-1}(0) \cap \Omega_M\big\}$ and is not contained in any compact subset of Ω. In particular, if M is closed in $\mathbb{R}^k$ and $\Omega = [0,\infty) \times C_T(M)$, then Γ is unbounded.*

Similarly, by inspection of the proof of Theorem 2.3 of [24] one obtains the following:

Theorem 3.2. *Let $\mathfrak{f}$ be as in* (11)*, define $\mathfrak{v}\colon M \to \mathbb{R}^k$ be the autonomous vector field given by*

$$\mathfrak{v}(p) := \frac{1}{T}\int_0^T \mathfrak{f}(t,p,0)\,\mathrm{d}t.$$

Let Ω be an open subset of $[0,\infty) \times C_T(M)$, and put $\Omega_M = \big\{p \in M : (0,\overline{p}) \in \Omega\big\}$. Assume that $\deg(\mathfrak{v}, \Omega_M)$ *is well defined and nonzero. Then there exists a connected set Γ of nontrivial T-pairs in Ω whose closure in $[0,\infty) \times C_T(M)$ meets $\big\{(0,\overline{p}) \in [0,\infty) \times C_T(M) : p \in \mathfrak{v}^{-1}(0) \cap \Omega_M\big\}$ and is not contained in any compact subset of Ω. In particular, if M is closed in $\mathbb{R}^k$ and $\Omega = [0,\infty) \times C_T(M)$, then Γ is unbounded.*

Observe that Theorem 3.2 concerns the case not covered in general by Theorem 3.1 of perturbations of the zero vector field. In fact, unless the manifold M is compact, Theorem 3.1 above is not applicable, $\mathfrak{g}$ not being admissible.

Finally, consider the following system of coupled equations, depending on the parameter $\lambda \geq 0$, on the manifold $M \times N$, where $M \subseteq \mathbb{R}^k$ and $N \subseteq \mathbb{R}^s$ are differentiable manifolds

$$\left\{\begin{array}{l} \dot\xi(t) = \lambda\mathfrak{f}\big(t,\xi(t),\eta(t),\lambda\big), \\ \dot\eta(t) = \mathfrak{g}\big(\xi(t),\eta(t)\big) + \lambda\mathfrak{h}\big(t,\xi(t),\eta(t),\lambda\big), \end{array}\right. \tag{12}$$

where $\mathfrak{f}\colon \mathbb{R}\times M\times N\times[0,\infty) \to \mathbb{R}^k$, $\mathfrak{h}\colon \mathbb{R}\times M\times N\times[0,\infty) \to \mathbb{R}^s$ and $\mathfrak{g}\colon M\times N \to \mathbb{R}^s$ are continuous vector fields tangent to $M \times N$, $\mathfrak{f}$ and $\mathfrak{h}$ being T-periodic in the first variable.

A triple $(\lambda,x,y) \in [0,\infty) \times C_T(M \times N)$, such that (12) holds identically is a T-*triple* for (12). A T-triple (λ,x,y) is called *trivial* if (x,y) is constant and $\lambda = 0$.

As above, given $(p,q) \in M \times N$, by $\overline{p}$ and $\overline{q}$ we denote the functions constantly equal to p and q, respectively. Thus, a T-triple is trivial if and only if it is of the form $(0,\overline{p},\overline{q})$ with $(p,q) \in \mathfrak{g}^{-1}(0)$.

Let $\nu : M\times N \to \mathbb{R}^{k+s}$ be the vector field, tangent to the manifold $M\times N \subseteq \mathbb{R}^{k+s}$, given by

$$\nu(p,q) = \left(\frac{1}{T}\int_0^T \mathfrak{f}(t,p,q,0)\,\mathrm{d}t\ ,\ \mathfrak{g}(p,q)\right).$$

As for the previous two theorems, an inspection of the argument of Theorem 4 of [35] shows that the following statement holds:

Theorem 3.3. *Let $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$ be as in equation* (12)*, and let Ω be an open subset of $[0,\infty) \times C_T(M \times N)$. Assume that* $\deg\big(\nu, \Omega_{M\times N}\big)$ *is well-defined and nonzero.*

Then there exists a connected set Γ *of nontrivial* T*-triples in* Ω *of* (12) *whose closure in* $[0,\infty)\times C_T(M\times N)$ *intersects* $\big\{(0,\overline{p},\overline{q})\in[0,\infty)\times C_T(M\times N):(p,q)\in\nu^{-1}(0)\cap\Omega_{M\times N}\big\}$ *and is not contained in any compact subset of* Ω*. In particular, if* $M\times N$ *is closed in* $\mathbb{R}^{k+s}$ *and* $\Omega=[0,\infty)\times C_T(M\times N)$*, then* Γ *is unbounded.*

4. Main results

In this section we investigate the set of T-pairs for equations (4) and (5). A crucial step is the following simple observation that essentially is a version of Hadamard's Lemma, see e.g. [2]:

Remark 4.1. Let ϕ be as in equations (4) and (5). Let also $u\in U$, $v,w\in\mathbb{R}^n$ and $\lambda\in[0,\infty)$ be such that $w=\phi_\lambda(u,v)$; then there exists a continuous function h such that $v=\phi_0^{-1}(w)+\lambda h(\lambda,u,w)$. To see that this is the case, let us write $v=\psi(\lambda,u,w)$ and then, in order to isolate λ, we observe that

$$\begin{aligned}v-\phi_0^{-1}(w)=v-\psi(0,u,w)&=\int_0^1\frac{\mathrm{d}}{\mathrm{d}s}\,\psi(s\lambda,u,w)\,\mathrm{d}s\\&=\int_0^1\lambda\partial_1\psi(s\lambda,u,w)\,\mathrm{d}s=\lambda h(\lambda,u,w),\end{aligned}$$

where $h(u,w,\lambda)=\int_0^1\partial_1\psi(s\lambda,u,w)\,\mathrm{d}s$.

We consider separately the cases of equations (4) and (5).

4.1. **On the set of T-forced pairs for equation** (4)**.** Introducing the new function $y(t)=\phi_\lambda\big(x(t),x'(t)\big)$ and using Remark 4.1 we are led to the following system:

$$\left\{\begin{array}{l}x'(t)=\phi_0^{-1}\big(y(t)\big)+\lambda h\big(\lambda,x(t),y(t)\big),\\ y'(t)=\lambda f\Big(t,x(t),\phi_0^{-1}\big(y(t)\big)+\lambda h\big(\lambda,x(t),y(t)\big)\Big),\end{array}\right.\tag{13}$$

which is of the form (12). We relate the set $X\subseteq[0,\infty)\times C^1_T(U)$ of T-forced pairs of (4) with te set $Y\subseteq[0,\infty)\times C_T(U\times\mathbb{R}^n)$ of T-triples of (13).

Define the map $\mathfrak{H}\colon Y\to X$ by $\mathfrak{H}\colon(\lambda,x,y)\mapsto(\lambda,x)$, and observe that $\mathfrak{H}$ is continuous.

Lemma 4.2. *The map* $\mathfrak{H}$ *is a homeomorphisms that respects the notion of triviality, in the sense that it makes trivial* T*-triples correspond to trivial* T*-forced pairs and vice versa.*

Proof. First notice that, if (λ,x,y) is a T-triple of (13), then x is a T-periodic solution of (4) corresponding to λ. In particular, when (λ,x,y) is trivial, $\lambda=0$ and x is constant: hence (λ,x) is a trivial T-forced pair.

Conversely, take any $(\lambda,x)\in X$, and let $y(t)=\phi_\lambda\big(x(t),x'(t)\big)$. We immediately see that y is T-periodic and that (x,y) is a T-periodic solution of (13) corresponding to λ. In other words

$$\mathfrak{H}^{-1}(\lambda,x)=\big(\lambda,x,\phi_\lambda\big(x,x'\big)\big)=(\lambda,x,y)$$

and this map is obviously continuous. Observe, in particular, that when x is constant so is y, hence the image under $\mathfrak{H}^{-1}$ of any trivial T-pair is a trivial T-forced pair. □

Consider the following vector field, defined on $U\times\mathbb{R}^n$:

$$\nu(p,q):=\left(\frac{1}{T}\int_0^T f\big(t,p,\phi_0^{-1}(q)\big)\,\mathrm{d}t\ ,\ \phi_0^{-1}(q)\right).$$

Taking into account Theorem 3.3 and Lemma 4.2 one sees that the degree of ν plays a crucial role in our investigation. Furthermore, since ϕ_0^{-1} is a homeomorphism, the computation of the degree of ν can be reduced to the degree of the "average wind":

$$w(p) := \frac{1}{T}\int_0^T f(t,p,0)\,\mathrm{d}t, \tag{14}$$

as shown by the following technical lemma:

Lemma 4.3. *Let $W \subseteq U$ be open. The vector field ν is admissible for the degree in $W\times\mathbb{R}^n$ if and only if so is w in W, and*

$$\deg\big(\nu, W\times\mathbb{R}^n\big) = \pm\deg\big(w,W\big). \tag{15}$$

Proof. Since ϕ_0^{-1} is a homeomorphism let $q_0\in\mathbb{R}^n$ be the (unique) point such that $\phi_0^{-1}(q_0)=0$. We have

$$\nu^{-1}(0) = \big\{(p,q_0) : p\in w^{-1}(0)\big\}, \tag{16}$$

so that $w^{-1}(0)\cap W$ is compact if and only if so is $\nu^{-1}(0)\cap(W\times\mathbb{R}^n)$. This implies that ν is admissible in $W\times\mathbb{R}^n$ if and only if so is w in W.

We now prove formula (15). Let W_1 and W_2 be open and bounded subets of W and $\mathbb{R}^n$ respectively such that $\nu^{-1}(0)\subseteq W_1\times W_2$. By the excision property of the degree, we find

$$\deg\big(\nu, W_1\times\mathbb{R}^n\big) = \deg\big(\nu, W_1\times W_2\big). \tag{17}$$

Denote by ν_1 the first component of ν. By known transversality theorems, we can approximate ν_1 with a smooth map $\hat\nu_1$ with only isolated zeros and also approximate ϕ_0 with a smooth diffeomorphism $\hat\phi_0$. Let

$$\hat\nu(p,q) = \big(\hat\nu_1(p,q)\ ,\ \hat\phi_0^{-1}(q)\big).$$

We can take the approximations so close, that the homotopy

$$\begin{aligned} H(s,p,q) &= s\hat\nu(p,q) + (1-s)\nu(p,q)\\ &= \Big(s\hat\nu_1(p,q) + (1-s)\nu_1(p,q)\ ,\ s\hat\phi_0^{-1}(q) + (1-s)\phi_0^{-1}(q)\Big), \end{aligned}$$

has no zeros on the boundary of the bounded open set $W_1\times W_2$ for $s\in[0,1]$. Thus H is admissible in $W_1\times W_2$. The homotopy property of the degree yields

$$\deg\big(\nu, W_1\times W_2\big) = \deg\big(\hat\nu, W_1\times W_2\big). \tag{18}$$

Similarly, defining $\hat w(p)=\hat\nu_1(p,0)$, we have

$$\deg(w,W_1) = \deg(\hat w, W_1). \tag{19}$$

Given an isolated zero (p_0,q_0) of $\hat\nu$, we have

$$\det\hat\nu'(p_0,q_0) = \det\begin{pmatrix}\partial_1\hat\nu_1(p_0,q_0) & 0\\ \partial_2\hat\nu_1(p_0,q_0) & (\hat\phi_0^{-1})'(q_0)\end{pmatrix} = \det\hat w'(p_0)\det(\hat\phi_0^{-1})'(q_0).$$

As in (16) we have

$$\hat\nu^{-1}(0) = \big\{(p,q_0) : p\in\hat w^{-1}(0)\big\}.$$

Also, $\hat\phi_0^{-1}$ being a diffeomorphism, $\operatorname{sign}\det(\hat\phi_0^{-1})'(q_0)=\pm1$. Thus,

$$\begin{aligned}\deg(\hat\nu, W_1\times W_2) &= \sum_{(p_0,q_0)\in\hat\nu^{-1}(0)\cap W_1\times W_2} \operatorname{sign}\det\hat\nu'(p_0,q_0)\\ &= \operatorname{sign}\det(\hat\phi_0^{-1})'(q_0)\sum_{p_0\in\hat w^{-1}(0)\cap W_1}\operatorname{sign}\det\hat w'(p_0)\\ &= \pm\deg(\hat w, W_1).\end{aligned}$$

The assertion now follows from equations (19), (18) and (17). $\square$

Given an open subset Ω of $[0,\infty)\times C_T(U)$, the set $\Omega^* = \Omega \times C_T(\mathbb{R}^n)$ is open in $[0,\infty)\times C_T(U\times\mathbb{R}^n)$. As in Section 3 we define

$$\Omega_U = \big\{p \in U : (0,\overline{p}) \in \Omega\big\}$$

that is clearly open in U. Similarly, we consider the open set $\Omega^*_{U\times\mathbb{R}^n} \subseteq U \times \mathbb{R}^n$ given by

$$\Omega^*_{U\times\mathbb{R}^n} = \big\{(p,q) \in U\times\mathbb{R}^n : (0,\overline{p},\overline{q}) \in \Omega^*\big\} = \Omega_U \times \mathbb{R}^n.$$

By Lemma 4.3 we have that, when $\deg(w,\Omega_U)$ is well defined and nonzero, then so is $\deg(\nu,\Omega^*_{U\times\mathbb{R}^n})$. Thus, by Theorem 3.3 applied to (13) we find that there exists a connected set, say Θ, of nontrivial T-triples in Ω of (13) whose closure in $[0,\infty)\times C_T(U\times\mathbb{R}^n)$ meets $\nu^{-1}(0)\cap\Omega^*_{U\times\mathbb{R}^n}$ and is not contained in any compact subset of $\Omega^*_{U\times\mathbb{R}^n}$. According to Lemma 4.2, one finds that $\Gamma = \mathfrak{H}(\Theta)$ is a set of T-forced pairs for (4) in Ω such that trivial T-triples of (13) correspond to trivial T-forced pairs. Since $\mathfrak{H}$ is a homeomorphism, we have the following result:

Theorem 4.4. *Let Ω be an open subset of $[0,\infty)\times C_T(U)$. Let w be as in* (14) *and assume that $\deg(w,\Omega_U)$ is well-defined and nonzero. Then there exists a connected set Γ of nontrivial T-forced pairs in Ω of* (4) *whose closure in $[0,\infty)\times C_T(U)$ intersects the set $\big\{(0,\overline{p}) \in [0,\infty)\times C_T(U) : p \in w^{-1}(0)\cap\Omega_U\big\}$ and is not contained in any compact subset of Ω. In particular, when $U = \mathbb{R}^n$ and $\Omega = [0,\infty)\times C_T(\mathbb{R}^n)$ then Γ is unbounded.*

4.2. **On the set of T-forced pairs for equation** (5)**.** Proceeding analogously to the above case, system (7) can be written as

$$\text{(20)}\qquad \begin{cases} x'(t) = \phi_0^{-1}\big(y(t)\big) + \lambda h(\lambda, x(t), y(t)\big), \\ y'(t) = g\Big(x(t), \phi_0^{-1}\big(y(t)\big) + \lambda h(\lambda, x(t), y(t))\Big) + \\ \qquad +\lambda f\Big(t, x(t), \phi_0^{-1}\big(y(t)\big) + \lambda h(\lambda, x(t), y(t))\Big). \end{cases}$$

With the same procedure used in Remark 4.1, we can isolate λ from inside g as well and get, for some appropriate function $\widehat{h}$, the equality:

$$g\Big(x(t), \phi_0^{-1}\big(y(t)\big) + \lambda h(\lambda, x(t), y(t))\Big) = g\Big(x(t), \phi_0^{-1}\big(y(t)\big)\Big) + \lambda\widehat{h}(\lambda, x(t), y(t)\big).$$

So, collecting all the λ-dependent terms in the second equation into an appropriate function F we can rewrite (20) as

$$\text{(21)}\qquad \begin{cases} x'(t) = \phi_0^{-1}\big(y(t)\big) + \lambda h(\lambda, x(t), y(t)\big), \\ y'(t) = g\Big(x(t), \phi_0^{-1}\big(y(t)\big)\Big) + \lambda F\big(t, x(t), y(t), \lambda\big). \end{cases}$$

This is of the form (10) with $M := U\times\mathbb{R}^n$, so that, in analogy with Section 4.1, we will speak about T-triples rather than T-pairs. Accordingly, we will relate the set $\Xi \subseteq [0,\infty)\times C^1_T(U)$ of T-forced pairs of (5) with the set $\Upsilon \subseteq [0,\infty)\times C_T(U\times\mathbb{R}^n)$ of T-triples of (21).

Define the map $\mathfrak{G}\colon \Upsilon \to \Xi$ by $\mathfrak{H}\colon (\lambda,x,y) \mapsto (\lambda,x)$. A result analogous to Lemma 4.2 holds in this case too.

Lemma 4.5. *$\mathfrak{G}$ is a homeomorphisms that respects the notion of triviality, in the sense that it makes trivial T-triples correspond to trivial T-forced pairs and vice versa.*

In this case our result will be a consequence of Theorem 3.1. Let G be the vector field defined on $U\times\mathbb{R}^n$:

$$(p,q) \mapsto \Big(\phi_0^{-1}(q)\ ,\ g\big(p, \phi_0^{-1}(q)\big)\Big).$$

In analogy with the previous section, since ϕ_0^{-1} is a homeomorphism, the degree of G reduces to the degree of the vector field:

$$\gamma(p) := g(p,0). \tag{22}$$

The proof of the next lemma can be carried out as in Lemma 4.3 above and, therefore, is omitted.

Lemma 4.6. *Let $W \subseteq U$ be open. The vector field G is admissible for the degree in $W \times \mathbb{R}^n$ if and only if so is γ in W, and*

$$\deg\big(G, W \times \mathbb{R}^n\big) = \pm \deg\big(\gamma, W\big). \tag{23}$$

We conclude as in Section 4.1. Given an open subset Ω of $[0,\infty) \times C_T(U)$, we observe that the set $\Omega^* = \Omega \times C_T(\mathbb{R}^n)$ is open in $[0,\infty) \times C_T(U \times \mathbb{R}^n)$ and we define the open subset of U:

$$\Omega_U = \big\{p \in U : (0,\overline{p}) \in \Omega\big\}.$$

Furthermore, we consider the open set $\Omega^*_{U\times\mathbb{R}^n} \subseteq U \times \mathbb{R}^n$ given by

$$\Omega^*_{U\times\mathbb{R}^n} = \big\{(p,q) \in U \times \mathbb{R}^n : (0,\overline{p},\overline{q}) \in \Omega^*\big\} = \Omega_U \times \mathbb{R}^n.$$

By Lemma 4.6 it follows that, if $\deg(\gamma, \Omega_U)$ is well defined and nonzero, then so is $\deg(G, \Omega^*_{U\times\mathbb{R}^n})$. Hence, Theorem 3.1 applied to (21) yields the existence of a connected set, say Θ, of nontrivial T-triples in Ω of (21) whose closure in $[0,\infty) \times C_T(U \times \mathbb{R}^n)$ meets $G^{-1}(0) \cap \Omega^*_{U\times\mathbb{R}^n}$ and is not contained in any compact subset of $\Omega^*_{U\times\mathbb{R}^n}$. By Lemma 4.5, we get that $\Gamma = \mathfrak{G}(\Theta)$ is a set of T-forced pairs of (5) in Ω since $\mathfrak{G}$ is a homeomorphism. As above, observe that a T-triple $(0,x,y)$ of (21) is trivial if and only if $(0,x)$ is a trivial T-pair of (5). Thus, we have proved the following result:

Theorem 4.7. *Let Ω be an open subset of $[0,\infty) \times C_T(U)$. Let γ be as in (22) and assume that $\deg\big(\gamma, \Omega_U\big)$ is well-defined and nonzero. Then there exists a connected set Γ of nontrivial T-forced pairs in Ω of (5) whose closure in $[0,\infty) \times C_T(U)$ intersects the set $\big\{(0,\overline{p}) \in [0,\infty) \times C_T(U) : p \in \gamma^{-1}(0) \cap \Omega_U\big\}$ and is not contained in any compact subset of Ω. In particular, when $U = \mathbb{R}^n$ and $\Omega = [0,\infty) \times C_T(\mathbb{R}^n)$ then Γ is unbounded.*

As already pointed out, this theorem cannot be obtained as a direct consequence of Theorem 4.4.

5. Final remarks and perspectives

In this paper we have investigated the structure of the set of T-periodic solutions of equations (1) and (2). Although the former can be seen as a particular case of the latter, and despite the fact that the results concerning the set of T-forced pairs of either equations have a similar form, we notice an important difference. Namely, the degree of the average wind w, that is crucial for Theorem 4.4, plays no role in Theorem 4.7: it could, in principle, be not even defined. Conversely, for equation (1), the degree of γ does not even make sense. It seems natural to compare Theorems 3.2 and 3.1 with Theorems 4.4 and 4.7, respectively.

An interesting question, that we postpone to further research, is whether it would be possible to find a bridge between Theorems 4.4 and 4.7. After all, Theorem 3.3 does something similar for Theorems 3.1 and 3.2, see the pertinent discussion in [35].

A further attractive line of study, not addressed here, is the investigation of the set of T-periodic solutions of equations (1) and (2) when a dependence on delayed arguments is introduced in ϕ and f. Namely if equations of the form

$$\begin{aligned}[\phi_\lambda\big(x(t), x(t-r), x'(t)\big)]' &= g\big(x(t), x'(t)\big) \\ &\quad + \lambda f\big(t, x(t), x(t-r), x'(t), x(t-r)\big), \quad \lambda \geq 0\end{aligned}$$

are considered. In fact, it seems reasonable to combine the method used here to prove Theorem 3.1 with the arguments developed in [27]. Actually, at the price of some increase in the technical details it seems possible to use the arguments of [14] in conjunction with the techniques of the present paper to further generalize our results to the functional delay case.

Finally, we mention another possible extension of the results obtained here. It seems plausible to use the results of [23, 34] in order to generalize Theorem 3.3 to the case when $\mathfrak{h}$ and $\mathfrak{f}$ are Carathéodory so that, following the argument of Section 4, one could extend Theorems 4.4 and 4.7 to equations of the form (1) and (2) where the forcing term f is allowed to be Carathéodory.

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