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Restricting Dyck Paths and 312-Avoiding Permutations

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Abstract

We interpret Dyck paths of height at most h and without valleys at height $h-1$ combinatorially, by means of 312-avoiding permutations with some restrictions on their left-to-right maxima. We obtain our results by analyzing a restriction of a well-known bijection between the sets of Dyck paths and 312-avoiding permutations. We also provide a recursive formula enumerating these two structures by using the ECO method and the theory of production matrices. As a further result we obtain a family of combinatorial identities involving Catalan numbers.

1 Introduction

Dyck paths have been widely used in several combinatorial applications. Here, we only recall their involvement in theory of codes [1, 9], cryptography [19], and partially ordered structures [8]. Dyck path enumeration has also received much attention in recent decades. An interesting paper dealing with this matter is the one by Deutsch [15] where the author enumerates Dyck paths according to various parameters.

A subclass of these paths has been considered thanks to the simple behavior of the recursive relations describing them and the rational nature of the related generating function. More precisely, the generating function associated with Dyck paths is algebraic, and it is rational when the paths are bounded [12, 13], for example with respect to the height. Kallipoliti et al. [17] consider Dyck paths of height less or equal to a precise value k. Moreover, in the same paper a further restriction is considered: the authors analyze some characteristics of Dyck paths avoiding valleys at specified height.

In our work, we consider Dyck paths of height equal or less than h and with no valley at height $h-1$. We obtain an interesting relation with a subclass of 312-avoiding permutations (actually, we obtain a bijection) having some constraints on the left-to-right maxima.

The paper structure is the following. In Section 2 some preliminaries on Dyck paths and pattern avoiding permutations are introduced and moreover we recall a well-known bijection between the sets of Dyck paths and 312-avoiding permutations. We are going to largely use this bijection in the whole paper. Section 3 and Section 4 are devoted to the generation of the considered Dyck paths (of height equal or less than h and with no valley at height $h-1$) and the corresponding 312-avoiding permutations with some restriction on their left-to-right maxima. The enumerative results are presented in Section 5. We provide the generating functions for the above mentioned classes and a recurrence relation for their enumeration according to their size.

Finally, we conclude the paper proposing some further developments on the present topics.

2 Preliminaries

A *Dyck path* is a lattice path in the discrete plane $\mathbb{Z}^2$ from $(0,0)$ to $(2n,0)$ with up and down steps in $\{(1,1),(1,-1)\}$, never crossing the x-axis. The number of up steps in every prefix of a Dyck path is greater or equal to the number of down steps, and the total number of steps (the *length* of the path) is $2n$. We denote the set of Dyck paths of length $2n$ (or equivalently semilength n) by $\mathcal{D}_n$. A Dyck path can be codified by a string over the alphabet $\{U,D\}$, where U and D replace the up and down steps, respectively. The empty Dyck path is denoted by ε.

The *height* of a Dyck path P is the maximum ordinate reached by one of its steps. A *valley* of P is an occurrence of the substring DU, while a *peak* is an occurrence of the substring UD. The height of a valley (peak) is the ordinate reached by D (U).

We denote by $\mathcal{D}_n^{(h,k)}$ the set of Dyck paths having semilength n and height at most h, and avoiding $k-1$ consecutive valleys at height $h-1$. The set of Dyck paths having semilength n with height at most h (without restriction on the number of valleys) is denoted by $\mathcal{D}_n^{(h)}$. Moreover,

$$\mathcal{D}^{(h)} = \sum_{n\geq 0} \mathcal{D}_n^{(h)} \text{ and } \mathcal{D}^{(h,k)} = \sum_{n\geq 0} \mathcal{D}_n^{(h,k)}.$$

The cardinalities of $\mathcal{D}_n^{(h,k)}$ and $\mathcal{D}_n^{(h)}$ are indicated by $D_n^{(h,k)}$ and $D_n^{(h)}$, respectively. Finally, the set $\mathcal{D}_n$ of unrestricted Dyck paths having semilength $n \geq 0$ is enumerated by the n-Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

When $k = 2$, the set $\mathcal{D}_n^{(h,2)}$ represents the set of Dyck paths with height at most h and without valleys at height $h-1$. In the present work we describe a combinatorial interpretation of $\mathcal{D}_n^{(h,2)}$ in terms of restricted permutations.

In our context, the above mentioned permutations are related to the notion of pattern avoidance which can be generally described as the absence of a substructure inside a larger structure. In particular, an occurrence of a pattern σ in a permutation π of length n is a subsequence (not necessarily constituted by consecutive entries) of π whose entries appear in the same relative order as those in σ. Otherwise, we say that π *avoids* the pattern σ, or that σ is a forbidden pattern for π. For example, the permutation $\pi = 352164$ contains two occurrences of $\sigma = 312$ in the subsequences 514 and 524, while $\pi = 34251$ avoids the pattern σ. The set $\mathcal{S}_n(312)$ denotes the set of 312-avoiding permutations of length n which is enumerated by the n-Catalan number.

We are going to briefly recall a well-known bijection φ, useful in the rest of the paper, between the classes $\mathcal{D}_n$ and $\mathcal{S}_n(312)$. For more details we refer to [15, 18]. Figure 1 shows an example of the bijection. We fix a Dyck path P, and label its up steps by enumerating them from left to right (so that the ℓ-th up step is labelled ℓ). Then, we draw one horizontal line starting from each up step U until it meets a down step D which is the down step corresponding to U. Next, we assign to each down step the same label of the up step it corresponds to. Now, let us consider the permutation whose entries are constituted by the labels of the down steps read from left to right. Such a permutation $\pi = \varphi(P)$ is easily seen to be 312-avoiding. As far as the inverse map φ^{-1} is concerned, once fixed a 312-avoiding permutation $\pi = \pi_1\pi_2\cdots\pi_n$ we can consider its factorization in terms of descending subsequences whose first elements coincide with the left-to-right maxima of π. A *left-to-right maximum* (l.r.M for short) is an element π_i which is greater than all the elements to its left, i.e., greater than all π_j with $j < i$. Denoting $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_\ell}$ the left-to-right maxima of π, the corresponding Dyck path $P = \varphi^{-1}(\pi)$ is obtained as follows:

- write as many U's as $\pi_{i_1}(= \pi_1)$ followed by as many D's as the cardinality of the first descending subsequence headed by π_{i_1};
- for each $j = 2, \ldots, \ell$, add as many U's as $\pi_{i_j} - \pi_{i_{j-1}}$ followed by as many D's as the cardinality of the subsequence headed by π_{i_j}.

Two easy properties of the l.r.M of $\pi \in S_n(312)$ and their corresponding steps in $P = \varphi^{-1}(\pi)$ are summarized in the following:

Proposition 1. *Let P denote a Dyck path in $\mathcal{D}_n$ and $\pi = \varphi(P) = \pi_1 \ldots \pi_n$ be the associated permutation in $S_n(312)$. Each label π_{i_j} corresponding to the first down step of a subsequence of consecutive down steps in P is a left to right maximum. Moreover, the number $\pi_{i_j} - i_j$ is the height reached by the down step corresponding to π_{i_j} in P .*

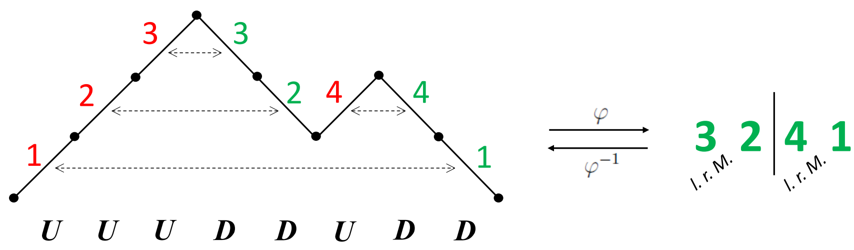

Figure 1: The bijection φ between the set of Dyck paths and the set of 312-avoiding permutations.

3 A generating algorithm

The set $\mathcal{D}^{(h,2)}$ can be exhaustively generated by means of an ECO operator [3] which allows constructing all the paths of a certain length $n+1$ (the size of the combinatorial objects) starting from the ones of size n.

To this aim, consider a Dyck path $P \in \mathcal{D}_n^{(h,2)}$ which, obviously, starts with $t \leq h$ up steps U. We mark these steps factorizing the path P as $P = U_1U_2\cdots U_tDP'$, where P' is a suitable Dyck suffix of length $n-t-1$. The idea is to consider some *sites* in $P \in \mathcal{D}_n^{(h,2)}$ where an insertion of the factor **UD** is allowed in order to obtain paths in $\mathcal{D}_{n+1}^{(h,2)}$ from P (so that the sites are called *active sites*).

Thus, we define an operator ϑ for the class $\mathcal{D}_n^{(h,2)}$ as follows:

- if $P = U_1U_2\cdots U_{t-1}U_tDP' \in \mathcal{D}_n^{(h,2)}$, with $t < h$, then

$$\begin{aligned}\vartheta(P) = \{&\mathbf{UD}U_1U_2\cdots U_{t-1}U_tDP',\\ &U_1\mathbf{UD}U_2\cdots U_{t-1}U_tDP',\\ &\cdots\\ &U_1U_2\cdots U_{t-1}\mathbf{UD}U_tDP',\\ &U_1U_2\cdots U_{t-1}U_t\mathbf{UD}DP'\ \};\end{aligned}$$

- if $P = U_1U_2\cdots U_{t-1}U_tDP' \in \mathcal{D}_n^{(h,2)}$, with $t = h$, then

$$\begin{aligned}\vartheta(P) = \{&\mathbf{UD}U_1U_2\cdots U_{t-1}U_tDP',\\ &U_1\mathbf{UD}U_2\cdots U_{t-1}U_tDP',\\ &\cdots\\ &U_1U_2\cdots\mathbf{UD}U_{t-1}U_tDP'\}.\end{aligned}$$

We note that the insertion of **UD** may create a valley in the paths of $\vartheta(P)$. In particular,

- the insertion of **UD** before the step U_j, with $j = 1, 2, \ldots, t-1$, gives the occurrence of the valley $\mathbf{D}U_j$ of height $j-1 < h-1$ in every case;

- the insertion of **UD** before the step U_t in the case $t < h$ gives the occurrence of the valley $\mathbf{D}U_t$ of height equal to $t-1 < h-1$;
- the insertion of **UD** after the step U_t in the case $t < h$ does not give the occurrence of a valley (since the next step is again a D step).

In other words, the valley possibly generated by the insertion of **UD** has height less than $h-1$. Therefore we have

Proposition 2. *If $x \in \vartheta(P)$, with $P \in \mathcal{D}_n^{(h,2)}$, then $x \in \mathcal{D}_{n+1}^{(h,2)}$.*

In the spirit of the ECO method, we have to prove the following proposition.

Proposition 3. *The operator ϑ is an ECO operator.*

Proof. The proof consists in the following steps:

i) If $x, y \in \mathcal{D}_n^{(h,2)}$ with $x \neq y$, then $\vartheta(x) \cap \vartheta(y) = \emptyset$.

ii) If $x \in \mathcal{D}_{n+1}^{(h,2)}$ then $\exists\ y \in \mathcal{D}_n^{(h,2)}$ such that $x \in \vartheta(y)$.

For case i), we suppose that a path P such that $P \in \vartheta(x)$ and $P \in \vartheta(y)$, with $x \neq y$, does exist. From the description of the operator ϑ it is easy to realize that the first peak of P is precisely generated by the insertion of the factor **UD**. By removing such a peak from P, we obtain a unique path. Thus, we would have $x = y$, against the hypothesis.

For case ii), being $x \in D_{n+1}^{(h,2)}$ then $x = U^j\mathbf{UD}T'$, with $j = 0, 1, \ldots, h-1$, where T' is a Dyck suffix of suitable length. Then, the path $y = U^jT'$ starts with at most h up steps U so that $y \in D_n^{(h,2)}$. Clearly, we have $x \in \vartheta(y)$ since y is obtained by the insertion of **UD** in x. □

A generating algorithm can be naturally described by means of the concept of *succession rule*. Such a concept was introduced by Chung et al. [14] to study reduced Baxter permutations. Recently, this technique has been successfully applied to other combinatorial objects [10, 11], and it has been recognized as an extremely useful tool for the ECO method [3]. In all these cases there is a common approach to the examined enumeration problem: a generating tree is associated with a certain combinatorial class according to some enumerative parameters, in such a way that the number of nodes appearing on level n of the tree gives the number of n-sized objects in the class.

A succession rule is a formal system constituted by an *axiom* (a) and some *productions* (possibly only one) having the form

$$(k) \rightsquigarrow \big(e_1(k)\big)\big(e_2(k)\big)\cdots\big(e_k(k)\big),$$

so that a succession rule Ω is often denoted by

$$\Omega : \begin{cases} (a) \\ \\ (k) \rightsquigarrow \big(e_1(k)\big)\big(e_2(k)\big)\cdots\big(e_k(k)\big). \end{cases}$$

The symbols (a), (k), and $e_i(k)$ are called *labels* (their values are positive integers), and play a crucial role when the succession rule Ω is represented by a *generating tree*. This is a rooted tree whose nodes are the labels of Ω. More precisely, the root is labelled with (a) and each node having label (k) has k children having labels $e_1(k), e_2(k), \ldots, e_k(k)$, according to the productions in Ω.

In our case, the generating algorithm for $\mathcal{D}^{(h,2)}$ is performed by the operator ϑ and from its definition it is easy to realize the following:

- the empty path ε can be labelled with the axiom (1) having production $(1) \rightsquigarrow (2)$: the path ε generates the path UD, having in its turn label (2);
- every other path P can have label (2), (3), $\ldots$, (h) depending on the number t of its starting up steps U. More precisely, if $1 \leq t \leq h-1$ then P is labelled $(t+1)$. Otherwise, if $t = h$ then P is labelled $(h-1)$.

In order to write the productions of the labels (k) of P, with $k = 2, 3, \ldots, h$ we observe the following:

- if $k < h$ then the k paths in $\vartheta(P)$ start, respectively, with $1, 2, \ldots, k$ up steps, so that, in their turn, they are labelled $(2), (3), \ldots, (k+1)$. Then we can write the production
$$(k) \rightsquigarrow (2)(3)\cdots(k)(k+1),\ \ 2 \leq k < h.$$
- if $k = h$ then the k paths in $\vartheta(P)$ start, respectively, with $1, 2, \ldots, h$ up steps. Since the path having h starting up steps is labelled $(h-1)$ then we can write the production
$$(h) \rightsquigarrow (2)(3)\cdots(h-1)^2(h).$$
The two paths having label $(h-1)$ are precisely the one starting with h up steps and the one starting with $h-2$ up steps.

Finally, the generating algorithm for $\mathcal{D}^{(h,2)}$ can be described by the succession rule (for $h \geq 3$) as follows:

$$\Omega_h : \begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (k) \rightsquigarrow (2)(3)\cdots(k)(k+1), & \text{for } 2 \leq k < h \\ (h) \rightsquigarrow (2)(3)\cdots(h-1)^2(h). \end{cases} \tag{1}$$

4 The bijection with a subset of 312-avoiding permutations

Let $\mathcal{S}_n^{(h)}(312)$ denote the subset of permutations $\pi \in S_n(312)$ such that $\pi_{i_j} - i_j \leq h-1$, for each l.r.M. π_{i_j} of π. The reader can easily check that the restriction $\varphi\big|_{D_n^{(h)}}$ of φ to the set $D_n^{(h)}$ is a bijection between $D_n^{(h)}$ and $S_n^{(h)}(312)$ (using Proposition 1).

We now consider the paths in $D_n^{(h,2)}$ and characterize the corresponding permutations via the restriction of φ to $D_n^{(h,2)}$. The following proposition holds.

Proposition 4. *Let P denote a Dyck path in $\mathcal{D}_n^{(h)}$. Then $P \in \mathcal{D}_n^{(h,2)}$ if and only if in the corresponding permutation $\pi = \varphi(P)$ there is not any left-to-right maximum π_{i_j} such that*

1. $\pi_{i_j} - i_j = h - 1$ *and*
2. $\pi_{i_{j+1}} = \pi_{i_j} + 1$.

Proof. Suppose that $\pi = \varphi(P)$ does not have any left-to-right maximum π_{i_j} such that $\pi_{i_j} - i_j = h - 1$ and $\pi_{i_{j+1}} = \pi_{i_j} + 1$. Let $P = \varphi^{-1}(\pi)$ denote the corresponding path. We have to prove that $P \in \mathcal{D}_n^{(h,2)}$.

- If P has height less than h then $P \in \mathcal{D}_n^{(h,2)}$ and the proof is completed.
- Let us suppose that P has height equal to h and suppose, ad absurdum, that $P \notin \mathcal{D}_n^{(h,2)}$. Therefore, there exists a valley of height $h-1$. Thus, the path P can be written as $P'U_iD_iU_{i+1}D_{i+1}P''$, where P' and P'' are, respectively, a Dyck prefix and a Dyck suffix of height $h-1$. Considering the permutation $\pi = \varphi(P) = \pi_1 \cdots \pi_i\pi_{i+1} \cdots \pi_n$ (where we highlighted the entries π_i and π_{i+1} corresponding to the steps D_i and D_{i+1}), thanks to Proposition 1, it is possible to observe that the elements π_i and π_{i+1} associated with U_i and U_{i+1}, respectively, are l.r.M. in π. Again from Proposition 1, we have $\pi_i - i = h-1$ and $\pi_{i+1} - (i+1) = h - 1$. Therefore, by substitution, it is $\pi_{i+1} = \pi_i + 1$ against the hypothesis, and then $P \in \mathcal{D}_n(h, 2)$.

On the other side, let us suppose that $P \in \mathcal{D}_n^{(h,2)}$ and suppose, ad absurdum, that the permutation $\pi = \varphi(P) = \pi_1 \cdots \pi_i\pi_{i+1} \cdots \pi_n \in \mathcal{S}_n^{(h)}(312)$ has a left to right maximum π_i with $\pi_{i+1} = \pi_i + 1$ and $\pi_i - i = h - 1$. Then, it is $\pi = \pi_1 \cdots \pi_i(\pi_i + 1) \cdots \pi_n$. Since $\pi_i < \pi_{i+1}$ and π is a 312-avoiding permutation, then there is no $\pi_l > \pi_i$ with $l < i$. Thus, both π_i and π_{i+1} are l.r.M. in π. From Proposition 1, the quantities $\pi_i - i$ and $\pi_{i+1} - (i + 1)$ are the heights reached by the corresponding descending steps in P. Moreover, from the two hypotheses $\pi_i - i = h - 1$ and $\pi_{i+1} = \pi_i + 1$, we deduce $\pi_{i+1} - (i + 1) = \pi_i + 1 - (i + 1) = h - 1$. Thus, the path $P = \varphi^{-1}(\pi)$ can be factorized as $P = P'U_iD_iU_{i+1}D_{i+1}P''$ showing that P admits a valley of height $h - 1$, against the hypothesis $P \in \mathcal{D}_n^{(h,2)}$. □

The permutations corresponding to the paths in $\mathcal{D}_n^{(h,2)}$ are denoted by $\mathcal{S}_n^{(h,2)}(312)$. By means of the above proposition, we prove the following one.

Proposition 5. *There exists a bijection between the classes $\mathcal{S}_n^{(h,2)}(312)$ and $\mathcal{D}_n^{(h,2)}$, which is the restriction $\varphi\big|_{D_n^{(h,2)}}$.*

A generating algorithm for the class $\mathcal{S}_n^{(h,2)}(312)$ according to the succession rule Ω_h can be obtained, thanks to Proposition 5. A combinatorial interpretation of Ω_h in terms of permutations is then desired.

First of all we note that, if $\pi = \pi_1 \cdots \pi_n \in \mathcal{S}_n^{(h,2)}(312)$, then $\pi_1 \leq h$. After that, we have to find an interpretation of the parameters appearing in the rule Ω_h. The axiom (1) at level 0 can be associated with the *empty* permutation and its production labelled with (2) can be associated with the permutation 1. The parameter (k) at level n in the rule Ω_h admits the following interpretation according to the value of π_1 in $\pi \in S_n^{(h,2)}(312)$:

$$(k) = \begin{cases} \pi_1 + 1, & \text{if } \pi_1 \neq h; \\ \pi_1 - 1, & \text{if } \pi_1 = h. \end{cases} \tag{2}$$

More precisely, if $\pi_1 < h$, a permutation $\pi = \pi_1 \cdots \pi_n \in \mathcal{S}_n^{(h,2)}(312)$ at level n produces $k = \pi_1 + 1$ sons at level $n + 1$ by inserting the element ℓ, with $\ell = 1, 2, \ldots, \pi_1 + 1$, before π_1 and rescaling the sequence $\ell\pi$ in order to obtain a permutation $\pi' \in \mathcal{S}_{n+1}^{(h,2)}(312)$ (for the sake of clearness, each entry π_i of π equal or greater than ℓ is increased by 1 in order to obtain π').

Otherwise, when $\pi_1 = h$, a given permutation $\pi = \pi_1 \cdots \pi_n \in \mathcal{S}_n^{(h,2)}(312)$ at level n produces $k = \pi_1 - 1 = h - 1$ sons at level $n + 1$ by inserting the element ℓ, whose values are $\ell = 1, 2, \ldots, h - 1$, before π. Analogously, the permutation $\pi' \in \mathcal{S}_{n+1}^{(h,2)}(312)$ is obtained by rescaling the sequence $\ell\pi$, for each ℓ.

As an example, fixed $h = 3$, the succession rule for $\mathcal{S}_n^{(3,2)}(312)$, or equivalently for $\mathcal{D}_n^{(3,2)}$, is as follows:

$$\Omega_3 : \begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (2) \rightsquigarrow (2)(3) \\ (3) \rightsquigarrow (2)(2)(3). \end{cases} \tag{3}$$

In Figure 2 a graphical representation of the first levels of Ω_3 is shown in terms of permutations in $\mathcal{S}_n^{(3,2)}$.

5 Enumeration

The case $h = 2$ is not included in the general formula (1) for the succession rules. However, it is easy to see that in this case it is

$$\Omega_2 : \begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (2) \rightsquigarrow (1)(2). \end{cases} \tag{4}$$

The succession rule (4) defines the Fibonacci numbers.

A given succession rule can be also represented by the *production matrix* $P = \left(p_{k,i}\right)_{k,i \geq 0}$ where each entry $p_{k,i}$ is the number of labels l_i produced by label l_k. For more detail we refer to the theory developed by Deutsch et al. [16].

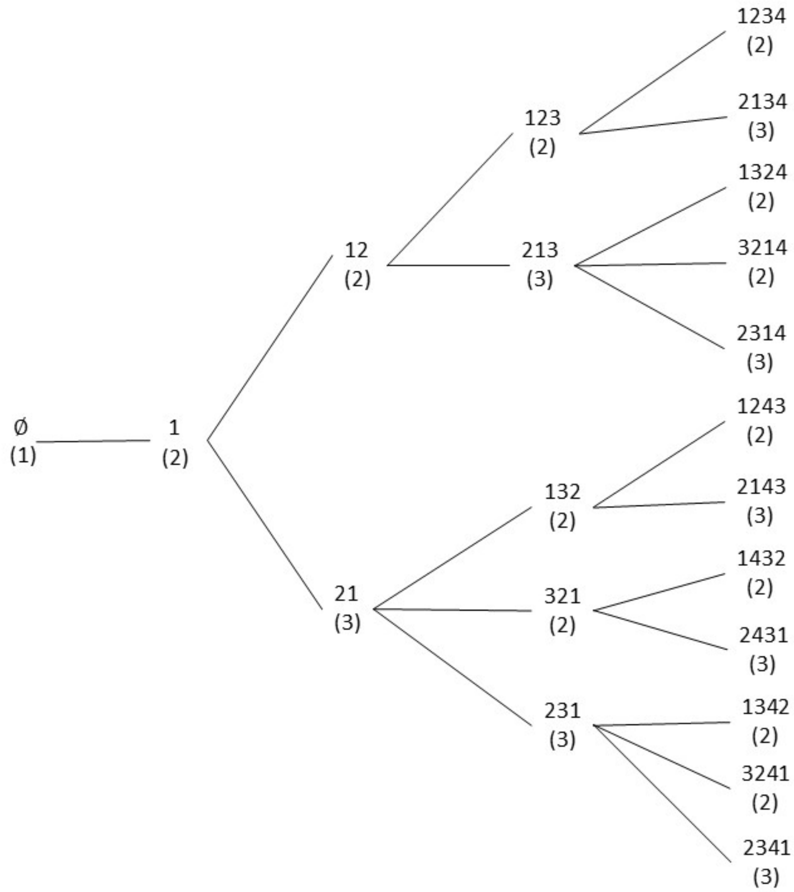

Figure 2: Graphical representation of the generating tree associated with $\mathcal{S}_n^{(3,2)}$ where the label associated with each permutation is shown between parentheses.

The production matrix P_2 associated with Ω_2 is

$$P_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \tag{5}$$

and, for $h \geq 3$, the production matrix P_h associated with Ω_h is

$$P_h = \begin{pmatrix} 0 & u^t \\ 0 & P_{h-1} + eu^t \end{pmatrix}, \tag{6}$$

where u^t is the row vector $(1\ 0\ 0 \cdots)$, and e is the column vector $(1\ 1\ 1\ \ \cdots)^t$ of appropriate size. For what the generating function $f_h(x)$ of the sequence corresponding to Ω_h is concerned, we have [16], for $h \geq 2$,

$$f_h(x) = \frac{1}{1 - x f_{h-1}(x)}. \tag{7}$$

When $h = 1$, clearly the unique paths in $\mathcal{D}_n^{(1,2)}$ are the empty path ε and UD, so that the sequence $(D_n^{(1,2)})_{n\geq 0}$ is $\{1, 1, 0, 0, \ldots\}$, whose generating function is $f_1(x) = 1 + x$ which is rational. Thanks to (7) it is possible to deduce that also $f_h(x)$ with $h \geq 2$ is rational, too. Therefore, we can consider its general form as follows:

$$f_h(x) = \frac{p_h(x)}{q_h(x)}, \tag{8}$$

where $p_h(x)$ and $q_h(x)$ are polynomials with suitable degrees. From (7) and (8) we obtain

$$\begin{aligned} p_h(x) &= q_{h-1}(x) \\ q_h(x) &= q_{h-1}(x) - xq_{h-2}(x). \end{aligned} \tag{9}$$

Since the degree of the polynomial $q_h(x)$ is $\left\lceil \frac{h+1}{2} \right\rceil$ (it can be easily seen by induction), we can assume

$$q_h(x) = a_{h,0} - a_{h,1}x - a_{h,2}x^2 - \cdots - a_{h,j}x^j \quad \text{with} \quad j = \left\lceil \frac{h+1}{2} \right\rceil.$$

Clearly, it is $a_{h,j} = 0$ if $j > \left\lceil \frac{h+1}{2} \right\rceil$.

As $a_{1,0} = 1$, thanks to (9) we have $a_{h,0} = a_{h-1}$, and

$$a_{h,0} = 1 \ \text{ for each } \ h \geq 1.$$

Moreover,

$$q_h(x) = 1 - a_{h,1}x - a_{h,2}x^2 - \cdots - a_{h,j}x^j \quad \text{with} \quad j = \left\lceil \frac{h+1}{2} \right\rceil. \tag{10}$$

Using the expression (9) for $q_h(x)$, we obtain

$$\begin{aligned} q_h(x) = 1 &- a_{h-1,1}x - a_{h-1,2}x^2 - \cdots - a_{h-1,j-1}x^{j-1} \\ &- x\big(1 - a_{h-2,1}x - a_{h-2,2}x^2 - \cdots - a_{h-2,j-2}x^{j-2}\big). \end{aligned} \tag{11}$$

Thanks to the identity theorem for polynomials, comparing formulas (10) and (11) for $q_h(x)$, it is

$$a_{h,j} = \begin{cases} a_{h-1,1} + 1, & \text{for } j = 1; \\ a_{h-1,j} - a_{h-2,j-1}, & \text{for } j = 2, 3, \ldots, \left\lceil \frac{h+1}{2} \right\rceil. \end{cases} \tag{12}$$

In Table 1 we list the first numbers of the coefficients $a_{h,j}$ for some fixed values of $h \geq 1$ and $j \geq 1$. On the diagonals, it is possible to observe a similarity with the A112467 sequence in The On-line Encyclopedia of Integer Sequences [20].

We have an explicit formula for the coefficients $a_{h,j}$ thanks to the following proposition.

Proposition 6. *For $h \geq 2$ and for $j = 1, 2, \ldots, \left\lceil \frac{h+1}{2} \right\rceil$ we have*

$$a_{h,j} = \frac{3j - h - 2}{j} \binom{h - j + 1}{j - 1} (-1)^j. \tag{13}$$

$h \backslash j$	1	2	3	4	5	6	7	8
1	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0
3	2	1	0	0	0	0	0	0
4	3	0	−1	0	0	0	0	0
5	4	−2	−2	0	0	0	0	0
6	5	−5	−2	1	0	0	0	0
7	6	−9	0	3	0	0	0	0
8	7	−14	5	5	−1	0	0	0
9	8	−20	14	5	−4	0	0	0
10	9	−27	28	0	−9	1	0	0
11	10	−35	48	−14	−14	5	0	0
12	11	−44	75	−42	−14	14	−1	0
13	12	−54	110	−90	0	28	−6	0
14	13	−65	154	−165	42	42	−20	1

Table 1: The coefficients $a_{h,j}$ for some fixed values of h and j.

Proof. We proceed by induction on h. For $h = 2$, it is $j = 1, 2$, and expression (13) gives $a_{2,1} = 1$ and $a_{2,2} = 1$, agreeing with the expression for $f_2(x) = \frac{1}{1-x-x^2}$ derived from (7) and $f_1(x) = 1 + x$.

For $h > 2$, we first analyze the case $j = 1$. Using $a_{h,1} = a_{h-1,1} + 1$ from (12) and the inductive hypothesis, we have

$$a_{h,1} = a_{h-1,1} + 1 = (2-h)(-1)^1 + 1 = h - 1$$

which matches the value of $a_{h,1}$ returned by (13).

For $j > 2$, we use $a_{h,j} = a_{h-1,j} - a_{h-2,j-1}$ from (12) and, again, the inductive hypothesis. We get

$$\begin{aligned} a_{h,j} &= a_{h-1,j} - a_{h-2,j-1} \\ &= \frac{3j-h-1}{j}\binom{h-j}{j-1}(-1)^j - \frac{3j-h-3}{j-1}\binom{h-j}{j-2}(-1)^{j-1} \\ &= \frac{3j-h-1}{j}\binom{h-j}{j-1}(-1)^j + \frac{3j-h-3}{j-1}\binom{h-j}{j-2}(-1)^j. \end{aligned}$$

Expanding the binomial coefficients and with some manipulations, it is

$$a_{h,j} = \frac{(-1)^j(h-j+1)!(3j-h-2)}{j(j-1)!(h-2j+2)!} = \frac{3j-h-2}{j}\binom{h-j+1}{j-1}(-1)^j,$$

as required. The proof is completed. □

In the sequel, we are going to evaluate a recurrence relation for the terms $D_n^{(h,2)}$ involving the series expansion at $x=0$ of the generating function

$$f_h(x) = \frac{p_h(x)}{q_h(x)} = \frac{q_{h-1}(x)}{q_h(x)} = \sum_{n\geq 0} D_n^{(h,2)} x^n.$$

The expression for $q_h(x)$ becomes

$$q_h(x) = 1 - \sum_{j=1}^{\lceil \frac{h+1}{2} \rceil} \frac{3j-h-2}{j} \binom{h-j+1}{j-1} (-1)^j x^j. \tag{14}$$

Thus, we obtain

$$f_h(x) = \frac{1 - \displaystyle\sum_{j=1}^{\lceil \frac{h}{2} \rceil} \frac{3j-h-1}{j} \binom{h-j}{j-1} (-1)^j x^j}{1 - \displaystyle\sum_{j=1}^{\lceil \frac{h+1}{2} \rceil} \frac{3j-h-2}{j} \binom{h-j+1}{j-1} (-1)^j x^j} \tag{15}$$

and

$$\left(1 - \sum_{j=1}^{\lceil \frac{h+1}{2} \rceil} \frac{3j-h-2}{j} \binom{h-j+1}{j-1} (-1)^j x^j\right) \left(\sum_{n\geq 0} D_n^{(h,2)} x^n\right) =$$
$$1 - \sum_{j=1}^{\lceil \frac{h}{2} \rceil} \frac{3j-h-1}{j} \binom{h-j}{j-1} (-1)^j x^j.$$

Sorting the first part according to the increasing powers of x we have

$$\sum_{n\geq 0} \left(D_n^{(h,2)} - \sum_{j=1}^{\lceil \frac{h+1}{2} \rceil} D_{n-j}^{(h,2)} \frac{3j-h-2}{j} \binom{h-j+1}{j-1} (-1)^j \right) x^n =$$
$$1 - \sum_{j=1}^{\lceil \frac{h}{2} \rceil} \frac{3j-h-1}{j} \binom{h-j}{j-1} (-1)^j x^j$$

where $D_\ell^{(h,2)} = 0$ whenever $\ell \leq 0$.

For the identity theorem for polynomials we can deduce the desired recurrence relation

$$D_n^{(h,2)} = \begin{cases} 1, & \text{for } n=0; \\ \displaystyle\sum_{j=1}^{\lceil \frac{h+1}{2} \rceil} D_{n-j}^{(h,2)} \frac{3j-h-2}{j} \binom{h-j+1}{j-1} (-1)^j - \frac{3n-h-1}{n} \binom{h-n}{n-1} (-1)^n, & \text{for } n\geq 1. \end{cases}$$

A very interesting note arises when, once h is fixed, we ask for the number $D_n^{(h,2)}$ of Dyck paths having semilength $n \leq h$. Clearly, in this case, it is $D_n^{(h,2)} = C_n$ since all the Dyck paths of a certain semilength $n \leq h$ have height at most equal to n. Thanks to the above argument it is possible to derive interesting relations involving Catalan numbers. Indeed, for the above remark, posing $h = n + \alpha$, we can write $D_n^{(n+\alpha,2)} = C_n$, where $\alpha \geq 0$ is integer. Then, it is possible to deduce a combinatorial identity involving Catalan numbers as follows:

$$C_n = \sum_{j=1}^{n} C_{n-j} \frac{3j - n - \alpha - 2}{j} \binom{n + \alpha - j + 1}{j - 1} (-1)^j - \frac{2n - \alpha - 1}{n} \binom{\alpha}{n - 1} (-1)^n. \quad (16)$$

6 Further developments

In this paper we analyzed the case $k = 2$ leading to bounded Dyck paths avoiding valleys at given height (i.e., $h - 1$) corresponding to the permutations in $\mathcal{S}_n^{(h,2)}(312)$. An interesting generalization could concern the cases $k > 2$ in order to investigate what are the arising constraints on the subclasses of 312-avoiding permutations. The number $k - 1$ of consecutive valleys allowed at height $h - 1$ clearly affects the value and position of the l.r.M., as we have seen in the $k = 2$ case. For values of k larger than 2, the permutations probably have a structure that can be described in terms of a suitable block decomposition.

The above combinatorial identity (16) is obtained by means of a purely combinatorial consideration. By virtue of this, similar relations are expected to arise even in cases $k > 2$. It might then be possible to derive a family of combinatorial identities as k varies.

Another further line of research could consider the possibility to list the paths of $\mathcal{D}_n^{(h,2)}$ in a Gray code sense using the tools developed by Barcucci et al. [1] and Bernini et al. [4, 5, 6, 7]. As mentioned in Section 2, these paths can be encoded by strings on the alphabet $\{U, D\}$, so the problem of defining a Gray code could be addressed by starting from the techniques developed by Vajnovszki et al. [21].

Moreover, the considered Dyck paths could be used for the construction of the strong non-overlapping code proposed by Barcucci et al. [2].

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