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Boundedness of functions in fractional Orlicz–Sobolev spaces

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A necessary and sufficient condition for fractional Orlicz–Sobolev spaces to be continuously embedded into $L^\infty(\mathbb{R}^n)$ is exhibited. Under the same assumption, any function from the relevant fractional-order spaces is shown to be continuous. Improvements of this result are also offered. They provide the optimal Orlicz target space, and the optimal rearrangement-invariant target space in the embedding in question. These results complement those already available in the subcritical case, where the embedding into $L^\infty(\mathbb{R}^n)$ fails. They also augment a classical embedding theorem for standard fractional Sobolev spaces.

1. Introduction

Fractional-order Orlicz–Sobolev spaces are associated with a positive non-integer smoothness parameter s, and with a Young function A which dictates a degree of integrability. They generalize the Gagliardo-Slobodeckii fractional Sobolev spaces, independently introduced in [36,52], and are defined in terms of a Luxemburg type seminorm $|\cdot|_{s,A,\mathbb{R}^n}$.

When $s \in (0,1)$, the seminorm $|\cdot|_{s,A,\mathbb{R}^n}$ is built upon the functional defined as

$$J_{s,A}(u) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n} \tag{1.1}$$

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for a measurable function u in $\mathbb{R}^n$. The space $V_d^{s,A}(\mathbb{R}^n)$ of those functions u decaying near infinity in a weakest possible sense and such that $|u|_{s,A,\mathbb{R}^n} < \infty$ will be considered. If $s \in (1,\infty) \setminus \mathbb{N}$, then the space $V_d^{s,A}(\mathbb{R}^n)$ consists of all functions u, whose weak derivatives up to the order $[s]$ decay near infinity, for which $\left|\nabla^{[s]}u\right|_{\{s\},A,\mathbb{R}^n} < \infty$. Here, $\nabla^{[s]}u$ denotes the vector of all weak derivatives of u of order $[s]$, the integer part of s, and $\{s\} = s - [s]$, the fractional part of s. In particular, $\nabla^{[s]}u = \nabla^0 u = u$ if $s \in (0,1)$. Precise definitions can be found in Section 2, where the necessary background is collected. Customary fractional Sobolev spaces, which will be denoted by $V_d^{s,p}(\mathbb{R}^n)$, are recovered via the choice $A(t) = t^p$, with $p \geq 1$.

As a part of a wealth of investigations on linear and nonlinear nonlocal equations of elliptic and parabolic type, the analysis of nonlocal problems driven by possibly non-polynomial type nonlinearities has recently started attracting the attention of researchers [6,7,39,51].

A sound theory of fractional Orlicz–Sobolev spaces – a natural functional framework for these problems – is of course crucial in connection with their study. Properties of fractional Orlicz–Sobolev spaces are the subject of the contributions [2–5,10,16,31,39]. They provide extensions of some aspects of the theory of the classical fractional Sobolev spaces, which has been developed over the years — see e.g. [8,11–15,17–19,21,22,27–30,32–35,38,40–43,45–50,54,55].

In particular, as in the case of integer-order spaces, embedding theorems are a central issue also in the fractional-order case. Optimal embeddings for the fractional Orlicz–Sobolev space $V_d^{s,A}(\mathbb{R}^n)$, of the form

$$V_d^{s,A}(\mathbb{R}^n) \to Y(\mathbb{R}^n), \tag{1.2}$$

where $Y(\mathbb{R}^n)$ is an Orlicz space, or, more generally, a rearrangement-invariant space, were established in [3]. Here, and in what follows, the arrow "→" stands for continuous embedding. The results of [3] deal with every

$$s \in (0,n) \setminus \mathbb{N}, \tag{1.3}$$

in the "subcritical" growth regime for A near infinity dictated by the condition

$$\int^{\infty} \left(\frac{t}{A(t)}\right)^{\frac{s}{n-s}} dt = \infty. \tag{1.4}$$

Note that, when $A(t) = t^p$ near infinity, condition (1.4) amounts to assuming that $1 \leq p \leq \frac{n}{s}$.

The present contribution is focused on the validity of embeddings (1.2) in the complementary "supercritical" growth dominion, corresponding to orders of smoothness s satisfying (1.3) and Young functions growing so fast near infinity that

$$\int^{\infty} \left(\frac{t}{A(t)}\right)^{\frac{s}{n-s}} dt < \infty. \tag{1.5}$$

We emphasize that restriction (1.3) is indispensable when embeddings of the form (1.2) are in question. In fact, our discussion begins by showing that, whatever the rearrangement-invariant target space $Y(\mathbb{R}^n)$ is, embedding (1.2) can only hold provided that s is as in (1.3) and A decays so slowly near zero that

$$\int_0 \left(\frac{t}{A(t)}\right)^{\frac{s}{n-s}} dt < \infty. \tag{1.6}$$

These assumptions will thus be kept in force throughout.

The core of this paper is a result asserting that condition (1.5) is necessary and sufficient for the space $V_d^{s,A}(\mathbb{R}^n)$ to be embedded into $L^\infty(\mathbb{R}^n)$, namely for (1.2) to hold with $Y(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. The same condition also turns out to be equivalent to the embedding of $V_d^{s,A}(\mathbb{R}^n)$ into the space of continuous functions. In particular, these conclusions provide us with an embedding into $L^\infty(\mathbb{R}^n)$ in case of Young functions which behave like t^p near infinity, with $p > \frac{n}{s}$. By contrast, no embedding as in (1.2) can hold for these values of p if $A(t) = t^p$ globally, namely for the standard homogeneous space $V_d^{s,p}(\mathbb{R}^n)$, because of the failure of condition (1.6). Such an embedding is classically restored, provided that the homogeneous space is replaced by its non-homogeneous version, which consists of those functions in $V^{s,p}(\mathbb{R}^n)$ all of whose derivatives up to the order $[s]$ belong to $L^p(\mathbb{R}^n)$.

Interestingly, condition (1.5) exactly matches an analogous necessary and sufficient condition for embeddings into $L^\infty(\mathbb{R}^n)$ of integer-order Orlicz–Sobolev spaces (see [20,23,44,53] for first-order spaces, and [26] for the higher-order case), which is reproduced by just setting s equal to the order of the latter spaces in (1.5).

Although fundamental, these conclusions merely provide information on "local" properties of functions in $V^{s,A}_d(\mathbb{R}^n)$. Since these functions are defined on the entire Euclidean space $\mathbb{R}^n$ – a domain with infinite Lebesgue measure – their integrability properties "near infinity" are also relevant. With this regard, results parallel to those obtained under assumption (1.4) in [3] are also offered. They provide us with the optimal Orlicz target space and the optimal rearrangement-invariant target space for embedding (1.2) to hold when condition (1.5) is current.

The results outlined so far are weaved with a general reduction principle for embedding (1.2), of independent interest, which is also established. This principle applies irrespective of whether (1.4) or (1.5) holds, and informs us about the equivalence of any embedding of this kind to a one-dimensional Hardy type inequality.

The approach to embeddings (1.2) exploited in [3] under assumption (1.4) relies upon an extension argument to a half-space in $\mathbb{R}^{n+1}$. It enables one to derive a Hardy type inequality, from which subcritical embeddings for $V^{s,A}_d(\mathbb{R}^n)$ into optimal Orlicz and rearrangement-invariant spaces follow. This method does not seem to be adaptable to deduce optimal conclusions when the opposite condition (1.5) is in force.

We have instead to resort to a strategy which consists in deriving optimal supercritical embeddings from subcritical ones. This technique can be developed thanks to embeddings available for Orlicz–Sobolev spaces built upon arbitrary subcritical Young functions, those of power type not being sufficient. In particular, unlike the usual argument exploited for classical non-homogeneous fractional Sobolev spaces, such an approach avoids embeddings into Campanato type spaces as an intermediate step. This is a critical point, since, although sharp embeddings of this type can be obtained [5], their use does not yield the optimal criterion for embeddings into $L^\infty(\mathbb{R}^n)$ mentioned above.

2. Background

Here, we recall basic definitions and classical properties concerning the function spaces involved in our discussion, as well as fractional Orlicz–Sobolev embeddings in the subcritical setting.

2.1. Orlicz spaces and rearrangement-invariant spaces

A function $A : [0,\infty) \to [0,\infty]$ is called a *Young function* if it is convex, non-constant, left-continuous and $A(0) = 0$. Any function enjoying these properties admits the representation

$$A(t) = \int_0^t a(\tau)d\tau \quad \text{for } t \geq 0 \tag{2.1}$$

for some non-decreasing, left-continuous function $a : [0,\infty) \to [0,\infty]$ which is neither identically equal to 0 nor to ∞. One has that

$$kA(t) \leq A(kt) \quad \text{for } k \geq 1 \text{ and } t \geq 0. \tag{2.2}$$

The *Young conjugate* $\widetilde{A}$ of A is the Young function obeying

$$\widetilde{A}(t) = \sup\{\tau t - A(\tau) : \tau \geq 0\} \quad \text{for } t \geq 0. \tag{2.3}$$

On denoting by a^{-1} the left-continuous (generalized) inverse of the function a appearing in (2.1), the following formula holds:

$$\widetilde{A}(t) = \int_0^t a^{-1}(\tau)d\tau \quad \text{for } t \geq 0. \tag{2.4}$$

A Young function A is said to *dominate* another Young function B *globally* [resp. *near zero*] [resp. *near infinity*] if there exist positive constants c and t_0 such that

$$B(t) \leq A(ct) \quad \text{for } t \geq 0 \quad [\text{for } 0 \leq t \leq t_0] \quad [\text{for } t \geq t_0]. \tag{2.5}$$

The functions A and B are called *equivalent globally*, or *near zero*, or *near infinity*, if they dominate each other in the respective range of values of their argument.

We shall write $B \lesssim A$ to denote that A dominates B, and $A \simeq B$ to denote that A is equivalent to B.

By contrast, the relation "$\approx$" between two expressions will be used to denote that they are bounded by each other, up to positive multiplicative constants depending on appropriate quantities.

The decay near 0 of a Young function A can be compared with that of a power function via its *Matuszewska–Orlicz index at zero*, defined as

$$I_0(A) = \lim_{\lambda \to 0^+} \frac{\log \lambda}{\log\Big(\limsup_{t \to 0^+} \frac{A^{-1}(\lambda t)}{A^{-1}(t)}\Big)}. \tag{2.6}$$

If A vanishes only at 0, then the following alternative expression for $I_0(A)$ holds:

$$I_0(A) = \lim_{\lambda \to \infty} \frac{\log\Big(\limsup_{t \to 0^+} \frac{A(\lambda t)}{A(t)}\Big)}{\log \lambda}. \tag{2.7}$$

Let us set

$$\mathcal{M}(\mathbb{R}^n) = \{u : \mathbb{R}^n \to \mathbb{R} : u \text{ is measurable}\}, \tag{2.8}$$

and define $\mathcal{M}_d(\mathbb{R}^n)$ as the subset of $\mathcal{M}(\mathbb{R}^n)$ of those functions u that decay near infinity, in the sense that all their level sets $\{|u| > t\}$ have finite Lebesgue measure for $t > 0$. Namely,

$$\mathcal{M}_d(\mathbb{R}^n) = \{u \in \mathcal{M}(\mathbb{R}^n) : |\{|u| > t\}| < \infty \text{ for every } t > 0\}, \tag{2.9}$$

where $|E|$ stands for the Lebesgue measure of a set $E \subset \mathbb{R}^n$.

The *Orlicz space* $L^A(\mathbb{R}^n)$, built upon a Young function A, is the Banach space of those functions $u \in \mathcal{M}(\mathbb{R}^n)$ making the *Luxemburg norm*

$$\|u\|_{L^A(\mathbb{R}^n)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} A\left(\frac{|u|}{\lambda}\right) dx \leq 1\right\} \tag{2.10}$$

finite. In particular, $L^A(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ if $A(t) = t^p$ for some $p \in [1, \infty)$, and $L^A(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$ if $A(t) = 0$ for $t \in [0, 1]$ and $A(t) = \infty$ for $t \in (1, \infty)$.

A version of Hölder's inequality in Orlicz spaces tells us that

$$\int_{\mathbb{R}^n} |uv|\, dx \leq 2\|u\|_{L^A(\mathbb{R}^n)} \|v\|_{L^{\widetilde{A}}(\mathbb{R}^n)} \tag{2.11}$$

for every $u \in L^A(\mathbb{R}^n)$ and $v \in L^{\widetilde{A}}(\mathbb{R}^n)$. Moreover,

$$\|v\|_{L^{\widetilde{A}}(\mathbb{R}^n)} \leq \sup_{u \in L^A(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} |uv|\, dx}{\|u\|_{L^A(\mathbb{R}^n)}}. \tag{2.12}$$

If A dominates B globally, then

$$\|u\|_{L^B(\mathbb{R}^n)} \leq c\|u\|_{L^A(\mathbb{R}^n)} \tag{2.13}$$

for every $u \in L^A(\mathbb{R}^n)$, where c is the constant appearing in inequality (2.5) (with $t_0 = 0$). Thus, if A is equivalent to B globally, then $L^A(\mathbb{R}^n) = L^B(\mathbb{R}^n)$, up to equivalent norms.

The Orlicz–Lorentz spaces provide us with a family of function spaces which generalizes the Orlicz spaces. Given a Young function A and a number $q \in \mathbb{R} \setminus \{0\}$, we denote by $L(A,q)(\mathbb{R}^n)$ the *Orlicz–Lorentz space* of those functions $u \in \mathcal{M}(\mathbb{R}^n)$ for which the quantity

$$\|u\|_{L(A,q)(\mathbb{R}^n)} = \left\|r^{-\frac{1}{q}} u^*(r)\right\|_{L^A(0,\infty)} \tag{2.14}$$

is finite. Under suitable assumptions on A and q, this quantity is a norm, which renders $L(A,q)(\mathbb{R}^n)$ a (non-trivial) Banach space. This is the case, for instance, if $q > 1$ and

$$\int^\infty \frac{A(t)}{t^{1+q}}\, dt < \infty\,, \tag{2.15}$$

see [25, Proposition 2.1].

The Orlicz spaces and the Orlicz–Lorentz spaces are special instances of rearrangement-invariant spaces, whose definition rests upon that of decreasing rearrangement. Recall that the *decreasing rearrangement* u^* of a function $u \in \mathcal{M}(\mathbb{R}^n)$ is given by

$$u^*(r) = \inf\{t \geq 0 : |\{x \in \mathbb{R}^n : |u(x)| > t\}| \leq r\} \quad \text{for } r \geq 0. \tag{2.16}$$

In other words, u^* is the (unique) non-increasing, right-continuous function from $[0,\infty)$ into $[0,\infty]$ which is equidistributed with u.

A *rearrangement-invariant space* is a Banach function space $X(\mathbb{R}^n)$, in the sense of Luxemburg [9, Chapter 1, Section 1], such that

$$\|u\|_{X(\mathbb{R}^n)} = \|v\|_{X(\mathbb{R}^n)} \quad \text{whenever } u^* = v^*. \tag{2.17}$$

The *representation space* $\overline{X}(0,\infty)$ of a rearrangement-invariant space $X(\mathbb{R}^n)$ is defined as the unique rearrangement-invariant space on $(0,\infty)$ such that

$$\|u\|_{X(\mathbb{R}^n)} = \|u^*\|_{\overline{X}(0,\infty)} \tag{2.18}$$

for every $u \in X(\mathbb{R}^n)$.

A basic property tells us that, if $X(\mathbb{R}^n)$ and $Y(\mathbb{R}^n)$ are rearrangement-invariant spaces, then

$$X(\mathbb{R}^n) \subset Y(\mathbb{R}^n) \quad \text{if and only if} \quad X(\mathbb{R}^n) \to Y(\mathbb{R}^n). \tag{2.19}$$

2.2. Fractional Orlicz-Sobolev spaces

Given $m \in \mathbb{N}$ and a Young function A, we denote by $V^{m,A}(\mathbb{R}^n)$ the integer-order *homogeneous Orlicz–Sobolev space* given by

$$V^{m,A}(\mathbb{R}^n) = \{u \in \mathcal{M}(\mathbb{R}^n) : u \text{ is } m\text{-times weakly differentiable in } \mathbb{R}^n \text{ and } |\nabla^m u| \in L^A(\mathbb{R}^n)\}. \tag{2.20}$$

The functional

$$\|\nabla^m u\|_{L^A(\mathbb{R}^n)}$$

defines a seminorm on the space $V^{m,A}(\mathbb{R}^n)$.

As for fractional-order spaces, given $s \in (0,1)$, the seminorm $|u|_{s,A,\mathbb{R}^n}$ is defined as

$$|u|_{s,A,\mathbb{R}^n} = \inf\left\{\lambda > 0 : J_{s,A}\left(\frac{u}{\lambda}\right) \leq 1\right\} \tag{2.21}$$

for $u \in \mathcal{M}(\mathbb{R}^n)$, where $J_{s,A}$ is the functional given by (1.1). The *homogeneous fractional Orlicz–Sobolev space* $V^{s,A}(\mathbb{R}^n)$ is defined as

$$V^{s,A}(\mathbb{R}^n) = \left\{ u \in \mathcal{M}(\mathbb{R}^n) : |u|_{s,A,\mathbb{R}^n} < \infty \right\}. \tag{2.22}$$

The definitions of the seminorm $|u|_{s,A,\mathbb{R}^n}$ and of the space $V^{s,A}(\mathbb{R}^n)$ carry over to vector-valued functions u just by replacing the absolute value of $u(x) - u(y)$ by the Euclidean norm of the same expression in the definition of the functional $J_{s,A}$.

The subspace of those functions in $V^{s,A}(\mathbb{R}^n)$ which decay near infinity is denoted by $V^{s,A}_d(\mathbb{R}^n)$. Thus,

$$V^{s,A}_d(\mathbb{R}^n) = \{ u \in V^{s,A}(\mathbb{R}^n) : |\{|u| > t\}| < \infty \text{ for every } t > 0 \}. \tag{2.23}$$

The definition of $V^{s,A}(\mathbb{R}^n)$ is extended to all $s \in (0,\infty) \setminus \mathbb{N}$ as follows. On denoting, as above, by $[s]$ and $\{s\}$ the integer and the fractional part of s, respectively, we define

$$V^{s,A}(\mathbb{R}^n) = \{ u \in \mathcal{M}(\mathbb{R}^n) : u \text{ is } [s]\text{-times weakly differentiable in } \mathbb{R}^n \text{ and } \nabla^{[s]} u \in V^{\{s\},A}(\mathbb{R}^n) \}. \tag{2.24}$$

In analogy with (2.23), for every $s \in (0,\infty) \setminus \mathbb{N}$ we set

$$V^{s,A}_d(\mathbb{R}^n) = \{ u \in V^{s,A}(\mathbb{R}^n) : |\{|\nabla^k u| > t\}| < \infty \text{ for every } t > 0 \text{ and for } k = 0, 1, \dots, [s] \}. \tag{2.25}$$

If s and A fulfill conditions (1.3) and (1.6), then the functional $\left|\nabla^{[s]} u\right|_{\{s\},A,\mathbb{R}^n}$ defines a norm on the space $V^{s,A}_d(\mathbb{R}^n)$, and the latter, equipped with this norm, is a Banach space. This is the content of Proposition 6.1, Section 6.

When the subcritical growth condition (1.4) is in force, optimal embeddings for $V^{s,A}_d(\mathbb{R}^n)$ take the following form. The optimal Orlicz target space for $V^{s,A}_d(\mathbb{R}^n)$ is built upon the Young function $A_{\frac{n}{s}}$ defined, for $n \in \mathbb{N}$ and $s \in (0,n) \setminus \mathbb{N}$, as

$$A_{\frac{n}{s}}(t) = A(H^{-1}(t)) \quad \text{for } t \geq 0, \tag{2.26}$$

where

$$H(t) = \left(\int_0^t \left(\frac{\tau}{A(\tau)} \right)^{\frac{s}{n-s}} d\tau \right)^{\frac{n-s}{n}} \qquad \text{for } t \geq 0. \tag{2.27}$$

Indeed, [3, Theorems 6.1 and 7.1] tell us that

$$V^{s,A}_d(\mathbb{R}^n) \to L^{A_{\frac{n}{s}}}(\mathbb{R}^n), \tag{2.28}$$

and there exists a constant c such that

$$\|u\|_{L^{A_{\frac{n}{s}}}(\mathbb{R}^n)} \leq c \left|\nabla^{[s]} u\right|_{\{s\},A,\mathbb{R}^n} \tag{2.29}$$

for every $u \in V^{s,A}_d(\mathbb{R}^n)$. Moreover, the space $L^{A_{\frac{n}{s}}}(\mathbb{R}^n)$ is the smallest possible target in (2.28) in the class of all Orlicz spaces.

The target space $L^{A_{\frac{n}{s}}}(\mathbb{R}^n)$ in embedding (2.28) can yet be enhanced if the realm of Orlicz spaces is abandoned, and all rearrangement-invariant spaces are allowed. Specifically, the Orlicz–Lorentz space $L(\widehat{A}, \frac{n}{s})(\mathbb{R}^n)$ comes into play, and is endowed with the norm defined, according to Eq. (2.14), by

$$\|u\|_{L(\widehat{A},\frac{n}{s})(\mathbb{R}^n)} = \left\| r^{-\frac{s}{n}} u^*(r) \right\|_{L^{\widehat{A}}(0,\infty)} \tag{2.30}$$

for $u \in \mathcal{M}(\mathbb{R}^n)$, where $\widehat{A}$ is the Young function given by

$$\widehat{A}(t) = \int_0^t \widehat{a}(\tau)\, d\tau \quad \text{for } t \geq 0, \tag{2.31}$$

and

$$\widehat{a}^{\,-1}(t) = \left(\int_{a^{-1}(t)}^{\infty} \left(\int_0^{\tau} \left(\frac{1}{a(\theta)}\right)^{\frac{s}{n-s}} d\theta\right)^{-\frac{n}{s}} \frac{d\tau}{a(\tau)^{\frac{n}{n-s}}}\right)^{\frac{s}{s-n}} \quad \text{for } t \geq 0. \tag{2.32}$$

One has that

$$V_d^{s,A}(\mathbb{R}^n) \to L(\widehat{A}, \tfrac{n}{s})(\mathbb{R}^n), \tag{2.33}$$

and there exists a constant c such that

$$\|u\|_{L(\widehat{A},\frac{n}{s})(\mathbb{R}^n)} \leq c\big|\nabla^{[s]}u\big|_{\{s\},A,\mathbb{R}^n} \tag{2.34}$$

for every $u \in V_d^{s,A}(\mathbb{R}^n)$. Furthermore, the space $L(\widehat{A}, \frac{n}{s})(\mathbb{R}^n)$ in embedding (2.33) is the smallest possible among all rearrangement-invariant spaces.

Hence,

$$L(\widehat{A}, \tfrac{n}{s})(\mathbb{R}^n) \to L^{A_{\frac{n}{s}}}(\mathbb{R}^n)\,. \tag{2.35}$$

By [3, Proposition 4.1], the norm of embedding (2.35) depends only on $\frac{n}{s}$.

In particular, the proof of the optimality of embeddings (2.28) and (2.33) rests upon the fact that, given any rearrangement-invariant space $Y(\mathbb{R}^n)$,

$$\text{if } V_d^{s,A}(\mathbb{R}^n) \to Y(\mathbb{R}^n), \text{ then } \quad \left\| \int_r^{\infty} f(\rho)\rho^{-1+\frac{s}{n}}\, d\rho \right\|_{\overline{Y}(0,\infty)} \leq c\|f\|_{L^A(0,\infty)} \text{ for every } f \in L^A(0,\infty), \tag{2.36}$$

for some constant c.

Property (2.36) is proved in [3, Lemmas 6.5 and 7.6]. It is one of the two implications of the reduction principle contained in Theorem 3.7, Section 3, the novelty in that theorem being the reverse one.

3. Main results

As a preliminary for the embeddings of $V_d^{s,A}(\mathbb{R}^n)$ to be offered, we state the necessity of conditions (1.3) and (1.6) on s and A.

Theorem 3.1 (*Admissible s and A*)**.** *Let $s \in (0,\infty)\setminus\mathbb{N}$ and let A be a Young function. Assume that embedding (1.2) holds for some rearrangement-invariant space $Y(\mathbb{R}^n)$. Then, s and A fulfill conditions (1.3) and (1.6).*

Having clarified the indispensability of assumptions (1.3) and (1.6), we are ready to state our first main result. It tells us that the supercritical growth condition (1.5) is necessary and sufficient for the space $V_d^{s,A}(\mathbb{R}^n)$ to be continuously embedded in $L^\infty(\mathbb{R}^n)$. The relevant condition turns also to be equivalent to the embedding of $V_d^{s,A}(\mathbb{R}^n)$ into the space $C^0(\mathbb{R}^n)$ of continuous functions in $\mathbb{R}^n$, equipped with the standard norm. The fact that a function in $V_d^{s,A}(\mathbb{R}^n)$ belongs to $C^0(\mathbb{R}^n)$ has to be interpreted, as usual, in the sense that u agrees a.e. in $\mathbb{R}^n$ with a continuous function in $\mathbb{R}^n$.

Theorem 3.2 (*Embeddings into $L^\infty(\mathbb{R}^n)$ and $C^0(\mathbb{R}^n)$*)**.** *Assume that s and A satisfy conditions (1.3) and (1.6). Then, the following statements are equivalent:*

(i) Condition (1.5) holds;

(ii) The embedding

$$V_d^{s,A}(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n) \tag{3.1}$$

holds;

(iii) The embedding

$$V_d^{s,A}(\mathbb{R}^n) \to C^0(\mathbb{R}^n) \tag{3.2}$$

holds.

Moreover, if condition (1.5) is in force, then there exists a constant c such that

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq c \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A\left(\frac{|\nabla^{[s]}u(x) - \nabla^{[s]}u(y)|}{|x-y|^{\{s\}}} \right) \frac{dxdy}{|x-y|^n} \right)^{\frac{s}{n}} \tag{3.3}$$

for every $u \in V_d^{s,A}(\mathbb{R}^n)$.

The norm in $L^\infty(\mathbb{R}^n)$, regarded as an Orlicz space, is, loosely speaking, the strongest one "locally" in $\mathbb{R}^n$, but the weakest one "near infinity". Indeed, membership of a function in $L^\infty(\mathbb{R}^n)$ does not entail any information on the behavior of the measure of its level sets when the levels approach zero. Under the same assumption (1.5), the next result augments embedding (3.1), and provides us with the optimal Orlicz target space on $\mathbb{R}^n$ into which the space $V_d^{s,A}(\mathbb{R}^n)$ is continuously embedded.

The Orlicz space in question is built upon the Young function $A_{\frac{n}{s}}$ defined as in (2.26)–(2.27) for the subcritical embedding (2.28). The novelty is that, since we are now assuming that condition (1.5) holds, the function H^{-1} has to be interpreted as the generalized left-continuous inverse of H. In particular,

$$H^{-1}(t) = \infty \quad \text{for} \quad t > \left(\int_0^\infty \left(\frac{\tau}{A(\tau)} \right)^{\frac{s}{n-s}} d\tau \right)^{\frac{n-s}{n}}, \tag{3.4}$$

and

$$A_{\frac{n}{s}}(t) = \infty \tag{3.5}$$

for t as in (3.4). In particular,

$$L^{A_{\frac{n}{s}}}(\mathbb{R}^n) \subsetneq L^\infty(\mathbb{R}^n). \tag{3.6}$$

Theorem 3.3 (*Optimal Orlicz Target Space*). *Assume that s and A satisfy conditions (1.3), (1.5) and (1.6). Let* $A_{\frac{n}{s}}$ *be the Young function defined by (2.26)–(2.27). Then,*

$$V_d^{s,A}(\mathbb{R}^n) \to L^{A_{\frac{n}{s}}}(\mathbb{R}^n), \tag{3.7}$$

and there exists a constant c such that

$$\|u\|_{L^{A_{\frac{n}{s}}}(\mathbb{R}^n)} \leq c \left|\nabla^{[s]}u\right|_{\{s\},A,\mathbb{R}^n} \tag{3.8}$$

for every $u \in V_d^{s,A}(\mathbb{R}^n)$. *Moreover, the space* $L^{A_{\frac{n}{s}}}(\mathbb{R}^n)$ *is optimal among all Orlicz target spaces in (3.7).*

Let us point out that, in view of property (3.5), only the asymptotic behavior of the function $A_{\frac{n}{s}}$ near zero has to be detected in identifying the space $L^{A_{\frac{n}{s}}}(\mathbb{R}^n)$ in concrete applications.

As recalled above, although it is just stated in the supercritical regime (1.5) focused in this paper, Theorem 3.3 also holds in the complementary situation when condition (1.4) is in force.

The embedding provided by the next result stands to embedding (3.7) as embedding (2.33) stands to embedding (2.28). Actually, it tells us that embedding (3.7) can still be improved, provided that the class of admissible target spaces is further broadened as to include all rearrangement-invariant spaces. The optimal target in this class for embeddings of the space $V_d^{s,A}(\mathbb{R}^n)$ can be obtained as the intersection of $L^\infty(\mathbb{R}^n)$ with the Orlicz–Lorentz space appearing in (2.33).

Theorem 3.4 (*Optimal Rearrangement-invariant Target Space*). *Assume that s and A satisfy conditions (1.3), (1.5) and (1.6). Let* $\widehat{A}$ *be the Young function associated with A as in (2.31)–(2.32). Then,*

$$V_d^{s,A}(\mathbb{R}^n) \to \big(L^\infty \cap L(\widehat{A}, \tfrac{n}{s})\big)(\mathbb{R}^n), \tag{3.9}$$

and there exists a constant c such that

$$\|u\|_{\left(L^\infty\cap L(\widehat{A},\frac{n}{s})\right)(\mathbb{R}^n)} \le c\left|\nabla^{[s]}u\right|_{\{s\},A,\mathbb{R}^n} \tag{3.10}$$

for every $u \in V_d^{s,A}(\mathbb{R}^n)$*. Moreover, the space* $\left(L^\infty \cap L\left(\widehat{A}, \frac{n}{s}\right)\right)(\mathbb{R}^n)$ *is optimal among all rearrangement-invariant target spaces in (3.9).*

Remark 3.5. Recall that a customary norm in $\left(L^\infty \cap L(\widehat{A}, \frac{n}{s})\right)(\mathbb{R}^n)$ is defined as

$$\|u\|_{L^\infty(\mathbb{R}^n)} + \|u\|_{L(\widehat{A},\frac{n}{s})(\mathbb{R}^n)} \tag{3.11}$$

for $u \in \mathcal{M}(\mathbb{R}^n)$. It turns out that, because of the presence of the norm in $L^\infty(\mathbb{R}^n)$, only the decay of the function $\widehat{A}$ at zero is relevant in the definition of $\left(L^\infty \cap L(\widehat{A}, \frac{n}{s})\right)(\mathbb{R}^n)$. This is apparent when making use of an equivalent norm in $\left(L^\infty \cap L(\widehat{A}, \frac{n}{s})\right)(\mathbb{R}^n)$, given by

$$\|u^*(r)\,\phi(r)\|_{L^{E_A}(0,\infty)} \tag{3.12}$$

for $u \in \mathcal{M}(\mathbb{R}^n)$, where E_A is a Young function such that

$$E_A(t) \simeq \begin{cases} \widehat{A}(t) & \text{near } 0 \\ \infty & \text{near infinity} \end{cases} \tag{3.13}$$

and the function $\phi : (0,\infty) \to [0,\infty)$ obeys

$$\phi(r) = \min\left\{1, r^{-\frac{s}{n}}\right\} \qquad \text{for } r > 0. \tag{3.14}$$

The equivalence of the norms (3.11) and (3.12) is established in [25, Proposition 2.1].

In this connection, let us also mention that $\widehat{A} \lesssim A$ in the sense of Young functions. Moreover,

$$\widehat{A}(t) \simeq A(t) \quad \text{near zero} \tag{3.15}$$

if and only if the upper Matuszewska–Orlicz index $I_0(A)$ of A at zero, defined by (2.6), satisfies

$$I_0(A) < \frac{n}{s}. \tag{3.16}$$

Hence, under assumption (3.16), the space $L\left(\widehat{A}, \frac{n}{s}\right)(\mathbb{R}^n)$ can be replaced by $L\left(A, \frac{n}{s}\right)(\mathbb{R}^n)$ in embedding (3.9); namely, one has that

$$V_d^{s,A}(\mathbb{R}^n) \to \left(L^\infty \cap L\left(A, \tfrac{n}{s}\right)\right)(\mathbb{R}^n). \tag{3.17}$$

Example 3.6. Consider the space $V_d^{s,A}(\mathbb{R}^n)$ associated with a Young function A such that

$$A(t) \simeq \begin{cases} t^{p_0}(\log\frac{1}{t})^{\alpha_0} & \text{near zero} \\ t^p(\log t)^\alpha & \text{near infinity.} \end{cases} \tag{3.18}$$

In order for A to be a Young function, the exponents appearing in Eq. (3.18) are such that either $p_0 > 1$ and $\alpha_0 \in \mathbb{R}$, or $p_0 = 1$ and $\alpha_0 \le 0$, and either $p > 1$ and $\alpha \in \mathbb{R}$ or $p = 1$ and $\alpha \ge 0$.

Let $s \in (0,n)\setminus\mathbb{N}$. The function A satisfies the necessary condition (1.6) from Theorem 3.1 provided that

$$\text{either } 1 \le p_0 < \frac{n}{s} \text{ and } \alpha_0 \in \mathbb{R}, \text{ or } p_0 = \frac{n}{s} \text{ and } \alpha_0 > \frac{n}{s} - 1.$$

Moreover, the supercritical growth assumption (1.5) amounts to requiring that

$$\text{either } p = \frac{n}{s} \text{ and } \alpha > \frac{n}{s} - 1, \text{ or } p > \frac{n}{s} \text{ and } \alpha \in \mathbb{R}.$$

Under these assumptions, Theorem 3.3 tells us that embedding (3.7) and inequality (3.8) hold, with

$$A_{\frac{n}{s}}(t) = \infty \quad \text{near infinity,} \quad \text{and} \quad A_{\frac{n}{s}}(t) \simeq \begin{cases} t^{\frac{np_0}{n-sp_0}} (\log \frac{1}{t})^{\frac{n\alpha_0}{n-sp_0}} & \text{if } 1 \le p_0 < \frac{n}{s} \\ e^{-t^{-\frac{n}{s(\alpha_0+1)-n}}} & \text{if } p_0 = \frac{n}{s} \text{ and } \alpha_0 > \frac{n}{s} - 1 \end{cases} \quad \text{near zero.} \tag{3.19}$$

Moreover, the target space $L^{A_{\frac{n}{s}}}(\mathbb{R}^n)$ is optimal among all Orlicz spaces.

On the other hand, Theorem 3.4 implies that embedding (3.9) and inequality (3.10) hold, where $\widehat{A}$ is any Young function such that

$$\widehat{A}(t) \simeq \begin{cases} t^{p_0} (\log \frac{1}{t})^{\alpha_0} & \text{if } 1 \le p_0 < \frac{n}{s} \\ t^{\frac{n}{s}} (\log \frac{1}{t})^{\alpha_0 - \frac{n}{s}} & \text{if } p_0 = \frac{n}{s} \text{ and } \alpha_0 > \frac{n}{s} - 1 \end{cases} \quad \text{near zero.} \tag{3.20}$$

Moreover, the target space $\big(L^\infty \cap L(\widehat{A}, \frac{n}{s})\big)(\mathbb{R}^n)$ is optimal among all rearrangement-invariant spaces.

The proof of the results stated above is intertwined with a general characterization, of independent interest, of embeddings of the space $V_d^{s,A}(\mathbb{R}^n)$ into rearrangement-invariant spaces $Y(\mathbb{R}^n)$. It amounts to a reduction principle for such an embedding to a considerably simpler one-dimensional inequality for a Hardy type operator, depending only on s and n, involving the norms in the spaces $L^A(0,\infty)$ and $Y(0,\infty)$. This is the content of our last main result.

Theorem 3.7 (*Reduction Principle*). *Let $s \in (0,n)\setminus\mathbb{N}$ and let A be a Young function. Assume that $Y(\mathbb{R}^n)$ is a rearrangement-invariant space. Then, the following statements are equivalent:*
(i) There exists a constant c such that

$$\|u\|_{Y(\mathbb{R}^n)} \le c\big|\nabla^{[s]} u\big|_{\{s\},A,\mathbb{R}^n} \tag{3.21}$$

for every $u \in V_d^{s,A}(\mathbb{R}^n)$;
(ii) There exists a constant c such that

$$\left\| \int_r^\infty f(\rho)\rho^{-1+\frac{s}{n}}\, d\rho \right\|_{\overline{Y}(0,\infty)} \le c\|f\|_{L^A(0,\infty)} \tag{3.22}$$

for every $f \in L^A(0,\infty)$.

As already mentioned, the fact that embedding (3.21) implies inequality (3.22), which is also stated in (2.36), was established in [3, Lemmas 6.5 and 7.6]. The novelty of Theorem 3.7 is the reverse implication.

4. Boundedness of fractional Orlicz-Sobolev functions

This section is devoted to the proof of the following key result on the plain boundedness of functions from the space $V_d^{s,A}(\mathbb{R}^n)$ in the supercritical regime.

Theorem 4.1 (*Boundedness of Fractional Orlicz–Sobolev Functions*)**.** *Assume that s and A satisfy conditions (1.3), (1.5) and (1.6). Then,*

$$V^{s,A}_d(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n). \tag{4.1}$$

The proof of Theorem 4.1 exploits the subcritical embedding (2.28) to show that a function u as in its statement belongs to any Orlicz space built upon a finite-valued Young function growing arbitrarily fast near infinity. This piece of information entails that, in fact, u has to be essentially bounded. The technical steps needed to implement this idea are distributed in some lemmas.

Lemma 4.2. *Let E be any finite-valued Young function. Then, there exists a continuously differentiable Young function F such that*

$$F \geq E, \tag{4.2}$$

and

$$\frac{1}{F} \quad \textit{is convex near infinity}. \tag{4.3}$$

Proof. Let $G : [0, \infty) \to [0, \infty)$ be a function of the form

$$G(t) = e^{\varphi(t)} - 1 \qquad \text{for } \ t \geq 0\,, \tag{4.4}$$

where φ is a twice continuously differentiable Young function such that $\varphi'(t) > 0$ for $t > 0$, and $\lim_{t\to\infty} \varphi'(t) = \infty$. One can choose the function φ in such a way that

$$G(t) \geq E(t) \quad \text{for } t \geq 0\ . \tag{4.5}$$

For instance, the choice

$$\varphi(t) = \int_0^{2t} \frac{1}{\tau} \int_0^{2\tau} \frac{E(\theta) + \theta^2}{\theta}\, d\theta\, d\tau \quad \text{for } t \geq 0$$

is admissible, inasmuch as $\varphi(t) \geq E(t) + t^2 > 0$ for $t > 0$, since the function $(E(t) + t^2)/t$ is non-decreasing. The latter property also ensures that the function φ' is non-decreasing.

Define the function $\zeta : (0, \infty) \to (0, \infty)$ as

$$\zeta(t) = \frac{1}{\varphi'(t)} \qquad \text{for } t > 0.$$

The function ζ is positive, non-increasing and continuously differentiable in $(0, \infty)$. Also, $\lim_{t\to\infty} \zeta(t) = 0$.

Denote by $\widehat{\zeta}$ the convex envelope of ζ. Such a function inherits properties from ζ. Specifically, $\widehat{\zeta} > 0$ and $\widehat{\zeta}$ is non-increasing in $(0, \infty)$. Moreover, $\lim_{t\to\infty} \widehat{\zeta}(t) = 0$. Inasmuch as ζ is continuously differentiable, $\widehat{\zeta}$ is continuously differentiable as well,

$$\widehat{\zeta}\,' \quad \text{is non-decreasing and} \quad \widehat{\zeta}\,' < 0, \tag{4.6}$$

and

$$\lim_{t\to\infty} \widehat{\zeta}\,'(t) = 0. \tag{4.7}$$

Let $\psi : [0, \infty) \to [0, \infty)$ be the function defined by

$$\psi(t) = \int_0^t \frac{d\tau}{\widehat{\zeta}(\tau)} \quad \text{for } t \geq 0.$$

Owing to the properties of the function $\widehat{\zeta}$, one has that ψ is a twice continuously differentiable Young function, and

$$\psi'(t) = \frac{1}{\widehat{\zeta}(t)} \qquad \text{for } t > 0\,. \tag{4.8}$$

Since $\widehat{\zeta} \leq \zeta$, we have that

$$\psi(t) \geq \varphi(t) \qquad \text{for } t \geq 0\,. \tag{4.9}$$

Moreover,

$$\psi''(t) = -\frac{\widehat{\zeta}\,'(t)}{\widehat{\zeta}(t)^2} = -\widehat{\zeta}\,'(t)\,\psi'(t)^2 \leq \psi'(t)^2 \qquad \text{near infinity,} \tag{4.10}$$

where the last inequality holds by property (4.7).

Now, define the Young function F as

$$F(t) = e^{\psi(t)} - 1 \quad \text{for } \ t \geq 0.$$

Owing to Eqs. (4.4), (4.5) and (4.9),

$$F(t) \geq E(t) \quad \text{for } t \geq 0,$$

whence property (4.2) holds.

We claim that property (4.3) is also fulfilled. Indeed, the latter is equivalent to

$$\left(\frac{1}{F}\right)'' \geq 0 \quad \text{near infinity,}$$

and this is in turn equivalent to

$$2\,(F')^2 - F\,F'' \geq 0 \quad \text{near infinity,}$$

namely

$$\frac{F}{F'} \leq \frac{2F'}{F''} \quad \text{near infinity.} \tag{4.11}$$

Since

$$\frac{F}{F'} = \frac{1}{\psi'} - \frac{1}{\psi'\,e^{\psi}} \leq \frac{1}{\psi'}$$

and

$$\frac{F'}{F''} = \frac{\psi'}{\psi'' + (\psi')^2},$$

inequality (4.11) follows from the fact that, by Eq. (4.10),

$$\frac{1}{\psi'} \leq \frac{2\,\psi'}{\psi'' + (\psi')^2} \quad \text{near infinity.} \quad \square$$

Lemma 4.3. *Assume that s and A satisfy conditions (1.3), (1.5) and (1.6). Let E be any finite-valued Young function. Then, there exists a finite-valued Young function B such that*

$$B \lesssim A \text{ globally and } A \simeq B \text{ near } 0\,; \tag{4.12}$$

$$\int^{\infty} \left(\frac{t}{B(t)}\right)^{\frac{s}{n-s}} dt = \infty\,; \tag{4.13}$$

$$E \lesssim B_{\frac{n}{s}} \text{ near infinity.} \tag{4.14}$$

Here, $B_{\frac{n}{s}}$ denotes the Sobolev conjugate of B, defined as in (2.26)–(2.27).

Proof. Let a be the function from Eq. (2.1), and let b be the function appearing in a parallel equation with A replaced by B. By [25, Lemma 2.3], assumption (1.6) is equivalent to

$$\int_0 \frac{a^{-1}(t)}{t^{\frac{n}{n-s}}}\, dt < \infty, \tag{4.15}$$

and condition (4.13) is equivalent to

$$\int^\infty \frac{b^{-1}(t)}{t^{\frac{n}{n-s}}}\, dt = \infty. \tag{4.16}$$

We make use of an equivalent expression for the function $B_{\frac{n}{s}}$, which tells us that

$$B_{\frac{n}{s}}(t) \simeq \Big(t\, M^{-1}\big(t^{\frac{n}{n-s}}\big)\Big)^{\frac{n}{n-s}} \quad \text{for } t \geq 0,$$

where the function $M : [0,\infty) \to [0,\infty)$ is defined by

$$M(t) = \int_0^t \frac{b^{-1}(\tau)}{\tau^{\frac{n}{n-s}}}\, d\tau \quad \text{for } t \geq 0,$$

see [24, Lemma 2]. Thus, it suffices to produce a finite-valued Young function B satisfying conditions (4.12) and (4.13), and the following inequality, equivalent to (4.14):

$$\Big(t\, M^{-1}\big(t^{\frac{n}{n-s}}\big)\Big)^{\frac{n}{n-s}} \geq E(ct) \qquad \text{near infinity} \tag{4.17}$$

for some positive constant c.

Now, observe that, if F is a finite-valued Young function such that

$$F(t) \geq E(t) \qquad \text{near infinity}, \tag{4.18}$$

then

$$t^{\frac{n}{n-s}}\, F\big(t^{\frac{n}{n-s}}\big)^{\frac{n}{s}} \geq E(t) \qquad \text{near infinity}.$$

Thereby, inequality (4.17) will follow if we show that

$$\Big(t\, M^{-1}\big(t^{\frac{n}{n-s}}\big)\Big)^{\frac{n}{n-s}} \geq t^{\frac{n}{n-s}}\, F\big(t^{\frac{n}{n-s}}\big)^{\frac{n}{s}} \qquad \text{near infinity} \tag{4.19}$$

for some Young function F satisfying inequality (4.18).

By Lemma 4.2, there exists a continuously differentiable Young function F for which properties (4.3) and (4.18) hold. Therefore, it suffices to exhibit a Young function B satisfying properties (4.12), (4.13) and (4.19) for some Young function F fulfilling (4.3). To this purpose, recall that

$$\widetilde{A}(t) = \int_0^t a^{-1}(\tau)\, d\tau \qquad \text{and} \qquad \widetilde{B}(t) = \int_0^t b^{-1}(\tau)\, d\tau \quad \text{for } t \geq 0. \tag{4.20}$$

We define the function B via (4.20), with

$$b^{-1}(t) = a^{-1}(t) + \eta(t) \qquad \text{for } t \geq 0,$$

where the function $\eta : [0,\infty) \to [0,\infty)$ is such that

$$\eta(t) \text{ is non-decreasing, and } \eta(t) = 0 \text{ near } 0. \tag{4.21}$$

This choice ensures that condition (4.12) is fulfilled, for this condition is equivalent to

$$\widetilde{A} \lesssim \widetilde{B} \quad \text{globally and} \quad \widetilde{A} \simeq \widetilde{B} \quad \text{near } 0. \tag{4.22}$$

The remaining desired conditions on the function B are satisfied, provided that the function η also obeys

$$\begin{cases} \eta(t) = 0 & \text{if } 0 \le t \le t_0 \\ \displaystyle\int_{t_0}^{t} \frac{\eta(\tau)}{\tau^{\frac{n}{n-s}}}\, d\tau = \frac{1}{2} F^{-1}\big(t^{\frac{s}{n-s}}\big) - \frac{1}{2} F^{-1}\big(t_0^{\frac{s}{n-s}}\big) & \text{if } t > t_0 \end{cases} \tag{4.23}$$

for some $t_0 > 0$ to be chosen later.

Indeed, Eq. (4.23) implies condition (4.16), and hence (4.13), since

$$\int_{t_0}^{\infty} \frac{b^{-1}(\tau)}{\tau^{\frac{n}{n-s}}}\, d\tau = \int_{t_0}^{\infty} \frac{a^{-1}(\tau)}{\tau^{\frac{n}{n-s}}}\, d\tau + \int_{t_0}^{\infty} \frac{\eta(\tau)}{\tau^{\frac{n}{n-s}}}\, d\tau \ge \lim_{t\to\infty} \frac{1}{2} F^{-1}\Big(t^{\frac{s}{n-s}}\Big) - \frac{1}{2} F^{-1}\big(t_0^{\frac{s}{n-s}}\big) = \infty\,. \tag{4.24}$$

Moreover, Eqs. (4.23), (4.24) and (4.15) entail that

$$M(t) = \int_0^t \frac{a^{-1}(\tau)}{\tau^{\frac{n}{n-s}}}\, d\tau + \int_0^t \frac{\eta(\tau)}{\tau^{\frac{n}{n-s}}}\, d\tau \le F^{-1}\Big(t^{\frac{s}{n-s}}\Big) \qquad \text{near infinity}\,, \tag{4.25}$$

whence

$$M^{-1}(t) \ge F(t)^{\frac{n-s}{s}} \qquad \text{near infinity}\,. \tag{4.26}$$

Raising both sides of inequality (4.26) to the power $\frac{n}{n-s}$, and multiplying through the resultant inequality by $t^{\frac{n}{n-s}}$ yield (4.19).

Thus, it is only left to show that a function η fulfilling (4.21) and (4.23) does exist. Eq. (4.23) is equivalent to

$$\eta(t) = \begin{cases} 0 & \text{if } 0 \le t \le t_0 \\ \displaystyle\frac{s}{2(n-s)} \, \frac{t^{\frac{2s}{n-s}}}{F'\Big(F^{-1}\big(t^{\frac{s}{n-s}}\big)\Big)} & \text{if } t > t_0. \end{cases} \tag{4.27}$$

On the other hand, Eq. (4.21) will be verified by showing that the function η is non-decreasing for sufficiently large t_0. In view of Eq. (4.27), this property will follow if we show that

$$\frac{F^2}{F'} \quad \text{is non-decreasing near infinity.} \tag{4.28}$$

Property (4.28) is trivially equivalent to the fact that $\frac{F'}{F^2}$ is non-increasing near infinity, namely to the fact that $\left(-\frac{1}{F}\right)'$ is non-increasing near infinity. The latter property is in turn equivalent to the concavity of the function $-\frac{1}{F}$ near infinity, and, hence, to the convexity of the function $\frac{1}{F}$ near infinity. This is true, thanks to Eq. (4.3). □

Lemma 4.4. *Let $u \in \mathcal{M}(\mathbb{R}^n)$ be such that $u \in L^E(\mathbb{R}^n)$ for every finite-valued Young function E vanishing near 0. Then, $u \in L^\infty(\mathbb{R}^n)$.*

Proof. Our assumption on the function u ensures that

$$\int_{\mathbb{R}^n} F(|u|)\, dx < \infty \tag{4.29}$$

for every finite-valued Young function F vanishing near 0. To verify this assertion, observe that, given any function F with this property, the function E defined as

$$E(t) = F(t^2) \quad \text{for } t \geq 0$$

is also a Young function vanishing near 0, and

$$\lim_{t\to\infty} \frac{E(\lambda t)}{F(t)} = \infty \tag{4.30}$$

for every $\lambda > 0$. Indeed, by property (2.2),

$$\frac{E(\lambda t)}{F(t)} = \frac{F(\lambda^2 t^2)}{F(t)} \geq \frac{\lambda^2 t F(t)}{F(t)} = \lambda^2 t \qquad \text{if } t \geq \frac{1}{\lambda^2}.$$

Since $u \in L^E(\mathbb{R}^n)$, there exists $\lambda > 0$ such that

$$\int_{\mathbb{R}^n} E(\lambda |u|)\, dx < \infty. \tag{4.31}$$

Inasmuch as the function E is finite-valued, there exists $t_0 > 0$ such that $|\{|u| > t\}| < \infty$ for $t \geq t_0$, and property (4.29) follows via Eqs. (4.30) and (4.31).

Now, assume, by contradiction, that $u \notin L^\infty(\mathbb{R}^n)$. Thus, the function $\mu : (0,\infty) \to [0,\infty)$, defined as

$$\mu(t) = |\{|u| > t\}| \quad \text{for } t > t_0,$$

is non-increasing and such that

$$\mu(t) > 0 \quad \text{for } t > t_0.$$

Hence, the function $F : [0,\infty) \to [0,\infty)$, given by

$$F(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq t_0 \\ \displaystyle\int_{t_0}^t \frac{d\tau}{\mu(\tau)} & \text{if } t > t_0, \end{cases}$$

is a finite-valued Young function vanishing in $[0, t_0]$. Also,

$$\int_{\mathbb{R}^n} F(|u|)\, dx = \int_0^\infty F'(t)\mu(t)\, dt = \int_{t_0}^\infty dt = \infty, \tag{4.32}$$

a contradiction to Eq. (4.29). $\square$

Proof of Theorem 4.1. Fix any finite-valued Young function E, and let B be any Young function as in the statement of Lemma 4.3. Define a Young function E_0 in such a way that

$$E_0(t) = \begin{cases} 0 & \text{near } 0 \\ E(t) & \text{near infinity}. \end{cases} \tag{4.33}$$

By Eq. (4.14),

$$B_{\frac{n}{s}} \quad \text{dominates} \quad E_0 \text{ globally}.$$

Hence,

$$L^{B_{\frac{n}{s}}}(\mathbb{R}^n) \to L^{E_0}(\mathbb{R}^n). \tag{4.34}$$

Thanks to embedding (2.28) with A replaced by B,

$$V_d^{s,B}(\mathbb{R}^n) \to L^{B_{\frac{n}{s}}}(\mathbb{R}^n)\,. \tag{4.35}$$

On the other hand, property (4.12) implies that

$$V_d^{s,A}(\mathbb{R}^n) \to V_d^{s,B}(\mathbb{R}^n)\,. \tag{4.36}$$

Combining embeddings (4.34)–(4.36) yields:

$$V_d^{s,A}(\mathbb{R}^n) \to L^{E_0}(\mathbb{R}^n)\,. \tag{4.37}$$

Owing to the arbitrariness of the Young function E, the latter embedding entails inclusion (4.1), via Lemma 4.4 . □

5. Smooth approximation

An approximation argument by continuous functions is needed in the proof of embedding (3.2) into the space of continuous functions. Approximation for functions $u \in V_d^{s,A}(\mathbb{R}^n)$ in the seminorm $\left|\nabla^{[s]}u\right|_{\{s\},A,\mathbb{R}^n}$ is only possible under the additional assumption that the Young function A satisfies the Δ_2-condition, an assumption which is not required in our results. By contrast, the approximation theorem to be established in this section shows that modular approximation, namely approximation of $\nabla^{[s]}u$ with respect to the functional $J_{\{s\},A}$ defined as in (1.1), is feasible for every finite-valued Young function A.

Theorem 5.1 (*Modular Smooth Approximation*). *Let $s \in (0,\infty)\setminus\mathbb{N}$ and let A be a finite-valued Young function. Let $J_{\{s\},A}$ be the functional defined as in (1.1). Assume that $u \in V_d^{s,A}(\mathbb{R}^n)$. Then, there exist $\lambda > 0$ and a sequence $\{u_j\} \subset C^\infty(\mathbb{R}^n) \cap V_{\rm d}^{s,A}(\mathbb{R}^n)$ such that*

$$\lim_{j\to\infty} J_{\{s\},A}\left(\frac{\nabla^{[s]}u_j - \nabla^{[s]}u}{\lambda}\right) = 0\,. \tag{5.1}$$

The approximating sequence announced in the statement of Theorem 5.1 will be obtained via convolutions defined as follows. Let $\varrho \in C_0^\infty(\mathbb{R}^n)$ be a nonnegative function such that

$$\int_{\mathbb{R}^n} \varrho(x)\,dx = 1 \qquad \text{and} \qquad \operatorname{supp}\varrho \subset \mathcal{B}, \tag{5.2}$$

where $\mathcal{B}$ denotes the unit ball in $\mathbb{R}^n$ centered at 0. For each $\varepsilon > 0$, set

$$\varrho_\varepsilon(x) = \varepsilon^{-n}\varrho\left(\frac{x}{\varepsilon}\right) \qquad \text{for } \; x \in \mathbb{R}^n\,. \tag{5.3}$$

Given a function $u \in L^1_{\rm loc}(\mathbb{R}^n)$, we define

$$u_\varepsilon = \varrho_\varepsilon * u, \tag{5.4}$$

the convolution of u with ϱ_ε.

The proof of property (5.1) makes use of the next two lemmas from [37] and [16], respectively.

Lemma A ([37]). *Let A be a finite-valued Young function. Assume that $u \in \mathcal{M}(\mathbb{R}^n)$ is such that*

$$\int_{\mathbb{R}^n} A(2\,|u(x)|)\,dx < \infty\,. \tag{5.5}$$

Then, $u_\varepsilon \in C^\infty(\mathbb{R}^n)$ for $\varepsilon > 0$, and

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n} A\left(|u_\varepsilon(x) - u(x)|\right)dx = 0\,. \tag{5.6}$$

Given a function $u : \mathbb{R}^n \to \mathbb{R}$ and $h > 0$, we set

$$\Delta_i^h u(x) = u(x + h\, e_i) - u(x) \quad \text{for } x \in \mathbb{R}^n, \tag{5.7}$$

and for $i \in \{1, \ldots, n\}$. Here, e_i denotes the ith coordinate unit vector in $\mathbb{R}^n$.

Lemma B *([16]). Let $s \in (0,1)$ and let A be a Young function. Then, there exist constants $c = c(n)$ and $c' = c'(n)$ such that*

$$\begin{aligned} &\sum_{i=1}^n \int_0^\infty \int_{\mathbb{R}^n} A\left(c\frac{|\Delta_i^h u(x)|}{h^s}\right) dx\,\frac{dh}{h} \\ &\quad \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n} \\ &\qquad \leq \sum_{i=1}^n \int_0^\infty \int_{\mathbb{R}^n} A\left(c'\frac{|\Delta_i^h u(x)|}{h^s}\right) dx\,\frac{dh}{h} \end{aligned} \tag{5.8}$$

for every $u \in \mathcal{M}(\mathbb{R}^n)$.

Proof of Theorem 5.1. Assume first that $s \in (0,1)$, and let $u \in V_d^{s,A}(\mathbb{R}^n)$. Thereby, there exists a constant $\lambda > 0$ such that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|u(x)-u(y)|}{\lambda\,|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n} < \infty\,. \tag{5.9}$$

Hence, by Lemma B, there exists a constant $c > 0$ such that

$$\int_0^\infty \int_{\mathbb{R}^n} A\left(c\frac{|\Delta_i^h u(x)|}{\lambda\, h^s}\right) dx\frac{dh}{h} < \infty \tag{5.10}$$

for $i \in \{1, \ldots, n\}$. Fix any $i \in \{1, \ldots, n\}$. Owing to inequality (5.10),

$$\int_{\mathbb{R}^n} A\left(c\frac{|\Delta_i^h u(x)|}{\lambda\, h^s}\right) dx < \infty \tag{5.11}$$

for a.e. $h > 0$. Define the function $g_i : (0,\infty) \to [0,\infty]$ as

$$g_i(h) = \frac{1}{h}\int_{\mathbb{R}^n} A\left(c\frac{|\Delta_i^h u(x)|}{\lambda\, h^s}\right) dx \quad \text{for } h > 0. \tag{5.12}$$

Inequality (5.10) implies that

$$g_i \in L^1(0,\infty). \tag{5.13}$$

For each $h > 0$, define the function $v_i : \mathbb{R}^n \to \mathbb{R}$ as

$$v_i(x) = \frac{c}{2}\frac{\Delta_i^h u(x)}{\lambda\, h^s} \qquad \text{for } x \in \mathbb{R}^n\,. \tag{5.14}$$

Owing to inequality (5.11), we have that

$$\int_{\mathbb{R}^n} A\left(2\,|v_i(x)|\right)\, dx < \infty \qquad \text{for a.e. } h > 0. \tag{5.15}$$

An application of Lemma A, with u replaced by v_i, ensures that

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n} A\left(|(v_i)_\varepsilon(x) - v_i(x)|\right)\, dx = 0 \qquad \text{for a.e. } h > 0, \tag{5.16}$$

where the function $(v_i)_\varepsilon$ is defined as in (5.4). For $\varepsilon > 0$, define the function $g_{i,\varepsilon}\colon (0,\infty) \to [0,\infty]$ as

$$g_{i,\varepsilon}(h) = \frac{1}{h} \int_{\mathbb{R}^n} A\left(|(v_i)_\varepsilon(x) - v_i(x)|\right)\, dx \qquad \text{for } h > 0. \tag{5.17}$$

As a consequence of Eq. (5.16), there exists a sequence $\{\varepsilon_j\}$ such that

$$\lim_{j\to\infty} g_{i,\varepsilon_j}(h) = 0 \qquad \text{for a.e. } h > 0, \tag{5.18}$$

and for every $i \in \{1, \dots, n\}$. Next, we have that

$$g_{i,\varepsilon_j}(h) \le \frac{1}{2h} \int_{\mathbb{R}^n} A(2|(v_i)_{\varepsilon_j}(x)|)\, dx + \frac{1}{2h} \int_{\mathbb{R}^n} A(2|v_i(x)|)\, dx \le \frac{1}{h} \int_{\mathbb{R}^n} A(2|v_i(x)|)\, dx \quad \text{for } h > 0, \tag{5.19}$$

for $j \in \mathbb{N}$, and $i \in \{1, \dots, n\}$. Notice that the last inequality holds inasmuch as, by Jensen's inequality and the properties of ϱ_{ε_j},

$$\int_{\mathbb{R}^n} A(2|(v_i)_{\varepsilon_j}(x)|)\, dx = \int_{\mathbb{R}^n} A(2|\varrho_{\varepsilon_j} * v_i(x)|)\, dx \le \int_{\mathbb{R}^n} A(2|v_i(x)|)\, dx \qquad \text{for } h > 0. \tag{5.20}$$

Eqs. (5.19) and (5.20) imply that

$$g_{i,\varepsilon_j}(h) \le g_i(h) \qquad \text{for } h > 0, \tag{5.21}$$

for $j \in \mathbb{N}$, and $i \in \{1, \dots, n\}$. Since, by Eq. (5.13), $g_i \in L^1(0,\infty)$, the dominated convergence theorem ensures that

$$\lim_{j\to\infty} \int_0^\infty g_{i,\varepsilon_j}(h)\, dh = 0 \tag{5.22}$$

for $i \in \{1, \dots, n\}$, whence

$$\begin{aligned} &\lim_{j\to\infty} \sum_{i=1}^n \int_0^\infty \int_{\mathbb{R}^n} A\left(\frac{c}{2} \frac{\left|\Delta_i^h \left(\varrho_{\varepsilon_j} * u - u\right)(x)\right|}{\lambda\, h^s}\right) dx\, \frac{dh}{h} \\ &\quad = \lim_{j\to\infty} \sum_{i=1}^n \int_0^\infty \int_{\mathbb{R}^n} A\left(\frac{c}{2} \frac{\left|\varrho_{\varepsilon_j} * \Delta_i^h u(x) - \Delta_i^h u(x)\right|}{\lambda\, h^s}\right) dx\, \frac{dh}{h} = 0\,. \end{aligned} \tag{5.23}$$

Hence, via another application of Lemma B, we conclude that there exists a constant c, depending on λ and n, such that

$$\lim_{j\to\infty} J_{s,A}(c(u_{\varepsilon_j} - u)) = 0\,. \tag{5.24}$$

Thus, Eq. (5.1) is established for $s \in (0,1)$.

When $s \in (1,\infty) \setminus \mathbb{N}$, the conclusion follows with the same argument applied to $\nabla^{[s]}u$ and $\{s\}$ in the place of u and s, respectively

The fact that u_{ε_j} not only belongs to $C^\infty(\mathbb{R}^n) \cap V^{s,A}(\mathbb{R}^n)$, but also to $V_{\mathrm{d}}^{s,A}(\mathbb{R}^n)$, is a consequence of the membership of $u_{\varepsilon_j} - u$ to the latter space. This membership in turn follows from embedding (6.8) below. □

6. Proofs of the main results

With the technical material of Sections 4 and 5 at disposal, we are in a position to accomplish the proofs of our main results.

Proof of Theorem 3.1. If assumption (1.3) is in force, namely $s \in (0,n) \setminus \mathbb{N}$, then [3, Proposition 6.3 and Remark 7.3] tell us that condition (1.6) is necessary for any embedding of the form (1.2) to hold.

Thus, it is sufficient to show that, if $s \in (n,\infty) \setminus \mathbb{N}$, then such an embedding fails for every rearrangement-invariant space $Y(\mathbb{R}^n)$. Assume, by contradiction, that there exists a rearrangement-invariant space $Y(\mathbb{R}^n)$ which renders embedding (1.2) true. Let $\xi \in C_0^\infty(\mathbb{R}^n)$ be a nonnegative function such that $\xi = 1$ in $\mathcal{B}$. For each $j \in \mathbb{N}$, consider the function $u_j : \mathbb{R}^n \to \mathbb{R}$ defined as

$$u_j(x) = j^{s-n}\xi\Big(\frac{x}{j}\Big) \qquad \text{for } x \in \mathbb{R}^n\,. \tag{6.1}$$

Since $u_j \in C_0^\infty(\mathbb{R}^n)$, we have that

$$|\{|\nabla^k u_j| > t\}| < \infty \qquad \text{for } t > 0,$$

for $k = 0, 1, \dots, [s]$.

We claim that there exists a constant c, independent of j, such that

$$|\nabla^{[s]} u_j|_{\{s\},A,\mathbb{R}^n} \le c. \tag{6.2}$$

To verify this claim, observe that

$$\nabla^{[s]} u_j(x) = j^{s-[s]-n}\nabla^{[s]}\xi\Big(\frac{x}{j}\Big) = j^{\{s\}-n}\nabla^{[s]}\xi\Big(\frac{x}{j}\Big) \quad \text{for } x \in \mathbb{R}^n, \tag{6.3}$$

for $j \in \mathbb{N}$. Therefore,

$$\frac{\Big|\nabla^{[s]}u_j(x) - \nabla^{[s]}u_j(y)\Big|}{|x-y|^{\{s\}}} = \frac{\Big|\nabla^{[s]}\xi\big(\frac{x}{j}\big) - \nabla^{[s]}\xi\big(\frac{y}{j}\big)\Big|}{\big|\frac{x-y}{j}\big|^{\{s\}}}\, j^{-n} \quad \text{for } x, y \in \mathbb{R}^n, \text{ with } x \ne y, \tag{6.4}$$

for $j \in \mathbb{N}$. Since ξ is smooth, and $j \ge 1$, the right-hand side of Eq. (6.4) is pointwise bounded by a constant t_0 independent of j.

Next, since A is a Young function, there exists a constant c such that $A(t) \le c\,t$ if $t \in [0,t_0]$. Hence,

$$\begin{aligned} |\nabla^{[s]} u_j|_{\{s\},A,\mathbb{R}^n} &\le c \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\big|\nabla^{[s]}u_j(x) - \nabla^{[s]}u_j(y)\big|}{|x-y|^{\{s\}}}\, \frac{dx\,dy}{|x-y|^n} \\ &= c\,j^{-2n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\Big|\nabla^{[s]}\xi\big(\frac{x}{j}\big) - \nabla^{[s]}\xi\big(\frac{y}{j}\big)\Big|}{\big|\frac{x-y}{j}\big|^{\{s\}}}\, \frac{dx\,dy}{\big|\frac{x-y}{j}\big|^n} \\ &= c \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\big|\nabla^{[s]}\xi(x) - \nabla^{[s]}\xi(y)\big|}{|x-y|^{\{s\}}}\, \frac{dx\,dy}{|x-y|^n}\,. \end{aligned} \tag{6.5}$$

The assumption that $\xi \in C_0^\infty(\mathbb{R}^n)$ ensures that the integral on the rightmost-hand side of Eq. (6.5) is convergent. Hence, inequality (6.2) follows.

Recall that any rearrangement-invariant space $Y(\mathbb{R}^n)$ is continuously embedded into $(L^1 + L^\infty)(\mathbb{R}^n)$ – see e.g. [9, Theorem 6.6, Chapter 2]. Thereby, assumption (1.2) entails that

$$V_d^{s,A}(\mathbb{R}^n) \to (L^1 + L^\infty)(\mathbb{R}^n).$$

Hence, from inequality (6.2) we deduce that

$$\|u_j\|_{(L^1+L^\infty)(\mathbb{R}^n)} \le c \tag{6.6}$$

for some constant c and for every $j \geq 1$. On the other hand, by [9, Theorem 6.2, Chapter 2],

$$\|u_j\|_{(L^1+L^\infty)(\mathbb{R}^n)} = \int_0^1 u_j^*(t)\,dt.$$

Thus,

$$\begin{aligned}\|u_j\|_{(L^1+L^\infty)(\mathbb{R}^n)} &= \int_0^1 u_j^*(t)\,dt = j^{s-n}\int_0^1 \xi^*\left(\frac{t}{j^n}\right)\,dt = j^s\int_0^{\frac{1}{j^n}} \xi^*(\tau)\,d\tau \\ &\geq j^s \xi^*\left(\frac{1}{j^n}\right)\int_0^{\frac{1}{j^n}} d\tau = j^{s-n}\,\xi^*\left(\frac{1}{j^n}\right) \geq j^{s-n} \qquad \text{for sufficiently large } j\,.\end{aligned} \tag{6.7}$$

Inasmuch as $s-n>0$, coupling inequality (6.6) with (6.7) yields a contradiction, for sufficiently large j. □

The following proposition substantiates the assertion from Section 2 that $V^{s,A}_d(\mathbb{R}^n)$ is actually a Banach space. This piece of information will be needed for an application of the closed graph theorem in the proof of Theorem 3.2.

Proposition 6.1. *Assume that s and A fulfill conditions (1.3) and (1.6). Then, the functional $|\nabla^{[s]}u|_{\{s\},A,\mathbb{R}^n}$ defines a norm on the space $V^{s,A}_d(\mathbb{R}^n)$. Moreover, the space $V^{s,A}_d(\mathbb{R}^n)$, equipped with this norm, is a Banach space.*

Proof. Checking that the functional $|\nabla^{[s]}u|_{\{s\},A,\mathbb{R}^n}$ is a norm on $V^{s,A}_d(\mathbb{R}^n)$ is standard. Let us show that $V^{s,A}_d(\mathbb{R}^n)$, equipped with this norm, is complete. Denote by $\overline{A}$ a Young function such that $\overline{A}=A$ if A fulfills condition (1.4), and $\overline{A}=B$, where B is any function satisfying properties (4.12) and (4.13), if A fulfills condition (1.5). Thus,

$$V^{s,A}_d(\mathbb{R}^n) \to V^{s,\overline{A}}_d(\mathbb{R}^n) \to \bigcap_{k=0}^{[s]} V^{k,\overline{A}_{\frac{n}{s-k}}}(\mathbb{R}^n), \tag{6.8}$$

where the second embedding holds thanks to embedding (2.28), applied with s replaced by $s-k$ for $k=0,1,\dots,[s]$. Here, $\overline{A}_{\frac{n}{s-k}}$ denotes the Young function defined as in (2.26)–(2.27), with A replaced by $\overline{A}$, and $V^{0,\overline{A}_{\frac{n}{s}}}(\mathbb{R}^n)$ is understood as $L^{\overline{A}_{\frac{n}{s}}}(\mathbb{R}^n)$.

Now, assume that $\{u_j\}$ is a Cauchy sequence in the space $V^{s,A}_d(\mathbb{R}^n)$. Thanks to embeddings (6.8), it is also a Cauchy sequence in the space $V^{s,A}(\mathbb{R}^n)\bigcap\Big(\bigcap_{k=0}^{[s]} V^{k,\overline{A}_{\frac{n}{s-k}}}(\mathbb{R}^n)\Big)$, endowed with the norm

$$\sum_{k=0}^{[s]} \|\nabla^k u\|_{\overline{A}_{\frac{n}{s-k}}(\mathbb{R}^n)} + |\nabla^{[s]}u|_{\{s\},A,\mathbb{R}^n}.$$

A customary argument, analogous to the one showing that the classical fractional Sobolev space is a Banach space, tells us that $V^{s,A}(\mathbb{R}^n)\bigcap\Big(\bigcap_{k=0}^{[s]} V^{k,\overline{A}_{\frac{n}{s-k}}}(\mathbb{R}^n)\Big)$ is a Banach space. Hence, the sequence $\{u_j\}$ converges to some function $u\in V^{s,A}(\mathbb{R}^n)\bigcap\Big(\bigcap_{k=0}^{[s]} V^{k,\overline{A}_{\frac{n}{s-k}}}(\mathbb{R}^n)\Big)$. In particular, the fact that $u\in\bigcap_{k=0}^{[s]} V^{k,\overline{A}_{\frac{n}{s-k}}}(\mathbb{R}^n)$ entails that

$$|\{|\nabla^k u|>t\}|<\infty \quad \text{for } t>0,$$

for $k=0,1,\dots,[s]$. Hence, $u\in V^{s,A}_d(\mathbb{R}^n)$, and $u_j\to u$ in $V^{s,A}_d(\mathbb{R}^n)$. □

Proof of Theorem 3.2. We begin by showing that assertion (i) implies (ii). Assume that condition (1.5) holds. Then, by Theorem 4.1, inclusion (4.1) holds. Now, observe that the identity map from $V^{s,A}_d(\mathbb{R}^n)$ into

$L^\infty(\mathbb{R}^n)$ has a closed graph. This claim is equivalent to the fact that, if $\{u_j\} \subset V_d^{s,A}(\mathbb{R}^n)$ is a sequence such that

$$u_j \to u \qquad \text{in } V_d^{s,A}(\mathbb{R}^n), \tag{6.9}$$

and

$$u_j \to v \qquad \text{in } L^\infty(\mathbb{R}^n) \tag{6.10}$$

for some functions $u \in V_d^{s,A}(\mathbb{R}^n)$ and $v \in L^\infty(\mathbb{R}^n)$, then

$$u = v\,. \tag{6.11}$$

To verify that the convergences in (6.9) and (6.10) imply Eq. (6.11), note that, if $\overline{A}$ is any Young function as in the proof of Proposition 6.1, then, by Eq. (6.8),

$$V_d^{s,A}(\mathbb{R}^n) \to L^{\overline{A}_{\frac{n}{s}}}(\mathbb{R}^n). \tag{6.12}$$

Eq. (6.9) and embedding (6.12) ensure that there exists a subsequence of $\{u_j\}$, still indexed by j, such that

$$u_j \to u \qquad \text{a.e. in } \mathbb{R}^n\,. \tag{6.13}$$

Coupling Eq. (6.10) with (6.13) implies (6.11).

By Proposition 6.1, $V_d^{s,A}(\mathbb{R}^n)$ is a Banach space. Since $L^\infty(\mathbb{R}^n)$ is also a Banach space, inclusion (4.1) yields, via the closed graph theorem, the continuous embedding (3.1).

Let us now prove that, conversely, assertion (ii) implies (i), namely that embedding (3.1) implies condition (1.5). Owing to property (2.36), embedding (3.1) entails that there exists a constant c such that

$$\int_0^\infty f(\rho)\,\rho^{-1+\frac{s}{n}}\,d\rho = \left\| \int_r^\infty f(\rho)\,\rho^{-1+\frac{s}{n}}\,d\rho \right\|_{L^\infty(0,\infty)} \le c\,\|f\|_{L^A(0,\infty)} \tag{6.14}$$

for every nonnegative function $f \in L^A(0,\infty)$. From Eq. (6.14) and inequality (2.12) we deduce that

$$\|r^{-1+\frac{s}{n}}\|_{L^{\widetilde{A}}(0,\infty)} \le c. \tag{6.15}$$

One can verify, by the very definition of Luxemburg norm, that the finiteness of the norm in (6.15) is equivalent to

$$\int_0^\infty \frac{\widetilde{A}(t)}{t^{1+\frac{n}{n-s}}}\,dt < \infty,$$

see e.g. [23, Proof of equation (3.10)]. As shown in [25, Lemma 2.3], this condition is in turn equivalent to

$$\int_0^\infty \left(\frac{t}{A(t)}\right)^{\frac{s}{n-s}} dt < \infty.$$

Hence, property (1.5) follows.

The fact that assertion (iii) implies (ii) is trivial. Thus, it remains to show that (ii) implies (iii). A proof of this implication relies upon inequality (3.3), which can be established as follows. Consider any function $u \in V_d^{s,A}(\mathbb{R}^n)$. If the right-hand side of inequality (3.3) is infinite, then the inequality holds trivially. Hence, we may assume that it is finite. Set

$$N = \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A\left(\frac{|\nabla^{[s]}u(x) - \nabla^{[s]}u(y)|}{|x-y|^{\{s\}}} \right) \frac{dx\,dy}{|x-y|^n} \right)^{-\frac{1}{n}}, \tag{6.16}$$

and define the function $u_N : \mathbb{R}^n \to \mathbb{R}$ as

$$u_N(x) = u\Big(\frac{x}{N}\Big) \qquad \text{for } x \in \mathbb{R}^n.$$

Hence,

$$\nabla^{[s]}u_N(x) = N^{-[s]}\nabla^{[s]}u\Big(\frac{x}{N}\Big) \qquad \text{for } x \in \mathbb{R}^n.$$

Clearly,

$$\|u_N\|_{L^\infty(\mathbb{R}^n)} = \|u\|_{L^\infty(\mathbb{R}^n)}. \tag{6.17}$$

Moreover, for every $\lambda > 0$, one has, by a change of variables,

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|\nabla^{[s]}u_N(x) - \nabla^{[s]}u_N(y)|}{\lambda\,|x-y|^{\{s\}}}\right)\frac{dx\,dy}{|x-y|^n} \\ = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|\nabla^{[s]}u\left(\frac{x}{N}\right) - \nabla^{[s]}u\left(\frac{y}{N}\right)|}{\lambda\,|\frac{x}{N}-\frac{y}{N}|^{\{s\}}}N^{-[s]-\{s\}}\right)\frac{dx\,dy}{|\frac{x}{N}-\frac{y}{N}|^n}N^{-n} \\ = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|\nabla^{[s]}u(x') - \nabla^{[s]}u(y')|}{\lambda\,|x'-y'|^{\{s\}}}N^{-s}\right)\frac{dx'\,dy'}{|x'-y'|^n}N^{n}\,.\end{aligned} \tag{6.18}$$

The choice $\lambda = N^{-s}$ in (6.18) yields

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|\nabla^{[s]}u_N(x) - \nabla^{[s]}u_N(y)|}{N^{-s}\,|x-y|^{\{s\}}}\right)\frac{dx\,dy}{|x-y|^n} = 1. \tag{6.19}$$

Hence, the definition of the seminorm $|\cdot|_{s,A,\mathbb{R}^n}$ implies that

$$|u_N|_{s,A,\mathbb{R}^n} \le N^{-s}\,. \tag{6.20}$$

From embedding (3.1) and Eqs. (6.17) and (6.20) one can infer that

$$\|u\|_{L^\infty(\mathbb{R}^n)} \le c\,N^{-s} \tag{6.21}$$

for some constant c. Hence, inequality (3.3) follows.

We are now ready to complete the proof by showing that assertion (iii) follows from (ii). This goal will be achieved on proving that any function $u \in V^{s,A}_d(\mathbb{R}^n)$ agrees with a continuous function a.e. in $\mathbb{R}^n$. To this purpose, we make use of Theorem 5.1, and pick a number $\lambda > 0$ and a sequence $\{u_j\} \subset C^\infty(\mathbb{R}^n)$ fulfilling property (5.1). Fix $\varepsilon > 0$. Then, there exists $j_\varepsilon \in \mathbb{N}$ such that

$$J_{\{s\},A}\left(\frac{\nabla^{[s]}u_j - \nabla^{[s]}u}{\lambda}\right) < \varepsilon \qquad \text{for } j \ge j_\varepsilon\,. \tag{6.22}$$

By inequality (3.3) applied with u replaced by $(u_i - u)/\lambda$ and $(u_j - u)/\lambda$, with $i, j \ge j_\varepsilon$,

$$\begin{aligned}\|u_i - u_j\|_{C^0(\mathbb{R}^n)} = \|u_i - u_j\|_{L^\infty(\mathbb{R}^n)} \le \|u_j - u\|_{L^\infty(\mathbb{R}^n)} + \|u_i - u\|_{L^\infty(\mathbb{R}^n)} \\ \le c\,\lambda\left(J_{\{s\},A}\left(\frac{\nabla^{[s]}u_i - \nabla^{[s]}u}{\lambda}\right)^{\frac{s}{n}} + J_{\{s\},A}\left(\frac{\nabla^{[s]}u_j - \nabla^{[s]}u}{\lambda}\right)^{\frac{s}{n}}\right) < 2c\lambda\varepsilon^{\frac{s}{n}}.\end{aligned} \tag{6.23}$$

Eq. (6.23) implies, in particular, that $u_j \to u$ in $L^\infty(\mathbb{R}^n)$. Moreover, Eq. (6.23) tells us that $\{u_j\}$ is a Cauchy sequence in the Banach space $C^0(\mathbb{R}^n)$. Thus, there exists a function $\overline{u} \in C^0(\mathbb{R}^n)$ such that $u_j \to \overline{u}$ in $C^0(\mathbb{R}^n)$, and, hence, in $L^\infty(\mathbb{R}^n)$. By the uniqueness of the limit, $\bar{u} = u$ a.e. on $\mathbb{R}^n$. □

Lemma 6.2. *Assume that s and A fulfill conditions (1.3) and (1.6). Let B be another Young function such that*

$$A(t) \simeq B(t) \qquad \textit{near } 0\,. \tag{6.24}$$

Then,

$$\big(L^\infty \cap L(A, \tfrac{n}{s})\big)\,(\mathbb{R}^n) = \big(L^\infty \cap L(B, \tfrac{n}{s})\big)\,(\mathbb{R}^n)\,, \tag{6.25}$$

up to equivalent norms.

Proof. Owing to assumptions (1.6) and (6.24), we also have that

$$\int_0 \left(\frac{t}{B(t)}\right)^{\frac{s}{n-s}} dt < \infty.$$

Define the Young functions E_A and E_B in such a way that

$$E_A(t) \simeq \begin{cases} A(t) & \text{near } 0 \\ \infty & \text{near infinity} \end{cases} \qquad E_B(t) \simeq \begin{cases} B(t) & \text{near } 0 \\ \infty & \text{near infinity,} \end{cases}$$

and the function $\phi : (0,\infty) \to [0,\infty)$ as in (3.14). By the equivalence of the norms (3.11) and (3.12),

$$\|u\|_{\left(L^\infty \cap L(A,\frac{n}{s})\right)(\mathbb{R}^n)} \approx \|u^*(r)\,\phi(r)\|_{L^{E_A}(0,\infty)}\,, \tag{6.26}$$

and

$$\|u\|_{\left(L^\infty \cap L(B,\frac{n}{s})\right)(\mathbb{R}^n)} \approx \|u^*(r)\,\phi(r)\|_{L^{E_B}(0,\infty)}, \tag{6.27}$$

with equivalence constants independent of $u \in \mathcal{M}(0,\infty)$. Owing to assumption (6.24), the Young functions E_A and E_B are globally equivalent. Thus, the norms on the right-hand sides of Eqs. (6.26) and (6.27) are equivalent. Hence,

$$\|u\|_{\left(L^\infty \cap L(A,\frac{n}{s})\right)(\mathbb{R}^n)} \approx \|u\|_{\left(L^\infty \cap L(B,\frac{n}{s})\right)(\mathbb{R}^n)} \tag{6.28}$$

as well, with equivalence constants independent of $u \in \mathcal{M}(0,\infty)$. Eq. (6.25) follows. □

Proof of Theorem 3.4. Let B be any Young function satisfying properties (4.12) and (4.13). By embedding (2.33),

$$V^{s,A}_d(\mathbb{R}^n) \to V^{s,B}_d(\mathbb{R}^n) \to L(\widehat{B},\tfrac{n}{s})(\mathbb{R}^n)\,, \tag{6.29}$$

where $\widehat{B}$ is the Young function defined as in (2.31)–(2.32), with $A, a, \widehat{a}$ replaced by $B, b, \widehat{b}$, respectively.

From embeddings (3.1) and (6.29) one infers that

$$V^{s,A}_d(\mathbb{R}^n) \to \left(L^\infty \cap L(\widehat{B},\tfrac{n}{s})\right)(\mathbb{R}^n)\,. \tag{6.30}$$

Since $B \simeq A$ near zero, one can verify that

$$\widehat{B} \simeq \widehat{A} \quad \text{near zero}\,. \tag{6.31}$$

Hence, Lemma 6.2, applied with A and B replaced with $\widehat{A}$ and $\widehat{B}$, tells us that

$$\left(L^\infty \cap L(\widehat{B},\tfrac{n}{s})\right)(\mathbb{R}^n) = \left(L^\infty \cap L(\widehat{A},\tfrac{n}{s})\right)(\mathbb{R}^n), \tag{6.32}$$

up to equivalent norms. Embedding (3.9) follows from Eqs. (6.30) and (6.32).

As far as the optimality of the space $\left(L^\infty \cap L(\widehat{A},\frac{n}{s})\right)(\mathbb{R}^n)$ is concerned, assume that embedding (1.2) holds for some rearrangement-invariant space $Y(\mathbb{R}^n)$. We have to show that

$$\left(L^\infty \cap L(\widehat{A},\tfrac{n}{s})\right)(\mathbb{R}^n) \to Y(\mathbb{R}^n)\,. \tag{6.33}$$

By property (2.36), embedding (1.2) implies that there exists a constant c such that

$$\left\| \int_r^\infty f(\rho)\,\rho^{-1+\frac{s}{n}}\,d\rho \right\|_{Y(0,\infty)} \le c\,\|f\|_{L^A(0,\infty)} \tag{6.34}$$

for every function $f \in L^A(0,\infty)$. Inequality (6.34) entails that

$$\left(L^\infty \cap L(\widehat{A},\tfrac{n}{s})\right)(\mathbb{R}^n) \subset Y(\mathbb{R}^n)\,. \tag{6.35}$$

This can be verified via the same argument as in the proof of optimality in [25, Theorem 1.1]. Thanks to property (2.19), inclusion (6.35) is equivalent to embedding (6.33). □

Proof of Theorem 3.3. Let B be a Young function as in the statement of Lemma 4.3. Embedding (2.35), with A replaced by B, and embedding (6.30) imply that

$$V_d^{s,A}(\mathbb{R}^n) \to \big(L^\infty \cap L^{B_{\frac{n}{s}}}\big)(\mathbb{R}^n)\,. \tag{6.36}$$

Since $B \simeq A$ near zero,

$$B_{\frac{n}{s}} \simeq A_{\frac{n}{s}} \quad \text{near zero}\,. \tag{6.37}$$

Hence, via an application of [1, Lemma 5.1] we deduce that

$$\big(L^\infty \cap L^{B_{\frac{n}{s}}}\big)(\mathbb{R}^n) \to L^{A_{\frac{n}{s}}}(\mathbb{R}^n)\,. \tag{6.38}$$

Embedding (3.7) follows from (6.36) and (6.38).

It remains to prove that $L^{A_{\frac{n}{s}}}(\mathbb{R}^n)$ is the optimal Orlicz target space in (3.7). Assume that E is a Young function such that

$$V_d^{s,A}(\mathbb{R}^n) \to L^E(\mathbb{R}^n)\,. \tag{6.39}$$

We have to show that

$$L^{A_{\frac{n}{s}}}(\mathbb{R}^n) \to L^E(\mathbb{R}^n). \tag{6.40}$$

Thanks to property (2.36), there exists a constant c such that

$$\left\| \int_r^\infty f(\rho)\, \rho^{-1+\frac{s}{n}}\, d\rho \right\|_{L^E(0,\infty)} \le c\, \|f\|_{L^A(0,\infty)} \tag{6.41}$$

for every function $f \in L^A(0,\infty)$. By [25, Lemma 1], $L^{A_{\frac{n}{s}}}(0,\infty)$ is the optimal Orlicz target space in (6.41). Thus, $L^{A_{\frac{n}{s}}}(0,\infty) \to L^E(0,\infty)$, and this embedding is equivalent to embedding (6.40). □

Proof of Theorem 3.7. The fact that inequality (3.21) implies inequality (3.22) is stated in property (2.36), and established in [3, Lemmas 6.5 and 7.6].

Let us prove the reverse implication. Assume that inequality (3.22) holds for some rearrangement-invariant space $Y(\mathbb{R}^n)$. We distinguish two cases, corresponding to the subcritical regime (1.4) and the supercritical regime (1.5).

If condition (1.4) is in force, then inequality (2.34) holds. Hence, by property (2.36),

$$\left\| \int_r^\infty f(\rho)\, \rho^{-1+\frac{s}{n}}\, d\rho \right\|_{L(\widehat{A},\frac{n}{s})(0,\infty)} \le C \|f\|_{L^A(0,\infty)} \tag{6.42}$$

for every $f \in L^A(0,\infty)$. Moreover, the target space $L(\widehat{A}, \frac{n}{s})(0,\infty)$ is optimal in inequality (6.42) among all rearrangement-invariant spaces – see the proof of the optimality in [25, Theorem 1.1, Part I]. This optimality ensures that $L(\widehat{A}, \frac{n}{s})(0,\infty) \to \overline{Y}(0,\infty)$. Hence,

$$\|u\|_{Y(\mathbb{R}^n)} \le c \|u\|_{L(\widehat{A},\frac{n}{s})(\mathbb{R}^n)} \tag{6.43}$$

for some constant c and for every $u \in L(\widehat{A}, \frac{n}{s})(\mathbb{R}^n)$. Inequality (3.21) follows from (2.34) and (6.43).

Assume next that condition (1.5) holds. Then, Theorem 3.4 provides us with inequality (3.10), which, coupled with property (2.36), yields the inequality

$$\left\| \int_r^\infty f(\rho)\, \rho^{-1+\frac{s}{n}}\, d\rho \right\|_{\left(L^\infty \cap L(\widehat{A},\frac{n}{s})\right)(0,\infty)} \le c\, \|f\|_{L^A(0,\infty)} \tag{6.44}$$

for some constant c and for every $f \in L^A(0,\infty)$. The space $\big(L^\infty \cap L(\widehat{A}, \frac{n}{s})\big)(0,\infty)$ is optimal in inequality (6.44) among all rearrangement-invariant spaces – see the proof of the optimality in [25, Theorem 1.1, Part II]. This optimality guarantees that $\big(L^\infty \cap L(\widehat{A}, \frac{n}{s})\big)(0,\infty) \to \overline{Y}(0,\infty)$, whence

$$\|u\|_{Y(\mathbb{R}^n)} \leq c\|u\|_{\left(L^\infty \cap L(\widehat{A}, \frac{n}{s})\right)(\mathbb{R}^n)} \tag{6.45}$$

for some constant c and for every $u \in \big(L^\infty \cap L(\widehat{A}, \frac{n}{s})\big)(\mathbb{R}^n)$. Inequality (3.21) is a consequence of (3.10) and (6.45). □

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