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Manipulation of social choice functions under incomplete information*

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Abstract

We propose a general framework to analyse how vulnerable to manipulation social choice functions are when a limited amount of information about individual preferences is available. We focus then on two properties, called WMG-strategy-proofness and group WMG-strategy-proofness, that are weaker than strategy-proofness and group strategy-proofness, respectively. A social choice function is [group] WMG-strategy-proof if it cannot be manipulated by an individual [a coalition of individuals] whose information about the preferences of the other individuals is limited to the knowledge, for every pair of alternatives, of the number of people preferring the first alternative to the second one. We prove that there are Pareto optimal, WMG-strategy-proof and non-dictatorial social choice functions. We also prove that, when at least three alternatives are considered, every Pareto optimal and anonymous social choice function is not WMG-strategy-proof. Finally, we show that every Pareto optimal and group WMG-strategy-proof is dictatorial, provided that the alternatives are three.

Keywords: social choice function; manipulability; strategy-proofness; group strategy-proofness; anonymity; Pareto optimality.

JEL Classification Numbers: D71, D72

1 Introduction

Consider a society whose purpose is to select a unique alternative among the ones in a given set. Assume that such a selection must be based only on individuals' preferences, expressed via rankings of the alternatives. Any procedure associating an alternative with each preference profile, namely a complete list of individuals' preferences, is called social choice function (SCF). Gibbard-Satterthwaite Theorem (Gibbard, 1973; Satterthwaite, 1975) is definitely one of the most celebrated results about SCFs. It states that, when at least three alternatives are considered, any strategy-proof and surjective SCF is dictatorial. Thus, even though highly desirable, strategy-proofness is a strong requirement.

Efforts were made by several authors to weaken strategy-proofness in interesting directions. Guided by the intuition that if a rule does not allow large enough gains from manipulating then individuals may decide not to support the cost for gathering the needed information about the others' preferences, Campbell and Kelly (2009) focus on the size of the potential gains an individual can achieve from manipulating Condorcet and scoring SCFs. Reffgen (2011) considers those manipulations

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that allow individuals to get a very good result and proves that every non-dictatorial and surjective SCF can be manipulated by an individual who can obtain her best or second best alternative via the manipulation. Sato (2013), in line with Kelly (1988), studies the implications of limiting the options for misrepresenting. He introduces the concept of AM-proofness, a condition weaker than strategy-proofness that imposes that individuals are reluctant to tell big lies, and proves an impossibility result on the universal domain (see also Carroll, 2012). Muto and Sato (2017) introduce the concept of bounded response that states that a switch between two alternatives in the preference order of an individual does not change the outcome or leads to select an alternative that is consecutively ranked to the original outcome in the (not modified) preference order of the considered individual. The authors prove that bounded response is strictly weaker than strategy-proofness but, along with Pareto optimality, still implies dictatorship. Following the ideas of Sato (2013), they also introduce a property weaker than AM-proofness mentioned above and prove that it still implies dictatorship when joined with Pareto optimality. Campbell et al. (2018) introduce a concept of stability that refers to the idea that a small change in one of the individual preferences should have a small impact in the outcome. Their concept of stability is strictly weaker than strategy-proofness (indeed, it is weaker than bounded response). They prove that the Plurality, the Condorcet, the Maximin and the Borda social choice correspondences[1] are stable when restricted to the set of preference profiles where they are resolute but none of these restrictions can be extended to the whole domain preserving stability. They also prove that if a SCF satisfies stability, Pareto optimality, monotonicity and tops-only, then it is dictatorial.

The mentioned results basically explore the effects of restricting the possible choice of false preferences for the individuals. That line of research seems to be unable to easily escape dictatorship. Nurmi (1987) suggests another possible way to weaken strategy-proofness.[2] First of all, recall that a SCF fails to be strategy-proof if there is an individual who could potentially misreport her own preferences by knowing the preferences that the others decide to report. The failure of strategy-proofness requires then the presence of an individual who has the capability to completely know the others' preferences. Of course, it is possible to conceive situations where that might happen. For instance, when the members of a small committee discuss a lot about the candidates before voting, their preferences can get precisely revealed to the others. However, in most circumstances, individuals cannot draw a clear picture of the preferences of the others. In particular, when the number of alternatives or individuals is large, the level of knowledge of private information that an individual can expect to get is in general limited. In these cases violating strategy-proofness is not that serious since nobody can really decide to deviate on the basis of the exact knowledge of the others' preferences, since nobody can get that knowledge. Nurmi (1987) interestingly observes, however, that an individual might decide to deviate on the basis of a smaller amount of information about the others' preferences. Thus, if the information needed to make an individual realize that it is profitable for her to report false preferences is small enough and easy to get, then manipulability issues become much more significant.

In the framework of social choice correspondences, Nurmi introduces a concept of degree of vulnerability "which is intended to reflect the type of knowledge one *typically* needs in order to benefit from preference misrepresentation. The more detailed the knowledge of the preference profile one needs, the less vulnerable is the procedure to misrepresentation" (Nurmi, 1987, p.119). We have to say that such a concept is not rigorously defined and, as stressed by Kelly (1993), no precise measure of the amount of knowledge is given. Anyway, on the basis of a number of examples and observations, Nurmi proposes a misrepresentation hierarchy. In particular, he states that the Plurality social choice correspondence has the highest level of manipulability as only the distribution of the first ranked alternatives is needed for a successful misrepresentation while the Coombs, the

[1] A social choice correspondence is a procedure associating a nonempty set of alternatives with each preference profile.

[2] There are other important lines of research that try to overcome impossibility results of the Gibbard-Satterthwaite type without looking for weak versions of strategy-proofness. Among them it is worth mentioning the analysis of the so-called dictatorial domains (Aswal et al., 2003; Pramanik, 2015) as well as the study of the informatively richer framework where voters not only rank alternatives but also evaluate them as acceptable or inacceptable (Erdamar et al., 2017).

Alternative Voting (Hare) and the Plurality with runoff social choice correspondences typically need the knowledge of the entire preference profile. The author concludes then that "there are marked differences between procedures and that therefore is not of much consequence to discover that all of them are manipulable. This finding conceals many important differences between the procedures and has thus very little practical bearing" (Nurmi, 1987, p.125).

In the light of Nurmi's observations, given a SCF, it becomes meaningful to study weak versions of strategy-proofness obtained by establishing, for every individual,

(a) a specific type of information about the preferences of the other individuals in the society,

(b) the conditions that make the individual have an incentive to misrepresent her preferences,

and then by imposing that every time an individual has the information described in (a), then conditions described in (b) never occur.[3] We stress that an individual may realize that there are several combinations of preferences of the others' that are consistent with the information described in (a). When such an uncertain situation occurs, it is important to know the conditions that really cause the individual to misrepresent her preferences. Establishing (b) just serves to clarify that point.

Of course, it is also meaningful to study weak versions of the classic concept of group strategy-proofness obtained by establishing, for every coalition of individuals,

(α) a specific type of information about the preferences of individuals outside the coalition,

(β) the conditions that make all the individuals in the coalition have an incentive to misrepresent their preferences,

and then by imposing that every time the individuals in a coalition have the information of the type described in (α), then conditions described in (β) never occur.[4]

Fix now some qualifications of (a) and (b). The corresponding weak version of strategy-proofness refers to individuals who exactly have the amount of information described in (a). That might sounds very strict and seem to limit the practical application of the property. However, the fact that a given SCF fulfils or fails to fulfil such a property allows to deduce interesting facts, provided that conditions in (b) imply that an increase in the level of information increases the incentives of individuals to report false preferences and that a decrease in the level of information makes people more reluctant to lie. Indeed, in that case, if a SCF satisfies the weak version of strategy-proofness corresponding to (a) and (b), then individuals never have incentive to report false preferences if they get a part or even all of the information described in (a) but not more than that; on the other hand, if a SCF fails to satisfy the weak version of strategy-proofness corresponding to (a) and (b), then there is an individual who could potentially decide to report false preferences by getting the information described by (a) and possibly some information more. Similar observations can be made for the considered weak versions of group strategy-proofness.[5]

Formalizations of the described weak versions of strategy-proofness can be found in Conitzer et al. (2011), Reijngoud and Endriss (2012) and Endriss et al. (2016). In order to describe the amount of information available to individuals, namely to qualify (a), Conitzer et al. (2011) associate with each individual a family of sets, called information sets, whose elements are lists of preferences of the others and assume that the individual is able to know which is the set having the true list of the others' preferences as element; Reijngoud and Endriss (2012) and Endriss et al. (2016) assume instead that individuals are given only some pieces of information extracted by an opinion poll and

[3]Since individuals know their own preferences, they can always translate any piece of information about the preferences of the others into a piece of information about the complete preference profile and vice versa. In some cases it will be convenient to qualify (a) referring to the complete preference profile.

[4]We assume that members of a coalition know each other's preferences. Thus, members of a coalition can translate any piece of information about the preferences of individuals outside the coalition into a piece of information about the complete preference profile and vice versa. In some cases it will be convenient to qualify (α) referring to the complete preference profile.

[5]As proved by Proposition 5, those remarks can be applied to (b_1) and (b_2) as well as (β_1) and (β_2), the main qualification of (b) and (β) considered in the paper.

described via a so-called poll information function.[6] For what concerns (b), all those authors assume a risk averse behaviour of individuals: an individual decides to report false preferences if, for every combination of preferences of the others consistent with her information, false preferences cannot make her worse off and, for at least one combination, make her better off. Thus, if there is one single possibility that false preferences can make an individual worse off, the individual will report sincere preferences. All those authors consider different qualifications of (a) and prove several results, most of which refer to specific SCFs.[7]

Further attempts to analyse the impact of incomplete information on the strategic behaviour of individuals are present in the literature. For instance, Osborne and Rubinstein (2003) assume that each individual deduces the distribution of preferences of all the other individuals from a small random sample that she knows and thinks to be representative; Chopra et al. (2004) use the concept of knowledge graph, a direct graph on the set of individuals, to describe the connections among individuals and assume that each individual knows the preferences of the individuals near her, namely the ones who are adjacent to her in the graph.

In this paper we first present a general formal framework, inspired to the one by Conitzer et al. (2011), for dealing with weak versions of strategy-proofness and group strategy-proofness based on the principle of limiting the amount of individuals' and coalitions' information. The framework we propose, which is more general than the ones by Conitzer et al. (2011), Reijngoud and Endriss (2012) and Endriss et al. (2016) previously mentioned, allows to easily formalize a variety of possible informational situations and compare them. We specialize then on the weak versions of strategy-proofness obtained by considering, for every individual, the following qualification of (a):

(a_0) for every pair of alternatives, the number of individuals in the society, other than the considered individual, who prefer the first alternative to the second one,

and one of the following qualifications of (b):

(b_1) the individual reports false preferences if, for every combination of preferences of the others consistent with her information, false preferences cannot make her worse off and, for at least one combination, make her better off,

(b_2) the individual reports false preferences if, for every combination of preferences of the others consistent with her information, false preferences make her better off.

Similarly, we focus on the weak versions of group strategy-proofness obtained by considering, for every coalition of individuals, the following qualification of (α):

(α_0) for every pair of alternatives, the number of individuals outside the coalition who prefer the first alternative to the second one,

and one of the following qualifications of (β):

(β_1) all the individuals in the coalition report false preferences if, for every combination of preferences of the others consistent with their information, false preferences cannot make one of them worse off and, for at least one combination, make all of them better off,

(β_2) all individuals in the coalition report false preferences if, for every combination of preferences of the others consistent with their information, false preferences make all of them better off.

Assumption (a_0) is certainly weaker than the one of complete knowledge of individuals' preferences implicitly used in the definition of strategy-proofness and definitely easier to obtain; assumption (b_1) refers to the already described risk averse behaviour of individuals; assumption (b_2) reduces the situations in which individuals would have incentives to manipulate. Assumptions (α_0), (β_1) and (β_2) are the natural counterparts of (a_0), (b_1) and (b_2) for coalitions. Note that Reijngoud and Endriss (2012) explicitly consider both (a_0) and (b_1) but do not prove any result about the corresponding

[6] See Section 3 for further details.

[7] Some of the results by Reijngoud and Endriss (2012) and Endriss et al. (2016) are collected in Theorem 7.

weak version of strategy-proofness. Note also that they introduce the property (a_0) referring to the weighted majority graph associated with the preference profile.

In order to give credit to Reijngoud and Endriss (2012), we denote the weak version of strategy-proofness obtained using (a_0) and (b_1) by WMG-strategy-proofness and the one obtained using (a_0) and (b_2) by weak WMG-strategy-proofness, where WMG stands for weighted majority graph. Moreover, we denote the weak version of group strategy-proofness obtained using (α_0) and (β_1) by group WMG-strategy-proofness and the one obtained using (α_0) and (β_2) by weak group WMG-strategy-proofness. Of course, [group] WMG-strategy-proofness implies weak [group] WMG-strategy-proofness and [weak] group WMG-strategy-proofness implies [weak] WMG-strategy-proofness.

As the first main result of the paper, we show that Gibbard-Satterthwaite theorem cannot be generalized by replacing strategy-proofness with WMG-strategy-proofness. Indeed, we are able to exhibit a family of Pareto optimal and WMG-strategy-proof SCFs that are non-dictatorial (Theorem 10). We also prove that such SCFs are not weakly group WMG-strategy-proof. As an example of a SCF in that family, simply consider three individuals, denoted by i_1, i_2 and i_3, who have to choose one among the alternatives a, b and c and let F be defined as follows: if individual i_1 has preferences xyz [8], i_2 has preferences $\bar{x}\bar{y}\bar{z}$ and i_3 has preferences $\hat{x}\hat{y}\hat{z}$, then

$$F(xyz, \bar{x}\bar{y}\bar{z}, \hat{x}\hat{y}\hat{z}) = \begin{cases} x & \text{if } \hat{x}\hat{y}\hat{z} \in \{abc, bca, cab\} \\ \bar{x} & \text{if } \hat{x}\hat{y}\hat{z} \in \{acb, bac, cba\} \end{cases}$$

Finding an appealing characterization of Pareto optimal and [weakly] WMG-strategy-proof SCFs is surely an interesting problem. Even though that problem is still open, as the second main result of the paper we prove that there is no weakly WMG-strategy-proof, Pareto optimal and anonymous SCF, provided that there are at least three alternatives (Theorem 11). Thus, given an anonymous and Pareto optimal SCF, we cannot exclude the possibility that an individual could decide to report false preferences on the basis of the knowledge of the pairwise comparison of alternatives and possibly some information more (like, for instance, the whole preferences of some individuals), even if individuals are suppose to manipulate under the conditions described in (b_2). Of course, in order to manipulate, an individual has to gather the described information about the others' preferences, and identify false preferences that satisfy the conditions in (b_2). That might turn out to be a very difficult task, especially when the number of individuals or alternatives is large. Nevertheless, Theorem 11 shows that the task is not in principle impossible.[9]

The proof of Theorem 11 is based on an induction argument on the number of voters and it is strongly inspired to the short proof of Gibbard-Satterthwaite theorem by Svensson and Reffgen (2014). As an intermediate step of the proof, we also prove that, when three alternatives are considered, every Pareto optimal and strategy-proof SCF defined on a special set of preference profiles is dictatorial (Proposition 19). Such a result does not seem to be a consequence of any of the known theorems about dictatorial domains (Aswal et al., 2003; Pramanik, 2015).

In the light of the results described, it is natural to wonder whether there exists a Pareto optimal and non-dictatorial SCF satisfying weak group WMG-strategy-proofness. As the third main result of the paper, we show that such a SCF does not exist if three alternatives are considered (Theorem 13). Thus, when the collective decision involves exactly three alternatives, we have that any Pareto optimal and non-dictatorial SCF can be manipulated by a coalition of individuals who know, for every pair of alternatives, how many individuals outside the coalition prefer the first alternative to the second one, even if individuals are suppose to manipulate under the conditions described in (β_2). At the moment, the existence of Pareto optimal, non-dictatorial and weakly group WMG-strategy-proof SCFs when the alternatives are four or more is an open problem.

[8] The writing xyz, where x, y and z are distinct elements of $\{a, b, c\}$, denotes the ranking where x is preferred to y and y is preferred to z.

[9] Similarly, manipulating a SCF on the basis of a complete knowledge of the others' preferences is in general very difficult, due to the obvious difficulties in gathering that type of information. However, Gibbard-Satterthwaite theorem shows that it is not in principle impossible, when surjective and non-dictatorial SCFs are considered and the alternatives are at least three.

In the last section of the paper, we finally show how our framework for managing incomplete information naturally leads to a general method for building indexes for measuring the degree of manipulability of SCFs. We focus then on a particular family of indexes and we propose very preliminary observations about them.

2 Preliminary definitions and notation

We assume $0 \notin \mathbb{N}$ and we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Given a finite set X, we denote by $|X|$ the size of X, by $\mathcal{P}_0(X)$ the set of the nonempty subsets of X, by X^2_* the set $\{(x, y) \in X^2 : x \neq y\}$ and by $\mathrm{Sym}(X)$ the set of bijective functions from X to X.

Let A be a nonempty and finite set. We denote by $\mathcal{L}(A)$ the set of complete, transitive and antisymmetric binary relations on A. Consider $q \in \mathcal{L}(A)$. If $x, y \in A$, we write $x \succeq_q y$ when $(x, y) \in q$ and we write $x \succ_q y$ when $(x, y) \in q$ and $(y, x) \notin q$. For every $k \in \{1, \ldots, |A|\}$, we denote by $r_k(q)$ the element of A that is ranked k-th in q.

The set of functions from a finite set X to $\mathcal{L}(A)$ is denoted by $\mathcal{L}(A)^X$. Note that $\varnothing$ is the unique element of $\mathcal{L}(A)^\varnothing$. Let I and J be finite sets. If I and J are disjoint, $p \in \mathcal{L}(A)^I$ and $p' \in \mathcal{L}(A)^J$, we denote by (p, p') the element of $\mathcal{L}(A)^{I \cup J}$ defined, for every $i \in I \cup J$, by

$$(p,p')(i) = \begin{cases} p(i) & \text{if } i \in I \\ p'(i) & \text{if } i \in J \end{cases}$$

Note that $(p, p') = (p', p)$ and that $J = \varnothing$ implies $(p, p') = p$.

Let $p \in \mathcal{L}(A)^I$. If $J \subseteq I$, we denote by $p_{|J}$ the restriction of p to J. Moreover, we denote by c_p the function from A^2_* to $\mathbb{N}_0$ defined, for every $(x, y) \in A^2_*$, by $c_p(x, y) = |\{i \in I : x \succ_{p(i)} y\}|$. We also consider the equivalence relation $\sim$ on $\mathcal{L}(A)^I$ defined as follows: for every $p, p' \in \mathcal{L}(A)^I$, $p \sim p'$ if $c_p = c_{p'}$. Of course, $\sim$ depends on A and I but those symbols are omitted in the writing as they will always be clear from the context. Note that if $|I| = 1$ and $p, p' \in \mathcal{L}(A)^I$, then $p \sim p'$ if and only if $p = p'$. Finally, if $\varphi \in \mathrm{Sym}(I)$, we denote by p^φ the element of $\mathcal{L}(A)^I$ defined, for every $i \in I$, by $p^\varphi(i) = p(\varphi^{-1}(i))$.

3 Main concepts

From now on, A and I are finite sets such that $|A| \geq 3$ and $|I| \geq 2$. We interpret A as the set of alternatives and I as the set of individuals.[10] Each $q \in \mathcal{L}(A)$ is interpreted as one of the possible individual preferences. Each element p in $\mathcal{L}(A)^I$ is called preference profile and represents the complete description of individual preferences: for every $i \in I$, $p(i)$ is interpreted as the preferences of individual i. Each nonempty subset of I is called coalition. The elements of $\mathcal{L}(A)^I$ are usually denoted by p, possibly with suitable superscripts. Given a coalition C, the elements of $\mathcal{L}(A)^C$ are usually denoted by $\hat{p}$, possibly with suitable superscripts, while the elements of $\mathcal{L}(A)^{I \setminus C}$ are usually denoted by $\bar{p}$, possibly with suitable superscripts.

A social choice function (SCF) is a function from $\mathcal{L}(A)^I$ to A. Given a SCF F, we say that F is Pareto optimal if, for every $p \in \mathcal{L}(A)^I$ and $x, y \in A$, if $x \succ_{p(i)} y$ for all $i \in I$, then $F(p) \neq y$; F is dictatorial with dictator $i \in I$ if, for every $p \in \mathcal{L}(A)^I$, $F(p) = r_1(p(i))$; F is dictatorial if there exists $i \in I$ such that F is dictatorial with dictator i; F is anonymous if, for every $p \in \mathcal{L}(A)^I$ and $\varphi \in \mathrm{Sym}(I)$, we have that $F(p^\varphi) = F(p)$; F is group strategy-proof[11] if, for every $C \in \mathcal{P}_0(I)$, $\hat{p}, \hat{p}' \in \mathcal{L}(A)^C$ with $\hat{p}(i) \neq \hat{p}'(i)$ for all $i \in C$ and $\bar{p} \in \mathcal{L}(A)^{I \setminus C}$, there exists $i \in C$ such that $F(\hat{p}, \bar{p}) \succeq_{\hat{p}(i)} F(\hat{p}', \bar{p})$; F is strategy-proof if, for every $i \in I$, $\hat{p}, \hat{p}' \in \mathcal{L}(A)^{\{i\}}$ and $\bar{p} \in \mathcal{L}(A)^{I \setminus \{i\}}$,

[10] When $|A| = 2$, the simple majority rule with ties broken according to an agenda is group strategy-proof, Pareto optimal and anonymous. That fact makes all the problems studied in this paper trivial. Thus, in order to simplify the discussion we assume from the beginning $|A| \geq 3$.

[11] See, for instance, Barberà et al. (2010).

$F(\hat{p}, \bar{p}) \succeq_{\hat{p}(i)} F(\hat{p}', \bar{p})$. Note that Pareto optimality implies surjectivity, anonymity implies non-dictatorship, group strategy-proofness implies strategy-proofness. Moreover, the next fundamental result holds true.

Theorem 1 (Gibbard-Satterthwaite). *If F is a surjective and strategy-proof* SCF*, then F is dictatorial.*

Let $C \in \mathcal{P}_0(I)$. An information function for the coalition C is a function

$$\Omega_C : \mathcal{L}(A)^C \to \mathcal{P}_0\left(\mathcal{P}_0\left(\mathcal{L}(A)^{I\setminus C}\right)\right).$$

Thus, for every $\hat{p} \in \mathcal{L}(A)^C$, $\Omega_C(\hat{p})$ is a nonempty set of nonempty subsets of $\mathcal{L}(A)^{I\setminus C}$. In what follows, we are going to interpret Ω_C as a complete description of the type of information that individuals in C can get, a type of information that potentially depends on the preferences of individuals in C. More precisely, if individuals in C have preferences represented by $\hat{p} \in \mathcal{L}(A)^C$, then those individuals are supposed to know that the preferences of the others can be represented by an element of $\mathcal{L}(A)^{I\setminus C}$ belonging to a suitable $\omega \in \Omega_C(\hat{p})$. An information function profile is a vector of the type

$$\Omega = (\Omega_C)_{C \in \mathcal{P}_0(I)}$$

where, for every $C \in \mathcal{P}_0(I)$, Ω_C is an information function for the coalition C. An information function profile encodes then all the informational aspects of the society. Finally, we denote the set of all the information function profiles by $\mathbf{\Omega}$.

Definition 2. *Let F be a* SCF *and $\Omega \in \mathbf{\Omega}$. We say that F is [weakly] group Ω-strategy-proof if, for every $C \in \mathcal{P}_0(I)$, $\hat{p}, \hat{p}' \in \mathcal{L}(A)^C$ with $\hat{p}(i) \neq \hat{p}'(i)$ for all $i \in C$, $\omega \in \Omega_C(\hat{p})$ and $\bar{p} \in \omega$, we have that $F(\hat{p}', \bar{p}) \succ_{\hat{p}(i)} F(\hat{p}, \bar{p})$ for all $i \in C$ implies that there exists $\bar{p}' \in \omega$ such that $F(\hat{p}, \bar{p}') \succ_{\hat{p}(i)} F(\hat{p}', \bar{p}')$ $\left[F(\hat{p}, \bar{p}') \succeq_{\hat{p}(i)} F(\hat{p}', \bar{p}')\right]$ for some $i \in C$.*

Thus, F is [weakly] group Ω-strategy-proof if, every time the members of a coalition C, whose preferences are described by $\hat{p}$, know that the preferences of the others are surely described by some element of ω and observe that for an element $\bar{p}$ of ω it is convenient for all of them to report the false preferences described by $\hat{p}'$, then there exists another element $\bar{p}'$ of ω for which at least one individual in C would be better off [wouldn't be worse off] if all the individuals in C told the truth.

Definition 3. *Let F be a* SCF *and $\Omega \in \mathbf{\Omega}$. We say that F is [weakly] Ω-strategy-proof if, for every $i \in I$, $\hat{p}, \hat{p}' \in \mathcal{L}(A)^{\{i\}}$, $\omega \in \Omega_{\{i\}}(\hat{p})$ and $\bar{p} \in \omega$, we have that $F(\hat{p}', \bar{p}) \succ_{\hat{p}(i)} F(\hat{p}, \bar{p})$ implies that there exists $\bar{p}' \in \omega$ such that $F(\hat{p}, \bar{p}') \succ_{\hat{p}(i)} F(\hat{p}', \bar{p}')$ $\left[F(\hat{p}, \bar{p}') \succeq_{\hat{p}(i)} F(\hat{p}', \bar{p}')\right]$.*

Thus, F is [weakly] Ω-strategy proof if, every time an individual i, whose preferences are described by $\hat{p}$, knows that the preferences of the others are surely described by some element of ω and observe that for an element $\bar{p}$ of ω it is convenient for her to report the false preferences described by $\hat{p}'$, then there exists another element $\bar{p}'$ of ω for which she would be better off [wouldn't be worse off] if she told the truth.

Of course, for every $\Omega \in \mathbf{\Omega}$, we have that

$$\begin{array}{lcl} \text{group } \Omega\text{-strategy-proofness} & \Rightarrow & \text{weak group } \Omega\text{-strategy-proofness} \\ \text{group } \Omega\text{-strategy-proofness} & \Rightarrow & \Omega\text{-strategy-proofness} \\ \text{weak group } \Omega\text{-strategy-proofness} & \Rightarrow & \text{weak } \Omega\text{-strategy-proofness} \\ \Omega\text{-strategy-proofness} & \Rightarrow & \text{weak } \Omega\text{-strategy-proofness.} \end{array} \tag{1}$$

Note also that, given a SCF F and $\Omega, \Omega' \in \mathbf{\Omega}$ such that, for every $i \in I$, $\Omega_{\{i\}} = \Omega'_{\{i\}}$, we have that F is [weakly] Ω-strategy-proof if and only if F is [weakly] Ω'-strategy-proof.

The described informational framework is strongly inspired to the one proposed by Conitzer et al. (2011). Indeed, for every $i \in I$, those authors associate with individual i a family of subsets of $\mathcal{L}(A)^{I\setminus\{i\}}$, called information sets, and suppose that individual i can identify an information set having the property that the preferences of the others are represented by one of its elements. Thus, the concept of information function we are considering in this paper shares the same rationale as the one of family of information sets. Moreover, given a family of information sets for each individual, we have that the concept of immunity to dominating manipulation introduced by Conitzer et al. (2011) for SCFs coincides with the one of Ω-strategy-proofness, where Ω is any information function profile such that, for every $i \in I$ and $\hat{p} \in \mathcal{L}(A)^{\{i\}}$, $\Omega_{\{i\}}(\hat{p})$ agrees with the family of information sets of individual i. Thus, we have that any result proved within the framework of Conitzer et al. (2011) can be rephrased within the one described in this paper. In addition, our framework based on information functions allows to consider a wider variety of informational scenarios due to the possibility to model situations where the information of individuals depends on their own preferences as well as to consider coalitions of individuals. It also generalizes the framework proposed by Reijngoud and Endriss (2012). Indeed, those authors, developing a model of opinion polls in elections, assume there exist a set $\mathcal{I}$, encoding all the possible pieces of poll information available to individuals, and a function $\pi : \mathcal{L}(A)^I \to \mathcal{I}$, called poll information function (PIF), having the property that every time an individual i having preferences represented by $\hat{p} \in \mathcal{L}(A)^{\{i\}}$ receives the signal $s \in \{\pi(\hat{p}, \bar{p}) \in \mathcal{I} : \bar{p} \in \mathcal{L}(A)^{I \setminus \{i\}}\}$, then she knows that the preferences of the other individuals in the society can be represented by an element in the set $\{\bar{p} \in \mathcal{L}(A)^{I \setminus \{i\}} : \pi(\hat{p}, \bar{p}) = s\}$. Consistently with that viewpoint, the authors introduce and study the concept of immunity to π-manipulation for SCFs.[12] It can be easily shown that, given a PIF π, the concept of immunity to π-manipulation coincides with the one of Ω-strategy-proofness, where Ω is any information function profile such that, for every $i \in I$ and $\hat{p} \in \mathcal{L}(A)^{\{i\}}$,

$$\Omega_{\{i\}}(\hat{p}) = \left\{ \left\{ \bar{p}' \in \mathcal{L}(A)^{I \setminus \{i\}} : \pi(\hat{p}, \bar{p}') = \pi(\hat{p}, \bar{p}) \right\} : \bar{p} \in \mathcal{L}(A)^{I \setminus \{i\}} \right\}.$$

As a consequence, we get that any result proved within the framework of Reijngoud and Endriss (2012) can be rephrased within ours. Finally, it is worth noticing that the use of information functions allows to model informational situations that cannot be modelled using PIFs. Consider, for instance, the situation where an individual i, independently on her own preferences, knows that the preferences of the others are surely represented by an element of a set ω, where ω is a nonempty and proper subset of $\mathcal{L}(A)^{I \setminus \{i\}}$, and no further information is available to her. We can model such a situation by considering $\Omega \in \mathbf{\Omega}$ such that, for every $\hat{p} \in \mathcal{L}(A)^{\{i\}}$, $\Omega_{\{i\}}(\hat{p}) = \{\omega\}$. On the other hand, in order to model that situation within the framework of Reijngoud and Endriss (2012), one should find a PIF π such that, for every $\hat{p} \in \mathcal{L}(A)^{\{i\}}$ and $s \in \{\pi(\hat{p}, \bar{p}) \in \mathcal{I} : \bar{p} \in \mathcal{L}(A)^{I \setminus \{i\}}\}$, $\{\bar{p} \in \mathcal{L}(A)^{I \setminus \{i\}} : \pi(\hat{p}, \bar{p}) = s\} = \omega$, but that is not possible.

The next definition is inspired to Definition 1 in Reijngoud and Endriss (2012).

Definition 4. *Let $\Omega, \Omega' \in \mathbf{\Omega}$. We say that Ω' is at least as informative as Ω, and we write $\Omega' \trianglerighteq \Omega$, if, for every $C \in \mathcal{P}_0(I)$, $\hat{p} \in \mathcal{L}(A)^C$ and $\omega \in \Omega_C(\hat{p})$, there exists $\mathcal{A} \subseteq \Omega'_C(\hat{p})$ with $\mathcal{A} \neq \varnothing$ such that $\omega = \bigcup_{\omega' \in \mathcal{A}} \omega'$.*

It is immediate to show that if $\Omega, \Omega', \Omega'' \in \mathbf{\Omega}$, then $\Omega'' \trianglerighteq \Omega'$ and $\Omega' \trianglerighteq \Omega$ imply $\Omega'' \trianglerighteq \Omega$. Proposition 5 is analogous to Lemma 1 in Reijngoud and Endriss (2012). It shows that, when we are dealing with information function profiles that are comparable with respect to the relation $\trianglerighteq$, we can also compare the corresponding versions of Ω-strategy-proofness.

Proposition 5. *Let F be a SCF and $\Omega, \Omega' \in \mathbf{\Omega}$ with $\Omega' \trianglerighteq \Omega$.*

(a) If F is [weakly] group Ω'-strategy-proof, then it is [weakly] group Ω-strategy-proof.

(b) If F is [weakly] Ω'-strategy-proof, then it is [weakly] Ω-strategy-proof.

Proof. Let us prove (a) proving that if F is not [weakly] group Ω-strategy-proof, then F is not [weakly] group Ω'-strategy-proof. Assume then that F is not [weakly] group Ω-strategy-proof. Then there

[12] That approach is shared by Endriss et al. (2016). Terzopoulou and Endriss (2019) introduce the analogous concept of judgement information function while studying the problem of manipulation under partial information in the judgement aggregation setting.

exist $C \in \mathcal{P}_0(I)$, $\hat{p}, \hat{p}' \in \mathcal{L}(A)^C$ with $\hat{p}(i) \neq \hat{p}'(i)$ for all $i \in C$, $\omega \in \Omega_C(\hat{p})$ and $\bar{p} \in \omega$ such that, for every $i \in C$ and $\bar{p}' \in \omega$, $F(\hat{p}', \bar{p}) \succ_{\hat{p}(i)} F(\hat{p}, \bar{p})$ and $F(\hat{p}', \bar{p}') \succeq_{\hat{p}(i)} F(\hat{p}, \bar{p}')$ $\left[F(\hat{p}', \bar{p}') \succ_{\hat{p}(i)} F(\hat{p}, \bar{p}')\right]$. Consider $\mathcal{A} \subseteq \Omega'_C(\hat{p})$ with $\mathcal{A} \neq \varnothing$ such that $\omega = \bigcup_{\omega' \in \mathcal{A}} \omega'$. Then there exists $\omega' \in \mathcal{A}$ such that $\bar{p} \in \omega'$. Since $\omega' \subseteq \omega$, we have that, for every $i \in C$ and $\bar{p}' \in \omega'$, $F(\hat{p}', \bar{p}) \succ_{\hat{p}(i)} F(\hat{p}, \bar{p})$ and $F(\hat{p}', \bar{p}') \succeq_{\hat{p}(i)} F(\hat{p}, \bar{p}')$ $\left[F(\hat{p}', \bar{p}') \succ_{\hat{p}(i)} F(\hat{p}, \bar{p}')\right]$. As a consequence, F is not [weakly] group Ω'-strategy-proof.

The proof of (b) is analogous and then omitted. □

4 Examples of information function profiles

Let $\Omega^{SP} \in \mathbf{\Omega}$ be such that, for every $C \in \mathcal{P}_0(I)$ and $\hat{p} \in \mathcal{L}(A)^C$,

$$\Omega_C^{SP}(\hat{p}) = \left\{ \{\bar{p}\} : \bar{p} \in \mathcal{L}(A)^{I \setminus C} \right\}.$$

Thus, according to Ω^{SP}, members of any coalition are able to know the preferences of the other individuals in the society or, equivalently, the complete preference profile. It is immediate to show that group Ω^{SP}-strategy-proofness and weak group Ω^{SP}-strategy-proofness both agree with group strategy-proofness and that Ω^{SP}-strategy-proofness and weak Ω^{SP}-strategy-proofness both agree with strategy-proofness.

Let $\Omega^{WMG} \in \mathbf{\Omega}$ be such that, for every $C \in \mathcal{P}_0(I)$ and $\hat{p} \in \mathcal{L}(A)^C$,

$$\Omega_C^{WMG}(\hat{p}) = \left\{ \left\{ \bar{p}' \in \mathcal{L}(A)^{I \setminus C} : c_{\bar{p}'} = c_{\bar{p}} \right\} : \bar{p} \in \mathcal{L}(A)^{I \setminus C} \right\}.$$

Observe that, for every $C \in \mathcal{P}_0(I)$ and $\hat{p} \in \mathcal{L}(A)^C$,

$$\Omega_C^{WMG}(\hat{p}) = \left\{ \left\{ \bar{p}' \in \mathcal{L}(A)^{I \setminus C} : c_{(\hat{p}, \bar{p}')} = c_{(\hat{p}, \bar{p})} \right\} : \bar{p} \in \mathcal{L}(A)^{I \setminus C} \right\}. \tag{2}$$

Thus, according to Ω^{WMG}, members of any coalition are able to know the weighted majority graph associated with the preferences of the others or, equivalently, the weighted majority graph associated with the complete preference profile.

For every $p \in \mathcal{L}(A)^I$, consider the majority graph associated with p, that is, the graph whose vertex set is A and whose arc set is

$$M(p) = \left\{ (x, y) \in A_*^2 : c_p(x, y) > \frac{|I|}{2} \right\}.$$

Let $\Omega^{MG} \in \mathbf{\Omega}$ be such that, for every $C \in \mathcal{P}_0(I)$ and $\hat{p} \in \mathcal{L}(A)^C$,

$$\Omega_C^{MG}(\hat{p}) = \left\{ \left\{ \bar{p}' \in \mathcal{L}(A)^{I \setminus C} : M(\hat{p}, \bar{p}') = M(\hat{p}, \bar{p}) \right\} : \bar{p} \in \mathcal{L}(A)^{I \setminus C} \right\}.$$

Thus, according to Ω^{MG}, members of any coalition are able to know the majority graph associated with the complete preference profile.

Finally, let $\Omega^{zero} \in \mathbf{\Omega}$ be such that, for every $C \in \mathcal{P}_0(I)$ and $\hat{p} \in \mathcal{L}(A)^C$,

$$\Omega_C^{zero}(\hat{p}) = \left\{ \mathcal{L}(A)^{I \setminus C} \right\}.$$

Thus, according to Ω^{zero}, members of any coalition have no idea about the preferences of the others.

In order to simplify the notation, from now on, we write [weak; group] WMG-strategy-proofness instead of [weak; group] Ω^{WMG}-strategy-proofness, [weak; group] MG-strategy-proofness instead of [weak; group] Ω^{MG}-strategy-proofness, [weak; group] zero-strategy-proofness instead of [weak; group] Ω^{zero}-strategy-proofness.

Note that the concepts of WMG-strategy-proofness, MG-strategy-proofness and zero-strategy-proofness coincide with the one of immunity to π-manipulation introduced by Reijngoud and Endriss (2012), where π is the WMG-PIF, the MG-PIF and the zero-PIF, respectively. Note also that, for every $C \in \mathcal{P}_0(I)$ and $\hat{p}, \hat{p}' \in \mathcal{L}(A)^C$, we have that $\Omega_C^{SP}(\hat{p}) = \Omega_C^{SP}(\hat{p}')$, $\Omega_C^{WMG}(\hat{p}) = \Omega_C^{WMG}(\hat{p}')$ and $\Omega_C^{zero}(\hat{p}) = \Omega_C^{zero}(\hat{p}')$. Such a property is not satisfied by Ω^{MG}.

Interestingly, the considered information function profiles are pairwise comparable with respect to the relation $\trianglerighteq$, as described by the next proposition.

Proposition 6. $\Omega^{SP} \trianglerighteq \Omega^{WMG} \trianglerighteq \Omega^{MG} \trianglerighteq \Omega^{zero}$.

Proof. The proof that $\Omega^{SP} \trianglerighteq \Omega^{WMG}$ and $\Omega^{MG} \trianglerighteq \Omega^{zero}$ is obvious.

We are then left with proving that $\Omega^{WMG} \trianglerighteq \Omega^{MG}$. Consider $C \in \mathcal{P}_0(I)$, $\hat{p} \in \mathcal{L}(A)^C$ and $\omega \in \Omega_C^{MG}(\hat{p})$. We have to show that there exists $\mathcal{A} \subseteq \Omega_C^{WMG}(\hat{p})$ such that $\omega = \bigcup_{\omega' \in \mathcal{A}} \omega'$. Since $\omega \in \Omega_C^{MG}(\hat{p})$, there exists $\bar{p}$ such that

$$\omega = \left\{ \bar{p}' \in \mathcal{L}(A)^{I \setminus C} : M(\hat{p}, \bar{p}') = M(\hat{p}, \bar{p}) \right\}.$$

Consider now the set

$$\mathcal{A} = \left\{ \left\{ \bar{p}'' \in \mathcal{L}(A)^{I \setminus C} : c_{(\hat{p}, \bar{p}'')} = c_{(\hat{p}, \bar{p}')} \right\} : \bar{p}' \in \omega \right\}.$$

Note that, since $\bar{p} \in \omega$, we have that $\mathcal{A} \neq \varnothing$. Moreover, by (2), it is immediate to observe that $\mathcal{A} \subseteq \Omega_C^{WMG}(\hat{p})$. Denoting by $\hat{\omega}$ the set $\bigcup_{\omega' \in \mathcal{A}} \omega'$, we are left with proving that $\omega = \hat{\omega}$. First of all, note that

$$\hat{\omega} = \bigcup_{\bar{p}' \in \omega} \left\{ \bar{p}'' \in \mathcal{L}(A)^{I \setminus C} : c_{(\hat{p}, \bar{p}'')} = c_{(\hat{p}, \bar{p}')} \right\}.$$

If $\bar{p}' \in \omega$, then $\bar{p}' \in \left\{ \bar{p}'' \in \mathcal{L}(A)^{I \setminus C} : c_{(\hat{p}, \bar{p}'')} = c_{(\hat{p}, \bar{p}')} \right\}$ so that $\bar{p}' \in \hat{\omega}$. Then we get $\omega \subseteq \hat{\omega}$. Consider now $\bar{p}'' \in \hat{\omega}$. Thus, there exists $\bar{p}' \in \omega$ such that $c_{(\hat{p}, \bar{p}'')} = c_{(\hat{p}, \bar{p}')}$ and then, in particular, $M(\hat{p}, \bar{p}'') = M(\hat{p}, \bar{p}')$. Since $M(\hat{p}, \bar{p}') = M(\hat{p}, \bar{p})$, we have that $M(\hat{p}, \bar{p}'') = M(\hat{p}, \bar{p})$ which implies $\bar{p}'' \in \omega$. Then we get $\hat{\omega} \subseteq \omega$. From $\omega \subseteq \hat{\omega}$ and $\hat{\omega} \subseteq \omega$, we finally deduce $\omega = \hat{\omega}$. □

Theorem 7 summarizes a number of results about MG-strategy-proofness and zero-strategy-proofness due Reijngoud and Endriss (2012) and Endriss et al. (2016).[13] Further results about [weak; group] MG-strategy-proofness and [weak; group] zero-strategy-proofness can be obtained from Theorem 7 by applying the implications in (1). Recall that a social choice correspondence (SCC) is a function from $\mathcal{L}(A)^I$ to $\mathcal{P}_0(A)$.

Theorem 7. *Let F be a* SCF *and $\alpha \in \mathcal{L}(A)$.*

(a) *If $|I|$ is even and F is a strongly Condorcet-consistent*[14] SCC *with ties broken according to α, then F is not MG-strategy-proof.*

(b) *If $|I| \geq 10$ and F is the Plurality* SCC *with ties broken according to α, then F is MG-strategy-proof.*

(c) *If F is the Borda* SCC *with ties broken according to α, then F is not MG-strategy-proof.*

(d) *If F is the Copeland* SCC *with ties broken according to α, then F is not MG-strategy-proof.*

(e) *For every $1 \leq k \leq |A| - 2$, there exists $n \in \mathbb{N}$ such that if $|I| \geq n$ and F is the k-approval* SCC *with ties broken according to α, then F is MG-strategy-proof.*

(f) *If $|I| \geq 3$ and F is a strongly Condorcet-consistent* SCC *with ties broken according to α, then F is zero-strategy-proof.*

[13] Parts (a), (b), (f) and (g) of Theorem 7 respectively correspond to Theorems 2, 7, 4, 5 in Reijngoud and Endriss (2012); parts (c) and (d) of Theorem 7 correspond to Theorem 4 in Endriss et al. (2016); part (e) of Theorem 7 corresponds to Theorem 5 in Endriss et al. (2016).

[14] A SCC is said to be strong Condorcet-consistent if it exactly selects the set of weak Condorcet winners whenever that set is nonempty. The Simpson SCC, a.k.a. the Maximin or the Condorcet SCC, fulfils that property.

(*g*) *If $|I| \geq 2|A|-2$ and F is a positional scoring* SCC *with ties broken according to α, then F is zero-strategy-proof.*

At the best of our knowledge, no results explicitly related to [weak; group] WMG-strategy-proofness are available in the literature. In the next section some possibility and impossibility results for those properties are presented.

5 Main results

Since in this section we are going to deal with the concepts of [weak] group WMG-strategy-proofness and [weak] WMG-strategy-proofness only, it is certainly useful to rephrase Definitions 2 and 3 for those special cases.

Definition 8. *Let F be a* SCF*. We say that F is [weakly] group WMG-strategy-proof if, for every $C \in \mathcal{P}_0(I)$, $\hat{p}, \hat{p}' \in \mathcal{L}(A)^C$ with $\hat{p}(i) \neq \hat{p}'(i)$ for all $i \in C$, and $\bar{p} \in \mathcal{L}(A)^{I\setminus C}$, we have that $F(\hat{p}', \bar{p}) \succ_{\hat{p}(i)} F(\hat{p}, \bar{p})$ for all $i \in C$ implies that there exists $\bar{p}' \in \mathcal{L}(A)^{I\setminus C}$ with $\bar{p}' \sim \bar{p}$ such that $F(\hat{p}, \bar{p}') \succ_{\hat{p}(i)} F(\hat{p}', \bar{p}')$ $\left[F(\hat{p}, \bar{p}') \succeq_{\hat{p}(i)} F(\hat{p}', \bar{p}')\right]$ for some $i \in C$.*

Definition 9. *Let F be a* SCF*. We say that F is [weakly] WMG-strategy-proof if, for every $i \in I$, $\hat{p}, \hat{p}' \in \mathcal{L}(A)^{\{i\}}$ and $\bar{p} \in \mathcal{L}(A)^{I\setminus\{i\}}$, we have that $F(\hat{p}', \bar{p}) \succ_{\hat{p}(i)} F(\hat{p}, \bar{p})$ implies that there exists $\bar{p}' \in \mathcal{L}(A)^{I\setminus\{i\}}$ with $\bar{p}' \sim \bar{p}$ such that $F(\hat{p}, \bar{p}') \succ_{\hat{p}(i)} F(\hat{p}', \bar{p}')$ $\left[F(\hat{p}, \bar{p}') \succeq_{\hat{p}(i)} F(\hat{p}', \bar{p}')\right]$.*

By means of Propositions 5 and 6 and Theorem 7, we deduce that the SCFs considered in parts (*a*), (*c*) and (*d*) of Theorem 7 are not WMG-strategy-proof. It is also easy to verify that strategy-proofness and [weak] WMG-strategy-proofness agree when $|I| = 2$. As a consequence, if $|I| = 2$, by Theorem 1, we have that, [weak] WMG-strategy-proofness and surjectivity imply dictatorship. However, that fact does not hold true in general. Indeed, as the next result immediately implies, there exist Pareto optimal, WMG-strategy-proof and non-dictatorial SCFs, provided that $|I| \geq 3$. Theorem 10 is the first main result of the paper.

Theorem 10. *Let J and K be nonempty subsets of I such that $I = J \cup K$ and $J \cap K = \varnothing$, and let $(L_i)_{i\in J}$ be a family of subsets of $\mathcal{L}(A)^K$ such that, for every $i, j \in J$ with $i \neq j$, $L_i \cap L_j = \varnothing$ and $\bigcup_{i\in J} L_i = \mathcal{L}(A)^K$. Let F be the* SCF *defined, for every $p \in \mathcal{L}(A)^I$, by $F(p) = r_1(p(i))$, where $i \in J$ is such that $p_{|K} \in L_i$. Then F is Pareto optimal and WMG-strategy-proof. Moreover, for the function F, the following conditions are equivalent:*

(*i*) *F is dictatorial,*

(*ii*) *F is weakly group WMG-strategy-proof,*

(*iii*) $|\{i \in J : L_i \neq \varnothing\}| = 1$.

We are not able, at the moment, to characterize either the surjective and [weakly] WMG-strategy-proof SCFs or the Pareto optimal and [weakly] WMG-strategy-proof SCFs. However, the next result shows that Pareto optimality, weak WMG-strategy-proofness and anonymity are surely inconsistent with each other. Theorem 11 is the second main result of the paper.

Theorem 11. *If F is a Pareto optimal and weakly WMG-strategy-proof* SCF*, then F is not anonymous.*

The major consequence of Theorem 11 is that any SCF that is obtained by a Pareto optimal and anonymous SCC[15], like the Plurality, the Borda, the Copeland, the Simpson and the Kemeny, endowed with an agenda for breaking ties fails to be weakly WMG-strategy-proof, provided there are at least three alternatives. By Theorem 11 and Proposition 5, we can also deduce the next

[15]Given a SCC F, we say that F is Pareto optimal if, for every $p \in \mathcal{L}(A)^I$ and $x, y \in A$, if $x \succ_{p(i)} y$ for all $i \in I$, then $y \notin F(p)$; F is anonymous if, for every $p \in \mathcal{L}(A)^I$ and $\varphi \in \mathrm{Sym}(I)$, we have that $F(p^\varphi) = F(p)$.

result. It states that each Pareto optimal and anonymous SCF can potentially be manipulated by an individual whose information about the preferences of the others is at least as informative as the knowledge of the number of individuals preferring an alternative to another one for all possible pairs of alternatives.

Corollary 12. *Let $\Omega \in \mathbf{\Omega}$ with $\Omega \trianglerighteq \Omega^{WMG}$. If F is a Pareto optimal and weakly Ω-strategy-proof* SCF*, then F is not anonymous.*

For the special family of Pareto optimal SCFs considered in Theorem 10, weak group WMG-strategy-proofness is equivalent to dictatorship. An interesting question is to understand whether weak group WMG-strategy-proofness and Pareto optimality are always enough to get dictatorship. We are able to prove that it is the case when three alternatives are considered. That is the content of Theorem 13, the third main result of the paper. It is an open problem proving whether the same result holds true for four alternatives or more.

Theorem 13. *Let $|A| = 3$. If F is a Pareto optimal and weakly group WMG-strategy-proof* SCF*, then F is dictatorial.*

Similarly as before, by Theorem 13 and Proposition 5, we can also get the next result.

Corollary 14. *Let $|A| = 3$ and $\Omega \in \mathbf{\Omega}$ with $\Omega \trianglerighteq \Omega^{WMG}$. If F is Pareto optimal and weakly group Ω-strategy-proof* SCF*, then F is dictatorial.*

Theorems 10, 11 and 13 are proved in the appendix.

6 Indexes for measuring the degree of manipulability

The evaluation of the degree of manipulability of social choice functions and correspondences is definitely an interesting problem and several indexes for measuring the degree of manipulability were proposed and studied. We mention here the contributions by Nitzan (1985), Kelly (1988, 1993), Aleskerov and Kurbanov (1999) and Aleskerov et al. (2012). The informational framework we are considering allows to define a variety of such indexes according to the following ideas.

Consider a finite sequence $\mathcal{S} = (\Omega^t)_{t\in\{1,\dots,k\}}$ in $\mathbf{\Omega}$ where $k \in \mathbb{N}$. Suppose that, for every $t_1, t_2 \in \{1,\dots,k\}$ with $t_1 \le t_2$, we have that $\Omega^{t_1} \trianglerighteq \Omega^{t_2}$. Given a SCF F, we define the following numbers

$$\tau^{wsp}_{\mathcal{S}}(F) = \min\big(\{t \in \{1,\dots,k\} : F \text{ is weakly } \Omega^t\text{-strategy-proof}\} \cup \{k+1\}\big),$$
$$\tau^{sp}_{\mathcal{S}}(F) = \min\big(\{t \in \{1,\dots,k\} : F \text{ is } \Omega^t\text{-strategy-proof}\} \cup \{k+1\}\big),$$
$$\tau^{wgsp}_{\mathcal{S}}(F) = \min\big(\{t \in \{1,\dots,k\} : F \text{ is weakly group } \Omega^t\text{-strategy-proof}\} \cup \{k+1\}\big),$$
$$\tau^{gsp}_{\mathcal{S}}(F) = \min\big(\{t \in \{1,\dots,k\} : F \text{ is group } \Omega^t\text{-strategy-proof}\} \cup \{k+1\}\big).$$

Each of these numbers can be seen as an index that measures the degree of manipulability of F: the smaller is the index, the more difficult is manipulating F. Indeed, consider two SCFs F_1 and F_2 and let $\tau_1 = \tau^{wsp}_{\mathcal{S}}(F_1)$ and $\tau_2 = \tau^{wsp}_{\mathcal{S}}(F_2)$. If $\tau_1 < \tau_2$, then F_1 is weakly Ω^{τ_1}-strategy-proof and F_2 is not; moreover, for every $t \in \{1,\dots,k\}$ for which F_2 is weakly Ω^t-strategy-proof, F_1 is weakly Ω^t-strategy-proof too. Thus, referring to the concept of manipulation underlying the definition of weak Ω-strategy-proofness, we can state that individuals can potentially manipulate F_2 more easily than F_1, namely using less information (among the type of information encoded in the information function profiles of $\mathcal{S}$). Similar observations hold true for the other indexes. Note also that $\tau^{wsp}_{\mathcal{S}}(F) \le \tau^{sp}_{\mathcal{S}}(F) \le \tau^{gsp}_{\mathcal{S}}(F)$ and $\tau^{wsp}_{\mathcal{S}}(F) \le \tau^{wgsp}_{\mathcal{S}}(F) \le \tau^{gsp}_{\mathcal{S}}(F)$.

As particular instances of the considered indexes, let us consider the ones associated with the finite sequence $\mathcal{S}^* = (\Omega^t)_{t\in\{1,\dots,|I|+1\}}$ in $\mathbf{\Omega}$ defined as follows. For every $t \in \{1,\dots,|I|+1\}$, let $V(t)$ be the set whose elements are the sets of t consecutive numbers taken in $\{0,\dots,|I|\}$. Thus, for instance, if $|I| = 3$ we have that

$$V(1) = \{\{0\},\{1\},\{2\},\{3\}\}, \quad V(2) = \{\{0,1\},\{1,2\},\{2,3\}\},$$

$$V(3)=\{\{0,1,2\},\{1,2,3\}\},\quad V(4)=\{\{0,1,2,3\}\}.$$

Fixed now $t\in\{1,\dots,|I|+1\}$, let $\mathcal{V}(t)$ be the set of functions v from A_*^2 to $V(t)$ such that, for every $(x,y)\in A_*^2$,

$$v(y,x)=\{i\in\{0,\dots,|I|\}:\text{ there exists } j\in v(x,y) \text{ such that } i=|I|-j\}, \tag{3}$$

and let $\Omega^t\in\mathbf{\Omega}$ be defined, for every $C\in\mathcal{P}_0(I)$ and $\hat{p}\in\mathcal{L}(A)^C$, by

$$\Omega^t_C(\hat{p})=\left\{\left\{\bar{p}\in\mathcal{L}(A)^{I\setminus C}:\forall(x,y)\in A_*^2, c_{(\hat{p},\bar{p})}(x,y)\in v(x,y)\right\}:v\in\mathcal{V}(t)\right\}\setminus\{\varnothing\}. \tag{4}$$

The information function profile Ω^t describes a situation where the members of each coalition are able to establish, for every pair of distinct alternatives, that the number of individuals in the society preferring the first alternative to the second one belongs to a suitable set of consecutive numbers having size t. We stress that condition (3) only requires that v is consistent with the fact that individual preferences are linear orders so that if an individual does not prefer x to y then she prefers y to x. Note also that, for every $C\in\mathcal{P}_0(I)$ and $\hat{p}\in\mathcal{L}(A)^C$, the set

$$\left\{\bar{p}\in\mathcal{L}(A)^{I\setminus C}:\forall(x,y)\in A_*^2, c_{(\hat{p},\bar{p})}(x,y)\in v(x,y)\right\}$$

cannot be empty for all $v\in\mathcal{V}(t)$. However, it might be empty for some $v\in\mathcal{V}(t)$ and (4) takes into account the fact. Moreover, we have that $\Omega^1=\Omega^{WMG}$, $\Omega^{|I|+1}=\Omega^{zero}$ and, for every $t\in\{1,\dots,|I|\}$, $\Omega^t\trianglerighteq\Omega^{t+1}$.

Some properties of the indexes associated with $\mathcal{S}^*$ are immediate. First of all, the value of all the indexes belongs to $\{1,\dots,|I|+2\}$. By Theorem 11, we have that if F is a Pareto optimal and anonymous SCF, then $\tau^{wsp}_{\mathcal{S}^*}(F)\ge 2$ so that $\tau^{sp}_{\mathcal{S}^*}(F)\ge 2$, $\tau^{wgsp}_{\mathcal{S}^*}(F)\ge 2$ and $\tau^{gsp}_{\mathcal{S}^*}(F)\ge 2$. By Theorem 13, we have that if $|A|=3$ and F is a Pareto optimal and non-dictatorial SCF, then $\tau^{wgsp}_{\mathcal{S}^*}(F)\ge 2$ so that $\tau^{gsp}_{\mathcal{S}^*}(F)\ge 2$.

Assume now that $A=\{a,b,c\}$ and $|I|=5$ and let BOR, PLU, NEG, SIM and AV be the SCFs respectively obtained by the Borda, the Plurality, the Negative Plurality, the Simpson and the Alternative Vote[16] SCCs with ties alphabetically broken. With the help of a computer[17], it can be shown that

$$\tau^{wsp}_{\mathcal{S}^*}(\mathrm{BOR})=4,\quad \tau^{wsp}_{\mathcal{S}^*}(\mathrm{PLU})=4,\quad \tau^{wsp}_{\mathcal{S}^*}(\mathrm{NEG})=3,\quad \tau^{wsp}_{\mathcal{S}^*}(\mathrm{SIM})=4,\quad \tau^{wsp}_{\mathcal{S}^*}(\mathrm{AV})=3,$$
$$\tau^{sp}_{\mathcal{S}^*}(\mathrm{BOR})=4,\quad \tau^{sp}_{\mathcal{S}^*}(\mathrm{PLU})=4,\quad \tau^{sp}_{\mathcal{S}^*}(\mathrm{NEG})=3,\quad \tau^{sp}_{\mathcal{S}^*}(\mathrm{SIM})=4,\quad \tau^{sp}_{\mathcal{S}^*}(\mathrm{AV})=4.$$

Thus, when five individuals and three alternatives are considered, $\tau^{wsp}_{\mathcal{S}^*}$ and $\tau^{sp}_{\mathcal{S}^*}$ are able to discriminate among classical SCFs. Moreover, the fact that BOR, PLU and SIM are not weakly Ω^3-strategy-proof seems to us a strong and quite unexpected result as well as the fact that none of the considered SCFs is weakly Ω^2-strategy-proof.

A Appendix

Recall that we are assuming $|A|\ge 3$. Let $q\in\mathcal{L}(A)$. We denote by q^r the element of $\mathcal{L}(A)$ such that, for every $x,y\in A$, $x\succeq_{q^r} y$ if and only if $y\succeq_q x$. If q is such that $x_1\succ_q\dots\succ_q x_{|A|}$, where $A=\{x_1,\dots,x_{|A|}\}$, we identify q with the writing $x_1\cdots x_{|A|}$. Note that if $q=x_1\cdots x_{|A|}$, then $q^r=x_{|A|}\cdots x_1$ and $q\neq q^r$. Given $i\in I$, we denote by $q[i]$ the element of $\mathcal{L}(A)^{\{i\}}$ such that

[16] The Alternative Vote SCC is also known as Hare's method. When five individuals and three alternatives are considered it works as follows. If there is an alternative that is first ranked at least three times, then that alternative is the unique selected alternative. If no alternative is first ranked at least three times, then there are exactly two alternatives, say x and y, that are both first ranked twice. If at least three individuals prefer x to y, then x is the unique selected alternative, else y is the unique selected alternative. Note that no tie can occur.

[17] The results are obtained using the CAS Maxima. The program is available upon request.

$q[i](i) = q$. Thus, for every $\bar{p} \in \mathcal{L}(A)^{I\setminus\{i\}}$, the writing $(q[i], \bar{p})$ represents the element of $\mathcal{L}(A)^I$ such that $(q[i], \bar{p})(i) = q$ and $(q[i], \bar{p})(j) = \bar{p}(j)$ for all $j \in I \setminus \{i\}$.

According to the described notation, we have that a SCF F is [weakly] WMG-strategy-proof if, for every $i \in I$, $q, q' \in \mathcal{L}(A)$ and $\bar{p} \in \mathcal{L}(A)^{I\setminus\{i\}}$, $F(q'[i], \bar{p}) \succ_q F(q[i], \bar{p})$ implies that there exists $\bar{p}' \in \mathcal{L}(A)^{I\setminus\{i\}}$ with $\bar{p}' \sim \bar{p}$ such that $F(q[i], \bar{p}') \succ_q F(q'[i], \bar{p}')$ $[F(q[i], \bar{p}') \succeq_q F(q'[i], \bar{p}')]$.

A.1 Proofs of Theorems 11 and 13: the three alternative case

In this section we assume $|A| = 3$ and $A = \{a, b, c\}$. Thus, we have that

$$\mathcal{L}(A) = \{abc, acb, bac, bca, cab, cba\}.$$

Proposition 15. *Let $p, p' \in \mathcal{L}(A)^I$. Then the following conditions are equivalent:*

(*a*) *there exists $\varphi \in \mathrm{Sym}(I)$ such that, $p' = p^\varphi$,*

(*b*) *for every $q \in \mathcal{L}(A)$, $|\{i \in I : p(i) = q\}| = |\{i \in I : p'(i) = q\}|$.*

Proof. $(a) \Rightarrow (b)$ Assume that there exists $\varphi \in \mathrm{Sym}(I)$ such that, $p' = p^\varphi$ and fix $q \in \mathcal{L}(A)$. Then

$$\{i \in I : p'(i) = q\} = \{i \in I : p^\varphi(i) = q\} = \{i \in I : p(\varphi^{-1}(i)) = q\} = \varphi(\{i \in I : p(i) = q\}).$$

Since φ is bijective, we get $|\{i \in I : p'(i) = q\}| = |\{i \in I : p(i) = q\}|$.

$(b) \Rightarrow (a)$ Assume now that, for every $q \in \mathcal{L}(A)$, $|\{i \in I : p(i) = q\}| = |\{i \in I : p'(i) = q\}|$. For every $q \in \mathcal{L}(A)$, let us define $I(q) = \{i \in I : p(i) = q\}$ and $I'(q) = \{i \in I : p'(i) = q\}$. Observe that

$$I = \bigcup_{q \in \mathcal{L}(A)} I(q) = \bigcup_{q \in \mathcal{L}(A)} I'(q)$$

and that, for every $q, q' \in \mathcal{L}(A)$ with $q \neq q'$, $I(q) \cap I(q') = \varnothing$ and $I'(q) \cap I'(q') = \varnothing$. Moreover, for every $q \in \mathcal{L}(A)$, $|I(q)| = |I'(q)|$, so that there exists a bijective function $\varphi_q : I(q) \to I'(q)$. Let now consider $\varphi : I \to I$ be defined, for every $i \in I$, by $\varphi(i) = \varphi_q(i)$ if $i \in I(q)$. The function φ is well-defined since, for every $i \in I$, there exists a unique $q \in \mathcal{L}(A)$, namely $q = p(i)$, such that $i \in I(q)$.

Let us prove that $\varphi \in \mathrm{Sym}(I)$. Since I is finite it is sufficient to prove that φ is injective. Consider $(i, j) \in I^2_*$. Assume first that there exists $q \in \mathcal{L}(A)$ such that $i, j \in I(q)$. Then $\varphi(i) = \varphi_q(i)$ and $\varphi(j) = \varphi_q(j)$. Since $\varphi_q : I(q) \to I'(q)$ is bijective, we get $\varphi_q(i) \neq \varphi_q(j)$ and then $\varphi(i) \neq \varphi(j)$. Assume now that there exist $q, q' \in \mathcal{L}(A)$ with $q \neq q'$ such that $i \in I(q)$ and $j \in I(q')$. Then $\varphi(i) = \varphi_q(i) \in I'(q)$ and $\varphi(j) = \varphi_{q'}(j) \in I'(q')$. Since $I'(q) \cap I'(q') = \varnothing$ we get $\varphi_q(i) \neq \varphi_q(j)$ and then $\varphi(i) \neq \varphi(j)$.

We complete the proof showing that $p' = p^\varphi$. Consider $i \in I$ and note that $i \in I(p(i))$. Since $\varphi(i) \in I'(p(i))$, we have that $p'(\varphi(i)) = p(i)$. As a consequence, for every $i \in I$, $p'(i) = p(\varphi^{-1}(i)) = p^\varphi(i)$, that is, $p' = p^\varphi$. □

Proposition 16. *Let $p \in \mathcal{L}(A)^I$. Then $\{p^\varphi \in \mathcal{L}(A)^I : \varphi \in \mathrm{Sym}(I)\} \subseteq \{p' \in \mathcal{L}(A)^I : p' \sim p\}$.*

Proof. Let $\varphi \in \mathrm{Sym}(I)$ and show that $p^\varphi \sim p$. Consider $(x, y) \in A^2_*$. Using the definition of p^φ and the fact that φ is bijective, we have that

$$c_{p^\varphi}(x, y) = |\{i \in I : x \succ_{(p^\varphi)(i)} y\}| = |\{i \in I : x \succ_{p(\varphi^{-1}(i))} y\}|$$

$$= |\varphi(\{i \in I : x \succ_{p(i)} y\})| = |\{i \in I : x \succ_{p(i)} y\}| = c_p(x, y).$$

Since the argument above does not depend on x and y, we get that $p^\varphi \sim p$, as desired. □

Theorem 17. *Let $p \in \mathcal{L}(A)^I$. Then the following conditions are equivalent:*

(*a*) $\{p^\varphi \in \mathcal{L}(A)^I : \varphi \in \mathrm{Sym}(I)\} = \{p' \in \mathcal{L}(A)^I : p' \sim p\}$, (*b*) *for every* $i, j \in I$, $p(i) \neq p(j)^r$.

Proof. $(a) \Rightarrow (b)$ Assume that $\{p^\varphi \in \mathcal{L}(A)^I : \varphi \in \mathrm{Sym}(I)\} = \{p' \in \mathcal{L}(A)^I : p' \sim p\}$ and suppose by contradiction that there are $i^*, j^* \in I$ such that $p(i^*) = p(j^*)^r$. Thus, $p(i^*) \neq p(j^*)$ and, in particular, $i^* \neq j^*$. Since $|\mathcal{L}(A)| = 6$, there exists $q \in \mathcal{L}(A)$ such that $q \notin \{p(i^*), p(j^*)\}$. As a consequence, $\{q, q^r\} \cap \{p(i^*), p(j^*)\} = \varnothing$, since $q^r = p(i^*)$ leads to the contradiction $q = p(i^*)^r = p(j^*)$ and $q^r = p(j^*)$ leads to the contradiction $q = p(j^*)^r = p(i^*)$.

Consider now $p' \in \mathcal{L}(A)^I$ defined, for every $i \in I$, by

$$p'(i) = \begin{cases} p(i) & \text{if } i \in I \setminus \{i^*, j^*\} \\ q & \text{if } i = i^* \\ q^r & \text{if } i = j^* \end{cases}$$

We claim that $p' \sim p$. Indeed, consider $(x, y) \in A^2_*$. Then

$$c_p(x, y) = |\{i \in I : x \succ_{p(i)} y\}| = |\{i \in I \setminus \{i^*, j^*\} : x \succ_{p(i)} y\}| + |\{i \in \{i^*, j^*\} : x \succ_{p(i)} y\}|$$

and

$$c_{p'}(x, y) = |\{i \in I : x \succ_{p'(i)} y\}| = |\{i \in I \setminus \{i^*, j^*\} : x \succ_{p'(i)} y\}| + |\{i \in \{i^*, j^*\} : x \succ_{p'(i)} y\}|.$$

Since, for every $i \in I \setminus \{i^*, j^*\}$, $p(i) = p'(i)$, we have that

$$|\{i \in I \setminus \{i^*, j^*\} : x \succ_{p(i)} y\}| = |\{i \in I \setminus \{i^*, j^*\} : x \succ_{p'(i)} y\}|.$$

Moreover, since $p(i^*) = p(j^*)^r$ and $p'(i^*) = p'(j^*)^r$, we have that

$$|\{i \in \{i^*, j^*\} : x \succ_{p(i)} y\}| = |\{i \in \{i^*, j^*\} : x \succ_{p'(i)} y\}| = 1.$$

Then we conclude that $c_p(x, y) = c_{p'}(x, y)$. As the argument does not depend on x and y we deduce that $p \sim p'$.

Observe now that $|\{i \in I : p'(i) = q\}| = |\{i \in I : p(i) = q\}| + 1$ so that, in particular, $|\{i \in I : p'(i) = q\}| \neq |\{i \in I : p(i) = q\}|$. Thus, by Proposition 15, we get that $p' \notin \{p^\varphi \in \mathcal{L}(A)^I : \varphi \in \mathrm{Sym}(I)\}$, which contradicts (a).

$(b) \Rightarrow (a)$ Assume now that, for every $i, j \in I$, $p(i) \neq p(j)^r$ and suppose by contradiction that (a) does not hold true. Thus, by Proposition 16, there exists $p' \in \mathcal{L}(A)^I$ such that $p' \sim p$ and such that $p' \notin \{p^\varphi \in \mathcal{L}(A)^I : \varphi \in \mathrm{Sym}(I)\}$. Let us define

$$\mathcal{I} = \{K \subseteq I : \exists \rho : K \to I \text{ injective such that } p'(\rho(i)) = p(i) \text{ for all } i \in K\}.$$

Note that $\varnothing \in \mathcal{I}$, since $\rho : \varnothing \to I$ satisfies the required properties, and that $I \notin \mathcal{I}$, since otherwise we would have $p' = p^\rho$ for a suitable $\rho \in \mathrm{Sym}(I)$. Consider now $K^* \in \mathcal{I}$ such that $|K^*| \geq |K|$ for all $K \in \mathcal{I}$ and let $\rho^* : K^* \to I$ be injective and such that $p'(\rho^*(i)) = p(i)$ for all $i \in K^*$. Since $K^* \neq I$, we have that $I \setminus K^* \neq \varnothing$ as well as $I \setminus \rho^*(K^*) \neq \varnothing$.

We claim now that, for every $i \in I \setminus K^*$ and $j \in I \setminus \rho^*(K^*)$, $p(i) \neq p'(j)$. Indeed, if by contradiction there were $i^* \in I \setminus K^*$ and $j^* \in I \setminus \rho^*(K^*)$ such that $p(i^*) = p'(j^*)$, we could consider $\rho' : K^* \cup \{i^*\} \to I$ defined, for every $i \in K^* \cup \{i^*\}$, as

$$\rho'(i) = \begin{cases} \rho^*(i) & \text{if } i \in K^* \\ j^* & \text{if } i = i^* \end{cases}$$

This function is clearly injective and $p'(\rho'(i)) = p(i)$ for all $i \in K^* \cup \{i^*\}$. Thus, $K^* \cup \{i^*\} \in \mathcal{I}$ and since $|K^* \cup \{i^*\}| > |K^*|$ the contradiction is found.

Consider then the nonempty sets

$$U = \{p(i) \in \mathcal{L}(A) : i \in I \setminus K^*\}, \quad V = \{p'(j) \in \mathcal{L}(A) : j \in I \setminus \rho^*(K^*)\}.$$

By the previous claim we have that $U \cap V = \varnothing$. By (b) we also know that if $q \in U$, then $q^r \notin U$ so that $|U| \leq \frac{1}{2}|\mathcal{L}(A)| = 3$. Define now, for every $(x, y) \in A^2_*$,

$$c^U_p(x, y) = |\{i \in I \setminus K^* : x \succ_{p(i)} y\}|, \quad c^V_{p'}(x, y) = |\{j \in I \setminus \rho^*(K^*) : x \succ_{p'(j)} y\}|.$$

We claim that $c^U_p(x, y) = c^V_{p'}(x, y)$. Indeed, since $p' \sim p$, we know that $c_p(x, y) = c_{p'}(x, y)$. Moreover

$$c_p(x, y) = |\{i \in K^* : x \succ_{p(i)} y\}| + c^U_p(x, y),$$

and

$$c_{p'}(x, y) = |\{j \in \rho^*(K^*) : x \succ_{p'(j)} y\}| + c^V_{p'}(x, y).$$

Since ρ^* is a bijection from K^* to $\rho^*(K^*)$, denoting by $(\rho^*)^{-1} : \rho^*(K^*) \to K^*$ its inverse, we also have that

$$\{i \in K^* : x \succ_{p(i)} y\} = \{i \in K^* : x \succ_{p'(\rho^*(i))} y\} = (\rho^*)^{-1}\left(\{j \in \rho^*(K^*) : x \succ_{p'(j)} y\}\right)$$

and then

$$|\{i \in K^* : x \succ_{p(i)} y\}| = |\{j \in \rho^*(K^*) : x \succ_{p'(j)} y\}|$$

that leads to $c^U_p(x, y) = c^V_{p'}(x, y)$.

Taking now into account the possible sizes of U, there are three cases to discuss. In what follows, we set $\tau = |I \setminus K^*| = |I \setminus \rho^*(K^*)| > 0$.

(i) Assume first that $|U| = 1$ and let xyz be its unique element. Then, $c^U_p(x, y) = c^U_p(y, z) = \tau$. Thus, $c^V_{p'}(x, y) = c^V_{p'}(y, z) = \tau$. Consider now $j \in I \setminus \rho^*(K^*)$. Then we necessarily have that $x \succ_{p'(j)} y$ and $y \succ_{p'(j)} z$, so that $p'(j) = xyz \in U$. Since $p'(j) \in V$ and $U \cap V = \varnothing$, a contradiction is found.

(ii) Assume now that $|U| = 2$ and let xyz be one of its elements. Then the other element of U is one among xzy, yxz, yzx, zxy since the reversal of xyz, that is zyx, cannot belong to U. Let us consider then all the possible cases.

$(ii.1)$ Assume that $U = \{xyz, xzy\}$. We have $c^U_p(x, y) = c^U_p(x, z) = \tau$ and then $c^V_{p'}(x, y) = c^V_{p'}(x, z) = \tau$. Considering now $j \in I \setminus \rho^*(K^*)$, we necessarily have that $x \succ_{p'(j)} y$ and $x \succ_{p'(j)} z$, so that $p'(j) \in U$. Since $p'(j) \in V$ and $U \cap V = \varnothing$, a contradiction is found.

$(ii.2)$ Assume that $U = \{xyz, yxz\}$. We have $c^U_p(x, z) = c^U_p(y, z) = \tau$ and then $c^V_{p'}(x, z) = c^V_{p'}(y, z) = \tau$. Considering now $j \in I \setminus \rho^*(K^*)$, we necessarily have that $x \succ_{p'(j)} z$ and $y \succ_{p'(j)} z$, so that $p'(j) \in U$. Since $p'(j) \in V$ and $U \cap V = \varnothing$, a contradiction is found.

$(ii.3)$ Assume that $U = \{xyz, yzx\}$. Let

$$|\{i \in I \setminus K^* : p(i) = xyz\}| = s, \quad |\{i \in I \setminus K^* : p(i) = yzx\}| = s'.$$

Of course, $s, s' \geq 1$ and $s + s' = \tau$. We have $c^U_p(y, z) = \tau$ and then $c^V_{p'}(y, z) = \tau$. Considering $j \in I \setminus \rho^*(K^*)$, we necessarily have that $y \succ_{p'(j)} z$, so that $p'(j) \in \{xyz, yxz, yzx\}$. Thus, since $p'(j) \notin U$, $p'(j) = yxz$. As a consequence, $c^V_{p'}(y, x) = \tau$ while $c^U_p(y, x) = s' < \tau$ since $s \geq 1$ and $s + s' = \tau$. In particular, $c^U_p(y, x) \neq c^V_{p'}(y, x)$ and the contradiction is found.

$(ii.4)$ Assume that $U = \{xyz, zxy\}$ and that

$$|\{i \in I \setminus K^* : p(i) = xyz\}| = s, \quad |\{i \in I \setminus K^* : p(i) = zxy\}| = s'.$$

Of course, $s, s' \geq 1$ and $s + s' = \tau$. We have $c^U_p(x, y) = \tau$ and then $c^V_{p'}(x, y) = \tau$. Considering $j \in I \setminus \rho^*(K^*)$, we necessarily have that $x \succ_{p'(j)} y$, so that $p'(j) \in \{xyz, xzy, zxy\}$. Thus, since $p'(j) \notin U$, $p'(j) = xzy$. As a consequence, $c^V_{p'}(x, z) = \tau$ while $c^U_p(x, z) = s < \tau$ since $s' \geq 1$ and $s + s' = \tau$. In particular, $c^U_p(x, z) \neq c^V_{p'}(x, z)$ and the contradiction is found.

(*iii*) Assume finally that $|U| = 3$ and that one of the elements of U is xyz. Then the other two elements of U are among xzy, yxz, yzx, zxy since the reversal of xyz, that is zyx, cannot belong to U. Moreover, $U \setminus \{xyz\} \neq \{xzy, yzx\}$ and $U \setminus \{xyz\} \neq \{yxz, zxy\}$, since U cannot contain a linear order and its reversal. There are then four cases to discuss.

(*iii*.1) Assume that $U = \{xyz, xzy, yxz\}$. Since $U \cap V = \varnothing$, we have that $V \subseteq \{yzx, zxy, zyx\}$. Since $c_p^U(x,z) = \tau$, $c_{p'}^V(x,z) = 0$ and $c_p^U(x,z) = c_{p'}^V(x,z)$, we get the contradiction $\tau = 0$.

(*iii*.2) Assume that $U = \{xyz, xzy, zxy\}$. Since $U \cap V = \varnothing$, we have that $V \subseteq \{yxz, yzx, zyx\}$. Since $c_p^U(x,y) = \tau$, $c_{p'}^V(x,y) = 0$ and $c_p^U(x,y) = c_{p'}^V(x,y)$, we get the contradiction $\tau = 0$.

(*iii*.3) Assume that $U = \{xyz, yxz, yzx\}$. Since $U \cap V = \varnothing$, we have that $V \subseteq \{xzy, zxy, zyx\}$. Since $c_p^U(y,z) = \tau$, $c_{p'}^V(y,z) = 0$ and $c_p^U(y,z) = c_{p'}^V(y,z) = 0$, we get the contradiction $\tau = 0$.

(*iii*.4) Assume that $U = \{xyz, yzx, zxy\}$. Since $U \cap V = \varnothing$, we have that $V \subseteq \{xzy, yxz, zyx\}$. Let us set

$$|\{i \in I \setminus K^* : p(i) = xyz\}| = s,$$
$$|\{i \in I \setminus K^* : p(i) = yzx\}| = s',$$
$$|\{i \in I \setminus K^* : p(i) = zxy\}| = s''$$

and

$$|\{j \in I \setminus \rho^*(K^*) : p'(j) = zyx\}| = t,$$
$$|\{j \in I \setminus \rho^*(K^*) : p'(j) = xzy\}| = t',$$
$$|\{j \in I \setminus \rho^*(K^*) : p'(j) = yxz\}| = t'',$$

where $s + s' + s'' = t + t' + t'' = \tau$ and $s, s', s'' \geq 1$. A simple computation shows that $c_p^U(x,y) = s + s''$, $c_p^U(y,z) = s + s'$, $c_p^U(z,x) = s' + s''$, $c_{p'}^V(x,y) = t'$, $c_{p'}^V(y,z) = t''$, $c_{p'}^V(z,x) = t$. Since $c_p^U(x,y) = c_{p'}^V(x,y)$, $c_p^U(y,z) = c_{p'}^V(y,z)$ and $c_p^U(z,x) = c_{p'}^V(z,x)$, we get $s + s'' = t'$, $s + s' = t''$ and $s' + s'' = t$. It follows then that $2(s + s' + s'') = t + t' + t''$, that is $2\tau = \tau$. That leads to the contradiction $\tau = 0$.

□

Define the set
$$\mathcal{Q}_I = \{p \in \mathcal{L}(A)^I : \forall i, j \in I,\ p(i) \neq p(j)^r\}.$$

Note that if $p \in \mathcal{Q}_I$ and $\varphi \in \mathrm{Sym}(I)$, then $p^\varphi \in \mathcal{Q}_I$. Consider now a function $F : \mathcal{Q}_I \to A$. We say that F is Pareto optimal if, for every $p \in \mathcal{Q}_I$ and $x, y \in A$, if $x \succ_{p(i)} y$ for all $i \in I$, then $F(p) \neq y$; dictatorial with dictator $i \in I$ if, for every $p \in \mathcal{Q}_I$, $F(p) = r_1(p(i))$; dictatorial if there exists $i \in I$ such that F is dictatorial with dictator i; strategy-proof if, for every $i \in I$, $q, q' \in \mathcal{L}(A)$ and $\bar{p} \in \mathcal{L}(A)^{I \setminus \{i\}}$ with $(q[i], \bar{p}) \in \mathcal{Q}_I$ and $(q'[i], \bar{p}) \in \mathcal{Q}_I$, $F(q[i], \bar{p}) \succeq_q F(q'[i], \bar{p})$. Moreover, we say that individual $i \in I$ is able to manipulate F at $p \in \mathcal{Q}_I$ via $q \in \mathcal{L}(A)$ if $(q[i], p_{|I \setminus \{i\}}) \in \mathcal{Q}_I$ and $F(q[i], p_{|I \setminus \{i\}}) \succ_{p(i)} F(p)$. Of course, F is strategy-proof if and only if, for every $i \in I$, $p \in \mathcal{Q}_I$ and $q \in \mathcal{L}(A)$, individual i is not able to manipulate F at p via q, that is, if $(q[i], p_{|I \setminus \{i\}}) \in \mathcal{Q}_I$ then $F(p) \succeq_{p(i)} F(q[i], p_{|I \setminus \{i\}})$.

The next proposition is inspired to Theorem 1 in Svensson and Reffgen (2014).

Proposition 18. *Let $|I| = 2$ and let $F : \mathcal{Q}_I \to A$. If F is Pareto optimal and strategy-proof, then F is dictatorial.*

Proof. Assume that $I = \{i_1, i_2\}$ and, given $p \in \mathcal{Q}_I$, identify p with the vector $(p(i_1), p(i_2))$. Define then the sets

$$A_1 = \{x \in A : \forall p \in \mathcal{Q}_I \text{ such that } r_1(p(i_1)) = x, F(p) = x\},$$

$$A_2 = \{x \in A : \forall p \in \mathcal{Q}_I \text{ such that } r_1(p(i_2)) = x, F(p) = x\}.$$

Consider distinct $x, y, z \in A$ and $(xyz, yxz) \in \mathcal{Q}_I$ and note that, by Pareto optimality $F(xyz, yxz) \neq z$. We claim that

$$F(xyz, yxz) = x \text{ implies } x \in A_1, \tag{5}$$

$$F(xyz, yxz) = y \text{ implies } y \in A_2. \tag{6}$$

Let us first prove (5). Set $p^1 = (xyz, yxz)$ and assume $F(p^1) = x$. Consider then the following elements of $\mathcal{Q}_I$

$$p^2 = (xyz, yzx), \quad p^3 = (xyz, zxy), \quad p^4 = (xyz, xyz), \quad p^5 = (xyz, xzy), \quad p^6 = (xzy, xyz)$$

$$p^7 = (xzy, xzy), \quad p^8 = (xzy, yxz), \quad p^9 = (xzy, zxy), \quad p^{10} = (xzy, zyx),$$

and note that $\{p \in \mathcal{Q}_I : r_1(p(i_1)) = x\} = \{p^1, p^2, \ldots, p^{10}\}$. Thus, we get $x \in A_1$ proving that, for every $i \in \{2, \ldots, 10\}$, $F(p^i) = x$. We proceed by a case-by-case analysis. By Pareto optimality $F(p^2) \neq z$. If $F(p^2) = y$ then individual i_2 would be able to manipulate F at p^1 via yzx. As a consequence, $F(p^2) = x$. By Pareto optimality $F(p^3) \neq y$. If $F(p^3) = z$ then individual i_2 would be able to manipulate F at p^2 via zxy. As a consequence, $F(p^3) = x$. By Pareto optimality, $F(p^4) = F(p^5) = F(p^6) = F(p^7) = x$. $F(p^8) = x$ otherwise individual i_1 would be able to manipulate F at p^8 via xyz. $F(p^9) = x$ otherwise individual i_1 would be able to manipulate F at p^9 via xyz. By Pareto optimality, $F(p^{10}) \neq y$. If $F(p^{10}) = z$ then individual i_2 would be able to manipulate F at p^9 via zyx. As a consequence, $F(p^{10}) = x$ and the proof of (5) is completed. The proof of (6) is similar and then omitted.

Consider now $p^* = (abc, bac) \in \mathcal{Q}_I$ and note that, by Pareto optimality, $F(p^*) \neq c$ so that $F(p^*) \in \{a, b\}$. We complete the proof of the theorem showing that $F(p^*) = a$ implies $A_1 = A$, that is, F is dictatorial with dictator i_1, and that $F(p^*) = b$ implies that $A_2 = A$, that is, F is dictatorial with dictator i_2.

Assume first that $F(p^*) = a$. By (5), we have that $a \in A_1$. Considering now $(bca, cba) \in \mathcal{Q}_I$, by (5) and (6), we get that $b \in A_1$ or $c \in A_2$. However, $c \notin A_2$, otherwise, given $(acb, cab) \in \mathcal{Q}_I$, we would have both $F(acb, cab) = a$ and $F(acb, cab) = c$, a contradiction. Then we get $b \in A_1$. Since $\{a, b\} \subseteq A_1$, we also have that $a \notin A_2$. Indeed, if $a \in A_2$, then considering $(bac, abc) \in \mathcal{Q}_I$ we should have both $F(bac, abc) = b$ and $F(bac, abc) = a$, a contradiction. Let us finally prove that $c \in A_1$. Indeed, consider $(cab, acb) \in \mathcal{Q}_I$. By (5) and (6), we have that $c \in A_1$ or $a \in A_2$. Since we know that $a \notin A_2$, we get $c \in A_1$.

The proof that $F(p^*) = b$ implies $A_2 = A$ is analogous and then omitted. □

The next result shows that any SCF that is Pareto optimal and strategy-proof on $\mathcal{Q}_I$ is dictatorial. It is inspired to Lemma 3 and Theorem 2 in Svensson and Reffgen (2014). At the best of our knowledge, it is not a consequence of any of the known results within the theory of dictatorial domains (Aswal et al., 2003; Pramanik, 2015). In what follows, given three pairwise disjoint subsets I_1, I_2 and I_3 of I such that $I_1 \cup I_2 \cup I_3 = I$ and $p^1 \in \mathcal{L}(A)^{I_1}$, $p^2 \in \mathcal{L}(A)^{I_2}$ and $p^3 \in \mathcal{L}(A)^{I_3}$, we denote by (p^1, p^2, p^3) the element of $\mathcal{L}(A)^I$ defined, for every $i \in I$, by $(p^1, p^2, p^3)(i) = p^j(i)$ if $i \in I_j$ with $j \in \{1, 2, 3\}$.

Theorem 19. *Let $F : \mathcal{Q}_I \to A$. If F is Pareto optimal and strategy-proof, then F is dictatorial.*

Proof. Let us first rephrase the theorem as follows: for every $h \in \mathbb{N}$ with $h \geq 2$,

> if I is a finite set with $|I| = h$ and $F : \mathcal{Q}_I \to A$ is Pareto optimal and strategy-proof, then F is dictatorial. (7)

We are going to prove the above result working by induction on h. By Proposition 18 we know that (7) is true when $h = 2$. In order to complete the proof of the theorem, assume then that (7) is true for $h \geq 2$ and show that (7) is true for $h + 1$ too.

Consider then I such that $|I| = h + 1 \geq 3$ and $F : \mathcal{Q}_I \to A$ that is Pareto optimal and strategy-proof. For every $(i, j) \in I_*^2$, let us consider the function $\gamma_{(i,j)} : \mathcal{Q}_{I \setminus \{j\}} \to \mathcal{Q}_I$ such that, for every $p \in \mathcal{Q}_{I \setminus \{j\}}$,

$$\gamma_{(i,j)}(p) = (p, p(i)[j]).$$

Note that $\gamma_{(i,j)}$ is well defined, that is, if $p \in \mathcal{Q}_{I\setminus\{j\}}$, then $\gamma_{(i,j)}(p)$ is an element of $\mathcal{Q}_I$.

For every $(i,j) \in I^2_*$, let $G_{(i,j)} : \mathcal{Q}_{I\setminus\{j\}} \to A$ be defined, for every $p \in \mathcal{Q}_{I\setminus\{j\}}$, as

$$G_{(i,j)}(p) = F(\gamma_{(i,j)}(p)).$$

We are going to show that, for every $(i,j) \in I^2_*$, $G_{(i,j)}$ is dictatorial. Since $G_{(i,j)}$ refers to the set of individuals $I \setminus \{j\}$ whose size is h, we can get such a result by applying the inductive assumption, provided that $G_{(i,j)}$ is Pareto optimal and strategy-proof.

Let us prove then that $G_{(i,j)}$ is Pareto optimal. Indeed, consider $p \in \mathcal{Q}_{I\setminus\{j\}}$ and $x, y \in A$ such that, for every $k \in I \setminus \{j\}$, $x \succ_{p(k)} y$. Then, for every $k \in I$, $x \succ_{(\gamma_{(i,j)}(p))(k)} y$. Since F is Pareto optimal we get that $y \notin F(\gamma_{(i,j)}(p)) = G_{(i,j)}(p)$, and the Pareto optimality of $G_{(i,j)}$ is proved.

Let us prove now that $G_{(i,j)}$ is strategy-proof. Fix then $t \in I\setminus\{j\}$, $q, q' \in \mathcal{L}(A)$ and $\bar{p} \in \mathcal{L}(A)^{I\setminus\{j,t\}}$ such that $(q[t], \bar{p}), (q'[t], \bar{p}) \in \mathcal{Q}_{I\setminus\{j\}}$ and show that $G_{(i,j)}(q[t], \bar{p}) \succeq_q G_{(i,j)}(q'[t], \bar{p})$. Assume first that $t \neq i$. Using the fact that F is strategy-proof, we get

$$G_{(i,j)}(q[t], \bar{p}) = F(\gamma_{(i,j)}(q[t], \bar{p})) = F(q[t], \bar{p}, \bar{p}(i)[j])$$

$$\succeq_q F(q'[t], \bar{p}, \bar{p}(i)[j]) = F(\gamma_{(i,j)}(q'[t], \bar{p})) = G_{(i,j)}(q'[t], \bar{p}),$$

as desired. Assume now that $t = i$. Using the fact that F is strategy-proof, we get

$$G_{(i,j)}(q[i], \bar{p}) = F(\gamma_{(i,j)}(q[i], \bar{p})) = F(q[i], \bar{p}, q[j])$$

$$\succeq_q F(q'[i], \bar{p}, q[j]) \succeq_q F(q'[i], \bar{p}, q'[j]) = F(\gamma_{(i,j)}(q'[i], \bar{p})) = G_{(i,j)}(q'[i], \bar{p}),$$

as desired. Thus, we conclude that $G_{(i,j)}$ is strategy-proof.

Let us prove now that F is dictatorial. Assume first that there exist $(i^*, j^*) \in I^2_*$ and $t^* \in I\setminus\{j^*\}$ with $t^* \neq i^*$ such that $G_{(i^*,j^*)}$ is dictatorial with dictator t^* and prove that F is dictatorial with dictator t^*. Consider $p \in \mathcal{Q}_I$ and assume that $r_1(p(t^*)) = x$ and $F(p) = y$. Our purpose is then to show that $x = y$. Since F is strategy-proof and since

$$F\left(p_{|I\setminus\{i^*,j^*\}}, p(i^*)[i^*], p(j^*)[j^*]\right) = F(p) = y \tag{8}$$

and[18]

$$F\left(p_{|I\setminus\{i^*,j^*\}}, p(j^*)[i^*], p(j^*)[j^*]\right) = G_{(i^*,j^*)}\left(p_{|I\setminus\{i^*,j^*\}}, p(j^*)[i^*]\right) = r_1(p(t^*)) = x,$$

we get $y \succeq_{p(i^*)} x$. Moreover, since F is strategy-proof and since

$$F\left(p_{|I\setminus\{i^*,j^*\}}, p(i^*)[i^*], p(i^*)[j^*]\right) = G_{(i^*,j^*)}\left(p_{|I\setminus\{i^*,j^*\}}, p(i^*)[i^*]\right) = r_1(p(t^*)) = x$$

and (8), we also get $x \succeq_{p(i^*)} y$. Since $p(i^*)$ is a linear order, from $y \succeq_{p(i^*)} x$ and $x \succeq_{p(i^*)} y$ we finally obtain $x = y$.

We complete the proof of the theorem showing that it cannot happen that, for every $(i,j) \in I^2_*$, $G_{(i,j)}$ is dictatorial with dictator i. Indeed, assume by contradiction otherwise. Then, for every $p \in \mathcal{Q}_I$ such that $p(i) = p(j)$ for some $(i,j) \in I^2_*$, we have $F(p) = r_1(p(i))$ since

$$F(p) = F(p_{|I\setminus\{i,j\}}, p(i)[i], p(i)[j]) = G_{(i,j)}(p_{|I\setminus\{i,j\}}, p(i)[i]) = r_1(p(i)).$$

Thus, if $|I| = h + 1 = 3$ and $I = \{i_1, i_2, i_3\}$, consider $p \in \mathcal{Q}_I$ such that $p(i_1) = abc$, $p(i_2) = bca$, $p(i_3) = cab$. Note that if $F(p) = a$, then individual i_2 is able to manipulate F at p via cab; if $F(p) = b$, then individual i_3 is able to manipulate F at p via abc; if $F(p) = c$, then individual i_1 is able to manipulate F at p via bca[19]. Since F is strategy-proof we get then a contradiction. If $|I| = h + 1 \geq 4$, consider $p \in \mathcal{Q}_I$ such that there are distinct $i_1, i_2, i_3, i_4 \in I$ such that $p(i_1) = p(i_2) = abc$, $p(i_3) = p(i_4) = bac$. Then both $F(p) = a$ and $F(p) = b$, a contradiction. □

[18] Note that $p \in \mathcal{Q}_I$ implies $\left(p_{|I\setminus\{i^*,j^*\}}, p(j^*)[i^*], p(j^*)[j^*]\right) \in \mathcal{Q}_I$ and $\left(p_{|I\setminus\{i^*,j^*\}}, p(i^*)[i^*], p(i^*)[j^*]\right) \in \mathcal{Q}_I$.

[19] Observe that in each of the three considered cases, the preference profile obtained after manipulating F are in $\mathcal{Q}_I$.

Lemma 20. *Assume $F : \mathcal{L}(A)^I \to A$. If F is Pareto optimal and weakly WMG-strategy-proof, then F restricted to $\mathcal{Q}_I$ is dictatorial.*

Proof. Assume that F is Pareto optimal and weakly WMG-strategy-proof and let G be the restriction of F to the set $\mathcal{Q}_I$. We are going to prove that G is dictatorial proving that it is Pareto optimal and strategy-proof and applying Theorem 19.

Let us first prove that G is Pareto optimal. Consider $p \in \mathcal{Q}_I$ and $x, y \in A$ such that, for every $i \in I$, $x \succ_{p(i)} y$. Then $F(p) \neq y$ and since $G(p) = F(p)$ we also get $G(p) \neq y$.

Let us now prove that G is strategy-proof. Consider $i \in I$, $q, q' \in \mathcal{L}(A)$ and $\bar{p} \in \mathcal{L}(A)^{I \setminus \{i\}}$ with $(q[i], \bar{p}), (q'[i], \bar{p}) \in \mathcal{Q}_I$ and assume by contradiction that $G(q'[i], \bar{p}) \succ_q G(q[i], \bar{p})$, that is,

$$F(q'[i], \bar{p}) \succ_q F(q[i], \bar{p}). \tag{9}$$

Since F is weakly WMG-strategy-proof there exists $\bar{p}' \in \mathcal{L}(A)^{I \setminus \{i\}}$ with $\bar{p}' \sim \bar{p}$ such that

$$F(q[i], \bar{p}') \succeq_q F(q'[i], \bar{p}'). \tag{10}$$

Observe now that since $(q[i], \bar{p}) \in \mathcal{Q}_I$, we have that $\bar{p} \in \mathcal{Q}_{I \setminus \{i\}}$. Then, by Theorem 17, there exists $\varphi \in \mathrm{Sym}(I \setminus \{i\})$, such that $\bar{p}' = \bar{p}^{\varphi}$. Consider now $\hat{\varphi} \in \mathrm{Sym}(I)$ such that, for every $j \in I \setminus \{i\}$, $\hat{\varphi}(j) = \varphi(j)$ and $\hat{\varphi}(i) = i$. It is easily checked that

$$(q[i], \bar{p}') = (q[i], \bar{p})^{\hat{\varphi}} \quad \text{and} \quad (q'[i], \bar{p}') = (q'[i], \bar{p})^{\hat{\varphi}} \tag{11}$$

so that, in particular, both $(q[i], \bar{p}')$ and $(q'[i], \bar{p}')$ belong to $\mathcal{Q}_I$. By (11) and the fact that F is anonymous we have that

$$F(q[i], \bar{p}') = F\left((q[i], \bar{p})^{\hat{\varphi}}\right) = F(q[i], \bar{p}) \tag{12}$$

and

$$F(q'[i], \bar{p}') = F\left((q'[i], \bar{p})^{\hat{\varphi}}\right) = F(q'[i], \bar{p}). \tag{13}$$

By (10), (12) and (13), we get $F(q[i], \bar{p}) \succeq_q F(q'[i], \bar{p})$, which contradicts (9). □

Proof of Theorem 11: case $|A| = 3$. Assume by contradiction that there exists $F : \mathcal{L}(A)^I \to A$ that is Pareto optimal, weakly WMG-strategy-proof and anonymous. Let G be the restriction of F to the set $\mathcal{Q}_I$. By Lemma 20, we know that that G is dictatorial, so that there exists $i^* \in I$ such that, for every $p \in \mathcal{Q}_I$, $G(p) = r_1(p(i^*))$. Consider now $j^* \in I \setminus \{i^*\}$, $p \in \mathcal{Q}_I$ such that $p(i^*) = abc$ and $p(j^*) = bac$ and $\varphi \in \mathrm{Sym}(I)$ such that $\varphi(i^*) = j^*$, $\varphi(j^*) = i^*$ and $\varphi(i) = i$ for all $i \in I \setminus \{i^*, j^*\}$. Then, we have that

$$F(p) = G(p) = r_1(p(i^*)) = r_1(abc) = a$$

and, since also $p^{\varphi} \in \mathcal{Q}_I$,

$$F(p^{\varphi}) = G(p^{\varphi}) = r_1(p^{\varphi}(i^*)) = r_1(p(\varphi^{-1}(i^*))) = r_1(p(j^*)) = r_1(bac) = b.$$

However, since F is anonymous, we also have $F(p) = F(p^{\varphi})$, a contradiction. □

Proof of Theorem 13. Assume that F is Pareto optimal and weakly group WMG-strategy-proof. Since F is weakly WMG-strategy-proof, by Lemma 20 we know that there exists $i^* \in I$ such that, for every $p \in \mathcal{Q}_I$, $F(p) = r_1(p(i^*))$. Let us prove that F is dictatorial with dictator i^*.

Consider $p \in \mathcal{L}(A)^I$ and prove that $F(p) = r_1(p(i^*))$. Assume by contradiction that $F(p) \neq r_1(p(i^*))$. Let us suppose that $p(i^*) = xyz$, so that $F(p) \in \{y, z\}$. We complete the proof by showing that both $F(p) = y$ and $F(p) = z$ lead to a contradiction.

Assume first that $F(p) = y$. Let $\hat{p}' = p_{|I \setminus \{i^*\}} \in \mathcal{L}(A)^{I \setminus \{i^*\}}$ and $\bar{p} = p(i^*)[i^*] \in \mathcal{L}(A)^{\{i^*\}}$. Thus, $p = (\hat{p}', \bar{p})$ and $F(\hat{p}', \bar{p}) = y$. Consider then $\hat{p} \in \mathcal{L}(A)^{I \setminus \{i^*\}}$ defined, for every $i \in I \setminus \{i^*\}$, by

$$\hat{p}(i) = \begin{cases} yxz & \text{if } p(i) = yzx \\ yzx & \text{if } p(i) \neq yzx \end{cases}$$

Note that, for every $i \in I \setminus \{i^*\}$, $y \succ_{\hat{p}(i)} x$ and $\hat{p}(i) \neq \hat{p}'(i)$. Moreover, $(\hat{p}, \bar{p}) \in \mathcal{Q}_I$, since, for every $i \in I$, $(\hat{p}, \bar{p}) \in \{xyz, yxz, yzx\}$. As a consequence, $F(\hat{p}, \bar{p}) = r_1(\bar{p}(i^*)) = r_1(p(i^*)) = x$. For every $i \in I \setminus \{i^*\}$, we have that

$$F(\hat{p}', \bar{p}) = y \succ_{\hat{p}(i)} x = F(\hat{p}, \bar{p}).$$

Since F is weakly group WMG-strategy-proof, there exist $\bar{p}' \in \mathcal{L}(A)^{\{i^*\}}$ with $\bar{p}' \sim \bar{p}$ and $j \in I \setminus \{i^*\}$ such that

$$F(\hat{p}, \bar{p}') \succeq_{\hat{p}(j)} F(\hat{p}', \bar{p}').$$

However, since $\bar{p}' \sim \bar{p}$ implies $\bar{p}' = \bar{p}$, we have that $F(\hat{p}, \bar{p}') = x$ and $F(\hat{p}', \bar{p}') = y$. We conclude then that $x \succeq_{\hat{p}(j)} y$, a contradiction.

Assume now that $F(p) = z$. Let $\hat{p}' = p_{|I \setminus \{i^*\}} \in \mathcal{L}(A)^{I \setminus \{i^*\}}$ and $\bar{p} = p(i^*)[i^*] \in \mathcal{L}(A)^{\{i^*\}}$. Thus, $p = (\hat{p}', \bar{p})$ and $F(\hat{p}', \bar{p}) = z$. Consider then $\hat{p} \in \mathcal{L}(A)^{I \setminus \{i^*\}}$ defined, for every $i \in I \setminus \{i^*\}$, by

$$\hat{p}(i) = \begin{cases} zxy & \text{if } p(i) = yzx \\ yzx & \text{if } p(i) \neq yzx \end{cases}$$

Note that, for every $i \in I \setminus \{i^*\}$, $z \succ_{\hat{p}(i)} x$ and $\hat{p}(i) \neq \hat{p}'(i)$. Moreover, $(\hat{p}, \bar{p}) \in \mathcal{Q}_I$, since, for every $i \in I$, $(\hat{p}, \bar{p}) \in \{xyz, zxy, yzx\}$. As a consequence, $F(\hat{p}, \bar{p}) = r_1(\bar{p}(i^*)) = r_1(p(i^*)) = x$. For every $i \in I \setminus \{i^*\}$, we have that

$$F(\hat{p}', \bar{p}) = z \succ_{\hat{p}(i)} x = F(\hat{p}, \bar{p}).$$

Since F is weakly group WMG-strategy-proof, there exist $\bar{p}' \in \mathcal{L}(A)^{\{i^*\}}$ with $\bar{p}' \sim \bar{p}$ and $j \in I \setminus \{i^*\}$ such that

$$F(\hat{p}, \bar{p}') \succeq_{\hat{p}(j)} F(\hat{p}', \bar{p}').$$

However, since $\bar{p}' \sim \bar{p}$ implies $\bar{p}' = \bar{p}$, we have that $F(\hat{p}, \bar{p}') = x$ and $F(\hat{p}', \bar{p}') = z$. We conclude then that $x \succeq_{\hat{p}(j)} z$, a contradiction. □

A.2 Proof of Theorem 11: the general case

By Section A.1, we know that Theorem 11 holds true when $|A| = 3$. Consider now $|A| = m \geq 4$ and assume by contradiction that there exists $F : \mathcal{L}(A)^I \to A$ that is Pareto optimal, weakly WMG-strategy-proof and anonymous.

Let us fix an ordering $x_1, \ldots, x_m$ of the elements of A and let $B = \{x_1, x_2, x_3\}$ and

$$U = \{(x_r, x_t) \in A^2 : r, t \geq 4, r \leq t\}.$$

Of course, $B \subseteq A$, $|B| = 3$ and $U \subseteq (A \setminus B)^2$. Let us consider now $\varepsilon : \mathcal{L}(B) \to \mathcal{L}(A)$ associating with every $q \in \mathcal{L}(B)$ the element of $\mathcal{L}(A)$ given by

$$\varepsilon(q) = q \cup (B \times (A \setminus B)) \cup U.$$

In other words, if $q = xyz$, where $x, y, z \in B$, then $\varepsilon(q) = xyzx_4 \cdots x_m$. Given now $J \subseteq I$ with J nonempty and $p \in \mathcal{L}(B)^J$, we denote by $\varepsilon(p)$ the the element of $\mathcal{L}(A)^J$ such that, for every $i \in J$, $\varepsilon(p)(i) = \varepsilon(p(i))$. Finally let $G : \mathcal{L}(B)^I \to B$ be defined, for every $p \in \mathcal{L}(B)^I$, by $G(p) = F(\varepsilon(p))$. Note that G is well-defined, that is, for every $p \in \mathcal{L}(B)^I$, $F(\varepsilon(p)) \in B$. Indeed, let $p \in \mathcal{L}(B)^I$ and $y \in A \setminus B$. Picking any $x \in B$, we have that $x \succ_{\varepsilon(p)(i)} y$ for all $i \in I$ and then, since F is Pareto optimal, we get $F(\varepsilon(p)) \neq y$. We complete the proof of the theorem showing that G is Pareto optimal, weakly WMG-strategy-proof and anonymous. Indeed, since $|B| = 3$, that contradicts Theorem 11.

Let us first prove that G is Pareto optimal. Consider $p \in \mathcal{L}(B)^I$ and $x, y \in B$ and assume that, for every $i \in I$, $x \succ_{p(i)} y$. Then, for every $i \in I$, $x \succ_{\varepsilon(p(i))} y$, that is, $x \succ_{\varepsilon(p)(i)} y$. Since F is Pareto optimal, $G(p) = F(\varepsilon(p)) \neq y$.

Let us now prove that G is weakly WMG-strategy-proof. Consider $i \in I$, $q, q' \in \mathcal{L}(B)$ and $\bar{p} \in \mathcal{L}(B)^{I \setminus \{i\}}$ and assume that $G(q'[i], \bar{p}) \succ_q G(q[i], \bar{p})$. We have to show that there exists $\bar{p}' \in \mathcal{L}(B)^{I \setminus \{i\}}$ with $\bar{p}' \sim \bar{p}$ such that $G(q[i], \bar{p}') \succeq_q G(q'[i], \bar{p}')$. Since $G(q[i], \bar{p}) = F(\varepsilon(q)[i], \varepsilon(\bar{p})) \in B$ and

$G(q'[i],\bar{p}) = F(\varepsilon(q')[i],\varepsilon(\bar{p})) \in B$, we get $F(\varepsilon(q')[i],\varepsilon(\bar{p})) \succ_q F(\varepsilon(q)[i],\varepsilon(\bar{p}))$, so that $F(\varepsilon(q')[i],\varepsilon(\bar{p})) \succ_{\varepsilon(q)} F(\varepsilon(q)[i],\varepsilon(\bar{p}))$. By weak WMG-strategy-proofness of F, there exists $\bar{p}^* \in \mathcal{L}(A)^{I\setminus\{i\}}$ such that $\bar{p}^* \sim \varepsilon(\bar{p})$ and

$$F(\varepsilon(q)[i],\bar{p}^*) \succeq_{\varepsilon(q)} F(\varepsilon(q')[i],\bar{p}^*). \tag{14}$$

Consider now $\bar{p}' \in \mathcal{L}(B)^{I\setminus\{i\}}$ defined, for every $j \in I \setminus \{i\}$, by

$$\bar{p}'(j) = \{(x,y) \in B^2 : (x,y) \in \bar{p}^*(j)\}. \tag{15}$$

We claim that

$$\varepsilon(\bar{p}') = \bar{p}^*. \tag{16}$$

In order to prove (16) it is sufficient to prove that, for every $j \in I \setminus \{i\}$, $\varepsilon(\bar{p}'(j)) \subseteq \bar{p}^*(j)$. Indeed, since $\varepsilon(\bar{p}'(j)), \bar{p}^*(j) \in \mathcal{L}(A)$, $\varepsilon(\bar{p}'(j)) \subseteq \bar{p}^*(j)$ implies $\varepsilon(\bar{p}'(j)) = \bar{p}^*(j)$. Fix then $j \in I \setminus \{i\}$ and

$$(x^*,y^*) \in \varepsilon(\bar{p}'(j)) = \bar{p}'(j) \cup (B \times (A \setminus B)) \cup U.$$

If $(x^*,y^*) \in \bar{p}'(j)$, then by (15), $(x^*,y^*) \in \bar{p}^*(j)$. Assume now that $(x^*,y^*) \in B \times (A \setminus B)$. Note that, in particular, $x^* \neq y^*$ and that $c_{\varepsilon(\bar{p})}(x^*,y^*) = |I| - 1$. Since $\bar{p}^* \sim \varepsilon(\bar{p})$, we know that $c_{\bar{p}^*}(x^*,y^*) = c_{\varepsilon(\bar{p})}(x^*,y^*) = |I| - 1$ so that necessarily $(x^*,y^*) \in \bar{p}^*(j)$. Assume finally that $(x^*,y^*) \in U$. Then there exist $r,t \geq 4$ with $r \leq t$ such that $x^* = x_r$ and $y^* = x_t$. If $r = t$, then $x^* = y^*$ and $(x^*,y^*) \in \bar{p}^*(j)$ since $\bar{p}^*(j) \in \mathcal{L}(A)$ and so it is reflexive. If $r < t$ then $x^* \neq y^*$ and $c_{\varepsilon(\bar{p})}(x^*,y^*) = |I| - 1$. Since $\bar{p}^* \sim \varepsilon(\bar{p})$, we know that $c_{\bar{p}^*}(x^*,y^*) = c_{\varepsilon(\bar{p})}(x^*,y^*) = |I| - 1$ so that necessarily $(x^*,y^*) \in \bar{p}^*(j)$. That completes the proof of (16).

From (14) and (16), we get $F(\varepsilon(q)[i],\varepsilon(\bar{p}')) \succeq_{\varepsilon(q)} F(\varepsilon(q')[i],\varepsilon(\bar{p}'))$, that is, $G(q[i],\bar{p}') \succeq_{\varepsilon(q)} G(q'[i],\bar{p}')$. Thus, since $G(q[i],\bar{p}') \in B$ and $G(q'[i],\bar{p}') \in B$, we deduce $G(q[i],\bar{p}') \succeq_q G(q'[i],\bar{p}')$. We complete then the proof that G is weakly WMG-strategy-proof simply noticing that, by (16) and since $\bar{p}^* \sim \varepsilon(\bar{p})$, for every $(x,y) \in B^2_*$, we have that

$$c_{\bar{p}'}(x,y) = c_{\varepsilon(\bar{p}')}(x,y) = c_{\bar{p}^*}(x,y) = c_{\varepsilon(\bar{p})}(x,y) = c_{\bar{p}}(x,y),$$

so that $\bar{p}' \sim \bar{p}$.

We are finally left with proving that G is anonymous. Consider $p \in \mathcal{L}(B)^I$ and $\varphi \in \mathrm{Sym}(I)$. Then $\varepsilon(p^\varphi) = \varepsilon(p)^\varphi$. Indeed, for every $i \in I$

$$\varepsilon(p^\varphi)(i) = \varepsilon(p^\varphi(i)) = \varepsilon(p(\varphi^{-1}(i))) = \varepsilon(p)(\varphi^{-1}(i)) = \varepsilon(p)^\varphi(i).$$

Then, using the definition of G and the fact that F is anonymous, we get

$$G(p^\varphi) = F(\varepsilon(p^\varphi)) = F(\varepsilon(p)^\varphi) = F(\varepsilon(p)) = G(p),$$

and the anonymity of G is proved.

A.3 Proof of Theorem 10

First of all, note that F is well defined as, for every $p \in \mathcal{L}(A)^I$, there exists a unique $i \in J$ such that $p_{|K} \in L_i$.

Let us prove that F is Pareto optimal. Let $p \in \mathcal{L}(A)^I$ and $x,y \in A$ be such that, for every $i \in I$, $x \succ_{p(i)} y$. We have to prove that $F(p) \neq y$. Assume then by contradiction that $F(p) = y$. Consider $i \in J$ such that $p_{|K} \in L_i$. Then $F(p) = r_1(p(i)) = y$. Since $x \succ_{p(i)} y$, we get a contradiction.

Let us prove now that F is WMG-strategy-proof. Assume that there exist $i^* \in I$, $q,q' \in \mathcal{L}(A)$ and $\bar{p} \in \mathcal{L}(A)^{I\setminus\{i^*\}}$ such that

$$F(q'[i^*],\bar{p}) \succ_q F(q[i^*],\bar{p}).$$

We have to show that there exists $\bar{p}' \sim \bar{p}$ such that

$$F(q[i^*],\bar{p}') \succ_q F(q'[i^*],\bar{p}').$$

Let us prove first that $i^* \notin J$. Indeed, assume by contradiction that $i^* \in J$. Then $(q[i^*], \bar{p})_{|K} = (q'[i^*], \bar{p})_{|K} = \bar{p}_{|K}$. Consider now $i \in J$ such that $\bar{p}_{|K} \in L_i$. If $i = i^*$, then we get the contradiction

$$F(q'[i^*], \bar{p}) = r_1(q') \succ_q r_1(q) = F(q[i^*], \bar{p}).$$

If instead $i \neq i^*$, then we get the contradiction

$$F(q'[i^*], \bar{p}) = r_1(\bar{p}(i)) \succ_q r_1(\bar{p}(i)) = F(q[i^*], \bar{p}).$$

Thus $i^* \notin J$, so that $i^* \in K$. Consider now $j, j' \in J$ such that $(q[i^*], \bar{p})_{|K} \in L_j$ and $(q'[i^*], \bar{p})_{|K} \in L_{j'}$. Note that we cannot have $j = j'$, otherwise we get the contradiction

$$F(q'[i^*], \bar{p}) = r_1(\bar{p}(j)) \succ_q r_1(\bar{p}(j)) = F(q[i^*], \bar{p}).$$

Thus $j \neq j'$ and

$$F(q'[i^*], \bar{p}) = r_1(\bar{p}(j')) \succ_q r_1(\bar{p}(j)) = F(q[i^*], \bar{p}). \tag{17}$$

In particular, $|J| \geq 2$. Let $\varphi \in \mathrm{Sym}(I \setminus \{i^*\})$ be such that $\varphi(j) = j'$, $\varphi(j') = j$ and, for every $i \in I \setminus \{j, j', i^*\}$, $\varphi(i) = i$. Define then $\bar{p}' = \bar{p}^\varphi$. It is immediate to note that $\bar{p}' \sim \bar{p}$ and that $(q[i^*], \bar{p}')_{|K} \in L_j$ and $(q'[i^*], \bar{p}')_{|K} \in L_{j'}$. Thus,

$$F(q'[i^*], \bar{p}') = r_1(\bar{p}'(j')) = r_1(\bar{p}^\varphi(j')) = r_1(\bar{p}(\varphi^{-1}(j'))) = r_1(\bar{p}(j))$$

and

$$F(q[i^*], \bar{p}') = r_1(\bar{p}'(j)) = r_1(\bar{p}^\varphi(j)) = r_1(\bar{p}(\varphi^{-1}(j))) = r_1(\bar{p}(j')).$$

By (17), we conclude that $F(q[i^*], \bar{p}') \succ_q F(q'[i^*], \bar{p}')$, as desired.

Let us prove now that, for the function F, the conditions (i), (ii) and (iii) are equivalent. We are going to prove several implications involving (i), (ii) and (iii), some of them depending on the size of $|A|$. Considering them together, they imply the equivalence of (i), (ii) and (iii) for any size of A.

$(i) \Rightarrow (ii)$. Assume that F is dictatorial. Then F is group strategy-proof so that, by Propositions 5 and 6, it is weakly group WMG-strategy-proof too.

$|A| = 3 \wedge (ii) \Rightarrow (i)$. Assume that $|A| = 3$ and that F is weakly group WMG-strategy-proof. Since F is Pareto optimal, by Theorem 13, we conclude that F is dictatorial.

$(iii) \Rightarrow (i)$. Assume that $|\{i \in J : L_i \neq \varnothing\}| = 1$ and let i^* be the unique element in $\{i \in J : L_i \neq \varnothing\}$. Thus, $L_{i^*} = \mathcal{L}(A)^K$. As a consequence, for every $p \in \mathcal{L}(A)^I$, $p_{|K} \in L_{i^*}$ and then $F(p) = r_1(p(i^*))$. Thus, F is dictatorial with dictator i^*.

$(i) \Rightarrow (iii)$. Assume that $|\{i \in J : L_i \neq \varnothing\}| \geq 2$ and prove that F is not dictatorial. Let us prove first that, for every $i \in K$, F is not a dictatorship of dictator i. Indeed, fix $i^* \in K$. Consider then $q \in \mathcal{L}(A)$ and let $p \in \mathcal{L}(A)^I$ be such that, for every $i \in J$, $p(i) = q$ and, for every $i \in K$, $p(i) = q^r$. Thus, $F(p) = r_1(q)$. Since $r_1(p(i^*)) = r_1(q^r) \neq r_1(q)$, we deduce that F is not a dictatorship of dictator i^*.

Let us prove now that, for every $i \in J$, F is not a dictatorship of dictator i. Indeed, fix $i^* \in J$. Since $|\{i \in J : L_i \neq \varnothing\}| \geq 2$, there exists $j^* \in J$ such that $j^* \neq i^*$ and $L_{j^*} \neq \varnothing$. Consider then $p' \in L_{j^*}$ and $p'' \in \mathcal{L}(A)^J$ such that $p''(i^*) = p''(j^*)^r$ and set $p = (p', p'') \in \mathcal{L}(A)^I$. Since

$$F(p) = F(p', p'') = r_1(p''(j^*)) \neq r_1(p''(i^*)) = r_1(p(i^*)),$$

we deduce that F is not a dictatorship of dictator i^*.

$|A| \geq 4 \wedge (ii) \Rightarrow (iii)$. Assume that $|A| \geq 4$ and that $|\{i \in J : L_i \neq \varnothing\}| \geq 2$ and prove that F is not weakly group WMG-strategy-proof. Let us consider the subset of $\mathcal{L}(A)^K$ given by

$$\mathcal{U}_K(A) = \left\{ t \in \mathcal{L}(A)^K : \exists x, y \in A \text{ such that, for every } j \in K,\ x \succ_{t(j)} y \right\}.$$

The following claim holds true.

Claim. *Assume that $i \in J$ satisfies the following conditions:*

(a) *there exists $t^* \in L_i \cap \mathcal{U}_K(A)$,*

(b) *for every $t \in L_i \cap \mathcal{U}_K(A)$, $\{t' \in \mathcal{L}(A)^K : \forall j \in K\ t'(j) \neq t(j)\} \subseteq L_i$.*

Then $L_i = \mathcal{L}(A)^K$.

Proof. Since $L_i \subseteq \mathcal{L}(A)^K$, we are left with proving that if $\hat{t} \in \mathcal{L}(A)^K$, then $\hat{t} \in L_i$. Consider then $\hat{t} \in \mathcal{L}(A)^K$. By (a), we know that there exists $t^* \in L_i \cap \mathcal{U}_K(A)$. In particular, there are $x^*, y^* \in A$ such that, for every $j \in K$, $x^* \succ_{t^*(j)} y^*$. Since $|A| \geq 3$, there are at least three elements in $\mathcal{L}(A)$ having (x^*, y^*) as element. For every $j \in K$, pick then $q_j \in \mathcal{L}(A) \setminus \{t^*(j), \hat{t}(j)\}$ such that $x^* \succ_{q_j} y^*$ and let $\bar{t} \in \mathcal{L}(A)^K$ be defined, for every $j \in K$, by $\bar{t}(j) = q_j$. Since, for every $j \in K$, $\bar{t}(j) \neq t^*(j)$ and since $t^* \in L_i \cap \mathcal{U}_K(A)$, by (b) we have that $\bar{t} \in L_i$. Moreover, for every $j \in K$, $x^* \succ_{\bar{t}(j)} y^*$ so that $\bar{t} \in \mathcal{U}_K(A)$. Thus, $\bar{t} \in L_i \cap \mathcal{U}_K(A)$. Since, for every $j \in K$, $\bar{t}(j) \neq \hat{t}(j)$, by (b) we also have that $\hat{t} \in L_i$, as desired. □

By the claim and the fact that $|\{i \in J : L_i \neq \varnothing\}| \geq 2$ we deduce that, for every $i \in J$, i does not satisfy (a) and (b). There exist then $i^*, j^* \in J$ with $i^* \neq j^*$, $t^* \in L_{i^*} \cap \mathcal{U}_K(A)$ and $t' \in L_{j^*}$ such that, for every $j \in K$, $t'(j) \neq t^*(j)$. Let x^*, y^* such that, for every $j \in K$, $x^* \succ_{t^*(j)} y^*$.

Since $|A| \geq 4$, we can find $q, q' \in \mathcal{L}(A)$ such that $r_1(q) = r_1(q') = x^*$, $r_2(q) = r_2(q') = y^*$ and $q \neq q'$. Consider also $q'' \in \mathcal{L}(A)$ such that $r_1(q'') = y^*$ and $r_2(q'') = x^*$. Let $p \in \mathcal{L}(A)^I$ be defined, for every $i \in I$, by

$$p(i) = \begin{cases} q & \text{if } i \in J \setminus \{i^*\} \\ t^*(i) & \text{if } i \in K \\ q'' & \text{if } i = i^* \end{cases}$$

Thus, for every $i \in I \setminus \{i^*\}$, $x^* \succ_{p(i)} y^*$. Moreover, $p_{|K} \in L_{i^*}$ so that $F(p) = r_1(p(i^*)) = r_1(q'') = y^*$. Let $p' \in \mathcal{L}(A)^I$ be defined, for every $i \in I$, by

$$p'(i) = \begin{cases} q' & \text{if } i \in J \setminus \{i^*\} \\ t'(i) & \text{if } i \in K \\ q'' & \text{if } i = i^* \end{cases}$$

Thus, $p'_{|K} \in L_{j^*}$ so that $F(p') = r_1(p'(j^*)) = r_1(q') = x^*$. Define now

$$\hat{p} = p_{|I\setminus\{i^*\}} \in \mathcal{L}(A)^{I\setminus\{i^*\}}, \ \ \hat{p}' = p'_{|I\setminus\{i^*\}} \in \mathcal{L}(A)^{I\setminus\{i^*\}}, \ \ \bar{p} = q''[i^*] \in \mathcal{L}(A)^{\{i^*\}}.$$

We have that, for every $i \in I \setminus \{i^*\}$, $\hat{p}(i) \neq \hat{p}'(i)$. Moreover, for every $i \in I \setminus \{i^*\}$,

$$F(\hat{p}', \bar{p}) = F(p') = x^* \succ_{p(i)} y^* = F(p) = F(\hat{p}, \bar{p}).$$

Since, for every $\bar{p}' \in \mathcal{L}(A)^{\{i\}}$, $\bar{p}' \sim \bar{p}$ implies $\bar{p}' = \bar{p}$, we conclude that F fails to be weakly group WMG-strategy-proof.

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