Z. Angew. Math. Phys. (2022) 73:248

https://doi.org/10.1007/s00033-022-01887-6

Zeitschrift für angewandte Mathematik und Physik ZAMP

Global weak attractors in the dynamics of bodies with vector-type microstructure

Luca Bisconti and Paolo Maria Mariano

Abstract. We investigate the dynamics of bodies with vector-type microstructure. We consider linear constitutive relations and a nonlinear coupling between macroscopic and microscopic motions, determined by gyroscopic-type inertia. Based on an existence result obtained in the presence of viscous-type stress components, we determine the existence of a global attractor; its weak nature derives from the lack of uniqueness determined by the nonlinear coupling.

Mathematics Subject Classification. 35B41, 74H20, 74E99.

Keywords. Multifield theories, Microstructures, Weak attractors.

1. Introduction

Let $\mathscr{B}$ be a three-dimensional, bounded, regular domain. With $u : \mathscr{B}\times[0,T] \to \mathbb{R}^3$ and $\nu : \mathscr{B}\times[0,T] \to \mathbb{R}^3$ sufficiently smooth maps, consider the following system of partial differential equations, written with respect to an orthonormal frame of reference (the subscript t indicates derivative with respect to time):

$$\begin{aligned}
&\rho u_{tt} - \epsilon\Delta u_t = b + \mu\Delta u + \xi\nabla\operatorname{div} u + \kappa\Delta\nu + \bar{\xi}\nabla\operatorname{div}\nu,\\
&\text{ in } \mathscr{B}\times[0,T],\\
&\varsigma\nu_t - \delta\Delta\nu_t + \ell(\mathrm{curl} u_t)\times\nu_t = \zeta\Delta\nu + \gamma\nabla\operatorname{div}\nu\\
&\qquad\qquad\qquad\qquad\qquad\qquad + \kappa\Delta u + \xi\nabla\operatorname{div} u - \kappa_0\nu,\\
&\text{in } \mathscr{B}\times[0,T],\\
&u(t,x) = 0,\ \nu(t,x) = 0,\\
&\text{on } \partial\mathscr{B}\times[0,T],\\
&u|_{t=0} = u_0,\ u_t|_{t=0} = \dot{u}_0,\ \nu|_{t=0} = \nu_0, \qquad \text{on } \mathscr{B},
\end{aligned} \tag{1.1}$$

where u_0, $\dot{u}_0$, and ν_0 are assigned. The factors in front of the derivatives of u and ν are constants together with κ_0; b is a forcing term (a bulk force, indeed).

Previous equations describe aspects of the continuum dynamics of complex bodies with descriptors of microstructural shapes selected in $\mathbb{R}^3$.

In reference [3] we have proven global-in-time existence of weak solutions for system (1.1). However, we do not have uniqueness, due to the nonlinear coupling $(\mathrm{curl} u_t)\times\nu_t$. Consequently, as regards qualitative properties of the weak solutions, we can think of an attractor just in weak sense (see, e.g., [1,17,18]).

Specifically, for $\mathscr{W}$ the space of weak solutions (defined below) to the system (1.1), namely

$$\mathscr{W} = \Big\{\boldsymbol{w} = (u,\nu;v)\in L^2_{\mathrm{loc}}[0,+\infty;W^{1,2}(\mathscr{B}))^2\times L^2_{\mathrm{loc}}[0,+\infty;L^2(\mathscr{B}))\ |\\ v = u_t,\ \text{and}(u,\nu)\text{is a weak solution to } (1.1)\Big\},$$

with $u|_{\partial\mathscr{B}} = 0$, and $\nu|_{\partial\mathscr{B}} = 0$, where $L^2_{\mathrm{loc}}[0,+\infty;X)^2$ (X every time a different space) is a short-hand notation for $L^2_{\mathrm{loc}}([0,+\infty);X)^2$ our, main result reads as follows:

Published online: 05 November 2022

Theorem 1.1. *With* $b \in L^2(0,\infty; L^2(\mathscr{B}))$ *assigned, there exists a global weak attractor* $\mathscr{A} \subseteq \mathscr{W}$ *for the system* (1.1).

2. The ground from which the balance equations considered emerge

2.1. Macroscopic and microscopic shapes

Take $\hat{\mathbb{R}}^3$ and $\mathbb{R}^3$ as two copies of the $3D$ real space connected just by the identification map $\iota : \hat{\mathbb{R}}^3 \longrightarrow \mathbb{R}^3$. The latter is the physical space, while the former is a useful reference ambient. Consider in $\hat{\mathbb{R}}^3$ and $\mathbb{R}^3$ non-singular metrics $\hat{g}$ and g, respectively. We select in $\hat{\mathbb{R}}^3$ a simply connected region $\mathscr{B}$ with piecewise Lipschitz boundary and take it as a macroscopic reference shape for a continuum body. We consider (gross scale) deformations as orientation-preserving differentiable one-to-one maps $x \longmapsto y := \tilde{y}(x) \in \mathbb{R}^3$.

F indicates the derivative $D\tilde{y}(x)$. We have $\nabla \tilde{y}(x) = D\tilde{y}(x)\hat{g}^{-1}$, so that they coincide when $\hat{g}^{-1}$ is flat, i.e., it refers to an orthonormal frame. We adopt for F the common denomination and call it the *deformation gradient*. Two linear operators can be defined: the *formal adjoint* F^* and the *transpose* F^{T} of F. They are related as $F^{\mathsf{T}} = \hat{g}^{-1}F^*g$, so that they coincide when both metrics are flat.

The orientation-preserving condition implies $\det F > 0$.[1]

We define the *displacement field* by $u := \tilde{u}(x) := \tilde{y}(x) - \iota(x)$, distinguishing the map $\tilde{u}$ from its value $u := \tilde{u}$, so that $F = I + Du := I + D\tilde{u}(x)$, with I a second-rank identity tensor, precisely the shifter from $\hat{\mathbb{R}}^3$ to $\mathbb{R}^3$, with components $\delta^i{}_A$, where capital indices refer to the coordinates in $\hat{\mathbb{R}}^3$, the lowercase ones to the coordinates in $\mathbb{R}^3$.

A differentiable vector field $\tilde{\nu} : \mathscr{B} \longrightarrow \bar{\mathbb{R}}^3$, with $\bar{\mathbb{R}}^3$ a copy of the $3D$ real space in principle independent of the other two (a reason for this choice is to distinguish the representation of phenomena at different scales), brings at macroscopic spatial scale information on the microstructural morphology. We indicate by ν the value of the map $\tilde{\nu}$. N is a short-hand notation for the spatial derivative $D\nu := D\tilde{\nu}(x)$.

2.2. Motions (in generalized sense)

Previous description of the body morphology imply that a motion, intended in generalized sense, is now a pair of fields, namely

$$(x,t) \longmapsto y := \tilde{y}(x,t) \in \mathbb{R}^3$$

and

$$(x,t) \longmapsto \nu := \tilde{\nu}(x,t) \in \mathbb{R}^3,$$

with the time t ranging into, say, $[0,T]$. We assume differentiability with respect to time. A superposed dot indicates as usual total time derivative; in the Lagrangian representation adopted in this section, since x is fixed with respect to time, the total derivative coincides with the partial one. Pertinent rates of change of the body morphology are

$$\dot{y} = \frac{\mathrm{d}\tilde{y}(x,t)}{\mathrm{d}t} = \dot{u}$$

and

$$\dot{\nu} = \frac{\mathrm{d}\tilde{\nu}(x,t)}{\mathrm{d}t}\,.$$

[1] The specific analyses presented here refer to small strain regime where the orientation-preserving condition does not play an evident role. However, since we show how to derive the balance equations considered fro first principles in a fully nonlinear setting, along such a path the condition $\det F > 0$ plays a role.

2.3. Isometry-based changes in observers

An *observer* is a representation (i.e., a prescription of reference frames) over all the spaces adopted for representing the morphology of a body and its motion [11,12]. Here such spaces are the reference one (namely $\hat{\mathbb{R}}^3$), the physical space (i.e., $\mathbb{R}^3$), the one in which $\tilde{\nu}$ takes values (namely $\bar{\mathbb{R}}^3$), and the time interval. We consider changes in observers that leave invariant $\mathbb{R}^3$ and the time scale while they are of rigid-body type in $\mathbb{R}^3$. If y is a place evaluated by a first observer, a second one records $y' = a(t) + Q(t)(y - y_0) + y_0$, where $t \longmapsto a(t) \in \mathbb{R}^3$ is a vector-valued smooth map depending only on time, y_0 is an arbitrary fixed point, and $t \longmapsto Q(t) \in SO(3)$ is a orthogonal tensor-valued smooth map depending only on time. The first observer records a velocity $\dot{y}$ while for the second one it is $\dot{y}' = \dot{a} + \dot{Q}(y - y_0) + Q\dot{y}$. When we pull-back $\dot{y}'$ in the frame of the first observer, we get a velocity $\dot{y}^{\diamond} := Q^{\mathsf{T}}\dot{y}'$ (the superscript T means standard transposition) given by

$$\dot{y}^{\diamond} = \mathfrak{c} + q \times (y - y_0) + \dot{y}.$$

The vector $\mathfrak{c} := Q^{\mathsf{T}}\dot{a}$ is a relative translation velocity between the two observers, while q is the axial vector of the skew-symmetric second-rank tensor $Q^{\mathsf{T}}\dot{Q}$.

Rotating observers in the physical space perceive differently ν as it is a $3D$ vector, which is otherwise insensitive to relative translations of observers because it describes only events occurring inside the material element presumed at x in the reference configuration, i.e., properties *relative* to the element itself. So, the value ν recorded by a first observer changes into $\nu' = Q(t)\nu$, when the two observers are distinguished by rigid-body motions, and the time rate $\dot{\nu}$ becomes $\dot{\nu}' = \dot{\nu} + \dot{Q}\nu$. By pulling back this last vector in the frame of the first observer, we obtain a new vector $\dot{\nu}^{\diamond} := Q^{\mathsf{T}}\dot{\nu}'$ given by

$$\dot{\nu}^{\diamond} = \dot{\nu} + q \times \nu,$$

which we write as

$$\dot{\nu}^{\diamond} = \dot{\nu} + \mathscr{A}(\nu) q,$$

with the linear operator $\mathscr{A}(\nu)$ given by $\mathscr{A}(\nu) = -\nu\times$, using a symbology adopted in the general model-building framework for the mechanics of complex materials.[2]

2.4. A fundamental first principle: invariance of the external power

With $\mathfrak{b}$ an arbitrary (internal) part of $\mathscr{B}$, i.e., a subset of the reference place with non-null volume and a piecewise Lipschitz boundary, we subdivide external actions on it into bulk and contact families. They are defined by the power $\mathscr{P}_{\mathfrak{b}}^{\text{ext}}$ that they perform in the body shape rate of change, i.e., on any pair $(\dot{y}, \dot{\nu})$. Precisely, we write such a power as

$$\mathscr{P}_{\mathfrak{b}}^{\text{ext}}(\dot{y}, \dot{\nu}) := \int_{\mathfrak{b}} \left(b^{\ddagger} \cdot \dot{y} + \beta^{\ddagger} \cdot \dot{\nu}\right) \mathrm{d}\mu(x) + \int_{\partial\mathfrak{b}} \left(\mathfrak{t}_{\partial} \cdot \dot{y} + \tau_{\partial} \cdot \dot{\nu}\right) \mathrm{d}\mathscr{H}^2(x),$$

where $\mathrm{d}\mu(x)$ is the standard volume measure, $\mathrm{d}\mathscr{H}^2(x)$ the surface one, and the dot indicated duality pairing, identified with the scalar product when we consider flat metrics. The subscript ∂ indicates that the contact actions $\mathfrak{t}$ and τ depend on the boundary $\partial\mathfrak{b}$ besides x and t.

We define as *balanced* those actions for which $\mathscr{P}_{\mathfrak{b}}^{\text{ext}}$ is invariant under changes in observers above, namely

$$\mathscr{P}_{\mathfrak{b}}^{\text{ext}}(\dot{y}, \dot{\nu}) = \mathscr{P}_{\mathfrak{b}}^{\text{ext}}(\dot{y}^{\diamond}, \dot{\nu}^{\diamond})$$

[2] A more detailed analysis concerning changes of observers in the mechanics of complex bodies, i.e., those with active microstructure, is in reference [12]; it deals with the general case in which ν belongs to a finite-dimensional differentiable manifold generically not embedded into a linear space.

for any choice of $\mathfrak{c}$, q, and $\mathfrak{b}$. Such a request of invariance implies the standard balances of forces

$$\int_{\mathfrak{b}} b^{\ddagger} \ \mathrm{d}\mu(x) + \int_{\partial\mathfrak{b}} \mathfrak{t}_{\partial} \ \mathrm{d}\mathscr{H}^2(x) = 0,$$

and a non-standard balance of couples

$$\int_{\mathfrak{b}} \left((y-y_0)\times b^{\ddagger} + \mathscr{A}^*\beta^{\ddagger}\right)\mathrm{d}\mu(x) + \int_{\partial\mathfrak{b}} \left((y-y_0)\times \mathfrak{t}_{\partial} + \mathscr{A}^*\tau_{\partial}\right)\mathrm{d}\mathscr{H}^2(x) = 0,$$

where $\mathscr{A}^*$ is the formal adjoint of $\mathscr{A}$.

- If $\left|b^{\ddagger}\right|$ is bounded over $\mathscr{B}$ and $\mathfrak{t}_{\partial}$ depends continuously on x, the action–reaction principle holds first on flat boundaries, and, on its basis, one may further show that $\mathfrak{t}_{\partial}$ depends on $\partial\mathfrak{b}$ only through the normal n to it in all points where n is well-defined, i.e., $\mathfrak{t}_{\partial} = \mathfrak{t} := \tilde{\mathfrak{t}}(x,n) = -\tilde{\mathfrak{t}}(x,-n)$. Also, as a function of n, $\tilde{\mathfrak{t}}$ is homogeneous and additive, i.e., there exists a second-rank tensor field $x \longmapsto P(x)$ such that $\tilde{\mathfrak{t}}(x,n) = P(x)\,n(x)$. This is the standard Cauchy theorem preceded by the Hamel–Noll result; P is the *first Piola–Kirchhoff stress*.
- If in addition $\left|\mathscr{A}^*\beta^{\ddagger}\right|$ is bounded over $\mathscr{B}$ and τ_{∂} depends continuously on x, the boundedness of the region considered in space implies that we can choose y_0 such that the density of the bulk integral in the non-standard balance of couples is bounded too and the microstructural contact action τ_{∂} satisfies a non-standard action–reaction principle and depends on $\partial\mathfrak{b}$ only through the normal n to it in all points where n is well-defined; we have, in fact, $\mathscr{A}^*(\tilde{\tau}(x,n) + \tilde{\tau}(x,-n)) = 0$. Also, as a function of n, $\tilde{\tau}$ is homogeneous and additive, i.e., there exists a second-rank tensor field $x \longmapsto \mathscr{S}(x)$, so called *microstress*, such that $\tilde{\tau}(x,n) = \mathscr{S}(x)\,n(x)$.
- If both stress fields are in $C^1(\mathscr{B}) \cap C(\bar{\mathscr{B}})$ and the bulk actions $x \longmapsto b$, $x \longmapsto \beta^{\ddagger}$ are continuous over $\mathscr{B}$, the point-wise balance of forces

$$\mathrm{Div}P + b^{\ddagger} = 0 \tag{2.1}$$

holds and there exists a covector field $x \longmapsto z(x) \in \mathbb{R}^{3*}$ such that

$$\mathrm{Div}\mathscr{S} + \beta^{\ddagger} - z = 0, \qquad \mathrm{skw}PF^* = \frac{1}{2}\mathsf{e}\left(\mathscr{A}^*z + (D\mathscr{A}^*)\,\mathscr{S}\right); \tag{2.2}$$

moreover,

$$\mathscr{P}^{\mathrm{ext}}_{\mathfrak{b}}(\dot{y},\dot{\nu}) = \int_{\mathfrak{b}} \left(P\cdot\dot{F} + z\cdot\dot{\nu} + \mathscr{S}\cdot\dot{N}\right)\mathrm{d}\mu(x), \tag{2.3}$$

with the right-hand side integral called a *internal* (or *inner*) *power*. We interpret z as a self-action occurring within the material element at x. It emerges as a consequence of the insensitivity of ν to rigid translations of reference frames in the whole physical space.

2.5. Identification of the inertial terms

The bulk actions $b^{\ddagger}$ and $\beta^{\ddagger}$ include inertial and non-inertial components, the former indicated below by a superscript "in". Precisely, we presume validity of the additive decomposition

$$b^{\ddagger} = b^{\mathrm{in}} + b, \qquad \beta^{\ddagger} = \beta^{\mathrm{in}} + \beta.$$

We identify the inertial components b^{in} and β^{in} by imposing that the negative of their power equals the time rate of the kinetic energy for any choice of the rate fields. Here, we do not consider microstructural relative inertia (which is otherwise possible in general; see [4,5,11,14]) so that we write

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathfrak{b}} \frac{1}{2}\rho|\dot{y}|^2 \ \mathrm{d}\mu(x) = -\int_{\mathfrak{b}} \left(b^{\mathrm{in}}\cdot\dot{y} + \beta^{\mathrm{in}}\cdot\dot{\nu}\right)\mathrm{d}\mu(x),$$

presuming its validity for any choice of body part and rates considered. Such an imposed arbitrariness implies on one side the standard identification

$$b^{\text{in}} = -\rho\ddot{y} = -\rho\ddot{u},$$

but it also prescribes

$$\beta^{\text{in}} \cdot \dot{\nu} = 0.$$

This last identity is obviously satisfied when $\beta^{\text{in}} = 0$, but also, and this is here the point under question, when β^{in} is orthogonal to $\dot{\nu}$, i.e., when $\beta^{\text{in}} = h \times \dot{\nu}$, with h a generic vector. Specifically, we choose $h = \text{curl}\,\dot{y} = \text{curl}\,\dot{u}$ because the local macroscopic spin may in principle change orientation of the microstructural vector [13]. For this reason, we adopt the expression

$$\beta^{\text{in}} = (\text{curl}\,\dot{u}) \times \dot{\nu} = (\text{curl}\,u_t) \times \nu_t,$$

which is the coupling nonlinear term in system (1.1). In addition we set

$$\beta = 0\,,$$

excluding in this way direct bulk non-inertial actions on the microstructure. This choice is not general. For ferroelectric or magneto-elastic materials, cases in which ν represents the local polarization or the magnetization, respectively, β is determined by the external electric field. Consequently the previous choice for β is just specific of the system that we analyze here.

2.6. Constitutive restrictions

The last step in deriving system (1.1) deals with the constitutive choices. They are restricted by the Clausius-Duhem inequality, which reads in the present isothermal setting as

$$(\text{rate of the free energy on every } \mathfrak{b}) - (\text{power performed in } \mathfrak{b}) \leq 0,$$

presumed to hold for any time rate of the state variables considered, each rate assumed to be chosen independently of the others. Here, it is explicitly expressed by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathfrak{b}} \psi \;\, \mathrm{d}\mu(x) - \int_{\mathfrak{b}} \left(P \cdot \dot{F} + z \cdot \dot{\nu} + \mathscr{S} \cdot \dot{N}\right) \mathrm{d}\mu(x) \leq 0,$$

where ψ is the free energy density. We presume it holds true for any choice of the time rates involved. We also select the following constitutive functional dependence on the state variables:

$$\begin{aligned} \psi = \tilde{\psi}(F,\nu,N), \quad P &= \tilde{P}^e(F,\nu,N) + \tilde{P}^d(F,\nu,N;\dot{F},\dot{\nu},\dot{N}),\\ z &= \tilde{z}^e(F,\nu,N) + \tilde{z}^d(F,\nu,N;\dot{F},\dot{\nu},\dot{N}),\\ \mathscr{S} &= \tilde{\mathscr{S}}^e(F,\nu,N) + \tilde{\mathscr{S}}^d(F,\nu,N;\dot{F},\dot{\nu},\dot{N}), \end{aligned}$$

where e and d in superscript position indicate, respectively, energetic and dissipative components of the stresses P, $\mathscr{S}$, and the self-action z. By inserting into the inequality and exploiting the arbitrariness of $\dot{F}$, $\dot{\nu}$, and $\dot{N}$, we get

$$P^e = \frac{\partial\psi}{\partial F}, \quad z^e = \frac{\partial\psi}{\partial\nu}, \quad \mathscr{S}^e = \frac{\partial\psi}{\partial N}, \quad P^d \cdot \dot{F} + z^d \cdot \dot{\nu} + \mathscr{S}^d \cdot \dot{N} \geq 0.$$

The last inequality is compatible with

$$P = \frac{\partial\psi}{\partial F} + a_{\nabla u}(\dots)\nabla\dot{u}, \quad z = \frac{\partial\psi}{\partial\nu} + a_{\nu}(\dots)\dot{\nu}, \quad \mathscr{S} = \frac{\partial\psi}{\partial N} + a_{\nabla\nu}(\dots)\nabla\dot{\nu},$$

once again due to the arbitrariness of time rates considered. Furthermore, $a_{\nabla u}(\dots)$, $a_{\nu}(\dots)$, and $a_{\nabla\nu}(\dots)$ are positive-valued state function, here chosen to be scalars, namely ϵ, ς, and δ, respectively, only for the sake of simplicity. We write the gradient ∇ instead of the derivative D by referring only to an orthonormal frame here and in the rest of this paper for the sake of simplicity.

These actions have their counterparts that are values of fields defined in the current configuration $\mathscr{B}_{\mathrm{c}} := \tilde{y}(\mathscr{B}, t)$ given by

$$\sigma = \frac{1}{\det F} P F^*, \quad z_{\mathrm{c}} = \frac{1}{\det F} z, \quad \mathscr{S}_{\mathrm{c}} = \frac{1}{\det F} \mathscr{S} F^*,$$

where σ is the standard *Cauchy stress*.

In a *small strain* regime, defined by the condition $|\nabla u| \ll 1$, which we accept here, constraining the analysis to small strain range, we may avoid to distinguish between referential ($\mathscr{B}$) and current ($\mathscr{B}_{\mathrm{c}}$) configurations, so that

$$\sigma \approx P, \quad z_{\mathrm{c}} \approx z, \quad \mathscr{S}_{\mathrm{c}} \approx \mathscr{S}.$$

In a *large strain* regime, convexity of the free energy with respect to the deformation gradient is *incompatible* with its objectivity, i.e., its invariance under rigid-body type changes in observers. Such a limitation does not affects the small strain regime accepted here, in which we may also admit for ψ a quadratic dependence on its entries as

$$\begin{aligned}\psi = {} & \frac{1}{2}\lambda \left(\mathrm{sym}\nabla u \cdot I\right)^2 + \mu \mathrm{sym}\nabla u \cdot \mathrm{sym}\nabla u \\ & + \frac{1}{2} k_1 \left(\nabla \nu \cdot I\right)^2 + k_2 \mathrm{sym}\nabla \nu \cdot \mathrm{sym}\nabla \nu + k_2' \mathrm{skw}\nabla \nu \cdot \mathrm{skw}\nabla \nu \\ & + k_3 \left(\mathrm{sym}\nabla u \cdot I\right)\left(\nabla \nu \cdot I\right) + k_3' \mathrm{sym}\nabla \nu \cdot \mathrm{sym}\nabla u + \frac{1}{2}\kappa_0 \left|\nu\right|^2,\end{aligned}$$

where I is here the unit tensor, and the operators sym(.) and skw(.) extract, respectively, symmetric and skew-symmetric components of their arguments (for the derivation of such an expression as a consequence symmetry conditions see reference [13]). λ and μ are the standard Lamé constants, the k's are other constants such that the energy is positive definite, namely, we have

$$\mu > 0, \quad k_2 > 0, \quad 2k_2 + 3k_1 > 0, \quad k_2' > 0, \quad k_3' < 2\sqrt{\mu k_2},$$
$$3k_3 + k_3' < \sqrt{(2\mu + 3\lambda)(2k_2 + 3k_1)}, \quad \kappa_0 \geq 0.$$

The derivatives of ψ imply

$$\begin{aligned}\sigma &= \lambda \left(\mathrm{tr}\left(\mathrm{sym}\nabla u\right)\right) I + 2\mu \mathrm{sym}\nabla u + k_3 \left(\mathrm{tr}\nabla \nu\right) I + k_3' \mathrm{sym}\nabla \nu + \epsilon \nabla \dot{u}, \\ z_{\mathrm{c}} &= k_0 \nu + \varsigma \dot{\nu}, \\ \mathscr{S}_{\mathrm{c}} &= k_1 \left(\mathrm{tr}\nabla \nu\right) I + 2k_2 \mathrm{sym}\nabla \nu + 2k_2' \mathrm{skw}\nabla \nu + k_3 \left(\mathrm{tr}(\mathrm{sym}\nabla u)\right) I \\ & \quad + 2k_2' \mathrm{skw}\nabla \nu + k_3 \left(\mathrm{tr}(\mathrm{sym}\nabla u)\right) I + k_3' \mathrm{sym}\nabla u + \delta \nabla \dot{\nu}.\end{aligned}$$

When we insert these constitutive structures in the balance equations, we get the system (1.1), after setting $\xi = \lambda + \mu$, $\bar{\xi} = k_3 + \frac{1}{2}k_3'$, $\zeta = k_2 + k_2'$, $\gamma = k_1 + k_2 - k_2'$, $\kappa = \frac{1}{2}k_3'$.

3. A special case where system (1.1) directly applies

The system of balance equations (1.1) arises from the general model-building framework for the mechanics of *complex bodies* (or *materials*) [4,11,12], those suffering events at small spatial scales, which are driven by interactions hardly expressible only in terms of standard stresses.

Besides such general structure, and among other possible examples, system (1.1) applies directly to the linear mechanics of quasicrystals with nonlinear gyroscopic effects. Quasicrystals are alloys characterized by quasi-periodic atomic distributions not due to twin structures. Their initial recognition has been progressively accepted up to inducing a change in 2011 of the same definition of crystals by the International Union of Crystallography, it dates back 1982 [19].

A quasi-periodic lattice in $3D$ space can be viewed as the projection of a periodic lattice that fills a six-dimensional space (call it hyper-lattice) onto an incommensurate $3D$ subspace (see, e.g., [6–10,15]).

A displacement in the hyper-lattice admits a component in the space over which we project atoms and another component in its orthogonal complement in the $6D$ space. The latter component is represented by ν, the former by u.

Another evidence of the emergence of ν appears when we expand the mass of a three-dimensional quasi-periodic lattice in Fourier series. Six-dimensional wave vectors arise [9]. Components exceeding the ambient space dimension can be interpreted as inner degrees of freedom, those represented by ν in system (1.1) and exploited by atoms to shift relatively as to assure quasi-periodicity when boundary conditions vary.

Scattering experiments record only three sound-like branches in quasicrystals [16], so that we avoid assigning peculiar kinetic energy to $\dot{\nu}$ in this special case, while the derivation of balance equations does not exclude the presence of rotational inertia. We have no specific estimation of it. Our analysis here aims at helping the pertinent investigation. Incidentally, if we neglect rotational inertia (a nonlinear effects, due to its coupling with the gross motion, as proposed in reference [13]), and neglect also viscous-type effects in the standard stress (i.e., at gross scale) and the microstress, leaving them only to the self-action associated with ν, system (1.1) would simplify to

$$\begin{aligned} \rho u_{tt} &= b + \mu\Delta u + \xi\nabla \operatorname{div} u + \kappa\Delta\nu + \bar{\xi}\nabla \operatorname{div} \nu, \\ \varsigma\nu_t &= \zeta\Delta\nu + \gamma\nabla \operatorname{div} \nu + \kappa\Delta u + \xi\nabla \operatorname{div} u - \kappa_0\nu. \end{aligned} \tag{3.1}$$

However, we come back to system (1.1) and analyze below the properties of its solutions.

4. Notations adopted in the subsequent analyses

For $p \geq 1$, we indicate by $L^p(\mathscr{B})$ the usual Lebesgue space of p-power summable functions, endowed with norm $\|\cdot\|_p$. When $p = 2$, we use the notation $\|\cdot\| = \|\cdot\|_2$. Moreover, for k a non-negative integer and p as above, we denote, as usual, by $W^{k,p}(\mathscr{B})$ and $\|\cdot\|_{k,p}$ a Sobolev space and its norm, respectively. We write $W_0^{1,p}(\mathscr{B})$ for the closure of $C_0^\infty(\mathscr{B})$ in $W^{1,p}(\mathscr{B})$ and $W^{-1,p'}(\mathscr{B})$, $p' = p/(p-1)$, for its dual endowed with norm $\|\cdot\|_{-1,p'}$. Let X be a real Banach space with norm $\|\cdot\|_X$. We will use the common notation $W^{k,p}(0,T;X)$ to indicate those spaces of maps that are $W^{k,p}$ with respect to time and belong to the space X as functions of space variables. We will indicate by $\|\cdot\|_{W^{k,p}(0,T;X)}$ the pertinent norms. In particular, $W^{0,p}(0,T;X) = L^p(0,T;X)$ corresponds to a standard Bochner space. With f and g two square-integrable fields defined on $\mathscr{B}$, the symbol $(f\,,\,g)$ will indicate the standard L^2-product, i.e., the integral $\int_{\mathscr{B}} f\cdot g$, where the dot indicates duality pairing, which coincides with the scalar product when referred to orthonormal frames. In the sequel c or $\bar{c}$ will denote positive constants that may assume different values, even in the same equation. We also define the space $\mathscr{H}^1$ by

$$\mathscr{H}^1 := \left\{ v \in W^{1,2}(\mathscr{B}) \,:\, v_{|\partial\mathscr{B}} = 0 \right\}$$

and its dual by $\mathscr{H}^{-1}$. We have analogous definitions for the higher order spaces $\mathscr{H}^n$, $n \in \mathbb{N}$.

In order to keep the notation concise, we will use the same symbols $\mathscr{H}^n$, $n \in \mathbb{N}$, for the spaces related to the variables u and ν.

5. An essential tool

Let (W, d) be a metric space, d the metric.

A *semigroup* on (W, d) is a family of operators $\big(S(t)\big)_{t\geq 0}$, each one acting as $S(t)\colon W \to W$, with $S(0)w = w$ and $S(s)S(t)w = S(t+s)w$ for each $w \in W$ and for every $s, t \geq 0$.

A *semiflow* on (W, d) is a mapping $\mathfrak{s}\colon [0, +\infty) \times W \to W$ defined by $\mathfrak{s}(t, w) = S(t)w$, where $\big(S(t)\big)_{t\geq 0}$ is a semigroup such that the restriction $\mathfrak{s}\colon (0, +\infty) \times W \to W$ is continuous.

A bounded subset $\mathfrak{B} \subset W$ is called an *absorbing set* (or a *forward invariant set*) if, for any bounded set B of W, there exists $t_1 = t_1(B)$ such that $S(t)B \subseteq \mathfrak{B}$ for all $t \geq t_1$.

A semiflow is said to be a *compact* one if, for every bounded set $B \subset W$ and every $t > 0$, the set $S(t)B$ lies in compact subset of W.

A *global attractor* for S is a non-empty compact set $\mathscr{A}$ of W, which is forward invariant with respect to S and is such that, for all bounded sets $\mathscr{U}$ in W,

$$\lim_{t\to\infty} d(S(t)\mathscr{U}, \mathscr{A}) = 0.$$

Theorem 5.1. ([1,2,17,18,20]) *Let $S(t)$ be a compact semiflow admitting an absorbing set $\mathfrak{B}$ on a complete metric space W. Then, $S(t)$ has a global attractor $\mathscr{A}$ in W, given by*

$$\mathscr{A} = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} S(t)\mathfrak{B}},$$

where the closure is taken in W.

$\mathscr{A}$ is a ω-*limit set* (see, e.g., [1] and also [2]).

For the sake of conciseness, in what follows we will omit measures in the integrals, not paying in terms of clarity.

6. Weak solutions

Definition 6.1. We say that a pair (u, ν) is a *weak solution* of the system (1.1) if, for a given $T > 0$, the conditions defined below hold true:

(1) Regularity:

$$u \in L^\infty(0,T;\mathscr{H}^1) \cap W^{1,2}(0,T;\mathscr{H}^1),\ \nu \in L^\infty(0,T;\mathscr{H}^1) \cap W^{1,2}(0,T;\mathscr{H}^1), \tag{6.1}$$

$$\begin{aligned} &u_t \in L^2(0,T;\mathscr{H}^1) \cap W^{1,2}(0,T;\mathscr{H}^{-1}),\ u_{tt} \in L^2(0,T;\mathscr{H}^{-1}),\\ &\nu_t \in L^2(0,T;\mathscr{H}^1). \end{aligned} \tag{6.2}$$

(2) Weak formulation: For all $(w,h) \in C_0^\infty([0,T] \times \mathscr{B}) \times C_0^\infty([0,T[\times\mathscr{B})$,

$$\begin{aligned} &-\rho \int_0^T \int_{\mathscr{B}} u_t \cdot w_t + \int_0^T \int_{\mathscr{B}} (\epsilon\nabla u_t + \mu\nabla u) \cdot \nabla w + \kappa \int_0^T \int_{\mathscr{B}} \nabla\nu \cdot \nabla w \\ &\quad = -\xi \int_0^T \int_{\mathscr{B}} \operatorname{div} u \cdot \operatorname{div} w - \bar{\xi} \int_0^T \int_{\mathscr{B}} \operatorname{div} \nu \cdot \operatorname{div} w + \int_0^T \int_{\mathscr{B}} b \cdot w, \end{aligned} \tag{6.3}$$

$$\begin{aligned} &\int_0^T \int_{\mathscr{B}} (\varsigma\nu_t + \kappa_0\nu) \cdot h + \delta \int_0^T \int_{\mathscr{B}} \nabla\nu_t \cdot \nabla h + \ell \int_0^T \int_{\mathscr{B}} (\mathrm{curl} u_t) \times \nu_t \cdot h \\ &\quad + \int_0^T \int_{\mathscr{B}} (\zeta\nabla\nu + \kappa\nabla u) \cdot \nabla h \\ &\quad = -\gamma \int_0^T \int_{\mathscr{B}} \operatorname{div} \nu \cdot \operatorname{div} h - \bar{\xi} \int_0^T \int_{\mathscr{B}} \operatorname{div} u \cdot \operatorname{div} h. \end{aligned} \tag{6.4}$$

We proved existence of weak solutions to system (1.1) in reference [3] when $b = 0$ and initial data $u_0, \nu_0 \in \mathscr{H}^1$, and $\dot{u}_0 \in \mathscr{H}^1$. In the present case, we may directly adapt that proof with minor changes and an additional assumption about the regularity of b, namely $b \in L^2(0, \infty; L^2(\mathscr{B}))$.

We also find it expedient to write the system (1.1) as

$$\begin{aligned}
&u_t = v && \text{in } \mathscr{B} \times [0, T],\\
&\rho v_t - \epsilon\Delta v = \mu\Delta u + \xi\nabla \operatorname{div} u + \kappa\Delta\nu + \bar{\xi}\nabla \operatorname{div} \nu + b && \text{in } \mathscr{B} \times [0, T],\\
&\varsigma\nu_t - \delta\Delta\nu_t + \ell(\operatorname{curl} v) \times \nu_t = \zeta\Delta\nu + \gamma\nabla \operatorname{div} \nu + \kappa\Delta u + \bar{\xi}\nabla \operatorname{div} u - \kappa_0\nu && \text{in } \mathscr{B} \times [0, T],\\
&u(t, x) = 0,\ \nu(t, x) = 0, && \text{on } \partial\mathscr{B} \times [0, T],\\
&u|_{t=0} = u_0,\ v|_{t=0} = \dot{u}_0,\ \nu|_{t=0} = \nu_0, && \text{on } \mathscr{B},
\end{aligned} \tag{6.5}$$

As in reference [3, Theorem 4.1], we assume u_0, ν_0 bounded in $\mathscr{H}^1$. Moreover, we require $\dot{u}_0 \in \mathscr{H}^1$.

As already indicated in the Introduction, we denote by $\boldsymbol{w}(t)$ the triple $\boldsymbol{w}(t) := (u(t), \nu(t); v(t)) = (u(t), \nu(t); u_t(t))$. It belongs to the space $\mathscr{W}$ of weak solutions to the system (1.1) [(i.e. (6.5)]:

$$\begin{aligned}
\vec{\mathscr{W}} = \Big\{ \vec{\boldsymbol{w}} = (u, \nu; v) \in L^2_{\mathrm{loc}}[0, +\infty; (\mathscr{H}^1)^2) \times L^2_{\mathrm{loc}}[0, +\infty; L^2(\mathscr{B})) \mid \\
v = u_t,\ \text{and} (u, \nu) \quad \text{a weak solution to (1.1) in the sense of Definition 6.1} \Big\},
\end{aligned}$$

which are indeed those of system (6.5).

We consider $\mathscr{W}$ as a subset of $L^2_{\mathrm{loc}}[0, +\infty; (\mathscr{H}^1)^2) \times L^2_{\mathrm{loc}}[0, +\infty; L^2(\mathscr{B}))$ with metric d induced by this latter space, i.e.,

$$d(\boldsymbol{w}_1, \boldsymbol{w}_2) = \sum_{n=0}^{\infty} 2^{-n} \min\big\{1,\ |\!|\!| \boldsymbol{w}_1 - \boldsymbol{w}_2 |\!|\!|_{L^2(0,n)}\big\}, \tag{6.6}$$

where, given $\boldsymbol{w} = (u, v; \nu)$, we define

$$|\!|\!| \boldsymbol{w} |\!|\!|^2_{L^2(a,b)} := \int_a^b |\!|\!| \boldsymbol{w}(s) |\!|\!|^2$$

and

$$|\!|\!| \vec{\boldsymbol{w}}(t) |\!|\!|^2 := (\|u\|^2 + \|\nabla u\|^2 + \|\nu\|^2 + \|\nabla\nu\|^2 + \|v\|^2),$$

with $\|\cdot\|$, we repeat, the L^2-norm.

Let us recall that a set B in a linear topological space $\mathscr{X}$ is *bounded* when for every neighborhood U of the origin in $\mathscr{X}$ we find $r > 0$ such that $B \subset \{ru \colon u \in U\}$. For $\mathscr{X} = L^2_{\mathrm{loc}}[0, +\infty; (\mathscr{H}^1)^2) \times L^2_{\mathrm{loc}}[0, +\infty; L^2(\mathscr{B}))$, the notion of boundedness for a generic its subset B writes formally

$$\sup\big\{ |\!|\!| \boldsymbol{w} |\!|\!|_{L^2(0,n)} \,|\, \boldsymbol{w} \in B \big\} < +\infty,\ \ \forall\, n = 0, 1, 2, \ldots$$

Theorem 6.1. *The time-shift operator $S(t)\boldsymbol{w} = \boldsymbol{w}_{+t} := \boldsymbol{w}(\cdot + t)$, $\boldsymbol{w} \in \mathscr{W}$, associated with (6.5) admits a weak global attractor $\mathscr{A}$ in $\mathscr{W}$.*

To prove the statement we adapt Theorem 5.1 to the present situation. Then, Theorem 1.1 is a direct consequence. A difficulty is the lack of uniqueness of weak solutions to system (6.5). For this reason we adopt the approach developed by Sell in reference [17] (specifically, Lemma 7 in [17]; see also [18]).

Proposition 6.1. *The mapping $(0, +\infty) \times \mathscr{W} \to \mathscr{W}$ given by $S(t)\boldsymbol{w} = \boldsymbol{w}_{+t} = \boldsymbol{w}(\cdot + t)$ is a semiflow.*

Proof. $S(t)$ is a semigroup. We need to prove that $(\tau, \mathbf{w}) \to S(\tau)\mathbf{w} = \mathbf{w}_{+\tau}$ is continuous for $(\tau, \mathbf{w}) \in (0, +\infty) \times L^2_{\mathrm{loc}}[0, +\infty; (\mathscr{H}^1)^2) \times L^2_{\mathrm{loc}}[0, +\infty; L^2(\mathscr{B}))$. It is sufficient to prove that, if τ_n and $\boldsymbol{w}^n$ are sequences such that $\tau_n \to \tau$ in $(0, +\infty)$ and $\boldsymbol{w}^n \to \boldsymbol{w}$ in $L^2_{\mathrm{loc}}[0, +\infty; (\mathscr{H}^1)^2) \times L^2_{\mathrm{loc}}[0, +\infty; L^2(\mathscr{B}))$, we get $d(\boldsymbol{w}^n_{+\tau_n}, \boldsymbol{w}_{+\tau}) \to 0$ as $n \to +\infty$, which holds true provided that

$$\int_a^b \left|\left|\left| \boldsymbol{w}^n_{+\tau_n} - \boldsymbol{w}_{+\tau} \right|\right|\right|^2 \to 0 \quad \text{as } n \to +\infty$$

for any given pair (a, b) with $0 \le a < b < \infty$ (even if, here, we can always set $a = 0$). Since $\tau > 0$, we can assume $\frac{1}{2}\tau \le \tau_n \le 2\tau$, so that it suffices to show that

$$d(\boldsymbol{w}^n_{+\tau_n}, \boldsymbol{w}_{+\tau_n}) \to 0 \text{ and } d(\boldsymbol{w}_{+\tau_n}, \boldsymbol{w}_{+\tau}) \to 0, \text{ as} n \to +\infty. \tag{6.7}$$

By assumption, $\boldsymbol{w}^n \to \boldsymbol{w}$ in $L^2_{\mathrm{loc}}[0, +\infty; (\mathscr{H}^1)^2) \times L^2_{\mathrm{loc}}[0, +\infty; L^2(\mathscr{B}))$ as $n \to +\infty$, so that we compute

$$\int_a^b \left|\left|\left| \boldsymbol{w}^n_{+\tau_n} - \boldsymbol{w}_{+\tau_n} \right|\right|\right|^2 = \int_{a+\tau_n}^{b+\tau_n} |||\boldsymbol{w}^n - \boldsymbol{w}|||^2 \le \int_{a+\frac{1}{2}\tau}^{b+2\tau} |||\boldsymbol{w}^n - \boldsymbol{w}|||^2 \to 0,$$

a result implying the first convergence in the pair of limits (6.7).

To prove the second limit, let us fix $\varepsilon > 0$. Take $\vec{\boldsymbol{\psi}} \in \mathscr{C}^1([a+\frac{\tau}{2}, b+2\tau]; (\mathscr{H}^1)^3)$ such that $\int_a^b \left|\left|\left| \boldsymbol{w}_{+\ell} - \boldsymbol{\psi}_{+\ell} \right|\right|\right|^2 \le \varepsilon$ for all $\ell \in [\tau/2, 2\tau]$. Let K be an upper bound constant to $|||\partial_t \boldsymbol{\psi}(s)|||$ for $a + \frac{\tau}{2} \le s \le b + 2\tau$. Then, we have

$$|||\boldsymbol{\psi}(\tau_n + t) - \boldsymbol{\psi}(\tau + t)||| \le \left| \int_{\tau_n}^{\tau} |||\partial_t \boldsymbol{\psi}(s + t)||| \mathrm{d}s \right| \le K|\tau_n - \tau|.$$

Also, we find

$$\int_a^b \left|\left|\left| \boldsymbol{\psi}_{+\tau_n} - \boldsymbol{\psi}_{+\tau} \right|\right|\right|^2 \le K^2(b-a)|\tau_n - \tau|^2 \le \varepsilon$$

for $n \ge N$ sufficiently large. By using the triangular inequality twice and previous bounds, we infer

$$\begin{aligned}
\int_a^b |||\boldsymbol{w}_{+\tau_n} - \boldsymbol{w}_{+\tau}|||^2 &\le \int_a^b (\left|\left|\left| \boldsymbol{w}_{+\tau_n} - \boldsymbol{\psi}_{+\tau_n} \right|\right|\right| \\
&\quad + \left|\left|\left| \boldsymbol{\psi}_{+\tau_n} - \boldsymbol{\psi}_{+\tau} \right|\right|\right| + \left|\left|\left| \boldsymbol{\psi}_{+\tau} - \boldsymbol{w}_{+\tau} \right|\right|\right|)^2 \\
&\le 3 \int_a^b \left(\left|\left|\left| \boldsymbol{w}_{+\tau_n} - \boldsymbol{\psi}_{+\tau_n} \right|\right|\right|^2 + \left|\left|\left| \boldsymbol{\psi}_{+\tau_n} - \boldsymbol{\psi}_{+\tau} \right|\right|\right|^2 \right. \\
&\quad \left. + \left|\left|\left| \boldsymbol{\psi}_{+\tau} - \boldsymbol{w}_{+\tau} \right|\right|\right|^2 \right) \le 9\varepsilon
\end{aligned}$$

for all $n \ge N$, which proves that $d(\boldsymbol{w}_{+\tau_n}, \boldsymbol{w}_{+\tau}) \to 0$, as $n \to +\infty$, and this ends the proof. $\square$

For any given $\boldsymbol{w} \in \mathscr{W}$, with $\boldsymbol{w} = (u, \nu; v)$, define $\mathfrak{z}(t)$ as

$$\mathfrak{z}(t) := \int\limits_{\mathscr{B}} \Big(\rho|v + \alpha u|^2 + \alpha^2|u|^2 + (\mu - \epsilon\alpha)|\nabla u|^2 + (\kappa_0 + \alpha\varsigma)|\nu|^2 + (\zeta + \alpha\delta)|\nabla\nu|^2\Big), \tag{6.8}$$

which is equivalent to $\|\boldsymbol{w}(t)\|^2$, i.e., $\mathfrak{z}(t) \approx \|\boldsymbol{w}(t)\|^2$. In the definition above, α is a positive parameter; we take $\mu - \epsilon\alpha > 0$ and assume

$$\mu \geq \epsilon\alpha + k, \quad \varsigma > k. \tag{6.9}$$

Lemma 6.1. *By taking $\alpha > 0$ sufficiently small, we get*

$$\mathfrak{z}(t) \leq \mathfrak{z}(\tau)\exp\{-\tilde{\beta}(t-\tau)\} + \tilde{c}, \textit{ with } t \geq \tau \geq 0, \tag{6.10}$$

where $\tilde{\beta}$ and $\tilde{c}$ are positive constants, with $\tilde{c} = \tilde{c}(\|b\|^2_{L^2(0,+\infty;L^2(\mathscr{B}))})$,

$$\mathfrak{z}(t) \leq \kappa_1 \exp\{-\tilde{\beta}t\} + \tilde{c}, \tag{6.11}$$

with $t \geq 0$, and

$$\kappa_1 = \int\limits_{\mathscr{B}} \Big(\rho|\dot{u}_0 + \alpha u_0|^2 + \alpha^2|u_0|^2 + (\mu - \epsilon\alpha)|\nabla u_0|^2 + (\kappa_0 + \alpha\varsigma)|\nu_0|^2 + (\zeta + \alpha\delta)|\nabla\nu_0|^2\Big).$$

Proof. First, we multiply $(1.1)_1$ and $(1.1)_2$ respectively by $v = u_t$ and ν_t in $L^2(\mathscr{B})$. Then, after integrating in time on the interval $(0, t)$, we infer

$$\rho\|u_t\|^2 + \kappa_0\|\nu\|^2 + \mu\|\nabla u\|^2 + \zeta\|\nabla\nu\|^2 + 2\varsigma\int\limits_0^t \|\nu_t\|^2 + \epsilon\int\limits_0^t \|\nabla u_t\|^2 + 2\delta\int\limits_0^t \|\nabla\nu_t\|^2 \leq \bar{c} + c_2\|b\|^2_{L^2(L^2(\mathscr{B}))} =: \bar{\bar{c}}, \tag{6.12}$$

where $\bar{c} = \bar{c}(\|u_0\|_{1,2}, \|\dot{u}_0\|, \|\nu_0\|_{1,2}, \rho, \kappa_0, \mu, \zeta, \xi, \bar{\xi}, \gamma)$.

By inserting $\vartheta := u_t + \alpha u$ into $(1.1)_1$, we write

$$\rho\vartheta_t - \alpha\vartheta + \alpha^2 u - \epsilon\Delta\vartheta - (\mu - \epsilon\alpha)\Delta u = F(u, \nu), \tag{6.13}$$

with $\mu - \epsilon\alpha > 0$ and $F(u, \nu) := \xi\nabla \operatorname{div} u + \kappa\Delta\nu + \bar{\xi}\nabla \operatorname{div} \nu + b$. Computation of the L^2-product of Eq. (6.13), after multiplication by ϑ, leads to the inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\rho\|\vartheta\|^2 + \alpha^2\|u\|^2 + (\mu - \epsilon\alpha)\|\nabla u\|^2\big) + \alpha^3\|u\|^2 + \alpha(\mu - \epsilon\alpha)\|\nabla u\|^2 + \epsilon\|\nabla\vartheta\|^2 \leq \big(F(u, \nu), \vartheta\big) + \alpha\|\vartheta\|^2. \tag{6.14}$$

Again, taking the L^2-product of $(1.1)_2$, after multiplication by $\nu_t + \alpha\nu$, adding the resulting equation with (6.14), and proceeding as in the case of inequality (6.12), we get

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\rho\|\vartheta\|^2+\alpha^2\|u\|^2+(\mu-\epsilon\alpha)\|\nabla u\|^2+(\kappa_0+\alpha\varsigma)\|\nu\|^2+(\zeta+\alpha\delta)\|\nabla\nu\|^2\big)\\
&\qquad+\alpha^3\|u\|^2+\alpha(\mu-\epsilon\alpha)\|\nabla u\|^2+\alpha\kappa_0\|\nu\|^2+\alpha\zeta\|\nabla\nu\|^2\\
&\qquad+\varsigma\|\nu_t\|^2+\epsilon\|\nabla\vartheta\|^2+\delta\|\nabla\nu_t\|^2+\alpha\xi\int_{\mathscr{B}}|\operatorname{div}u|^2+\alpha\gamma\int_{\mathscr{B}}|\operatorname{div}\nu|^2\\
&\quad\le\alpha 2\kappa\int_{\mathscr{B}}\nabla u\cdot\nabla\nu+2\alpha\bar{\xi}\int_{\mathscr{B}}\operatorname{div}u\,(\operatorname{div}\nu)+\alpha\|\vartheta\|^2\\
&\qquad+\|b\|\|\vartheta\|+\ell\int_{\mathscr{B}}|\mathrm{curl}\vartheta||\nu_t||\nu|.
\end{aligned}
$$

Thus, by reabsorbing terms in the left-hand side, as done for the inequality (6.12), and using the Hölder inequality, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\rho\|\vartheta\|^2+\alpha^2\|u\|^2+(\mu-\epsilon\alpha)\|\nabla u\|^2+(\kappa_0+\alpha\varsigma)\|\nu\|^2+(\zeta+\alpha\delta)\|\nabla\nu\|^2\Big)\\
&\qquad+\alpha^3\|u\|^2+\alpha(\mu-\epsilon\alpha-\kappa)\|\nabla u\|^2+\alpha\kappa_0\|\nu\|^2+\alpha(\zeta-\kappa)\|\nabla\nu\|^2\\
&\qquad+\varsigma\|\nu_t\|^2+\epsilon\|\nabla\vartheta\|^2+\delta\|\nabla\nu_t\|^2\\
&\quad\le\ell\|\mathrm{curl}v\|\|\nu_t\|_{L^4}\|\nu\|_{L^4}+(\alpha+\epsilon)\|\vartheta\|^2+c_\epsilon\|b\|^2.
\end{aligned}
\tag{6.15}
$$

Ladyzhenskaya's and Young's inequalities imply also

$$
\begin{aligned}
\ell\|\mathrm{curl}\vartheta\|\|\nu_t\|_{L^4}\|\nu\|_{L^4}&\le\varepsilon\|\nabla\vartheta\|^2+c_\varepsilon\big(\|\nu_t\|^{\frac12}\|\nabla\nu_t\|^{\frac32}\big)\big(\|\nu\|^{\frac12}\|\nabla\nu\|^{\frac32}\big)\\
&\le\varepsilon\|\nabla\vartheta\|^2+c_{\varepsilon,\bar\varepsilon}\|\nu\|^2\|\nu_t\|^2+\bar\varepsilon\|\nabla\nu\|^2\|\nabla\nu_t\|^2\\
&\le\varepsilon\|\nabla\vartheta\|^2+c_{\varepsilon,\bar\varepsilon}\|\nu\|^2_{L^\infty(L^2)}\|\nu_t\|^2+\bar\varepsilon\|\nabla\nu\|^2_{L^\infty(L^2)}\|\nabla\nu_t\|^2\\
&\le\varepsilon\|\nabla\vartheta\|^2+c_{\varepsilon,\bar\varepsilon}\bar{\bar{c}}\|\nu_t\|^2+\bar\varepsilon\bar{\bar{c}}\|\nabla\nu_t\|^2\ ,
\end{aligned}
$$

where the Young inequality, with $p=4$ and $q=4/3$, plays once again a role in the second step, while inequality (6.12) implies the last bound. The parameters $\varepsilon,\bar\varepsilon>0$ are small as needed. Relation (6.15) and the previous inequality imply

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\rho\|\vartheta\|^2+\alpha^2\|u\|^2+(\mu-\epsilon\alpha)\|\nabla u\|^2+(\kappa_0+\alpha\varsigma)\|\nu\|^2\\
&\quad+(\zeta+\alpha\delta)\|\nabla\nu\|^2\Big)+\alpha^3\|u\|^2\\
&\quad+\alpha(\mu-\epsilon\alpha-\kappa)\|\nabla u\|^2+\alpha\kappa_0\|\nu\|^2\\
&\quad+\alpha(\zeta-\kappa)\|\nabla\nu\|^2+(\epsilon-\varepsilon)\|\nabla\vartheta\|^2\\
&\quad+(\varsigma-c_{\varepsilon,\bar\varepsilon}\bar{\bar{c}})\|\nu_t\|^2+(\delta-\bar\varepsilon\bar{\bar{c}})\|\nabla\nu_t\|^2\\
&\quad\le(\alpha+\varepsilon)\|\vartheta\|^2+c_\varepsilon\|b\|^2.
\end{aligned}
\tag{6.16}
$$

We find also

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\rho\|\vartheta\|^2+\alpha^2\|u\|^2+(\mu-\epsilon\alpha)\|\nabla u\|^2+(\kappa_0+\alpha\varsigma)\|\nu\|^2+(\zeta+\alpha\delta)\|\nabla\nu\|^2\Big)\\
&\qquad+\alpha^3\|u\|^2+\alpha(\mu-\epsilon\alpha-\kappa)\|\nabla u\|^2+\alpha\kappa_0\|\nu\|^2+\alpha(\zeta-\kappa)\|\nabla\nu\|^2\\
&\qquad+[(\epsilon-\varepsilon)\lambda_1-(\alpha+\varepsilon)]\|\vartheta\|^2\le c_\varepsilon\|b\|^2,
\end{aligned}
$$

where λ_1 is the best (smallest) constant for which the Poincaré inequality exploited here [3, §4] holds true. We also assume that $\epsilon > \alpha + \varepsilon(1+\lambda_1)$, so that we compute

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\big(\rho\|\vartheta\|^2 + \alpha^2\|u\|^2 + (\mu-\epsilon\alpha)\|\nabla u\|^2 + (\kappa_0+\alpha\varsigma)\|\nu\|^2\big)\\ &\quad + \tilde{\beta}\big(\rho\|\vartheta\|^2 + \alpha^2\|u\|^2 + (\mu-\epsilon\alpha)\|\nabla u\|^2\\ &\quad + (\kappa_0+\alpha\varsigma)\|\nu\|^2\big) \le 2c_\varepsilon\|b\|^2,\end{aligned} \tag{6.17}$$

where $\tilde{\beta} = \tilde{\beta}(\alpha, \epsilon, \lambda_1, \mu)$ is a suitable positive constant, provided that

$$\rho > [(\epsilon-\varepsilon)\lambda_1 - (\alpha+\varepsilon)],$$

with δ and ς large enough. In this last inequality we have omitted some terms in the left-hand side, which are however not directly useful to reach the bound (6.10). By integrating the inequality (6.17) in the interval (t_0, t), $t_0 \ge 0$, and setting $\tilde{c} := 2c_\varepsilon\|b\|^2_{L^2(L^2(\mathscr{B}))}$, we get the result. □

Proposition 6.2. *$S(t)$ admits a bounded absorbing set $\mathfrak{B} \subset \mathscr{W}$.*

Proof. Define $\mathfrak{B}$ as the subset of $\mathscr{W}$ containing all $\boldsymbol{w}$ such that

$$|\!|\!|\boldsymbol{w}(t)|\!|\!|^2 \le 2\tilde{c} \tag{6.18}$$

for every $t \ge t_0(\mathfrak{B})$, with $t_0(\mathfrak{B})$ to be determined. Due to the definition (6), as a consequence of relation (6.10), $\mathfrak{B}$ is bounded in $\mathscr{W}$. Actually, the dissipative nature evidenced by relation (6.10) (for any fixed $\tau > 0$) implies the existence of $t_1 > 0$ such that the inequality (6.18) holds for all $t \ge t_1$, as soon as $|\!|\!|\boldsymbol{w}(\tau)|\!|\!|^2 \exp\{-\tilde{\beta}(t_1-\tau)\} \le \tilde{c}$. Thus, $S(t)\boldsymbol{w}$ belongs to $\mathfrak{B}$ for each $t \ge t_1$, and $\mathfrak{B}$ is an absorbing set. To conclude, it is enough to take $t_0(\mathfrak{B})$ as the smallest t_1 of the type above. □

In light of the previous preliminary results, Theorem 6.1 is a consequence of the following proposition:

Proposition 6.3. *The semiflow defined by $S(t)$ on $\mathscr{W}$ is compact, i.e, for every bounded set B in $\mathscr{W}$ and for each $t > 0$, the trajectory $S(t)B$ lies in a compact subset of $\mathscr{W}$.*

Proof. Let B be a bounded subset of $\mathscr{W}$. If $S(t)B$ is contained in a compact set of $\mathscr{W}$ for some $t > 0$, $S(t+s)B$ lies in a compact set of $\mathscr{W}$ too, as a consequence of the semigroup property of $S(t)$. Thus, to prove the claim, it suffices to show that $S(t)B$ lies in a compact set of $\mathscr{W}$ for $0 < t \le 1$.

Since $\mathscr{W}$ is a metric space, it is enough to show that $S(t)B$ is sequentially compact. Let $\{\boldsymbol{w}^n\} = \{(u^n, \nu^n; v^n)\}$, $v^n = u^n_t$, be a bounded sequence in $\mathscr{W}$. Consider

$$\begin{aligned}\mathfrak{z}^n(t) = \int_{\mathscr{B}} \Big(&\rho|u^n_t + \alpha u^n|^2 + \alpha^2|u^n|^2 + (\mu-\epsilon\alpha)|\nabla u^n|^2\\ &+ (\kappa_0+\alpha\varsigma)|\nu^n|^2 + (\zeta+\alpha\delta)|\nabla\nu^n|^2\Big) \approx |\!|\!|\boldsymbol{w}^n(t)|\!|\!|^2.\end{aligned} \tag{6.19}$$

There is a positive constant M_0 such that $\int_0^1 |\!|\!|\boldsymbol{w}^n(s)|\!|\!|^2 = c\int_0^1 \mathfrak{z}^n(s) \le M_0^2$, where we have used inequality (6.11). By recalling that $S(t)\boldsymbol{w}^n(\tau) = \boldsymbol{w}^n_{+t}(\tau) = \boldsymbol{w}^n(\tau+t)$, due to the estimate (6.10), for $s_0 \in (0, t)$ and $s \ge 0$ we compute

$$\begin{aligned}\left|\!\left|\!\left|\boldsymbol{w}^n_{+t}(s)\right|\!\right|\!\right|^2 = |\!|\!|\boldsymbol{w}^n(s+t)|\!|\!|^2 &= c\mathfrak{z}^n(t+s)\\ &\le c\mathfrak{z}^n(s_0)\exp\{-\tilde{\beta}(t+s-s_0)\} + \tilde{c}\\ &\le c\mathfrak{z}^n(s_0) + \tilde{c},\end{aligned}$$

and, by integrating on $(0,1)$, we get

$$\left|\left|\left|\boldsymbol{w}_{+t}^n(s)\right|\right|\right|^2 \leq \int_0^1 \left(c\mathfrak{z}^n(s_0)+\tilde{c}\right) \leq M_0^2+\tilde{c}$$

i.e., $\sup_{t\geq 0}\left|\left|\left|\boldsymbol{w}_{+t}^n(s)\right|\right|\right|^2 \leq M_0^2+\tilde{c}$, for all n. Also, by Lemma 4.1, we obtain

$$\begin{aligned}\int_m^{m+1} \left|\left|\left|\boldsymbol{w}_{+t}^n(s)\right|\right|\right|^2 &= \int_m^{m+1} |||\boldsymbol{w}^n(s+t)|||^2 \\ &\leq |||\boldsymbol{w}^n(t)|||^2 + \left(1+\tilde{c}\right) \\ &\leq |||\boldsymbol{w}^n(s_0)|||^2 e^{-\tilde{\beta}(t-s_0)} + \left(1+2\tilde{c}\right)\end{aligned}$$

for $s_0 \in (0,t)$. By integrating this inequality on the interval $(0,1)$, we get

$$\int_m^{m+1} \left|\left|\left|\boldsymbol{w}_{+t}^n(s)\right|\right|\right|^2 \leq M_0^2 e^{-\tilde{\beta}(t-s_0)} + \left(1+3\tilde{c}\right), \tag{6.20}$$

for any $n=0,1,2\ldots$ and for any $m=0,1,2\ldots$

The above estimates imply that $S(t)\boldsymbol{w}^n = S(t)(u^n,\nu^n,v^n)$ is bounded in $L^2_{\text{loc}}[0,+\infty;(\mathscr{H}^1)^2)\times L^2_{\text{loc}}[0,+\infty;L^2(\mathscr{B}))$.

In a similar way, by using the estimate (6.16), we obtain

$$\begin{aligned}&\int_m^{m+1}\Big((\varsigma - c_{\varepsilon,\bar{\varepsilon}}\bar{\bar{c}})\|\partial_t\nu_{+t}^n\|^2 + (\delta-\bar{\varepsilon}\bar{\bar{c}})\|\nabla\partial_t\nu_{+t}^n\|^2 + \hat{\eta}\|\nabla v_{+t}^n\|^2\Big) \\ &\quad= \int_m^{m+1}\Big(\hat{\varsigma}\|\partial_t\nu^n(s+t)\|^2 + \hat{\delta}\|\nabla\partial_t\nu^n(s+t)\|^2 + \hat{\eta}\|\nabla v_{+t}^n\|^2\Big) \\ &\quad\leq |||\boldsymbol{w}^n(t)|||^2 + (1+\tilde{c}) \\ &\quad\leq |||\boldsymbol{w}^n(s_0)|||^2 e^{-\tilde{\beta}(t-s_0)} + (1+2\tilde{c}),\end{aligned}$$

for $s_0\in(0,t)$, and $\hat{\eta}=\min\{\alpha(\mu-\epsilon\alpha-\kappa),\epsilon-\varepsilon\}$. Therefore, by integrating the above inequality on the interval $(0,1)$, we get

$$\begin{aligned}&\int_m^{m+1}\big(\varsigma - c_{\varepsilon,\bar{\varepsilon}}\bar{\bar{c}}\big)\|\partial_t\nu_{+t}^n\|^2 + (\delta-\bar{\varepsilon}\bar{\bar{c}})\|\nabla\partial_t\nu_{+t}^n\|^2 \\ &\qquad + \hat{\eta}\|\nabla v_{+t}^n\|^2\big) \leq M_0^2 + (1+2\tilde{c}).\end{aligned} \tag{6.21}$$

These estimates imply the boundedness of $S(t)(\partial_t\nu^n,v^n)=S(t)(\partial_t\nu^n,\partial_t u^n)$ in the space $L^2_{\text{loc}}[0,+\infty;(\mathscr{H}^1)^2)$. Along the same path followed to prove existence in reference [3], we can also prove that $S(t)\partial_t v^n \in W^{1,2}_{\text{loc}}(0,+\infty;\mathscr{H}^{-1})$.

Thus, to estimate $\int_m^{m+1}\|\partial_t v_{+t}^n(s)\|^2_{\mathscr{H}^{-1}} = \int_m^{m+1}\|\partial^2_{tt}u_{+t}^n(s)\|^2_{\mathscr{H}^{-1}}$ the only term to be controlled is $\ell(\text{curl}v^m)\times \nu_t^m$. In fact, for all $\varphi\in\mathscr{H}^1$, we compute

$$\ell \int_m^{m+1} \Big((\mathrm{curl} v^m) \times \nu_t^m \big)(s), \varphi \Big)$$
$$\leq C \Big[\int_m^{m+1} \|(\mathrm{curl} v^m) \times \nu_t^m)(s)\|_{H^{-1}} \mathrm{d}s \Big] \|\nabla \varphi\|$$
$$\leq C \Big[\int_m^{m+1} \|\mathrm{curl} v^m\| \|\nabla \nu_t^m\| \Big] \|\nabla \varphi\|$$
$$\leq \Big[\frac{C}{2} \int_m^{m+1} \|\nabla v^m\|^2 + \frac{C}{2} \int_m^{m+1} \|\nabla \nu_t^m\|^2 \Big] \|\nabla \varphi\|$$

and the conclusion follows by exploiting inequality (6.21).

The Aubin–Lions compactness theorem implies that $S(t)\boldsymbol{w}^n$ converges strongly to $\hat{\boldsymbol{w}}(t)$ in $L^2_{\mathrm{loc}}[0, +\infty; L^2(\mathscr{B})^3)$ and weakly in $L^2_{\mathrm{loc}}[0, +\infty; (\mathscr{H}^1)^3)$, as $n \to +\infty$, up to a subsequence. Moreover, since $S(t)\boldsymbol{w}^n(\tau) = \boldsymbol{w}^n(\tau+t)$, the same compactness argument implies that $\boldsymbol{w}^n$ converges strongly to $\boldsymbol{w}$ in $L^2_{\mathrm{loc}}[0, +\infty; L^2(\mathscr{B})^3)$ and weakly in $L^2_{\mathrm{loc}}[0, +\infty; (\mathscr{H}^1)^3)$, up to a subsequence.

By the continuity of $(\tau, \boldsymbol{w}) \mapsto S(t)\boldsymbol{w}$ in $(0, +\infty) \times L^2_{\mathrm{loc}}[0, +\infty; (\mathscr{H}^1)^3)$, we also realize that $S(t)\boldsymbol{w}^n \to S(t)\boldsymbol{w}$ and the limit uniqueness implies an inclusion $\hat{\boldsymbol{w}}(t) = S(t)\boldsymbol{w} \in \mathscr{W}$. Thus, the limiting function $\hat{\boldsymbol{w}}$ is a weak solution to system (6.5). This concludes the proof. □

Eventually, we prove Theorem 1.1 by applying directly Theorem 6.1, Lemma 6.1, Propositions 6.1, 6.2, and 6.3.

7. Closing remark

Besides its pertinence to the dynamics of quasicrystals, our result can be referred to other classes of bodies with vector-type microstructure, provided that they undergo mechanical processes appropriately described by system (1.1), within the limits of its validity.

Sure, the presence of viscous-type interactions, which have a regularizing role from an analytical viewpoint, suggests a priori the possible existence of an attractor; however, the lack of uniqueness induced by the nonlinear coupling complicate the scenario and calls into play weak attractors rather than the strong ones.

Acknowledgements

This work has been developed within activities of the research group in “Theoretical Mechanics” of the “Centro di Ricerca Matematica Ennio De Giorgi” of the Scuola Normale Superiore in Pisa. The support of GNAMPA-INDAM, GNFM-INDAM and GNAFA-INDAM is acknowledged.

Funding Information Open access funding provided by Universit[Pleaseinsertintopreamble] degli Studi di Firenze within the CRUI-CARE Agreement.

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Luca Bisconti
DIMAI
Università di Firenze
via Santa Marta 3
50139 Florence
Italy
e-mail: luca.bisconti@unifi.it

Paolo Maria Mariano
DICEA
Università di Firenze
via Santa Marta 3
50139 Florence
Italy
e-mail: paolomaria.mariano@unifi.it

(Received: September 4, 2021; revised: July 24, 2022; accepted: October 18, 2022)