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Classes of Dyck paths associated with numerical semigroups*†

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Abstract

We investigate the relationship between numerical semigroups and Dyck paths discovered by Bras-Amorós and de Mier. More specifically, we consider some classes of Dyck paths and characterize those paths giving rise to numerical semigroups.

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1 Introduction

A *numerical semigroup* is a subset S of the set of natural numbers $\mathbb{N}$ such that:

(i) $0 \in S$;

(ii) $\mathbb{N} \setminus S$ is finite;

(iii) for every $x, y \in S$, $x + y \in S$.

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In other words, a numerical semigroup is a cofinite subset of $\mathbb{N}$ containing 0 and closed under addition.

Numerical semigroups probably owe their origin to the study of nonnegative integer solutions of Diophantine equations. Indeed, given positive integers $a_1, \dots, a_n$ such that $\gcd(a_1 \dots, a_n) = 1$, the set of natural numbers b such that the Diophantine equation $a_1x_1 + \cdots + a_nx_n = b$ has nonnegative solutions is a numerical semigroup. More specifically, it is the numerical semigroup *generated* by the set $G = \{a_1, \dots, a_n\}$ (meaning that all the elements of the semigroup can be written as linear combinations of elements of G with nonnegative integer coefficients). The celebrated *Frobenius problem* asks for the minimum integer b for which the above equation has *no* solution. Such a quantity is also called the *Frobenius number* of the associated semigroup, and corresponds to the largest element not belonging to the semigroup. The Frobenius problem has been solved, for instance, for semigroups whose minimal generating set has cardinality 2 (by Sylvester) and in other special cases, but it is still open in its full generality. There are many other *enumerative* problems about numerical semigroups, see for instance [B-A, E, K].

Given a numerical semigroup S, the elements of the (finite) set $\mathbb{N} \setminus S$ are called *gaps*. The total number of gaps of S is the *genus* of S, and the largest gap is its *Frobenius number*. The smallest integer c such that all numbers $n \geq c$ belongs to S is called the *conductor* of S, and the smallest element of S other than 0 is its *multiplicity*. Notice that the Frobenius number of S is $c-1$. When representing a numerical semigroup, we will write $S = \{0, m, a_1, a_2, \dots a_k, c \rightarrow\}$ to express the fact that all elements bigger than c belong to S. An *ordinary semigroup* is a numerical semigroup of the form $\{0, c \rightarrow\}$, for some $c \in \mathbb{N}$.

Numerical semigroups can be described using Dyck paths. A *Dyck path* is a lattice path starting from the origin of a fixed Cartesian coordinate system, ending on the x-axis, never falling below the x-axis and using two kind of steps, namely *up steps* $U = (1, 1)$ and *down steps* $D = (1, -1)$. Dyck paths are a very well studied object in classical (enumerative and algebraic) combinatorics, and they are counted by the Catalan numbers (with respect to the semilength, i.e. half of the number of its steps). A *peak* of a Dyck path is a sequence UD made of an up step followed by a down step. Similarly, a *valley* consists of a down step followed by an up step. A *descent* is a maximal sequence of consecutive down steps, whereas an *ascent* is a maximal sequence of consecutive up steps.

In [B-AdM], Bras-Amorós and de Mier define a very natural function f mapping each numerical semigroup into a Dyck path. More precisely, given a numerical semigroup S of genus g, $f(S)$ is the Dyck path of length $2g$ whose i-th step is U (respectively, D) whenever $i \notin S$ (respectively, $i \in S$). See Figure 1 for an instance of the map f. They show that f is a well defined injection, i.e. $f(S)$ is indeed a Dyck path. This result gives the opportunity of addressing problems concerning numerical semigroups by using the language and the machinery developed for the study of lattice paths and their combinatorics. Unfortunately, f is not a bijection, which means that there exist Dyck paths that do not represent any numerical semigroup. It is thus natural to investigate the subclass of

Dyck paths that come from semigroups or, in other words, the image of f. A full characterization of such a class of Dyck paths is still missing. Aim of the present paper is to contribute to this line of research, by considering some special subsets of Dyck paths and characterizing, among them, those which correspond to numerical semigroups through the map f. Specifically, we will deal with Dyck paths having exactly two peaks, Dyck paths of the form $U^{h-1}(UD)^p D^{h-1}$ and Dyck paths avoiding DUU. We will also discuss the enumeration of numerical semigroups in terms of the height of the first valley of the associated Dyck path.

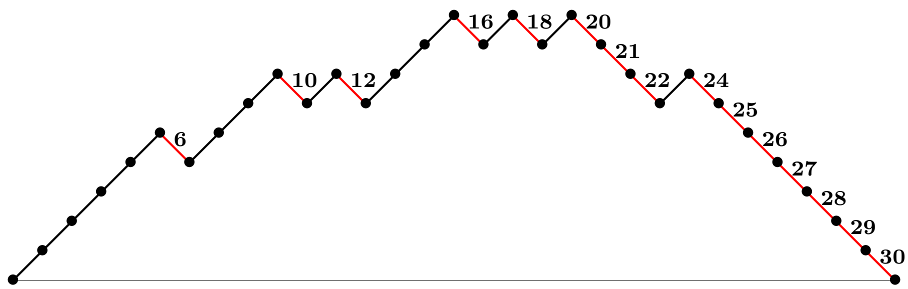

Figure 1: The Dyck path corresponding (through the map f) to the numerical semigroup $\{0, 6, 10, 12, 16, 18, 20, 21, 22, 24 \rightarrow\}$.

2 Dyck paths with two peaks

As it is easily shown in [B-AdM], Dyck paths having exactly one peak correspond to ordinary semigroups. The next step is thus to consider Dyck paths having two peaks.

Let $P = U^\alpha D^\beta U^\gamma D^\delta$ be a Dyck path, with $\alpha, \beta, \gamma, \delta > 0$. If there exists S such that $f(S) = P$, then S is of the form $S = \{0\} \cup \{m, m+1, \ldots, b, c \rightarrow\}$, where m (the multiplicity of S) corresponds to the first down step of P, b corresponds to the final down step of the first descent, and c (the conductor of S) corresponds to the first down step of the final descent of P. Moreover, $b + 2 \leq c$. Our first result provides a characterization of such Dyck paths corresponding to numerical semigroups, both in terms of parameters of the Dyck paths and of the numerical semigroups.

Proposition 1 *Let $P = U^\alpha D^\beta U^\gamma D^\delta$ as above. Then there exists S such that $f(S) = P$ if and only if $\alpha + 1 \geq \beta + \gamma$.*

Proof. Suppose first that S is a semigroup such that $f(S) = P$, so that $S = \{0\} \cup \{m, m+1, \ldots, b, c \rightarrow\}$. Then necessarily $2m = m + m \in S$. It is not difficult to realize that it cannot be $2m \leq b$, otherwise the Dyck path P would fall below the x-axis after its first peak (and S would be an ordinary semigroup). So necessarily $c \leq 2m$. On the other hand, given any set $S = \{0\} \cup \{m, m+1, \dots, b, c \rightarrow\}$, if $c \leq 2m$, then S is certainly a semigroup, since the smallest element which can be obtained as a sum of two nonzero elements of S is $2m \geq c$, which implies that every sum of elements of S is bigger than c, and so it belongs to S. We have thus shown that there exists S such that $f(S) = P$ if and only if $c \leq 2m$. Translating such a condition into parameters on the Dyck path P, since clearly $c = \alpha + \beta + \gamma + 1$ and $m = \alpha + 1$, we get the desired inequality $\alpha + 1 \geq \beta + \gamma$. □

Remark. As we have partially seen in the above proof, the main parameters of the numerical semigroup S can be expressed in terms of the parameters of the associated Dyck path, and vice versa. More specifically, we have

$$\begin{aligned} \alpha &= m - 1, & c &= \alpha + \beta + \gamma + 1, \\ \beta &= c - g - 1, & g &= \alpha + \gamma, \\ \gamma &= g - m + 1, & m &= \alpha + 1. \\ \delta &= 2g - c + 1. & & \end{aligned}$$

3 Dyck paths of the form $U^{h-1}(UD)^p D^{h-1}$

In this section we consider Dyck paths of minimal length having all peaks at the same height h. There exists precisely one such path having exactly p peaks, which is $U^{h-1}(UD)^p D^{h-1}$ for some $h > 1$ (see Figure 2). A Dyck path having the above form will be called a *comb Dyck path*. Our next result is a characterization of comb Dyck paths representing numerical semigroups (in terms of the parameters h and p).

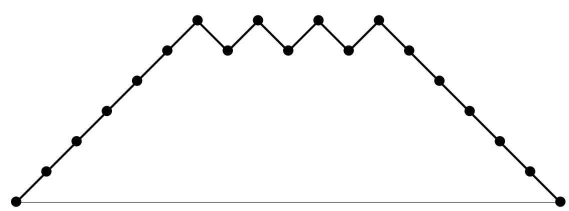

Figure 2: The comb Dyck path corresponding to $h = 5$ and $p = 4$.

Proposition 2 *Let $P = U^{h-1}(UD)^p D^{h-1}$, with $h > 1$ and $p \geq 2$. If h is odd, then for all p there exists a numerical semigroup S such that $f(S) = P$. If h is even, then such an S exists if and only if $p \leq \frac{h+2}{2}$.*

Proof. Looking at the down steps of the path P, the candidate S for which (possibly) $f(S) = P$ is the set $S = \{0, h+1, h+3, \dots, h+(2p-3), h+(2p-1) \rightarrow\}$.

Suppose first that h is odd. This means that S contains all the even numbers starting from $h+1$, and *all* the numbers starting from $h+2p-1$ (which is the conductor). As a consequence, we observe that, if $x, y \in S$ are both even, then certainly $x+y \in S$ (since the sum of two even numbers is even), whereas, if at least one between x and y is odd, then $x+y \geq h+2p-1$, and so again $x+y \in S$. This shows that S is indeed a semigroup (independently from p).

Now suppose that h is even, and assume that S is indeed a numerical semigroup. Since $p \geq 2$ and the path P is symmetric, recalling that the length of P is $2g$, we have that $h+1 \leq g$, and so $2(h+1) \leq 2g$. Since $2(h+1)$is an even number belonging to S, then necessarily $2(h+1) = 2h+2 \geq h+2p$. Moreover, it is clear that $2h+1 \in S$, since all odd numbers bigger than or equal to $h+1$ belong to S. We thus have that both $2h+1$ and $2h+2$ are elements of S, and in particular $2h+1, 2h+2 \geq h+2p$. In terms of the comb Dyck path P, this means that the corresponding down steps both belong to the last descent of P. Hence, the maximum number p of peaks P occurs when $2h+2 = h+2p$, hence $p = \frac{h+2}{2}$. Summing up, we have that the inequality $p \leq \frac{h+2}{2}$ gives a necessary condition for S to be a numerical semigroup. It is now easy to realize that such a condition is also sufficient. In fact, since $h+1$ is minimum nonzero element of S, the smallest element that can be expressed as the sum of two nonzero elements of S is $2h+2$. However, S contains all elements greater than $h+2p-1$, and our inequality implies that $h+2p-1 \leq 2h+1$, hence $2h+2 \in S$ is bigger than the (candidate) conductor of S. This is enough to ensure that any sum of elements of S still belongs to S, i.e. S is a semigroup. $\square$

Remark. As we did in the previous section, if S is a semigroup such that $f(S)$ is a comb Dyck path, we can express parameters of S in terms of P and vice versa. In this case, we have

$$\begin{aligned} p &= c-g, & c &= h+2p-1, \\ h &= 2g-c+1, & g &= h+p-1. \end{aligned}$$

In particular notice that, when S is of the above form and its smallest nonzero element is odd, then a necessary and sufficient condition for S to be a numerical semigroup is that its (candidate) conductor and genus satisfy the inequality $4g-3c+3 \geq 0$.

4 Enumerating semigroups in terms of the height of the first valley of the associated Dyck path

Let P be a Dyck path having its first valley at height k. Under some specific conditions, it is possible to enumerate all numerical semigroups S such that $f(S) = P$.

Suppose that the first down step of P is step m. Then the last step of the first descent of p is step $b = 2m-k-2$. If P represents some semigroup S, then necessarily all numbers between $2m$ and $2b$ must belong to S. This implies that, after step $b+1$ (that necessarily does not belong to S), there are k steps that can be chosen arbitrarily, then the successive step must be a down step, since $(b+1)+(k+1) = 2m-k-1+k+1 = 2m \in S$. It is now clear that, if the path P terminates before reaching step $2b$, then any choice of steps between step $b+1$ and step $2m$ gives a legal semigroup, and these are all semigroups that can be obtained. The above condition on P can be expressed by saying that the height of P after step $2m-1$ must be smaller than the number of steps starting with $2m$ and ending with $2b$, that is $2k+1 < 2b-2m+1$, which gives $m > 2(k+1)$. We collect these facts in the following proposition.

PROPOSITION 3 *Let S be a numerical semigroup having multiplicity m, and let $P = f(S)$ be the associated Dyck path. Suppose that the height of the first valley of P is k. If $m > 2(k+1)$, then there are precisely 2^k semigroups having multiplicity m and such that the first set of consecutive elements belonging to the semigroups terminates with $2m-2-k$.*

5 Dyck paths avoiding DUU

Our last result concerns the class of Dyck paths avoiding DUU. We say that a Dyck path P *avoids* DUU when P does not contain any down step immediately followed by two up steps. We remark that this notion of pattern, which is somehow classical in the context of formal languages (see for instance [ABBG, ABR, STT]), is different from the notion of *nonconsecutive* pattern studied in [BBFGPW, BBCF]. The class of numerical semigroups corresponding to Dyck paths avoiding DUU are somewhat related to the class of *Arf semigroups* [B-AB, RG-SG-GB]; this justifies our interest in such a class of Dyck paths.

In the following, we will use the term *unrestricted path* to denote *any* finite sequence of down and up steps (without any further constraint).

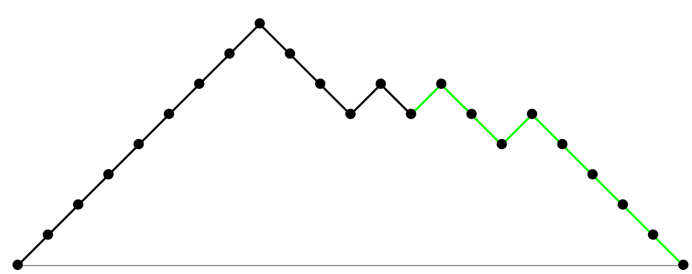

Figure 3: A Dyck path avoiding DUU, with $m = 9$ and $j = 3$. The green part corresponds to the unrestricted path α.

Let P be a Dyck path avoiding DUU. Then P consists of a starting ascent, followed by several descents separated by single up steps. More formally, it will be convenient to write P as $P = U^{m-1}D^jUD\alpha$, where $m > 1$, $j > 0$ and α is an unrestricted path avoiding UU (i.e., not containing two consecutive up steps). See Figure 3 for an example. Notice that, with such a notation, we exclude all Dyck paths having a single peak (i.e. of the form $U^k D^k$); however, as we already know, these corresponds to ordinary semigroups. We first consider the case $j > 1$, that is when the first descent of P contains at least two steps. The next proposition gives a necessary and sufficient condition for P to be associated with some semigroup.

Proposition 4 *Let P be a Dyck path as above, with $j > 1$. Then there exists a numerical semigroup S such that $f(S) = P$ if and only if $\alpha = \beta D^h$, where $h \geq 0$ and β is an unrestricted path of length $m - j - 2$ avoiding UU.*

Proof. We start by supposing that S is a numerical semigroup such that $f(S) = P$. Since we know that $m, m+1, \ldots, m+j-1 \in S$, then necessarily we also have that $2m, 2m+1, \ldots, 2m+2j-2 \in S$. For this reason, it is convenient to factor the unrestricted path α as a concatenation $\alpha = \beta\gamma$, in such a way that γ starts with step $2m$ of P. This is equivalent to say that the length of β is $m - j - 2$. Moreover, it is obvious that both β and γ avoids UU. We now wish to understand what are the steps of P from $2m + 2j - 1$ to $3m$. So let $x \in \{2m + 2j - 1, 2m + 2j, \ldots 3m\}$. Observe that it is possible to find $y, z \in \{m, m + 1, \ldots 2m - 1\}$ such that $x = y + z$, since any such x can be expressed as the sum of $m + j - 1$ and some number between m and $m + j - 1$ (and of course we have $m+j-1 \leq 2m-1$, because $j \leq m-1 < m$). Depending on whether y and z are down steps or not, we have three cases to analyze.

- If both y, z are down steps, then $y, z \in S$, and so $x \in S$.
- If both y, z are up steps, then necessarily $y - 1$ and $z + 1$ are down steps, since there cannot be two consecutive up steps after the first ascent (and $y \neq m$, since we are supposing that y is an up step). Therefore $x = (y - 1) + (z + 1) \in S$.
- If y is an up step and z is a down step, we consider all pairs $(y-k, z+k)$, such that $m \leq y - k, z + k \leq 2m - 1$. Since P avoids UU after the first ascent, and there are at least two consecutive down steps between m and $2m - 1$ (because $j > 1$), there must be some k such that both $y - k$ and $z + k$ are down steps. As a consequence, $x \in S$.

Therefore we can conclude that all elements in $\{2m+2j-1, 2m+2j, \ldots 3m\}$ belong to S. As a consequence, in the path P all steps between $2m$ and $3m$ are down steps, i.e. P contains a descent of length $m + 1$. This is however impossible, since P has a unique peak (the first one) having maximum height $m-1$, unless P terminates *before* step $3m$. We thus have that $\gamma = D^h$, for some $h \geq 0$, as desired.

On the other hand, if P is a Dyck path as in the statement of the proposition, then consider the set

$$S = \{0, m, m + 1, \ldots m + j - 1, m + j + 1, \star, \star, \ldots, \star, 2m \rightarrow\} \subseteq \mathbb{N},$$

where the elements denoted with $\star$ correspond to the steps of P between step $m+j+2$ and step $2m-1$. It is immediate to see that S is a semigroup and that $f(S) = P$. □

The above proposition helps us also to count numerical semigroups avoiding DUU when the first descent has length at least 2. We first need an easy lemma concerning unrestricted paths. Recall that the sequence F_n of *Fibonacci numbers* is defined by the recurrence $F_{n+2} = F_{n+1} + F_n$ with initial conditions $F_0 = F_1 = 1$.

LEMMA 1 *The number of unrestricted paths of length n avoiding UU is F_{n+1}.*

Proof. The statement is easily verified when $n = 0, 1$. For $n \geq 2$, the set of unrestricted paths of length n avoiding UU can be split into those ending with D and those ending with U. In the former case, we have a bijection with unrestricted paths avoiding UU of length $n-1$ by just removing the last up step. In the latter case, the second-to-last step has to be a down step (since the path has to avoid UU), so we have a bijection with unrestricted paths avoiding UU of length $n-2$ by removing the last two steps. □

PROPOSITION 5 *Let $k, j > 0$, with $j \leq k-2$. Denote with $\Delta^{(k,j)}$ the set of all Dyck paths of the form $U^k D^{j+1} UD\alpha$, where α avoids UU. Then there are F_{k-j-1} Dyck paths in $\Delta^{(k,j)}$ which represent numerical semigroups through the map f.*

Proof. Let $P \in \Delta^{(k,j)}$. As we have seen in Proposition 4, if P represents a numerical semigroup, then its multiplicity is $k+1$, hence all steps from $2k+2$ on are necessarily down steps. On the other hand, the set of steps between $k+j+4$ and $2k+1$ can be chosen arbitrarily, provided that no occurrence of UU shows up, and any such choice (which corresponds to exactly one semigroup) is an unrestricted path avoiding UU of length $(2k+1)-(k+j+3) = k-j-2$. As a consequence, thanks to the above lemma, the number of semigroups represented by Dyck paths in $\Delta^{(k,j)}$ is F_{k-j-1}, as desired. □

To conclude the enumerative discussion of this class of Dyck paths, we first observe that, when $j = k-1$, there are exactly two allowed Dyck paths, namely $U^k D^k$ (corresponding to an ordinary semigroup) and $U^k D^k UD$. On the other hand, it is a little bit more involved to understand what happens when $j = 0$. This corresponds to Dyck path whose first descent has length 1. It is convenient to distinguish two cases, that need separate analysis.

Suppose first that k is even, and let $P \in \Delta^{(k,0)}$. If there exists S such that $f(S) = P$, then the multiplicity of S is $k+1$, which is odd. Let i be the smallest index such that $k+1+2i, k+2+2i \in S$. Then clearly $1 \leq i \leq \frac{k}{2}$. It is not difficult to realize that the steps between $k+3+2i$ and $2k+1$ can be chosen arbitrarily, provided that the pattern UU does not occur. Since these are precisely $(2k+1)-(k+2+2i) = k-2i-1$ steps, from Lemma 1 we have that the are exactly F_{k-2i} possible choices. We next observe that all even steps between

$2k+2$ and $2k+2+2i$ must be down steps, whereas the odd steps between them can be chosen arbitrarily. The total number of possible choices of such odd steps is then 2^i. Finally, using an argument similar to that of Proposition 4, we have that all steps from $2k+2+2i$ on have to be down steps. Summing up, when k is even, the total number of elements of $\Delta^{(k,0)}$ representing numerical semigroups is

$$\sum_{i=1}^{k/2} 2^i F_{k-2i}.$$

Using a standard generating function argument (that we omit here), it is possible to simplify the above expression, which turns out to be equal to $2(F_{2k+1} - 2^k)$.

Now suppose that k is odd and let $P \in \Delta^{(k,0)}$. Again, assume that a numerical semigroup S exists such that $f(S) = P$, and define i as the smallest index such that $k+1+2i, k+2+2i \in S$. In this situation, however, it may well happen that $i \geq \frac{k+1}{2}$, so we have to distinguish two cases. If $i < \frac{k+1}{2}$, then we can argue as in the previous case, and we get that there are $2^i F_{k-2i}$ Dyck paths representing some numerical semigroup. Instead, if $i \geq \frac{k+1}{2}$, the set of steps between $k+3+2i$ and $2k+1+2i$ has even cardinality $k-1$, and half of such steps (the even ones) are down steps, whereas the remaining ones (the odd ones) can be chosen arbitrarily. Since all steps from $2k+2+2i$ on have to be down steps, we get that, for each $i \geq \frac{k+1}{2}$, the number of Dyck paths representing numerical semigroups is $2^{\frac{k-1}{2}}$.

6 Further work

The remarkable relationship between numerical semigroups and Dyck paths discovered by Bras-Amorós and De Mier is still far from being fully understood. In the present paper we have contributed to shed some light, by discussing some classes of Dyck paths, in order to find conditions that guarantee the existence of a numerical semigroup represented by a path of the class. Another important direction of research concerns a better understanding of the semigroup tree. The *semigroup tree* (see for instance [B-AB]) is the infinite rooted tree whose nodes are all numerical semigroups and such that the father of a given semigroup S is obtained by adding to S its Frobenius number. When representing numerical semigroups using Dyck paths through the map f, the semigroup tree can be seen as a subtree of a very well known *Catalan tree*, which describes the growth of Dyck path according to a specific local expansion rule, consisting of adding a new peak into the last descent. This generating tree for Dyck paths is described, for instance, in [BDLPP], and is an important tool in the context of the so-called *ECO method*. To be more precise, if P is a Dyck path representing the semigroup S (so that $f(S) = P$), then the sons of P in the semigroup tree are obtained by adding a new peak into *some* positions of the last descent of P, and it is still unclear which positions are allowed. It would then be extremely interesting to investigate specific classes of Dyck paths, trying to describe completely the set of their sons in the semigroup tree.

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