Reconstruction of thin conductivity imperfections

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Reconstruction of Thin Conductivity Imperfections*

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We consider the case of a uniform plane conductor containing a thin curve-like inhomogeneity of finite conductivity. In this article we prove that the imperfection can be uniquely determined from the boundary measurements of the first order correction term in the asymptotic expansion of the steady state voltage potential as the thickness goes to zero.

Keywords: Thin dielectric inhomogeneities; Uniqueness; Reconstruction

AMS Subject Classifications: 35R30

1. INTRODUCTION

Let $\Omega \subset \mathbf{R}^2$ be a smooth bounded domain that represents a uniform conductor. This conductor may contain a thin conductivity imperfection localized in the neighborhood of a curve. To be more precise, we consider inhomogeneities of the form

$$\omega_\varepsilon = \{x + \eta n(x)\colon x \in \sigma, \eta \in (-\varepsilon, \varepsilon)\},$$

where σ is a simple C^3 curve contained in Ω, with positive distance from the boundary, $\partial\Omega$, $n(x)$ is a unit vectorfield normal to σ at x, and ε is a positive small parameter that represents the thickness of the imperfection.

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We set the background conductivity equal to 1. The constant finite conductivity of the imperfection ω_ε, will be denoted by k, with $0 < k < +\infty$ and $k \neq 1$.

Let u_ε be the steady state voltage potential in the presence of the inhomogeneity ω_ε, that is, the unique solution to

$$\begin{cases} \nabla \cdot \big((1 + (k-1)\mathbf{1}_{\omega_\varepsilon})\nabla u_\varepsilon\big) = 0 & \text{in } \Omega, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = h & \text{on } \partial\Omega, \\ \displaystyle\int_{\partial\Omega} u_\varepsilon = 0. \end{cases}$$

Here $\mathbf{1}_{\omega_\varepsilon}$ is the characteristic function of ω_ε, and ν is the unit outer normal to $\partial\Omega$. The function $h \in H^{-1/2}(\partial\Omega)$ represents the applied boundary current and satisfies the compatibility condition $\int_{\partial\Omega} h = 0$.

Let u_0 be the potential induced by the current h in the domain Ω without the imperfection ω_ε, that is the solution to

$$\begin{cases} \Delta u_0 = 0 & \text{in } \Omega, \\ \dfrac{\partial u_0}{\partial \nu} = h & \text{on } \partial\Omega, \\ \displaystyle\int_{\partial\Omega} u_0 = 0. \end{cases}$$

For every $x \in \sigma$, let $\tau(x)$ and $n(x)$ be unit vectorfields that are respectively tangent and normal to σ at x. We define the symmetric matrix $A(x)$ by:

$$\begin{aligned} &A(x) \text{ has eigenvectors } \tau(x) \text{ and } n(x), \\ &\text{the eigenvalue corresponding to } \tau(x) \text{ is } 2(k-1), \\ &\text{the eigenvalue corresponding to } n(x) \text{ is } 2\left(1 - \frac{1}{k}\right). \end{aligned} \tag{1}$$

The following asymptotic expansion has been derived in [3, p. 544] (see [2] for a rigorous error analysis):

$$u_\varepsilon(y) - u_0(y) = \varepsilon u_\sigma(y) + o(\varepsilon), \quad y \in \partial\Omega, \tag{2}$$

where the correction term u_σ is given by

$$u_\sigma(y) = -\int_\sigma A(x)\nabla u_0(x)\nabla_x \Phi_0(x,y)\, d\sigma_x, \quad y \in \overline{\Omega} \setminus \sigma.$$

Here Φ_0 is the Neumann's function related to the domain Ω. We will write the correction term in a slightly different but equivalent way. We denote by

$\Phi(x, y) = -(1/2\pi) \ln |x - y|$ the free-space fundamental solution for the Laplace equation. Since the function $\Phi_0 - \Phi$ is smooth, we can write

$$u_\sigma(y) = -\int_\sigma A(x)\nabla u_0(x)\nabla_x \Phi(x, y)\, d\sigma_x + w_\sigma(y), \quad y \in \overline{\Omega} \setminus \sigma, \tag{3}$$

where $w_\sigma(y) = -\int_\sigma A(x)\nabla u_0(x)\nabla_x(\Phi_0(x, y) - \Phi(x, y))\, d\sigma_x$ is the harmonic function in Ω such that

$$\frac{\partial u_\sigma}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$

and

$$\int_{\partial\Omega} u_\sigma = 0.$$

In this note we are interested in solving the following inverse problem: given the trace of the correction term u_σ on the boundary $\partial\Omega$ determine (uniquely) the curve σ.

As one clearly sees from (2), our data u_σ is only known approximately.

For any unit vector $a \in \mathbf{R}^2$, let u_σ^a denote the correction term that corresponds to the linear background solution

$$u_0^a(x) = a \cdot x - \frac{1}{|\partial\Omega|} \int_{\partial\Omega} a \cdot y\, ds_y,$$

i.e. to the Neumann data $h = a \cdot \nu$.

We first prove (Section 2) that given two C^3 open curves σ and σ', that are graphs with respect to a common coordinate system, if for some unit vector a, $u_\sigma^a = u_{\sigma'}^a$ on an open arc Γ contained in $\partial\Omega$, then $\sigma = \sigma'$. Then (Section 3), in the special case that σ is a straight line segment, we construct an algorithm for finding σ from $u_\sigma^{a_1}|_{\partial\Omega}$ and $u_\sigma^{a_2}|_{\partial\Omega}$, where a_1 and a_2 are two orthogonal unit vectors of $\mathbf{R}^2$.

2. UNIQUENESS

In this section we prove a uniqueness result in the class of curves that are graphs with respect to the same coordinate system.

Given a C^3 function $f : [\alpha, \beta] \to \mathbf{R}$, we consider

$$\sigma_f = \{x = (x_1, x_2) : x_1 \in (\alpha, \beta),\ x_2 = f(x_1)\},$$

such that σ_f is contained in Ω and has positive distance from the boundary. We also set

$$\tau_f(x) = \left(\frac{1}{\sqrt{1 + f'(x_1)^2}}, \frac{f'(x_1)}{\sqrt{1 + f'(x_1)^2}} \right),$$

and

$$n_f(x) = \left(\frac{-f'(x_1)}{\sqrt{1+f'(x_1)^2}}, \frac{1}{\sqrt{1+f'(x_1)^2}} \right),$$

for $x = (x_1, x_2) \in \sigma_f$.

For any function u defined in $\Omega \setminus \sigma_f$, let $[u]_{\sigma_f}$ be the jump of u on σ_f, i.e.

$$[u]_{\sigma_f}(x) = \lim_{t \to 0^+} u(x + t\, n_f(x)) - u(x - t\, n_f(x)), \quad x \in \sigma_f.$$

By P_f and Q_f we denote the endpoints of f, i.e. $P_f = (\alpha, f(\alpha))$ and $Q_f = (\beta, f(\beta))$.

By integrating by parts in (3) we can write,

$$\begin{aligned} u_{\sigma_f}(y) = &-2\left(1 - \frac{1}{k}\right) \int_{\sigma_f} \frac{\partial u_0}{\partial n_f}(x) \frac{\partial \Phi}{\partial n_f(x)}(x, y)\, d\sigma_x \\ &+ 2(k-1) \int_{\sigma_f} \frac{\partial^2 u_0}{\partial {\tau_f}^2}(x) \Phi(x, y)\, d\sigma_x + w_{\sigma_f}(y) \\ &- 2(k-1) \frac{\partial u_0}{\partial \tau_f}(P_f) \Phi(P_f, y) + 2(k-1) \frac{\partial u_0}{\partial \tau_f}(Q_f) \Phi(Q_f, y). \end{aligned} \tag{4}$$

From the above expression we conclude the following

PROPOSITION 2.1 *The function u_{σ_f} defined in* (4) *is harmonic in $\Omega \setminus \sigma_f$, it has logarithmic singularities in P_f and Q_f and the jumps on σ_f are given by*

$$\left[u_{\sigma_f}\right]_{\sigma_f} = -2\left(1 - \frac{1}{k}\right) \frac{\partial u_0}{\partial n_f},$$

$$\left[\frac{\partial u_{\sigma_f}}{\partial n_f}\right]_{\sigma_f} = -2(k-1) \frac{\partial^2 u_0}{\partial \tau_f^2},$$

where $\partial^2/\partial \tau_f^2$ represents the second order derivative with respect to the arclength parameter of σ_f.

For any $x_0 \in \sigma_f$ and ρ such that

$$0 < \rho < \min\left\{ \left|x_0 - P_f\right|, \left|x_0 - Q_f\right| \right\},$$

$u_{\sigma_f} \in C^{1,\lambda}$ in $\left\{(x_1, x_2) \in \Omega \colon x_2 \geq f(x_1)\right\} \cap B_\rho(x_0)$ and in $\left\{(x_1, x_2) \in \Omega \colon x_2 \leq f(x_1)\right\} \cap B_\rho(x_0)$, where $B_\rho(x_0) = \{x \colon |x - x_0| \leq \rho\}$ and $0 < \lambda < 1$.

Moreover one has the following estimates for u_{σ_f}:

$$\left|u_{\sigma_f}(x)\right| \leq c \ln \frac{1}{\left|x-P_f\right|\left|x-Q_f\right|}, \quad x \in \Omega \setminus \sigma_f, \tag{5}$$

$$\left|\nabla u_{\sigma_f}(x)\right| \leq c\left(\frac{1}{\left|x-P_f\right|}+\frac{1}{\left|x-Q_f\right|}\right), \quad x \in \Omega \setminus \sigma_f, \tag{6}$$

where c is a positive constant.

Proof This proposition follows from classical regularity results in potential theory (see, for example, [5] or [6, Theorems 2.1 and 2.2]).

For sake of completeness, we outline here the proof of estimates (5) and (6) near the endpoints.

By well-known properties of single and double layer potentials on sufficiently smooth open curves, we have that

$$\left|\int_{\sigma_f} \frac{\partial u_0}{\partial n_f}(x) \frac{\partial \Phi}{\partial n_f(x)}(x, y)\, d\sigma_x\right| \leq C \quad \text{for } y \in \Omega,$$

$$\left|\int_{\sigma_f} \frac{\partial^2 u_0}{\partial \tau_f^2}(x) \Phi(x, y)\, d\sigma_x\right| \leq C \quad \text{for } y \in \Omega,$$

and so, by (4), estimate (5) follows immediately.

Let us now derive estimate (6) in a neighborhood of P_f. Let us write

$$\int_{\sigma_f} \frac{\partial u_0}{\partial n_f}(x) \frac{\partial \Phi}{\partial n_f(x)}(x, y)\, d\sigma_x = \int_{\sigma_f}\left(\frac{\partial u_0}{\partial n_f}(x)-\frac{\partial u_0}{\partial n_f}(P_f)\right) \frac{\partial \Phi}{\partial n_f(x)}(x, y)\, d\sigma_x + \frac{\partial u_0}{\partial n_f}(P_f) \int_{\sigma_f} \frac{\partial \Phi}{\partial n_f(x)}(x, y)\, d\sigma_x. \tag{7}$$

An easy computation gives

$$\int_{\sigma_f} \frac{\partial \Phi}{\partial n_f}(x, y)\, d\sigma_x = -\frac{1}{2\pi}\left(\arg(y-Q_f)-\arg(y-P_f)\right),$$

hence

$$\nabla_y\left(\int_{\sigma_f} \frac{\partial \Phi}{\partial n_f}(x, y)\, d\sigma_x\right) = \frac{1}{2\pi}\left(\left(\frac{y_2-f(\beta)}{|y-Q_f|^2}, -\frac{y_1-\beta}{|y-Q_f|^2}\right) - \left(\frac{y_2-f(\alpha)}{|y-P_f|^2}, -\frac{y_1-\alpha}{|y-P_f|^2}\right)\right). \tag{8}$$

On the other hand (by [6, Theorems 2.1]) one can see that, in a neighborhood of P_f

$$\nabla_y\left(\int_{\sigma_f}\left(\frac{\partial u_0}{\partial n_f}(x)-\frac{\partial u_0}{\partial n_f}(P_f)\right)\frac{\partial \Phi}{\partial n_f(x)}(x,y)\,d\sigma_x\right)\quad \text{is bounded.} \tag{9}$$

In the same way, we write

$$\int_{\sigma_f}\frac{\partial^2 u_0}{\partial \tau_f^2}(x)\Phi(x,y)\,d\sigma_x=\int_{\sigma_f}\left(\frac{\partial^2 u_0}{\partial \tau_f^2}(x)-\frac{\partial^2 u_0}{\partial \tau_f^2}(P_f)\right)\Phi(x,y)\,d\sigma_x$$

$$+\frac{\partial^2 u_0}{\partial \tau_f^2}(P_f)\int_{\sigma_f}\Phi(x,y)\,d\sigma_x. \tag{10}$$

Again by [6, Theorems 2.1], in a neighborhood of P_f

$$\nabla_y\left(\int_{\sigma_f}\left(\frac{\partial^2 u_0}{\partial \tau_f^2}(x)-\frac{\partial^2 u_0}{\partial \tau_f^2}(P_f)\right)\Phi(x,y)\,d\sigma_x\right)\quad \text{is bounded} \tag{11}$$

while

$$\left|\nabla_y\int_{\sigma_f}\Phi(x,y)\,d\sigma_x\right|\le C\big|\ln|y-P_f|\big|, \tag{12}$$

for a positive constant C. By putting together (7)–(12) and recalling (4) we get the desired estimate. ■

We are now ready to state our uniqueness result:

THEOREM 2.2 *Let f and g be two C^3 functions such that σ_f and σ_g are contained in Ω. Let $a\in\mathbf{R}^2$ be a unit vector and Γ be an open subset of $\partial\Omega$. If*

$$u^a_{\sigma_f}=u^a_{\sigma_g}\quad \text{on } \Gamma,$$

then

$$\sigma_f=\sigma_g.$$

Proof We will first prove that $P_f=P_g$ and $Q_f=Q_g$.

Assume the contrary. If $P_f\neq P_g$ there exists an open arc (for example $\Sigma_f\subset\sigma_f$) with P_f as an endpoint, which can be reached on both sides with a continuous curve starting on Γ and contained in $\Omega\setminus\overline{\sigma}_f\cup\overline{\sigma}_g$.

Let $v=u^a_{\sigma_f}-u^a_{\sigma_g}$. By assumption, $v=\partial v/\partial \nu=0$ on Γ and $\Delta v=0$ in $\Omega\setminus\overline{\sigma}_f\cup\overline{\sigma}_g$.

Then by the unique continuation property of harmonic functions and the uniqueness of the Cauchy problem, $v\equiv 0$ in the connected component of $\Omega\setminus\overline{\sigma}_f\cup\overline{\sigma}_g$ containing Γ.

In particular $v=0$ in a neighborhood of Σ_f on both sides, hence $0=[v]_{\Sigma_f}=[u^a_{\sigma_f}]_{\Sigma_f}=2(1-(1/k))a\cdot n_f$. This implies

$$a\cdot n_f(P_f)=0. \tag{13}$$

On the other side, the function v cannot have a logarithmic singularity in P_f, hence, by (4),

$$\frac{\partial u_0}{\partial \tau_f}(P_f)=a\cdot\tau_f(P_f)=0. \tag{14}$$

Since (13) and (14) are in contradiction, it follows that $P_f=P_g=P$ (and, similarly, $Q_f=Q_g=Q$).

Now let us show that

$$\tau_f(P)=\tau_g(P),\quad \tau_f(Q)=\tau_g(Q). \tag{15}$$

By using the expression (4) and the linearity of u^a_0, we can write, for $y\in\Omega\setminus\sigma_f$,

$$\begin{aligned} v(y) &= w_{\sigma_f}(y)-w_{\sigma_g}(y)\\ &\quad -2(k-1)\big((\tau_f(P)-\tau_g(P))\cdot a\,\Phi(P,y)-(\tau_f(Q)-\tau_g(Q))\cdot a\,\Phi(Q,y)\big)\\ &\quad +2(k-1)\left(\int_{\sigma_f}\frac{\partial^2 u^a_0}{\partial\tau_f^2}(x)\Phi(x,y)\,d\sigma_x-\int_{\sigma_g}\frac{\partial^2 u^a_0}{\partial\tau_g^2}(x)\Phi(x,y)\,d\sigma_x\right)\\ &\quad -2\left(1-\frac{1}{k}\right)\left(\int_{\sigma_f}\frac{\partial u^a_0}{\partial n_f}(x)\frac{\partial\Phi}{\partial n_f(x)}(x,y)\,d\sigma_x-\int_{\sigma_g}\frac{\partial u^a_0}{\partial n_g}(x)\frac{\partial\Phi}{\partial n_g(x)}(x,y)\,d\sigma_x\right). \end{aligned}$$

This means (see Proposition 2.1) that in a neighborhood of P, the function v has a logarithmic singularity given by

$$-2(k-1)\big((\tau_f(P)-\tau_g(P))\cdot a\big)\,\Phi(P,y).$$

Since $v\equiv 0$ in the connected component of $\Omega\setminus\sigma_f\cup\sigma_g$ containing Γ, this singularity cannot appear, hence

$$(\tau_f(P)-\tau_g(P))\cdot a=0. \tag{16}$$

By (8) in the proof of Proposition 2.1 and since (16) holds, the leading order singularity for ∇v at P has the form

$$-\frac{1}{\pi}\left(1-\frac{1}{k}\right)\big((n_f(P)-n_g(P))\cdot a\big)\left(\frac{y_2-f(\alpha)}{|y-P_f|^2},\ -\frac{y_1-\alpha}{|y-P_f|^2}\right).$$

Again, since $v \equiv 0$ in the connected component of $\Omega \setminus \sigma_f \cup \sigma_g$ containing Γ, this singularity cannot appear, hence

$$(n_f(P) - n_g(P)) \cdot a = 0. \tag{17}$$

By putting together (16) and (17) one gets (15).

Now, let us assume that σ_f and σ_g do not coincide. Since the two curves are regular and have common endpoints, there exist at most a countable set of subintervals (α_1, β_1) of (α, β) such that, say, $g(x_1) < f(x_1)$ for every $x_1 \in (\alpha_1, \beta_1)$, $f(\alpha_1) = g(\alpha_1)$ and $f(\beta_1) = g(\beta_1)$. Set

$$D = \Big\{(x_1, x_2)\colon x_1 \in (\alpha_1, \beta_1),\ g(x_1) < x_2 < f(x_1)\Big\},$$

$P' = (\alpha_1, f(\alpha_1))$, and $Q' = (\beta_1, f(\beta_1))$.

We want to show that

$$\tau_f(P') = \tau_g(P'), \qquad \tau_f(Q') = \tau_g(Q'). \tag{18}$$

Set $v^+ = v_{|_D}$. The function v^+ satisfies the following overdetermined problem in D

$$\begin{cases} \Delta v^+ = 0, \quad \text{in } D, \\ v^+ = 2\left(1 - \dfrac{1}{k}\right) n_f \cdot a, \quad \text{on } \sigma_f \cap \partial D, \\ v^+ = 2\left(1 - \dfrac{1}{k}\right) n_g \cdot a, \quad \text{on } \sigma_g \cap \partial D, \\ \dfrac{\partial v^+}{\partial n_f} = 2(k-1)\dfrac{\partial}{\partial \tau_f}(\tau_f \cdot a), \quad \text{on } \sigma_f \cap \partial D, \\ \dfrac{\partial v^+}{\partial n_g} = 2(k-1)\dfrac{\partial}{\partial \tau_g}(\tau_g \cdot a), \quad \text{on } \sigma_g \cap \partial D. \end{cases} \tag{19}$$

Let us prove (18) for P'. If $P' = P$, this is a consequence of (15), so let us assume that $P' \neq P$, that is $\alpha_1 > \alpha$.

Observe that both $u^a_{\sigma_f}$ and $u^a_{\sigma_g}$ are continuous in a neighborhood of P' relative to D. By (19) this implies that

$$n_f(P') \cdot a = n_g(P') \cdot a. \tag{20}$$

Now let us prove that

$$\int_{\partial D} \frac{\partial v^+}{\partial \nu} = 0, \tag{21}$$

where ν is the unit outer normal to ∂D.

If $P' \neq P$ and $Q' \neq Q$ this simply follows from the fact that v^+ is in $C^1(\overline{D})$, hence one can apply Green's formula and the fact that v^+ is harmonic.

If, for example $P' = P$, let us show that one can again apply Green's formula.

By (15), we approximate D by a sequence of smooth domains $\{D_n\} \subset D$. Let x_n^1 be a point on σ_f such that the arc joining P' and x_n^1 has length smaller than $1/n$, and let x_n^2 be a point on σ_g such that the arc joining P' and x_n^2 has length smaller than $1/n$. Analogously we define $\tilde{x}_n^1$ and $\tilde{x}_n^2$ on the side of Q' in case $Q' = Q$. Let Σ_n^1 and Σ_n^2 be two smooth arcs joining x_n^1 and x_n^2 and $\tilde{x}_n^1$ and $\tilde{x}_n^2$ respectively, and such that the region D_n enclosed by Σ_n^1, Σ_n^2, σ_f and σ_g is sufficiently smooth and such that length(Σ_n^1) and length(Σ_n^2) are bounded by $1/n^2$ (this is possible because of (15)).

Now we can apply Green's formula in D_n, since $v^+ \in C^1(\overline{D}_n)$, and get

$$0 = \int_{D_n} \Delta v^+ = \int_{\partial D_n} \frac{\partial v^+}{\partial \nu}. \tag{22}$$

By (6) and by the choice of Σ_n^1 and Σ_n^2, we have

$$\lim_{n\to\infty} \int_{\Sigma_n^1} \frac{\partial v^+}{\partial \nu} = \lim_{n\to\infty} \int_{\Sigma_n^2} \frac{\partial v^+}{\partial \nu} = 0.$$

Hence, passing to the limit in (22) we get (21).

Consider the closed curve determined by $\sigma_f \cap \overline{D}$ and $\sigma_g \cap \overline{D}$. Let $0 \leq s \leq s_f$ be the arclength parameter on σ_f from P' to Q' and $s_f \leq s \leq s_f + s_g$ be the arclength parameter on σ_g from Q' to P'.

By (19), and noticing that $\nu = n_f$, $(\partial/\partial s) = (\partial/\partial \tau_f)$ on σ_f and $\nu = -n_g$ and $(\partial/\partial s) = -(\partial/\partial \tau_g)$ on σ_g, we arrive at

$$\begin{aligned} 0 &= \int_{\partial D} \frac{\partial v^+}{\partial \nu} \\ &= 2(k-1)\left(\int_0^{s_f} \frac{\partial}{\partial s}(\tau_f \cdot a)\, ds + \int_{s_f}^{s_f+s_g} \frac{\partial}{\partial s}(\tau_g \cdot a)\, ds\right) \\ &= 2(k-1)\Big(\tau_f(Q') - \tau_f(P') + \tau_g(P') - \tau_g(Q')\Big) \cdot a. \end{aligned}$$

Hence

$$\big(\tau_f(Q') - \tau_g(Q')\big) \cdot a = \big(\tau_f(P') - \tau_g(P')\big) \cdot a.$$

Now we observe that for the endpoint P we know that $\tau_f(P) = \tau_g(P)$, so, by a recursive algorithm one can conclude that

$$\big(\tau_f(x) - \tau_g(x)\big) \cdot a = 0,$$

for every point x in $\sigma_f \cap \sigma_g$. This, together with (20), gives (18).

Now we show that

$$\int_D |\nabla v^+|^2 = \int_{\partial D} v^+ \frac{\partial v^+}{\partial \nu}. \tag{23}$$

As before, it is enough to notice that if P' and Q' do not coincide with P and Q, the function v^+ is regular enough to apply Green's theorem. Otherwise we approximate D by the sequence of smooth domains $\{D_n\} \subset D$ and observe that

$$\int_{D_n} |\nabla v^+|^2 = \int_{\partial D_n} v^+ \frac{\partial v^+}{\partial \nu}. \tag{24}$$

By (5) and (6) and by the choice of Σ_n^1 and Σ_n^2, we have

$$\lim_{n\to\infty} \int_{\Sigma_n^1} v^+ \frac{\partial v^+}{\partial \nu} = \lim_{n\to\infty} \int_{\Sigma_n^2} v^+ \frac{\partial v^+}{\partial \nu} = 0.$$

Hence, passing to the limit in (24) we get (23).

Next step consists of proving that

$$\int_{\partial D} v^+ \frac{\partial v^+}{\partial \nu} = 0. \tag{25}$$

By (19),

$$\int_{\partial D} v^+ \frac{\partial v^+}{\partial \nu} = \frac{4(k-1)^2}{k} \left(\int_0^{s_f} (n_f(s) \cdot a) \frac{\partial}{\partial s} (\tau_f \cdot a)\, ds + \int_{s_f}^{s_f+s_g} (n_g(s) \cdot a) \frac{\partial}{\partial s} (\tau_g \cdot a)\, ds \right). \tag{26}$$

Since $\tau_f(s)$ is a regular function of s, there is a regular function $\theta_f(s)$ such that

$$\tau_f(s) \cdot a = \cos\theta_f(s)$$

and

$$n_f(s) \cdot a = -\sin\theta_f(s)$$

Hence

$$\int_0^{s_f} (n_f(s) \cdot a) \frac{\partial}{\partial s} (\tau_f \cdot a)\, ds = \int_0^{s_f} \sin^2\theta_f(s) \frac{\partial \theta_f(s)}{\partial s}\, ds = \int_{\theta_f(0)}^{\theta_f(s_f)} \sin^2\theta\, d\theta. \tag{27}$$

In the same way we define $\theta_g(s)$ such that

$$\tau_g(s) \cdot a = \cos\theta_g(s) \quad \text{and} \quad n_g(s) \cdot a = -\sin\theta_g(s),$$

and get

$$\int_{s_f}^{s_f+s_g} (n_g(s) \cdot a) \frac{\partial}{\partial s} (\tau_g \cdot a)\, ds = \int_{s_f}^{s_f+s_g} \sin^2\theta_g(s) \frac{\partial \theta_g(s)}{\partial s}\, ds = \int_{\theta_g(s_f)}^{\theta_g(s_f+s_g)} \sin^2\theta\, d\theta. \tag{28}$$

Condition (18) can be read as

$$\theta_f(0) = \theta_g(s_f + s_g) \quad \text{and} \quad \theta_f(s_f) = \theta_g(s_f).$$

From (26), (27), and (28) we get

$$\int_{\partial D} v^+ \frac{\partial v^+}{\partial \nu} = \frac{4(k-1)^2}{k} \left(\int_{\theta_f(0)}^{\theta_f(s_f)} \sin^2\theta \, d\theta + \int_{\theta_f(s_f)}^{\theta_f(0)} \sin^2\theta \, d\theta \right) = 0.$$

By putting together (23) and (25) we get that

$$v^+ = \text{constant} \quad \text{in } \overline{D},$$

which leads to

$$n_f \cdot a = n_g \cdot a = \text{constant} \quad \text{on } (\alpha_1, \beta_1).$$

This, means that $\sin\theta_f = \sin\theta_g = \mathit{constant}$ on (α_1, β_1). Since θ_f and θ_g are regular function this implies that both θ_f and θ_g are constant on (c, d). We already know that they coincide at the endpoints, hence the tangent fields to σ_f and σ_g coincide in all the interval. From this and the fact that the endpoints coincide we conclude that $\sigma_f = \sigma_g$. ∎

Remark 2.1 Notice that the assumption that σ_f and σ_g are graph with respect to the same coordinate system can be slightly relaxed. What we really need to perform the proof is that any point of $\sigma_f \cup \sigma_g$ can be reached from Γ with a continuous curve contained in the connected component of $\Omega \setminus \sigma_f \cup \sigma_g$ containing Γ.

3. ALGORITHM

In this section we present an algorithm for reconstructing a straight curve σ from the measurements on the boundary of the correction terms $u_\sigma^{a_1}|_{\partial\Omega}$ and $u_\sigma^{a_2}|_{\partial\Omega}$ generated by two constant current sources.

In fact in this case we expect good reconstructions since the inverse problem restricted to line segments should be well posed (Lipschitz continuous dependence). This is not true any more for arbitrary curves. In that case we are confident that the problem is severly ill-posed (logarithmic continuous dependence) as was shown in [4] for a similar problem.

We first note that, if we denote by e_1 and e_2 the unit vectors $(1, 0)$ and $(1, 0)$, for any unit vector a we have

$$u_\sigma^a = (a \cdot e_1) u_\sigma^{e_1} + (a \cdot e_2) u_\sigma^{e_2}, \tag{29}$$

so, the knowledge of $u_\sigma^{a_1}$ and $u_\sigma^{a_2}$ for any two orthogonal vectors a_1 and a_2 corresponds to the knowledge of $u_\sigma^{e_i}$, for $i = 1, 2$.

We assume that σ is a straight line segment and reconstruct its endpoint P and Q from $u_\sigma^{e_1}|_{\partial\Omega}$ and $u_\sigma^{e_2}|_{\partial\Omega}$. This generalizes the algorithm described in [1] for the reconstruction of small diameter imperfections to the more difficult case of thin imperfections. The major difficulty comes from the fact that the conductivity inhomogeneity is thin (i.e. with a bad Lipschitz character) and so, results from [1] are not applicable to our case.

Notice that in the case of a straight line segment, the matrix A defined in (1) is constant along σ. Let $\tilde{e}_1 = \overrightarrow{PQ}/|\overrightarrow{PQ}|$ and $\tilde{e}_2$ a unit vector orthogonal to $\tilde{e}_1$. Consider an open domain S containing $\overline{\Omega}$ with C^2 boundary ∂S.

Let us denote by $\mathcal{D}_\Omega v$ the double layer potential of the density function v, defined by

$$\mathcal{D}_\Omega v(x) = \int_{\partial\Omega} v(y) \frac{\partial \Phi}{\partial \nu(y)}(x, y)\, ds_y, \quad x \in \mathbf{R}^2 \setminus \partial\Omega.$$

Then, the following lemma holds.

LEMMA 3.1 *For any pair of unit vectors a and a^* we have*

$$\int_{\partial S} \frac{\partial}{\partial \nu} \mathcal{D}_\Omega\big(u_\sigma^a|_{\partial\Omega}\big)(y)\ a^* \cdot y\ ds_y - \int_{\partial S} \mathcal{D}_\Omega\big(u_\sigma^a|_{\partial\Omega}\big)(y)\ a^* \cdot \nu(y)\ ds_y = |\sigma|\ a^* \cdot A \cdot a,$$

where ν denotes the normal to ∂S pointing outside S

Proof Let us consider $y \in \mathbf{R}^2 \setminus \overline{\Omega}$ and, by using the definition in (3), let us evaluate

$$\begin{aligned}
&\int_{\partial\Omega} \frac{\partial u_\sigma}{\partial \nu(x)} \Phi(x, y)\, ds_x - \int_{\partial\Omega} u_\sigma(x) \frac{\partial \Phi}{\partial \nu(x)}(x, y)\, ds_x \\
&\quad = -\int_{\partial\Omega} \frac{\partial}{\partial \nu(x)} \left(\int_\sigma A(z) \nabla u_0(z) \nabla_z \Phi(z, x)\, d\sigma_z \right) \Phi(x, y)\, ds_x \\
&\quad\quad + \int_{\partial\Omega} \left(\int_\sigma A(z) \nabla u_0(z) \nabla_z \Phi(z, x)\, d\sigma_z \right) \frac{\partial \Phi}{\partial \nu(x)}(x, y)\, ds_x \\
&\quad\quad + \int_{\partial\Omega} \frac{\partial w_\sigma}{\partial \nu}(x) \Phi(x, y)\, ds_x - \int_{\partial\Omega} w_\sigma(x) \frac{\partial \Phi}{\partial \nu(x)}(x, y)\, ds_x.
\end{aligned}$$

Since w_σ is harmonic in Ω and $y \notin \Omega$, the sum of the last two integrals is zero. Notice that the points x, y and z are well separated and y is outside Ω. Hence, we can interchange the order of the integrations and get

$$\begin{aligned}
&\int_{\partial\Omega} \frac{\partial u_\sigma}{\partial \nu}(x) \Phi(x, y)\, ds_x - \int_{\partial\Omega} u_\sigma(x) \frac{\partial \Phi}{\partial \nu(x)}(x, y)\, ds_x \\
&\quad = -\int_\sigma A(z) \nabla u_0(z) \nabla_z \left(\int_{\partial\Omega} \left(\frac{\partial \Phi}{\partial \nu(x)}(z, x) \Phi(x, y) - \Phi(z, x) \frac{\partial \Phi}{\partial \nu(x)}(x, y) \right) ds_x \right) d\sigma_z \\
&\quad = -\int_\sigma A(z) \nabla u_0(z) \nabla_z \left(\int_\Omega (\Delta_x \Phi(z, x) \Phi(x, y) - \Phi(z, x) \Delta_x \Phi(x, y))\, dx \right) d\sigma_z \\
&\quad = -\int_\sigma A(z) \nabla u_0(z) \nabla_z \Phi(z, y) d\sigma_z.
\end{aligned}$$

In the special case $u_0 = u_0^a$ is linear, and observing that $(\partial u_\sigma^a/\partial\nu|_{\partial\Omega} = 0)$ and A is constant on σ, one gets, for $y \in \mathbf{R}^2 \setminus \overline{\Omega}$,

$$\mathcal{D}_\Omega(u_\sigma^a)(y) = \int_{\partial\Omega} u_\sigma^a(x)\frac{\partial\Phi}{\partial\nu(x)}(x,y)\, ds_x = A\cdot a\int_\sigma \nabla_x\Phi(x,y)\, d\sigma_x.$$

By using this expression we can calculate

$$\begin{aligned}
&\int_{\partial S}\frac{\partial}{\partial\nu(y)}\mathcal{D}_\Omega(u_\sigma^a|_{\partial\Omega})(y)\, a^*\cdot y\, ds_y - \int_{\partial S}\mathcal{D}_\Omega(u_\sigma^a|_{\partial\Omega})(y)\, a^*\cdot\nu(y)\, ds_y\\
&= \int_{\partial S} A\cdot a\Big(\frac{\partial}{\partial\nu(y)}\int_\sigma \nabla_x\Phi(x,y)\, d\sigma_x a^*\cdot y - \int_\sigma \nabla_x\Phi(x,y)\, d\sigma_x\, a^*\cdot\nu(y)\Big)ds_y\\
&= A\cdot a\int_\sigma \nabla_x\int_{\partial S}\left(\frac{\partial\Phi}{\partial\nu(y)}(x,y)\, a^*\cdot y - \Phi(x,y)\, a^*\cdot\nu(y)\right) ds_y\, d\sigma_x\\
&= A\cdot a\int_\sigma \nabla_x(a^*\cdot x)\, d\sigma_x = |\sigma|\, a^*\cdot A\cdot a.
\end{aligned}$$

■

From Lemma 3.1 it follows that the components of the matrix A in the basis (e_1, e_2) are given by

$$A_{ij} = \frac{1}{|\sigma|}\int_{\partial S}\left[\frac{\partial}{\partial\nu}\mathcal{D}_\Omega(u_\sigma^{e_j}|_{\partial\Omega})(y)e_i\cdot y - \mathcal{D}_\Omega(u_\sigma^{e_j}|_{\partial\Omega})(y)e_i\cdot\nu(y)\right] ds_y.$$

This expression only involves $|\sigma|$ and the boundary values of $u_\sigma^{e_j}$ for $j=1,2$.

In particular, the knowledge of $u_\sigma^{e_1}|_{\partial\Omega}$ and $u_\sigma^{e_2}|_{\partial\Omega}$ is enough to reconstruct the eigenvectors of A.

Notice that to distinguish between $\tilde{e}_1$ and $\tilde{e}_2$ (which one is tangent and which is orthogonal to σ) it is not necessary to know the value of k. The sign of the eigenvalues of A_{ij} tells us if k is bigger or smaller than 1: if the eigenvalues are positive, then $k > 1$ and $\tilde{e}_1$ corresponds to the bigger eigenvalue. If the eigenvalues are negative, then $k < 1$, and $\tilde{e}_1$ corresponds to the eigenvalue with smaller absolute value.

Now, let us express everything in the coordinate system $(\tilde{e}_1, \tilde{e}_2)$. In this way, $P = (p_1, \lambda)$ and $Q = (q_1, \lambda)$. By straightforward calculations, one can see that

$$\mathcal{D}_\Omega(u_\sigma^{\tilde{e}_2})(0, y_2) = -\frac{1}{\pi}\left(1 - \frac{1}{k}\right)(\lambda - y_2)\int_\sigma \frac{1}{|x - (0, y_2)|^2}\, ds_x. \tag{30}$$

Notice that, once $\tilde{e}_2$ is known, $u_\sigma^{\tilde{e}_2}|_{\partial\Omega}$ is obtained from $u_\sigma^{e_1}|_{\partial\Omega}$ and $u_\sigma^{e_2}|_{\partial\Omega}$ according to (29). Hence, the left-hand side of (30) is known in $\mathbf{R}^2 \setminus \Omega$ and λ can be identified as the value where it changes sign.

Finally, assuming $p_1 < q_1$,

$$\mathcal{D}_\Omega(u_\sigma^{\tilde{e}_1})(y_1, \lambda) = -\frac{1}{\pi}(k-1)\ln\frac{y_1 - q_1}{y_1 - p_1}, \quad \text{for any } y_1 < p_1 \text{ or } y_1 > q_1,$$

and from this we can determine p_1 and q_1.

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