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Consumption of Private Goods as Substitutes for Environmental Goods in an Economic Growth Model

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Abstract. We analyze growth dynamics in an economy where a private good can be consumed as a substitute for a free access environmental good. In this context we show that environmental deterioration may be an engine of economic growth. To protect themselves against environmental deterioration, economic agents are forced to increase their labour supply to increase the production and consumption of the private good. This, in turn, further depletes the environmental good, leading economic agents to further increase their labour supply and private consumption and so on. This substitution process may give rise to self-enforcing growth dynamics characterized by a lack of correlation between capital accumulation and private consumption levels, on one side, and economic agents' welfare, on the other.

Furthermore, we show that agents' self-protection consumption choices can generate indeterminacy; that is, they can give rise to the existence of a continuum of (Nash) equilibrium orbits leading to the same attracting fixed point or periodic orbit.

Keywords: self-protection choices, indeterminacy, undesirable economic growth.

1 Introduction

Private goods can be consumed by individuals to defend themselves from the deterioration of environmental resources. Classic examples of these types of goods are water or air filters, mineral water, devices for protection from noise pollution generated by industrial activities or urban traffic (for example, double-glazing) and drugs to treat respiratory illnesses caused by air pollution. Certain consumption expenditures on the part of city dwellers are conditioned, at least in part, by defensive reasoning. Consider the choice of using a car as a means of transport, a choice which may be caused by air pollution: individuals who would have preferred to go by bicycle are forced to use their car because the air is unbreathable. The shortage of parks and areas where children can play without the constant supervision of adults imposes further consumption expenditures. The massive use of home entertainment by children is partly a result of the lack of such areas, of the degradation of the urban environment and of the need to protect children from the dangers of urban traffic. The shortage of green space can induce individuals to purchase an entrance ticket for a protected nature area or to spend money on a day out to find a place to enjoy nature and leisure activities. Another possible expenditure could be joining a gym, substituting physical activity in a park in the open air with exercise carried out in a sports centre.

However, these are only “textbook” examples; the literature of environmental economics suggests that the category of defensive environmental spending can be interpreted in a very broad manner, comprising a vast range of consumptions that derive in part from environmental degradation, but are not merely a response to it. The literature on this argument (see, e.g., [1–8]) supports the idea that individuals may react to environmental deterioration in a great variety of ways. When the environment deteriorates, individuals are more incentivated to adopt consumption patterns based on the use of private goods rather than on the use of free access environmental goods: spending a day on an uncontaminated beach close to home can be more rewarding (and generally requires the consumption of a lesser quantity of private goods) than spending a day in town; nevertheless, the latter option becomes relatively more remunerative if the quality of the beach is compromised.

In this paper we analyze economic growth dynamics in an economy where there is an infinite number (a continuum) of identical economic agents, whose welfare depends on three goods: leisure, a free access environmental good and a private consumption good. Each agent produces the private good through his own work and accumulated (physical) capital. The private good can be consumed as a substitute for the environmental good and can be saved and accumulated as capital. The environmental good is deteriorated by the pollution caused by the average consumption activity of the private good in the economy.

At every instant of time each economic agent has to choose the allocation of his endowment of time between leisure and the production process of the private good and the allocation of the output between present consumption and accumulation of capital. Since the negative impact on the environmental good of each agent's consumption choice is negligible (agents being a continuum), he doesn't take this into account in his consumption choices.

In this context, by working harder, economic agents can consume more in the present and/or in the future (via accumulation of capital) and consequently can benefit from a better self-protection against environmental deterioration in the present and/or in the future. Thus, economic agents may react to the deterioration of the environmental good by increasing the production and consumption of private goods; by doing so, they cause a further depletion of the environmental resources, which can, in turn, force agents to further increase private consumption and accumulation.

The paper is organized as follows. Section 2 illustrates the main results of the paper, comparing them with those of related literature. Section 3 defines the model. Sections 4–9 analyze the dynamics. Section 10 outlines the conclusions.

2 Related literature

The mechanism of economic growth that we intend to analyze is based on the hypothesis that the consumption of the private good by each economic agent contributes to the depletion of the environmental good, and therefore generates a negative externality (in the model the agents do not take into consideration the negative impact of their choices on the environmental good) on the other agents. Consequently, in our model, defensive consumption choices may be classified as self-protective choices "transferring" the negative externalities to other individuals [9]; that is, each victim of negative environmental externalities defends himself by implementing the defensive consumption which generates further negative externalities for other individuals.

This situation has been analyzed, in a static model, by Shogren and Crocker

[10], who demonstrated that, in a context where individuals do not cooperate (i.e., they do not "internalize" the externalities), the outcome is a degree of self-protection that exceeds the socially optimal level. It implies that, if the individuals protect themselves by consuming private goods, the expected outcome is an excess in the consumption of private goods.

Such analysis has been extended to a dynamic context in several works[1]; Antoci and Bartolini [1,2,12], have studied the dynamics of labor supply under the assumption of bounded rationality (i.e., agents don't have perfect foresight about the future evolution of state variables), neglecting the accumulation of capital; in particular, they have analyzed evolutionary games where individuals have to choose their labor efforts from among a finite number of options and where the better performing choices become widespread in the population of individuals at the expense of those that are less rewarding. The analysis of these models shows that economic dynamics can present two or more locally attracting fixed points characterized by an inverse correlation between labor effort (and private consumption level) and individuals' welfare.

Bartolini and Bonatti [3] assume perfect foresight and analyze a model without capital accumulation where economic agents choose their (identical) labor efforts from among a continuum of values. In this context, they obtain results which are analogous to those obtained in the above-mentioned evolutionary games, showing that such results do not depend on the bounded rationality assumption.

Bartolini and Bonatti [4] analyze a discrete time perfect foresight dynamic with capital accumulation; however, their model shows a single fixed point which is a saddle point. Their analysis limits itself to the sensitivity analysis of the fixed point with respect to the variations of the parameters of the model.

The present work intends to contribute to this line of research by showing that, even in a model of capital accumulation and perfect foresight, there may exist multiplicity of fixed points and that there may be no correlation between private consumption and capital accumulation levels in such states and the welfare of the economic agents; fixed points with high consumption and accumulation levels

[1]All this works build on the well known work of Hirsch [11] who suggests that individuals' reactions to negative externalities (defensive consumptions) due to economic growth can be an engine of economic growth. However, he doesn't introduce his intuition in a mathematical model of economic growth.

can be Pareto-dominated by others characterized by lower levels. Furthermore the substitution process between environmental and produced goods may have effects on the stability of fixed points and may generate closed orbits; self-protection choices can produce indeterminacy, that is the existence of an infinite number of equilibrium orbits leading to the same (locally) attractive fixed point or periodic orbit. When indeterminacy occurs, given the initial values of the capital stock and the environmental good, the economy can reach the attracting fixed point (or the periodic orbit) by following an infinite number (a continuum) of growth paths, each characterized by different consumption patterns and welfare levels. Consequently, the economy may experience very different welfare situations[2]. Starting from different initial values of the state variables, it can reach different fixed points (characterized by different welfare levels). Furthermore, when indeterminacy occurs, each fixed point (or periodic orbit) can be reached along an infinite number of possible orbits, each of them giving rise to possibly different welfare levels[3].

Finally, there is a strand of literature (see, e.g., [15–19]) that highlights other mechanisms according to which natural resources abundance may inhibit economic growth (for a review of this literature see [20]). However, according to the mechanisms analyzed in the above-mentioned literature, economic growth always generates welfare improvements.

3 The model

There exists a large number (a continuum) of identical economic agents. Since all agents are identical, we can consider the choice process of a representative agent. We assume that, at each instant of time t, representative agent's welfare depends on three goods:

1. Leisure $1 - l(t)$, where $l(t)$ is representative agent's labor input.

2. A free access (renewable) environmental good $E(t)$.

[2]For a review of macroeconomic models featuring indeterminacy see [13].

[3]Antoci, Sacco and Vanin [14] analyse a growth model where economic agents can consume private goods as a defensive device against the deterioration of social rather than environmental capital.

3. A private good which can be consumed either as a substitute for the environmental good $\big(c_2(t)\big)$, i.e., as a self-protection device against environmental deterioration, or in order to satisfy needs different from those satisfied by the environmental resource $\big(c_1(t)\big)$.

   We assume that the representative agent's decision problem is

$$\max_{c_1, c_2, l} \int_0^\infty \big(\ln c_1 + a\ln(E + bc_2) + d\ln(1-l)\big)e^{-rt}dt, \tag{1}$$

$$\dot{k} = l^\alpha k^{1-\alpha}\Omega - c_1 - c_2, \tag{2}$$

$$\dot{E} = \beta E(\bar{E} - E) - \gamma(\bar{c}_1 + \bar{c}_2)E, \tag{3}$$

where a, b, d, r, α, β, γ and $\bar{E}$ are strictly positive parameters, $k(t)$ represents physical capital accumulated by the representative agent and $l(t)$ is the representative agent's labor input; $\dot{k}$ and $\dot{E}$ denote the time derivatives of k and E.

The representative agent has to choose the functions $c_1(t)$, $c_2(t)$ and $l(t)$ to maximize the integral in (1). Note that, according to the (instantaneous) utility function $\ln c_1 + a\ln(E + bc_2) + d\ln(1-l)$, an increase of substitutive consumption $c_2(t)$ compensates the negative effect deriving from a reduction of $E(t)$.

Equation (3) describes the dynamics of $E(t)$; note that the value of the parameter $\bar{E}$ can be interpreted as the endowment of the environmental good in the economy, i.e., the state variable E would reach such a value without the negative effect due to the average economy-wide consumption $\bar{c}_1 + \bar{c}_2$. The assumption that the renewable natural resource is depleted only by the consumption of the private good and not by the production of the same can be motivated by several real life examples; a paradigmatic example is that concerning the use of cars: the negative impact of their production process is negligible in comparison to the pollution generated by their daily use. This assumption is made for the sake of analytical simplicity. However, it is reasonable to suppose that the predictions of the model would be confirmed assuming that the production activity also depletes the environment; in fact, the self-enforcing nature of the substitution process described above, on which the results of the model are based, is fueled by the negative impact of economic activity on the environmental resource; the higher such impact, the greater the incentive to consume substitutes for the environmental good.

At each instant of time, the representative agent produces the quantity of output $l^{\alpha}k^{1-\alpha}\Omega$ and, according to the equation (2), the difference between the production $l^{\alpha}k^{1-\alpha}\Omega$ and the consumption $c_1 + c_2$ is accumulated as productive capital.

In the production function $l^{\alpha}k^{1-\alpha}\Omega$, Ω represents a positive externality due to the economy-wide production activity. We assume that $\alpha < 1$; so, with Ω constant, the production function exhibits a constant return-to-scale technology (i.e., it is a homogeneous function of degree 1). In accordance with the literature on economic growth with externalities (see, e.g., [13, 21]), we model the positive externality as follows

$$\Omega := \bar{l}^{\delta}\bar{k}^{\varepsilon},$$

where δ and ε are strictly positive parameters and $\bar{l}(t)$ and $\bar{k}(t)$ represent the average use of labor and capital in the economy, respectively. When the average capital or labor input goes up, the productivity of l and k grows in that Ω is increasing in $\bar{l}$ and in $\bar{k}$. We assume that average values are considered as exogenously given by the representative agent when optimizing. This assumption is plausible in a context in which there is a very large number of agents (in particular, we have assumed that they are a continuum); so, each agent considers as negligible the impact that his own choices may have on the average values of economic variables. A consequence of this assumption is that the growth dynamics we shall analyze are not optimal; however, each growth path followed by the economy represents a Nash equilibrium; that is, no agent has an incentive to modify his choices if the choices of the others are fixed.

Since all agents are identical, they make the same choices; consequently, the average values $\bar{c}_1(t)$, $\bar{c}_2(t)$, $\bar{l}(t)$, $\bar{k}(t)$ coincide (ex post) with the values of $c_1(t)$, $c_2(t)$, $l(t)$, $k(t)$ chosen by the representative agent. Note that, by substituting $\bar{l}(t) = l(t)$ and $\bar{k}(t) = k(t)$ in the production function $l^{\alpha}k^{1-\alpha}\Omega$, we obtain the function $l^{\alpha+\delta}k^{1-\alpha+\varepsilon}$, i.e., the function that would be considered by a social planner who had the possibility of coordinating agents' choices. Such function exhibits decreasing (respectively, constant and increasing) marginal productivity of the labor input l if $\alpha + \delta < 1$ (respectively, $\alpha + \delta = 1$ and $\alpha + \delta > 1$); an analogous consideration holds for k and its exponent $1 - \alpha + \varepsilon$.

For space constraints, we restrict our analysis here to the study of case $\varepsilon < \alpha$; this assumption rules out the possibility of unbounded growth of k; the case $\varepsilon \geq \alpha$ will be considered in a future study.

The Hamiltonian function for our problem is

$$\begin{aligned} H(E, k, \lambda, \theta, l, c_1, c_2) = & \ln c_1 + a \ln(E + bc_2) + d \ln(1 - l) \\ & + \lambda(l^\alpha k^{1-\alpha}\Omega - c_1 - c_2) \\ & + \theta\big(\beta E(\bar{E} - E) - \gamma(\bar{c}_1 + \bar{c}_2)E\big), \end{aligned}$$

where λ and θ are the co-state variables associated with k and E respectively. By applying the maximum principle we obtain that the dynamics of $c_1(t)$, $c_2(t)$, $l(t)$, $k(t)$, $E(t)$ must satisfy the following conditions

$$\frac{\partial H}{\partial l} = -\frac{d}{1-l} + \alpha\lambda l^{\alpha-1}k^{1-\alpha}\Omega = 0, \tag{4}$$

$$\frac{\partial H}{\partial c_1} = \frac{1}{c_1} - \lambda = 0, \tag{5}$$

$$\frac{\partial H}{\partial c_2} = \frac{ab}{E + bc_2} - \lambda \leq 0, \quad c_2 \geq 0, \quad c_2\frac{\partial H}{\partial c_2} = 0, \tag{6}$$

$$\dot{k} = \frac{\partial H}{\partial \lambda} = l^\alpha k^{1-\alpha}\Omega - c_1 - c_2, \tag{7}$$

$$\dot{\lambda} = r\lambda - \frac{\partial H}{\partial k} = \lambda\big(r - (1-\alpha)l^\alpha k^{-\alpha}\Omega\big), \tag{8}$$

$$\dot{E} = \frac{\partial H}{\partial \theta} = \beta E(\bar{E} - E) - \gamma(\bar{c}_1 + \bar{c}_2)E, \tag{9}$$

where $\bar{c}_1$, $\bar{c}_2$, $\bar{l}$, $\bar{k}$ must be replaced by c_1, c_2, l, k in expressions (4)–(9) and the control variables c_1, c_2, l are determined by conditions (4)–(6). Notice that, in our model the control variables c_1 and l are always strictly positive and $l < 1$.

We omit the dynamics of the co-state variable θ since equations (7)–(9) do not depend on it (precisely because $\bar{c}_1$ and $\bar{c}_2$ are considered exogenous by the representative agent). Furthermore, we assume the usual transversality condition

$$\lim_{t\to\infty} k(t)\lambda(t)e^{-rt} = 0,$$

which is satisfied by every orbit approaching a fixed point or a periodic orbit.

4 Dynamics with $c_2 = 0$

From (6) it follows that, if the condition

$$\frac{ab}{E} - \lambda \le 0 \tag{10}$$

is met, then the representative agent chooses $c_2 = 0$; i.e., he doesn't consume the private good as a substitute for the environmental good. Otherwise, he chooses $c_2 > 0$. Condition (10) is satisfied if, given λ, the value of E is high enough. The dynamics with $c_2 = 0$ are analyzed in [22], where the possibility of substitution between the private good and the environmental good is not considered. In this section, the basic analytical results of the said paper are illustrated.

When $c_2 = 0$, system (7)–(9) is decoupled in the planar system given by (7) and (8) and the non-autonomous differential equation (9). We can easily observe that at most one fixed point, say S', exists, with the following coordinates

$$k' = \left[\frac{1-\alpha}{r}\left(\frac{\alpha}{\alpha+d}\right)^{\alpha+\delta}\right]^{\frac{1}{\alpha-\varepsilon}},$$
$$\lambda' = \frac{1-\alpha}{rk'},$$
$$E' = \bar{E} - \frac{\gamma r}{\beta(1-\alpha)}k'.$$

Such a fixed point exists only if (10) is satisfied, which requires, coeteris paribus, the endowment of the environmental good, $\bar{E}$, to be sufficiently high and the negative impact, γ, of average consumption on the environmental good to be sufficiently low.

The stability of the fixed point is described by

Theorem 1. *Let*

$$p := \frac{\alpha-\varepsilon}{d(1-\alpha-\delta)+\alpha},$$
$$q := \frac{(1-\alpha)\big[d(1-\alpha-\delta)+\alpha\big]+(\alpha+d)\varepsilon}{d(1-\alpha-\delta)+\alpha}.$$

Then:

(i) *If $p > 0$, S' is a saddle with a bi-dimensional stable manifold.*

(ii) *If* $p < 0$ *and* $q > 0$, S' *is a saddle with a one-dimensional stable manifold.*

(iii) *If* $p < 0$ *and* $q < 0$, S' *is a sink.*

If case (i) holds, given the initial values of k and E, there exists (at least locally) a single initial value of λ (determined by the representative agent) from which the economy approaches the fixed point.

Note that condition (i) is satisfied if $\alpha + \delta \leq 1$, where α and δ are the exponents of l and $\bar{l}$, respectively, in the production function.

The fixed point cannot be (generically) reached if (ii) holds.

Vice versa, when (iii) is satisfied, given the initial values of k and E, there exists a continuum of initial values of λ leading to the fixed point. In other words there exist an infinite number of (Nash) equilibrium orbits that the economy may follow to reach the fixed point. Along each orbit no economic agent has an incentive to change his choices, given other agents' choices.

Observe that the parameters r (discount rate), $\bar{E}$ (endowment of the environmental good), γ (impact of average consumption on the environmental good) play a role in the existence of the fixed point (condition (10)), but don't affect its stability properties.

Theorem 2. *When* δ *crosses the value*

$$\bar{\delta} := 1 - \alpha + \frac{\alpha}{d} + \frac{(\alpha + d)}{1 - \alpha}$$

an attracting limit cycle (through a Hopf supercritical bifurcation) arises for $\delta > \bar{\delta}$.

Proof. Proofs of the above theorems are given in [22]. □

When an attracting orbit exists, by following such an orbit the economy may enter the region of the plane (λ, E) where $c_2 > 0$. However, this is not the case if the periodic orbit is small enough. Antoci, Brugnano and Galeotti [22] show, through numerical simulations, that the periodic orbit expands as the bifurcation parameter δ increases.

In the next section we analyze dynamics in the subset of the positive orthant of the space (k, λ, E) where condition (10) is not satisfied, and consequently economic agents consume the private good also as a substitute for the environmental good (i.e., $c_2 > 0$).

5 Fixed points in the regime $c_2 > 0$

It is easy to check that, in the regime $c_2 > 0$, there always exists a fixed point at which the environmental good is completely depleted, that is $\widetilde{S} = (k, c_1, E) = (\widetilde{k}, \widetilde{c}_1, 0)$, $c_1 = \frac{1}{\lambda}$ with

$$\widetilde{k} = \left(\frac{r}{1-\alpha} \frac{\alpha(1+a)+d}{\alpha(1+a)} \right)^{\frac{1}{\varepsilon-\alpha}}, \quad \widetilde{c}_1 = \frac{\alpha(\widetilde{k})^{1-\alpha+\varepsilon}}{\alpha(1+a)+d}.$$

Denote by $S = (k^*, c_1^*, E^*)$ a fixed point satisfying the conditions $E > 0$ and $c_2 > 0$. Then $E^*, k^*, c_1^* > 0$ and $bc_2^* = abc_1^* - E^* > 0$ [see (6)].

Theorem 3. *$S = (k^*, c_1^*, E^*)$ is a fixed point satisfying the conditions $E > 0$ and $c_2 > 0$, if and only if*

$$\bar{E} = \psi(k^*) := m(k^*)^{\frac{\varepsilon+\delta}{\alpha+\delta}} - nk^*, \tag{11}$$

where

$$m := \frac{\alpha b(a+1)}{d} \Big(\frac{r}{1-\alpha} \Big)^{\frac{\alpha+\delta-1}{\alpha+\delta}}, \quad n := \frac{br(1+a)}{d(1-\alpha)} \Big(\alpha + \frac{d(b\beta-\gamma)}{b\beta(a+1)} \Big),$$

and, furthermore, $k^* \in (k_1, k_2)$, k_1 and k_2 being determined by the intersections of the curve $\bar{E} = \psi(k^*)$ with the lines

$$\bar{E} = m_1 k^* := \frac{(ab\beta+\gamma)r}{(1-\alpha)\beta} k^*, \quad \bar{E} = m_2 k^* := \frac{\gamma r}{(1-\alpha)\beta} k^*.$$

Proof. See Appendix A. $\square$

Remark. *S is unique whenever $\psi'(k^*)$ does not change sign in (k_1, k_2) (see* Figs. 1, 2*). So, from* (11)*, it follows that there exists at most one fixed point S if $n \leq 0$ (implying $\psi'(k^*) > 0$ for $k^* > 0$), whereas two fixed points (with c_2^*, $E^* > 0$) can exist if $\psi(k^*)$ has a maximum in (k_1, k_2) (see* Fig. 3*). Straightforward computations yield that the latter case holds, if and only if*

$$\gamma < \sigma < ab\beta + \gamma, \tag{12}$$

where

$$\sigma := \frac{\big[\alpha(a+1)b\beta + (b\beta-\gamma)d\big](\alpha-\varepsilon)}{d(\varepsilon+\delta)}.$$

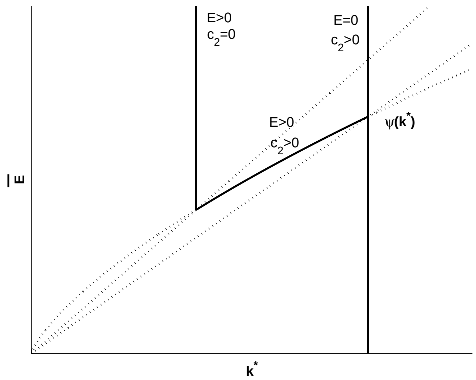

Fig. 1. Case $\sigma \leq \gamma$.

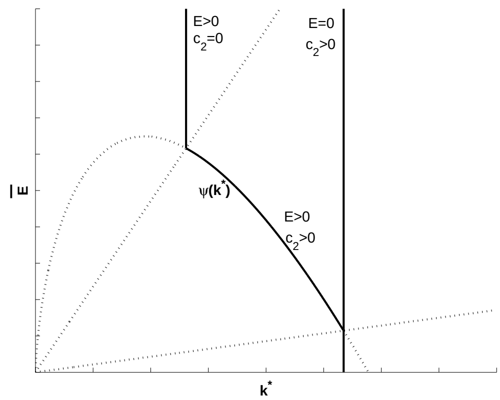

Fig. 2. Case $\sigma \geq ab\beta + \gamma$.

Then, if (12) is verified, an interval $(\bar{E}_l, \bar{E}_u)$ is given, with

$$\bar{E}_l := \max\left[\psi^*(k_1), \psi^*(k_2)\right],$$
$$\bar{E}_u := \psi(k_0), \quad \text{where} \quad \psi'(k_0) = 0, \quad k_1 < k_0 < k_2,$$

such that for any $\bar{E} \in (\bar{E}_l, \bar{E}_u)$ there exist two fixed points with a strictly positive E in the regime $c_2 > 0$.

Observe that condition (12) is never satisfied if coeteris paribus, the negative impact γ, on the environmental good of average consumption is high enough.

Figs. 1–3 illustrate the possible configurations of fixed points, in dependence of the parameters. With any point in the positive $\bar{E}$-axis being fixed (that is, given the endowment of the environmental good), the intersections between the horizontal straight line passing through it and the continuous lines drawn in each figure give the number of existing fixed points and the corresponding values of k^*.

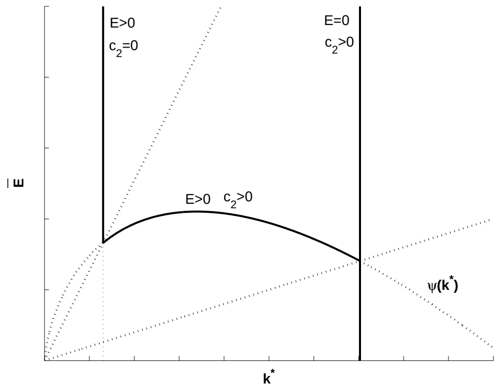

Fig. 3. Case $\gamma < \sigma < ab\beta + \gamma$.

Note that, in Figs. 1–3, the fixed point with the lowest level of capital accumulation is the one with $E > 0$ and $c_2 = 0$ (when existing). Such a fixed point would be unique if the private good could not be consumed as a substitute for the environmental good.

Fig. 1 illustrates the case $\sigma \leq \gamma$ (that is, the case where the negative impact on the environmental good of average consumption is sufficiently high). In such a case at most three fixed points can exist. If the endowment $\bar{E}$ of the environmental good is high enough, then there exist two fixed points: the one with $E = 0$ and $c_2 > 0$ and the one with $E > 0$ and $c_2 = 0$. As $\bar{E}$ decreases, then three fixed points appear: those with $E = 0$, $c_2 > 0$ and $E > 0$, $c_2 = 0$ and a fixed point where $E > 0$ and $c_2 > 0$. Finally, if $\bar{E}$ is sufficiently low, then only the fixed point with $E = 0$ exists.

Fig. 2 can be interpreted in a similar way. Unlike in Fig. 1, at most two fixed points can coexist.

Fig. 3 shows the more interesting case, where the highest number of fixed points can exist, i.e., one with $E > 0$ and $c_2 = 0$, two with $E > 0$ and $c_2 > 0$, one with $E = 0$ and $c_2 > 0$. Such a regime exists if, coeteris paribus, $\bar{E}$ and γ are not "too" high or "too" low.

6 Stability analysis

6.1 Stability of the fixed point with $E = 0$

It is easy to check that at $\widetilde{S} = (\widetilde{k}, \widetilde{c}_1, 0)$ the E-axis is an eigenspace of the Jacobian matrix, whose associated eigenvalue has the sign of $\bar{E} - \psi(\widetilde{k})$.

The stability of $\widetilde{S}$ as well as the dynamics in the invariant $E = 0$ plane can be reconducted to the projection on the (k, λ) plane of the $c_2 = 0$ regime, by replacing d with $d' := \frac{d}{a+1}$ and λ with $\lambda' := \frac{\lambda}{a+1}$.

Thus, in particular:

Case 1. If $\delta < 1 - \alpha + \frac{\alpha}{d'}$, $\widetilde{S}$ is a saddle in the invariant plane $E = 0$.

Case 2. If $\delta > 1 - \alpha + \frac{\alpha}{d'} + \frac{(\alpha + d')\varepsilon}{(1-\alpha)d'}$, $\widetilde{S}$ is a source in the invariant plane $E = 0$.

Case 3. If $1 - \alpha + \frac{\alpha}{d'} < \delta < 1 - \alpha + \frac{\alpha}{d'} + \frac{(\alpha + d')\varepsilon}{(1-\alpha)d'}$, $\widetilde{S}$ is a sink in the invariant plane $E = 0$.

Therefore, starting from a strictly positive value of E, the fixed point can be (generically) reached by suitably choosing λ if $\bar{E} - \psi(\widetilde{k}) < 0$ is satisfied and Case 1 or Case 3 holds. In particular, when Case 3 holds, there exists a continuum of orbits approaching the fixed point, i.e., indeterminacy occurs.

6.2 Stability of the fixed points with $E > 0$ in the regime $c_2 > 0$

Straightforward calculations enable the definition of the Jacobian matrix $J(S)$

$$J(S) = \begin{pmatrix} A & B - (a+1) & \frac{1}{b} \\ \frac{c_1^*}{k^*}\big((1-\alpha)A - r\big) & \frac{c_1^*}{k^*}(1-\alpha)B & 0 \\ 0 & -\gamma(a+1)E^* & \frac{\gamma - b\beta}{b}E^* \end{pmatrix}$$

where A, B, and $\det\big(J(S)\big)$ are computed in Appendix B.

We can distinguish the following subcases.

6.2.1 Case $\alpha + \delta \leq 1$

Theorem 4. *If $\alpha + \delta \leq 1$ and $\psi'(k^*) \neq 0$, $J(S)$ has at least one eigenvalue with positive real part. In particular, if $\psi'(k^*) > 0$, S is a saddle with a* *one-dimensional stable manifold; if $\psi'(k^*) < 0$, S is either a saddle with a bi-dimensional stable manifold or a repellor. In the last case, when k^* approaches k_2, a Hopf bifurcation, generically, takes place: S is transformed from a repellor into a saddle with a bi-dimensional stable manifold.*

Proof. See Appendices C and D. □

Remember that, in the production function of the representative agent, α is the exponent of labor input l and δ is the exponent of average labor input $\bar{l}$. The above theorem says that, if $\alpha + \delta \leq 1$, then the fixed points in the regime $E > 0$ and $c_2 > 0$ cannot be attractive: that is, indeterminacy cannot occur.

Furthermore, if a fixed point satisfies $\psi'(k^*) > 0$, then it cannot be reached (generically) by the economy. If instead the condition $\psi'(k^*) < 0$ holds, then the fixed point has a bi-dimensional stable manifold (and can be reached by the economy) if k^* is near enough to k_2; otherwise, it may be a repellor. In the latter case, such a fixed point might be "surrounded", via a Hopf bifurcation, by a periodic orbit with a bi-dimensional stable manifold.

Remember that, if $\alpha + \delta \leq 1$, the fixed point S' in the $c_2 = 0$ regime (when existing) is always a saddle with a bi-dimensional stable manifold. If $\alpha + \delta \leq 1$ and $\sigma \leq \gamma$ (see Fig. 1), the fixed point satisfying $E > 0$ and $c_2 > 0$ cannot be (generically) reached by the economy, being a saddle with a one-dimensional stable manifold. The fixed point with $E = 0$ cannot be reached (starting from a strictly positive value of E) if $\bar{E}$ is high enough, that is, when $\bar{E} > \psi(k^*)$ and the Jacobian matrix has a strictly positive eigenvalue in the E-axis direction.

Observe that both the fixed point with $E = 0$ and the one with $c_2 = 0$ can be saddles with bi-dimensional stable manifolds. In such a case a bi-stable dynamic regime occurs: the economy can approach either fixed point depending on the initial values of E and k.

Analogous observations can be made about Figs. 2 and 3.

6.2.2 Case $\alpha + \delta > 1$

Theorem 5. *Whenever $\alpha + \delta > 1$, an attractor can exist, with $E > 0$, in the $c_2 > 0$ regime.*

Proof. Applying formulae (C.2)–(C.5), it is easy to see that the fixed point S is an asymptotic attractor, if and only if

$$\operatorname{tr} J(S),\ \det J(S) < 0,\ H(S) > 0,\ |\det J(S)| < H(S) |\operatorname{tr} J(S)|, \tag{13}$$

where $H(S)$ is defined by formula (C.3).

In particular conditions (13) imply

$$\frac{r}{1-\alpha} < (\alpha+\delta-1)\frac{d}{\alpha}\frac{c_1^*}{k^*}, \tag{14}$$

and

$$A + \frac{c_1^*}{k^*}(1-\alpha)B < 0, \tag{15}$$

where A and B are computed in Appendix B.

From the expression of $\det\big(J(S)\big)$ (formula (B.1)) and from (14) it follows that
$\psi'(k^*) < 0$ if S is an attractor.

Two subcases are then to be examined:

1. $\gamma < \sigma < ab\beta + \gamma$, $k^* \in (k_0, k_2)$;
2. $ab\beta + \gamma \leq \sigma$, $k^* \in (k_1, k_2)$.

Subcase 1. Since

$$\bar{E} = \psi(k^*) \quad \text{and} \quad \frac{c_1^*}{k^*} = \frac{1}{(a+1)b}\frac{\psi(k^*)}{k^*} + \frac{r(b\beta-\gamma)}{(a+1)(1-\alpha)b\beta},$$

$\frac{c_1^*}{k^*}$ decreases as $k^* \in (k_0, k_2)$ increases. Therefore (14) holds in a subinterval of (k_0, k_2), if and only if it holds at the fixed point $S_0 = (k_0, c_{1_0}, E_0)$. If $\frac{d}{\alpha} = \frac{1}{\alpha+\delta-1}$, it is easily computed that

$$\frac{r}{1-\alpha} - \frac{c_{1_0}}{k_0} = \frac{r}{(1-\alpha)(a+1)b\beta}(ab\beta+\gamma-\sigma) > 0. \tag{16}$$

Hence (16) implies $d > \dfrac{\alpha}{\alpha+\delta-1}$, i.e.,

$$1 - \alpha + \frac{\alpha}{d} < \delta. \tag{17}$$

For example, fixed α and δ so that $\alpha + \delta > 1$, the other parameters can be chosen to satisfy

$$a = b = \beta = 1, \quad \gamma = \frac{\alpha(\alpha - \varepsilon)}{d(\varepsilon + \delta)},$$
$$r = \frac{1-\alpha}{2\left(1 + \frac{d}{\alpha}\right)^{\alpha+\delta}}, \quad \frac{(\alpha + \delta - 1)d}{2\alpha} = 1 + \frac{\varepsilon}{2(1-\alpha)}.$$

Then, if $\alpha - \varepsilon > 0$ is sufficiently small, the conditions

$$\operatorname{tr} J_0 < 0, \quad H_0 > 0$$

are seen to hold.

Hence, when k^* belongs to a suitable right neighborhood of k_0, the corresponding fixed point S is attractive.

Subcase 2. In such a case $\sigma \geq ab\beta + \gamma$ and $\frac{c_1^*}{k^*}$ decreases along (k_1, k_2). Hence (14) holds in a subinterval of (k_1, k_2), if and only if it holds at k_1.

Denote by S_1 the fixed point (k_1, c_1, E_1). We have

$$\bar{E}_1 = \psi^*(k_1) = \frac{r(ab\beta + \gamma)}{(1-\alpha)\beta} k_1,$$
$$\frac{c_1}{k_1} = \frac{1}{(a+1)b} \frac{\bar{E}_1}{k_1} + \frac{r(ab\beta - \gamma)}{(a+1)(1-\alpha)b\beta} = \frac{r}{1-\alpha}. \tag{18}$$

Therefore, again, (14) implies (17).

Let us check, next, condition (15). Exploiting (14), through easy calculations, (15) implies

$$\frac{r}{1-\alpha}(1 - \alpha + \varepsilon) - \left[(1-\alpha)(\alpha + \delta - 1) - \varepsilon\right] \frac{dc_1}{\alpha k_1} > 0,$$

i.e., because of (18),

$$(1-\alpha)(\alpha+\delta)\frac{d}{\alpha} < (1 - \alpha + \varepsilon)\Big(1 + \frac{d}{\alpha}\Big),$$

or

$$\delta < 1 - \alpha + \frac{\alpha}{d} + \frac{\varepsilon(\alpha + d)}{d(1-\alpha)}. \tag{19}$$

Letting $J_1 = J(S_1)$ and writing the characteristic polynomial $P_1(\lambda)$ of J_1, it follows that S is an attractor for k^* belonging to a suitable right neighborhood of k_1, if and only if

$$\det J_1 < 0, \ \operatorname{tr} J_1 < 0, \ H_1 > 0, \ |\det J_1| < |\operatorname{tr} J_1| H_1.$$

For example, let $\alpha + \delta > 1$, d satisfying (17) and (19), $\alpha - \varepsilon > 0$ sufficiently small. Furthermore set

$$b = \beta = 1, \quad a = \frac{\alpha - \varepsilon}{2(\varepsilon + \delta)}, \quad \gamma = (\alpha - \varepsilon)^2, \quad r = \frac{1-\alpha}{2}\left(\frac{\frac{\alpha}{d}}{\frac{\alpha}{d} + 1}\right)^{\alpha+\delta}.$$

Then it can be checked that, when $\alpha - \varepsilon$ is small enough, for k^* belonging to a suitable right neighborhood of k_1, the corresponding fixed point S is an attractor. $\square$

Remark. *Let S be the attractor of the example in Subcase 1. Then, for the same values of the parameters, two more fixed points with a positive E can exist, i.e., $S' = (k', c_1', E')$ in the $c_2 = 0$ regime and $S'' = (k'', c_1'', E'')$ in the $c_2 > 0$ regime, $k'' \in (k_1, k_0)$. Since* (17) *holds, it follows:*

(i) *S' is either an attractor or a saddle with a one-dimensional stable manifold. In fact, considering the above example, S' is an attractor when, for instance, both α and ε are sufficiently close to 1, while it can be a saddle when α and ε are themselves "small".*

(ii) *S'' is a saddle with a bi-dimensional stable manifold. Such a manifold is locally a separatrix.*

Note that, when $\alpha + \delta > 1$, there is the possibility of three reachable fixed points; this case occurs if, for example, the fixed point in $c_2 = 0$ is attractive, the one with $E > 0$, $c_2 > 0$, $\psi'(k^*) > 0$ is a saddle with a bi-dimensional stable manifold and the fixed point satisfying $E > 0$, $c_2 > 0$, $\psi'(k^*) < 0$ is also attractive.

7 Welfare analysis: numerical examples

Through two numerical examples, we show that capital accumulation level (and, consequently, private consumption level) and economic agents' welfare may be not positively correlated; that is, economic agents' welfare at a fixed point with a high accumulation level can be lower than at a fixed point with a low accumulation level. Firstly, we assume: $\alpha = 0.8$, $\beta = 0.05$, $\gamma = 0.1$, $\delta = 0.2$, $\varepsilon = 0.75$, $a = 2.5$, $b = 0.1$, $d = 0.05$, $r = 0.1$; in Fig. 4 we represent capital accumulation values k^*, evaluated at the fixed points A, B, C and D, as functions of the parameter $\bar{E}$ (remember that $\bar{E}$ is the endowment of the natural resource in the economy). The context we consider is that of Fig. 3, so, in the fixed point A it holds that $c_2 = 0$ (i.e., the private good is not consumed as a substitute for the environmental good) while in B, C and D we have $c_2 > 0$; among these, D is the fixed point where $E = 0$, that is where the natural resource is completely depleted.

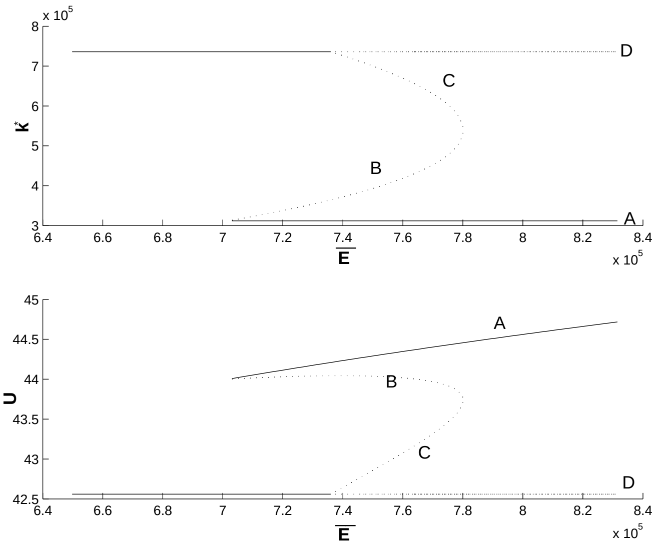

Fig. 4. Case $\alpha = 0.8$, $\beta = 0.05$, $\gamma = 0.1$, $\delta = 0.2$, $\varepsilon = 0.75$, $a = 2.5$, $b = 0.1$, $d = 0.05$, $r = 0.1$.

Reachable fixed points (i.e., those having at least two negative eigenvalues) are indicated by continuous lines, the others by dotted lines. Note that, for sufficiently low values of $\bar{E}$, only D is reachable; for sufficiently high values of $\bar{E}$, only A is reachable; finally, for intermediate values of $\bar{E}$, both A and D are reachable. Therefore, as shown in the above analysis, our model predicts that a consequence of a reduction of $\bar{E}$ (due, for example, to an exogenous shock) can be the convergence of the economy to a fixed point with a higher accumulation level.

In Fig. 4 we also represent the values assumed by the utility function U at the fixed points A, B, C and D, as functions of the parameter $\bar{E}$. Observe that the highest utility value is obtained at the fixed point A with the lowest accumulation level; the opposite holds for the fixed point D where the highest accumulation level is associated with the lowest utility level.

In Fig. 5 we modify the preceding example by assuming (coeteris paribus) $\delta = 0.5$. In Fig. 5a, we can see that only the fixed point D (respectively, only the fixed point A) is reachable if $\bar{E}$ is low enough (respectively, if $\bar{E}$ is high enough). For intermediate values of $\bar{E}$, two fixed points are reachable: A and D or A and C. Note that the effects of an increase of $\bar{E}$ on the values of k^* at the reachable fixed points are similar to those of the former example; however, in the latter example, we can see that in D and C the utility function assumes values higher than in A; consequently, in this context, economic growth is desirable.

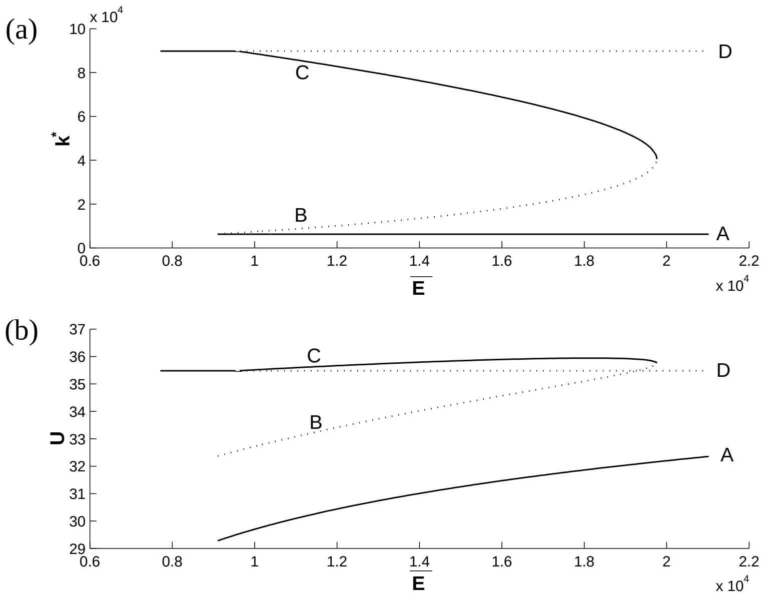

Fig. 5. Case $\alpha = 0.8$, $\beta = 0.05$, $\gamma = 0.1$, $\delta = 0.5$, $\varepsilon = 0.75$, $a = 2.5$, $b = 0.1$, $d = 0.05$, $r = 0.1$.

These results are due to the fact that the productive activity of each agent also generates positive externalities on the productive activity of the others; consequently, the level of welfare obtained is the result of the interaction of both positive and negative effects deriving from the choices of each agent. In the former example, negative externalities overcome positive externalities; vice versa in the latter.

8 Hopf bifurcations

Our interest in the existence of periodic orbits is motivated by the fact that oscillations of the state variables k and E produce a reduction in welfare compared with a state of the economy where the values of k and E are equal to the time averages of k and E along the periodic orbit, if economic agents are risk-averse (see, e.g., [13]).

The existence of periodic orbits in the $c_2 = 0$ regime was analyzed in [22], where a supercritical Hopf bifurcation was shown to give rise to a locally attracting periodic orbit (i.e., with a three-dimensional stable manifold). Let us now investigate, in the $c_2 > 0$ regime, local bifurcations taking place at the equilibrium $S = (k^*, c_1^*, E^*)$, with $E^* > 0$, when k^* varies in (k_1, k_2).

Case 1. Assume $\alpha + \delta \leq 1$ and S is a repellor for k^* belonging to a sub-interval I of (k_1, k_2).

Then $\frac{c_1^*}{k^*}$ is decreasing and $\psi'(k^*) < 0$ in I. It follows that, when k^* approaches k_2, generically a Hopf bifurcation occurs, as the real part of two complex conjugate eigenvalues turns from positive into negative. In other words, for k^* belonging to a suitable left neighborhood of k_2, S has a bi-dimensional stable and a one-dimensional unstable manifold.

Case 2. Assume $\alpha + \delta > 1$ and S is an attractor for k^* belonging to a sub-interval $I = (k_3, k_4)$ of (k_1, k_2).

Then $\psi'(k^*) < 0$ and $\frac{c_1^*}{k^*}$ is decreasing in I. Furthermore, recalling $d' := \frac{d}{a+1}$,

$$1 - \alpha + \frac{\alpha}{d} < \delta < 1 - \alpha + \frac{\alpha}{d'} + \frac{(\alpha + d')\varepsilon}{(1 - \alpha)d'},$$

as conditions (14) and (15) are checked to imply.

It is easily seen that $k_4 = k_2$, if

$$1 - \alpha + \frac{\alpha}{d'} < \delta < 1 - \alpha + \frac{\alpha}{d'} + \frac{(\alpha + d')\varepsilon}{(1-\alpha)d'},$$

while $k_4 < k_2$, if

$$1 - \alpha + \frac{\alpha}{d} < \delta < 1 - \alpha + \frac{\alpha}{d'}.$$

In the latter case, when k^* crosses k_4, one real negative eigenvalue becomes positive, passing through ∞, and S has a bi-dimensional stable and a one-dimensional unstable manifold as $k^* \in (k_4, k_2)$.

Furthermore it may happen that $k_3 > k_m$, where $k_m = k_0$ or $k_m = k_1$, in Cases 1 and 2, respectively.

If this occurs, then, generically, a Hopf bifurcation takes place when k^* crosses k_3, as S becomes an attractor from a saddle with a one-dimensional stable manifold.

Case 3. Consider the case

$$\delta > 1 - \alpha + \frac{\alpha}{d'} + \frac{(\alpha + d')\varepsilon}{(1-\alpha)d'}. \tag{20}$$

Then no bifurcation occurs in the possible interval $J \subseteq (k_1, k_2)$, where $\psi'(k^*) < 0$.

In such an interval S has a one-dimensional stable and a bi-dimensional unstable manifold.

If, furthermore, $\gamma < \sigma < ab\beta + \gamma$, then, passing through k_0 (recall $\psi'(k_0) = 0$), one real eigenvalue changes sign: it may turn either from positive into negative or vice versa.

Case 4. Finally a generic Hopf bifurcation can take place in the possible interval $H \subseteq (k_1, k_2)$, where $\psi'(k^*) > 0$.

Example. In the following numerical example (12) and (20) hold and (k_1, k_2) is divided into three sub-intervals: (k_1, k_h), (k_h, k_0), (k_0, k_2). As $k^* \in (k_1, k_h)$, S is a repellor. Then at k_h a Hopf bifurcation occurs and S has a bi-dimensional stable and a one-dimensional unstable manifold for $k^* \in (k_h, k_0)$.

Finally, when k^* crosses k_0, a real negative eigenvalue becomes positive and S has a one-dimensional stable and a bi-dimensional unstable manifold as $k^* \in (k_0, k_2)$.

The example is

$$\alpha = \frac{1}{2}, \quad a = b = \beta = 1, \quad d = 4, \quad \delta = 2, \quad r = \frac{1}{2\sqrt{6^5}},$$
$$\alpha - \varepsilon \quad \text{sufficiently small,}$$
$$\gamma = (\alpha - \varepsilon)^2.$$

In this section we have shown all the Hopf bifurcations which can occur in our model. In Fig. 6, a locally attractive periodic orbit and one orbit approaching it are plotted . In such a case, given the initial values of k and E [near enough to the projection of the orbit on the (k, E) plane], there exists a continuum of initial values of λ (or, alternatively, of c_1 or c_2) by which the economy can reach the periodic orbit. Consequently, an indeterminacy problem occurs.

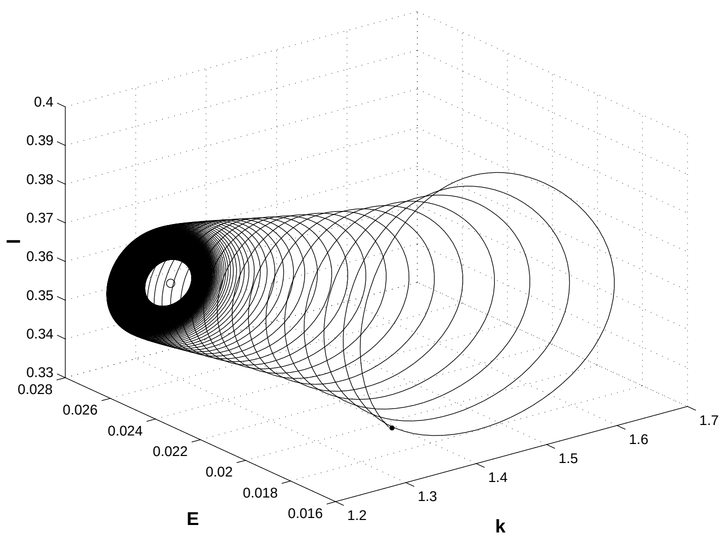

Fig. 6. Attracting limit cycle in case $\alpha = 0.8,\ \beta = 0.05,\ \gamma = 0.1,\ \delta = 0.2,$ $\varepsilon = 0.75,\ a = 2.5,\ b = 1,\ d = 0.05,\ r = 0.1.$

9 Behavior of orbits for high values of $\bar{E}$

Theorem 6. *When $\bar{E}$ is sufficiently high, orbits starting in the region $c_2 > 0$ enter and remain, after a finite time, in the regime $c_2 = 0$.*

Proof. To this end, let us replace, first, the variables (k, λ, E) in system (7)–(9) by (l, c_1, E), where $l \in (0,1)$ is defined by (4) and $c_1 = \frac{1}{\lambda}$. It follows from (4) that $k = \left(\frac{dl^{1-\alpha-\delta}}{\alpha(1-l)}c_1\right)^{\frac{1}{1-\alpha+\varepsilon}}$.

Hence we get, after simple steps,

$$\dot{c}_1 = c_1\left[(1-\alpha)\left(\frac{\alpha}{d}\right)^{\frac{\alpha-\varepsilon}{1-\alpha+\varepsilon}} \frac{l^{\frac{\varepsilon+\delta}{1-\alpha+\varepsilon}}(1-l)^{\frac{\alpha-\varepsilon}{1-\alpha+\varepsilon}}}{c_1^{\frac{\alpha-\varepsilon}{1-\alpha+\varepsilon}}} - r\right]. \tag{21}$$

This implies $\dot{c}_1 < 0$, if

$$\begin{aligned} c_1 > c_1^M &:= \left(\frac{1-\alpha}{r}\right)^{\frac{1-\alpha+\varepsilon}{\alpha-\varepsilon}} \max_{l\in(0,1)} l^{\frac{\varepsilon+\delta}{\alpha-\varepsilon}}(1-l) \\ &= \frac{\alpha(\alpha-\varepsilon)}{d(\alpha+\delta)}\left(\frac{1-\alpha}{r}\right)^{\frac{1-\alpha+\varepsilon}{\alpha-\varepsilon}}\left(\frac{\varepsilon+\delta}{\alpha+\delta}\right)^{\frac{\varepsilon+\delta}{\alpha-\varepsilon}}. \end{aligned} \tag{22}$$

Now assume

$$\bar{E} > b(a+1)c_1^M, \quad \text{if} \quad b\beta > \gamma, \tag{23}$$

$$\bar{E} > \frac{\gamma}{\beta}(a+1)c_1^M, \quad \text{if} \quad b\beta \le \gamma. \tag{24}$$

It follows, in particular, as is easily checked, that no equilibrium exists in the $c_2 > 0$ regime.

Furthermore, consider an orbit starting in the region $c_2 > 0$, i.e., such that $ac_1(0) > \frac{E(0)}{b}$.

Due to (21) and (22), and since no equilibrium exists when $c_2 > 0$, there is a time t_1 such that either

$$c_2(t_1) = 0 \tag{25}$$

or

$$c_1(t) \le c_1^M \text{ when } t \ge t_1. \tag{26}$$

If $c_2(t_1) > 0$, then $\dot{E}(t_1) > 0$, because of assumptions (23) and (24). Hence E increases, by a speed bounded away from zero, whereas $c_1 \le c_1^M$. Consequently there exists a time $t_2 > t_1$ such that

$$c_2(t_2) = 0.$$

Finally, we can choose $\bar{t}$, $\bar{t} \geq t_i$, $i = 1, 2$, in such a way that $c_1(t) \leq c_1^M$ for $t \geq \bar{t}$ and $c_2(t) = 0$ for t being in some right neighborhood of $\bar{t}$ (in the particular case $\frac{\alpha}{\alpha+d} = \frac{\varepsilon+\delta}{\alpha+\delta}$, the (22) value of c_1^M can be replaced by any *slightly* larger value). Otherwise, effectively, E would continue to increase, in line with our assumptions, in the region $c_2 > 0$, until we would obtain

$$E > abc_1^M \geq abc_1,$$

thus contradicting the condition $c_2 > 0$.

Now it is easily checked that assumptions (23) and (24) imply that, in the $c_2 = 0$ regime, $\dot{E}(t) \leq 0$, $t \geq \bar{t}$, requires $E(t)$ to be larger than abc_1^M. However, in this case, the orbit would never cross back the plane $ac_1 = \frac{E}{b}$. Therefore, should the orbit keep crossing such a plane forwards and back , i.e., moving indefinitely from the regime $c_2 > 0$ to the regime $c_2 = 0$ and vice versa, $\dot{E}(t)$ would be positive and bounded away from zero as $t \geq \bar{t}$, until at some time, in the $c_2 > 0$ regime,

$$E > abc_1^M \geq abc_1,$$

yielding a contradiction.

Hence we can choose the previous $\bar{t}$ as the time at which the orbit *enters* into and then *remains* in the $c_2 = 0$ regime. □

10 Conclusions

In order to better evaluate the relevance of the results obtained by our work, it may be useful to bear in mind what the dynamics of economic growth would be if the private good produced in the economy could not be consumed by the economic agents as a self-protection device against the deterioration of the environmental resource. In this case, whatever the values of E and k, the dynamics of the economy would only be described by the dynamics with $c_2 = 0$, that is there would be at most one fixed point with $E > 0$ (the fixed point A in Figs. 4 and 5). The possibility of consuming the private good to alleviate the negative effects deriving from environmental degradation generates considerably more complex dynamics. More specifically, the analysis performed shows that, as well as the fixed point with $c_2 = 0$, we can also have another three fixed points at which we have $c_2 > 0$; in these the levels of capital accumulation and consumption are higher than in the fixed point with $c_2 = 0$.

We have showed that the fixed point with $c_2 = 0$ exists only if, coeteris paribus, the endowment of the environmental good $\bar{E}$ is sufficiently high and the negative impact γ of average consumption on the environmental good is sufficiently low; however, the values of $\bar{E}$ and γ play no role in its stability properties. Furthermore, we have showed that when $\bar{E}$ is sufficiently high, orbits starting in the region $c_2 > 0$ enter and remain, after a finite time, in the regime $c_2 = 0$. Consequently, for high values of $\bar{E}$, the dynamics of the economy rule out definitively substitutive consumptions.

When $\bar{E}$ is lower, dynamics become more interesting; in particular, bi-stable regimes can occur where the economy may reach the fixed point (or a periodic orbit) where $c_2 = 0$ or a fixed point (or a periodic orbit) where $c_2 > 0$ depending on the initial values of E and k. Furthermore, when $\alpha + \delta > 1$, there is the possibility of three reachable fixed points (see remark concerning Subcase 1 in Section 6); for example, it can happen that the fixed point with $c_2 = 0$ is attractive, the one with $E > 0$, $c_2 > 0$ and $\psi'(k^*) > 0$ is a saddle with a bi-dimensional stable manifold and the fixed point satisfying $E > 0$, $c_2 > 0$ and $\psi'(k^*) < 0$ is attractive.

The complexity of the scenario described above is enhanced if we consider that indeterminacy can occur; in this case, even economies characterized by identical technologies, preferences and endowments of environmental goods, starting from the same initial values of E and k, may follow different growth dynamics choosing different initial values of λ (the multiplier associated with the state variable k). Note that the dynamics we have analyzed can have an attracting fixed point in the regime $c_2 > 0$ and a (reachable) saddle point in $c_2 = 0$; this implies that self-protection choices can cause indeterminacy.

The basic prediction of the model is that if there exist private goods that can be consumed as substitutes for environmental goods, then environmental deterioration may play the role of engine of economic growth. As regards economic agents' welfare, the analysis of the model has demonstrated that there is not necessarily a positive correlation between the level of accumulation of capital (and, consequently, the level of consumption of the private good) and economic agents' welfare. In particular, at the fixed point where capital accumulation is relatively low and $c_2 = 0$, welfare may be greater than in the other fixed points. Section 7 gives a numerical example where economic growth is undesirable and shows, by another example, that increasing (coeteris paribus) the value of the parameter δ (the exponent of average labour input), desirable economic growth can be obtained. These results are due to the fact that the productive activity of each agent also generates positive externalities on the productive activity of the others; consequently, the level of welfare obtained is the result of the interplay between positive and negative effects deriving from the choices of each agent. In the former example, negative externalities overcome positive externalities; vice versa in the latter.

The basic lesson emerging from the analysis of the model is that the aggregate level of consumption of the private goods is a distorted index of individuals' welfare. This paper suggests that economic growth policies which are capable of achieving their goals, but at high environmental costs, should be treated with great caution. Economic policies ought to guarantee the growth of the values of appropriate welfare indices, which take into consideration not only the level of aggregate consumption but also that of environmental degradation and of self-protection consumption.

The question which all the public administrators ought to ask themselves is the following: how many opportunities do individuals have at their disposal for enjoying their leisure at no cost? This question acquires particular significance if we consider the problem of the management of the cities. The negative effects caused by the interaction between individuals are in fact particularly evident in densely populated environments, such as the urban areas, which consequently prove to be the places in which most of the self-protective choices are implemented. The cities feature the advantage of offering a wide variety of opportunities for spending leisure; at the same time they also often feature the disadvantage that almost nothing which is on offer can be used without spending money. From this point of view, the measures for reducing urban pollution by blocking traffic during the weekend constitute an example of public intervention that is extremely efficacious, providing an incentive for the citizens to change their models of consumption. By reducing the acoustic and atmospheric pollution, such measures contribute greatly to extending the offer of free access sites where the citizens can enjoy their free time, and therefore contribute to reducing self-protective consumption. In general, on the basis of the results of our model, it would appear desirable for the public administration to identify and classify all the activities which may generate self-protective consumption, with reference to the types of entity involved (individuals, firms, public sector), the sites in which such activities are carried out and, finally, the possible solutions which can be offered by the public sector. The public administrators are the only entities which can implement efficacious intervention in that, as the model shows, even perfectly rational individuals may select inefficient models of consumption.

Appendix A

From $\dot{c}_1 = 0$ and $\dfrac{\dot{E}}{E} - \gamma \dot{k} = 0$ it follows

$$E^* = \bar{E} - \frac{\gamma r}{(1-\alpha)\beta} k^*. \tag{A.1}$$

Then, from $\dot{E} = 0$,

$$c_1^* = c_1^* = \frac{\bar{E}}{(a+1)b} + \frac{(b\beta - \gamma)r}{(a+1)(1-\alpha)b\beta} k^*. \tag{A.2}$$

Since $E^*, c_2^* > 0$, (A.1) and (A.2) imply

$$\frac{(1-\alpha)\beta}{(ab\beta+\gamma)r}\bar{E} < k^* < \frac{(1-\alpha)\beta}{\gamma r}\bar{E}.$$

Furthermore $\dot{c}_1 = 0$ and $\dfrac{\partial H}{\partial l} = 0$ imply

$$\frac{dl^*}{1-l^*} = \frac{\alpha r}{1-\alpha}\frac{k^*}{c_1^*} \tag{A.3}$$

and, putting $A = l^\delta k^\epsilon$ in $\dot{c}_1 = 0$,

$$(l^*)^{\alpha+\delta} = \frac{r}{1-\alpha}(k^*)^{\alpha-\epsilon}. \tag{A.4}$$

From (A.3) and (A.4) it follows

$$\frac{(1-\alpha)d}{\alpha r}c_1^* + k^* = \left(\frac{1-\alpha}{r}\right)^{\frac{1}{\alpha+\delta}}(k^*)^{\frac{\varepsilon+\delta}{\alpha+\delta}}$$

and, finally, from (A.2) we get (11).

Appendix B

It is easily computed that

$$A = \frac{(1-\alpha+\varepsilon)(\alpha+\delta)\frac{dr}{\alpha(1-\alpha)}\frac{c_1^*}{k^*}}{\frac{r}{1-\alpha} - (\alpha+\delta-1)\frac{d}{\alpha}\frac{c_1^*}{k^*}} + (1-\alpha+\varepsilon)\frac{r}{1-\alpha},$$

$$B = \frac{-(\alpha+\delta)\frac{dr}{\alpha(1-\alpha)}}{\frac{r}{1-\alpha} - (\alpha+\delta-1)\frac{d}{\alpha}\frac{c_1^*}{k^*}}.$$

Recalling the form of $\psi(k^*)$ in (11), one obtains

$$\det\big(J(S)\big) = \frac{-(\alpha+\delta)\beta d r c_1^* E^* \psi'(k^*)}{\alpha\beta k^*\left(\frac{r}{1-\alpha} - (\alpha+\delta-1)\frac{d}{\alpha}\frac{c_1^*}{k^*}\right)}. \tag{B.1}$$

Appendix C

Letting $\alpha+\delta \le 1$, (B.1) implies

$$\det\big(J(S)\big)\psi'(k^*) < 0, \quad \text{when} \quad \psi'(k^*) \neq 0.$$

Hence, if $\psi'(k^*) < 0$, $\det J(S) > 0$ and the proposition follows. Then, let $\psi'(k^*) > 0$ and consequently $\det J(S) < 0$.

Denote by g_{ik}, $i, k = 1, 2, 3$, the entries of $J(S)$. Observe, firstly, that

$$g_{11} + g_{22} = \frac{\varepsilon(\alpha+\delta)\frac{dr}{\alpha(1-\alpha)}\frac{c_1^*}{k^*}}{\frac{r}{1-\alpha} + (1-\alpha-\delta)\frac{d}{\alpha}\frac{c_1^*}{k^*}} + (1-\alpha+\varepsilon)\frac{r}{1-\alpha} > 0. \tag{C.1}$$

The characteristic polynomial of $J(S)$ is

$$P(\lambda) = \lambda^3 - \big(\operatorname{tr}(J)\big)\lambda^2 + H\lambda - \det J, \tag{C.2}$$

where

$$H = g_{11}g_{22} + g_{33}(g_{11} + g_{22}) - g_{12}g_{21}. \tag{C.3}$$

From elementary algebra a cubic polynomial

$$\lambda^3 + a\lambda^2 + b\lambda + c \tag{C.4}$$

has all non-positive real part roots, if and only if

$$a, b, c, ab - c \geq 0. \tag{C.5}$$

It follows from (C.1) that

$$\mathrm{tr}\,(J) = g_{11} + g_{22} + g_{33} \leq 0 \text{ implies } g_{33} < 0,$$

while $H \geq 0$, being $g_{11}g_{22}$, $g_{33}(g_{11} + g_{22})$, $g_{12} < 0$, requires $g_{21} > 0$. Finally the condition $ab - c \geq 0$ means $|\det J| \leq H|\,\mathrm{tr}\,J|$. Through simple calculations, though,

$$\begin{aligned} |\det J| &= |g_{33}|(g_{11}g_{22} - g_{12}g_{21}) \\ &> \big(|g_{33}| - (g_{11} + g_{22})\big)\big(g_{11}g_{22} + g_{33}(g_{11} + g_{22}) - g_{12}g_{21}\big) = |\,\mathrm{tr}\,J|H. \end{aligned}$$

Hence we arrive at a contradiction.

We can conclude that, when $\psi'(k^*) > 0$, $J(S)$ has only one eigenvalue with negative real part, and therefore negative.

Appendix D

Let us show that for any value $\alpha + \delta \leq 1$ it is possible to have a repellor S, in the $c_2 > 0$ regime, with $E^* > 0$ and $\psi'(k^*) < 0$.

Let $\gamma < \sigma < ab\beta + \gamma$, so that there exist $k_0 \in (k_1, k_2)$ satisfying $\psi'(k_0) = 0$. Call S_0 the corresponding equilibrium.

In order that S be a repelling equilibrium for k^* lying in a suitable right neighborhood of k_0, it suffices that the two non-zero eigenvalues of $J(S_0)$ have positive real part.

Posit $J_0 = J(S_0)$ and denote with $P_0(\lambda)$ its characteristic polynomial, i.e.,

$$P_0(\lambda) = \lambda^3 - (trJ_0)\lambda^2 + H_0\lambda.$$

The non-zero roots of $P_0(\lambda)$ have positive real part, if and only if

$$\mathrm{tr}\, J_0,\ H_0 > 0. \tag{D.1}$$

It is easy to check that (D.1) is verified, when $\alpha - \varepsilon > 0$ is sufficiently small and, furtherly,

$$\gamma = 2(\alpha - \varepsilon), \quad \beta = d = \alpha - \varepsilon, \quad b = 1,$$
$$a > 2\frac{(\epsilon + \delta)}{\alpha}, \quad \frac{(1-\alpha)(\varepsilon + \delta)}{r(\alpha + \delta)} > 1.$$

References

1. A. Antoci, S. Bartolini. Negative externalities as the engine of growth in an evolutionary context, working paper No. 83.99, Fondazione ENI Enrico Mattei, 1999.
2. A. Antoci, S. Bartolini. Negative externalities, defensive expenditures and labor supply in an evolutionary game, *Environment and Development Economics,* **9**, pp. 591–612, 2004.
3. S. Bartolini, L. Bonatti. Environmental and social degradation as the engine of economic growth, *Ecological Economics,* **43**, pp. 1–16, 2002.
4. S. Bartolini, L. Bonatti. Undesirable growth in a model with capital accumulation and environmental assets, *Environment and Development Economics,* **8**, pp. 11–30, 2003.
5. R. Hueting. *New scarcity and economic growth. More welfare through less production,* North Holland, 1980.
6. C. Leipert. National income and economic growth: the conceptual side of defensive expenditures, *Journal of Economic Issues,* **31**, pp. 843–856, 1989.
7. C. Leipert, U.E. Simonis. Environmental protection expenditures: the German example, *Rivista Internazionale di Scienze Economiche e Commerciali,* **36**, pp. 255–270, 1989.
8. J.R. Vincent. Green accounting: from theory to practice, *Environment and Development Economics,* **5**, pp. 13–24, 2000.
9. J. Bird. The transferability and depletability of externalities, *Journal of Environmental Economics and Management,* **14**, pp. 54–57, 1987.

10. J.F. Shogren, T.D. Crocker. Cooperative and non-cooperative protection against transferable and filterable externalities, *Environmental and Resource Economics,* **1**, pp. 195–213, 1991.

11. F. Hirsch. *Social limits to growth*, Harvard University Press, 1976.

12. A. Antoci. Negative externalities and growth of the activity level, working paper No. 9308 of the MURST Research Project *Nonlinear Dynamics and Applications to Economic and Social Sciences,* University of Florence, 1996.

13. J. Benhabib, R.E.A. Farmer. Indeterminacy and sunspots in macroeconomics, *Handbook of Macroeconomics*, North Holland, 1999.

14. A. Antoci, P.L. Sacco, P. Vanin. On the Possible Conflict Between Economic Growth and Social Development, in: *Economics and Social Interaction: Accounting for Interpersonal Relations,* B. Gui, R. Sugden (Eds.), Cambridge University Press, 2005 (forthcoming).

15. R.M. Auty. The political economy of the resource-driven growth, *European Economic Review,* **45**, pp. 839–846, 2001.

16. K. Matsuyama. Agricultural productivity, comparative advantage, and economic growth, *Journal of Economic Theory,* **58**, pp. 317–334, 1992.

17. F. Rodriguez, J.D. Sachs. Why do resource-abundant economies grow more slowly?, *Journal of Economic Growth,* **4**, pp. 277–303, 1999.

18. J.D. Sachs, A.M. Warner. The big push, natural resource booms and growth, *Journal of Development Economics,* **59**, pp. 43–76, 1999.

19. J.D. Sachs, A.M. Warner. The curse of natural resources, *European Economic Review,* **45**, pp. 827–838, 2001.

20. A. Antoci. Environmental resources depletion and interplay between negative and positive externalities in a growth model, working paper No. 9.05, Fondazione ENI Enrico Mattei, 2005.

21. R.J. Barro, X. Sala-i-Martin. *Economic Growth,* McGraw-Hill Inc., 1995.

22. A. Antoci, L. Brugnano, M. Galeotti. Oscillations, sustainability and indeterminacy in a growth model with environmental assets, *Nonlinear Dynamics: Real World Applications,* **5**, pp. 571–587, 2004.