Topological Degree, Jacobian Determinants and Relaxation

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Abstract

In English: a characterization of the total variation $TV(u,\Omega)$ of the Jacobian determinant $\det Du$ is obtained for some classes of functions $u:\Omega\to\mathbb{R}^n$ outside the traditional regularity space $W^{1,n}(\Omega;\mathbb{R}^n)$. In particular, explicit formulas are deduced for functions that are locally Lipschitz continuous away from a given one point singularity $x_0\in\Omega$. Relations between $TV(u,\Omega)$ and the distributional determinant $\mathrm{Det}Du$ are established, and an integral representation is obtained for the relaxed energy of certain polyconvex functionals at maps $u\in W^{1,p}(\Omega;\mathbb{R}^n)\cap W^{1,\infty}(\Omega\setminus\{x_0\};\mathbb{R}^n)$.

In Italian: si ottiene una caratterizzazione della variazione totale $TV(u,\Omega)$ del determinante Jacobiano $\det Du$ per alcune classi di applicazioni $u:\Omega\to\mathbb{R}^n$ che non fanno parte della tradizionale classe di Sobolev $W^{1,n}(\Omega;\mathbb{R}^n)$. In particolare, si forniscono formule esplicite per applicazioni localmente Lipschitziane al di fuori di un punto isolato $x_0\in\Omega$. Si stabiliscono anche alcune relazioni fra $TV(u,\Omega)$ e il determinante distribuzionale $\mathrm{Det}Du$. Inoltre si fornisce una rappresentazione integrale per l'energia rilassata di certi integrali policonvessi relativi ad applicazioni $u\in W^{1,p}(\Omega;\mathbb{R}^n)\cap W^{1,\infty}(\Omega\setminus\{x_0\};\mathbb{R}^n)$.

Contents

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1 Introduction

In this paper we address the study of the *Jacobian determinant* $\det Du$ of fields $u : \Omega \to \mathbb{R}^n$ outside the traditional regularity space $W^{1,n}(\Omega;\mathbb{R}^n)$. This issue surfaces regularly in a wide range of contemporary research in solid physics and in materials sciences. Indeed, applications of high-temperature superconducting magnetic materials have had a tremendous impact in the development of a whole mathematical theory based on Ginzburg-Landau model, and where vorticity plays a very important role (see [7], [16]). As pointed out by Jerrard and Soner in [43], the formation of vortices is accompanied by highly localized defectiveness at points or along rays, and the ability to extend and interpret the mechanism of change of volume dictated by the Jacobian to the range $p \in (n-1,n)$ may shed some light into this theory. Also, the formation of (radially symmetric) holes in rubber-like (nonlinear) elastic materials is studied in the theory of cavitation, and its advance is heavily hinged on the characterization of the *distributional Jacobian determinant* (see (1); see also (31) below) for certain ranges of $p < n$. This problem has attracted the attention of several mathematical researchers for the past twenty years, and although some progress has been made, pioneered by Ball [4], [5], and followed by James and Spector [42], Müller and Spector [56], Sivaloganathan [60], Marcellini [49] (the latter using an alternative, and closer to the point of view of the present paper, approach), and many others, we believe that we have only scratched the surface of a very rich field in the Calculus of Variations virtually unexplored until recently. In addition, the theoretical challenges presented by the understanding of the behavior of weak notions of the Jacobian determinant are relevant to the study of harmonic mappings with singularities (see [10]), and in the study of density results of smooth functions in $H^1(B(0,1);S^2)$, where $B(0,1) \subset \mathbb{R}^3$. Bethuel [6] showed that this density result holds for $u \in H^1(B(0,1);S^2)$ if $\det Du = 0$.

To fix the notation, we consider a vector-valued map $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, defined on an open set Ω of $\mathbb{R}^n$, for some $n \geq 2$. We denote by $Du = Du(x)$ the *gradient* of u at $x \equiv (x_1, x_2, \ldots, x_n) \in \Omega$, i.e., the $n \times n$ matrix (*Jacobian matrix*) of the partial derivatives of $u \equiv (u^1, u^2, \ldots, u^n)$ and by

$$\det Du(x) := \frac{\partial(u^1, u^2, \ldots, u^n)}{\partial(x_1, x_2, \ldots, x_n)}$$

its *determinant* (*Jacobian determinant*).

If $u \in W^{1,n}(\Omega;\mathbb{R}^n)$, since $|\det Du(x)| \leq n^{-n/2}|Du(x)|^n$, then the Jacobian determinant $\det Du$ is a function of class $L^1(\Omega;\mathbb{R}^n)$. In this case the set function

$$E \subset \Omega \longrightarrow m(E) := \int_E \det Du(x)\ dx$$

is a measure in Ω, whose *total variation* $|m|$ in Ω is given by

$$|m|(\Omega) := \int_\Omega |\det Du(x)|\ dx\,.$$

When $u \notin W^{1,n}(\Omega;\mathbb{R}^n)$ it may still be possible to consider the *distributional Jacobian determinant*

$$\operatorname{Det} Du := \sum_{i=1}^{n} (-1)^{i+1} \frac{\partial}{\partial x_i} \left(u^1 \frac{\partial (u^2, \ldots, u^n)}{\partial (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)} \right) \tag{1}$$

(or any other permutation in the set $\{u^1, u^2, \ldots, u^n\}$, with the sign of the permutation), which coincides almost everywhere with the pointwise Jacobian determinant $\det Du$ if $u \in W^{1,n}(\Omega;\mathbb{R}^n)$, but which may be different otherwise. The definition of the distributional Jacobian determinant $\operatorname{Det} Du$ is based on integration by parts of the formal expression in (1), after multiplication by a test function. To render the definition mathematically precise it is then necessary to make some assumptions on u. We may assume that u^1 (or, for symmetry reasons, also the full vector u) is bounded and the gradient Du is of class L^{n-1} (or, more generally, the $(n-1) \times n$ matrix $(Du^2, \ldots, Du^n)$ is of class L^{n-1}), i.e., $u \in L^\infty(\Omega;\mathbb{R}^n) \cap W^{1,n-1}(\Omega;\mathbb{R}^n)$. Another possibility is to require that $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ for some $p > n^2/(n+1)$ (the strict inequality is useful for compactness reasons); in fact, in this case by the Sobolev Imbedding Theorem $u \in L^{n^2}(\Omega;\mathbb{R}^n)$ and the products in (1) are well defined in L^1 because $1/n^2 + (n-1)\cdot(n+1)/n^2 = 1$. Local summability assumptions are also allowed. In this paper we assume that $u \in L^\infty_{\text{loc}}(\Omega;\mathbb{R}^n) \cap W^{1,p}(\Omega;\mathbb{R}^n)$ for some $p > n-1$. An extensive study of $\operatorname{Det} Du$ defined in (1) was carried out by Morrey [51], (see also Reshetnyak [59]). Later Ball pointed out in [4] some relevant applications of the Jacobian determinant to *nonlinear elasticity*, and sharp weak continuity properties of the Jacobian has been investigated in a series of papers by Müller, starting with [52]. More detailed description of the state of the art in this subject may be found in Section 4.

Several attempts have been made to establish relations between the distribution $\operatorname{Det} Du$ and the "*total variation*" of the Jacobian determinant $\det Du(x)$. One possible definition is based on the following limit formula. Given $u \in L^\infty_{\text{loc}}(\Omega;\mathbb{R}^n) \cap W^{1,p}(\Omega;\mathbb{R}^n)$ for some $p > n-1$, the *total variation* $TV(u,\Omega)$ *of the Jacobian determinant* is defined by

$$TV(u,\Omega) = \inf \Big\{ \liminf_{h \to +\infty} \int_\Omega |\det Du_h(x)| \, dx \; : \tag{2}$$

$$u_h \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega;\mathbb{R}^n), \; u_h \in W^{1,n}(\Omega;\mathbb{R}^n) \Big\}.$$

Although, a priori, definition (2) may depend on p, as it turns out that is not the case, and, moreover, surprisingly it can be shown that, for certain classes of functions u, *weak* convergence in $W^{1,p}(\Omega;\mathbb{R}^n)$ may be equivalently replaced by *strong* convergence (see (22)). Similar definitions may be proposed under other summability assumptions on u.

There is an extensive literature addressing the "relaxed" definition of the Jacobian determinant via (2). We refer, in particular, to the work by Marcellini [48], Giaquinta, Modica and Souček [37], [38], Fonseca and Marcellini [29], Bouchitté, Fonseca and Malý [9]. Marcellini [48] and Fonseca and Marcellini [29] showed that the total variation of the Jacobian determinant may have a nonzero singular part, and Bouchitté, Fonseca and Malý [9] proved that this singular part is a measure. Also, Giaquinta, Modica and Souček [37], [38], showed that the lower limit in (2) may be different from the total variation of the measure $\operatorname{Det} Du$. On the same vein, Malý [44] and Giaquinta, Modica and Souček [37] (see also Jerrard and Soner [43]) proved that, for some maps $u \in L^\infty(\Omega;\mathbb{R}^n) \cap W^{1,p}(\Omega;\mathbb{R}^n)$ with $p \in (n-1,n)$, it may happen that the distribution $\operatorname{Det} Du$ is *identically equal* to zero while the *total variation* of the Jacobian determinant is *different* from zero. When $\operatorname{Det} Du$ is a *measure*, it turns out that, in general, the total variation of the Jacobian determinant $\det Du(x)$ *is not* the total variation of the measure $\operatorname{Det} Du$. Some precise (from a quantitative point of view) examples illustrating this phenomenon are proposed in Section 10. More comments and references are given in Section 4.

In Section 9 we compute the total variation of a class of singular maps $u : \Omega \to S^{n-1} \subset \mathbb{R}^n$, playing a central role in the analysis of Jerrard and Soner [43], defined by

$$u(x) := \frac{w(x) - w(0)}{|w(x) - w(0)|}, \tag{3}$$

where $w : \Omega \to \mathbb{R}^n$ is a locally Lipschitz-continuous map, classically differentiable at $x = 0$ and such that $\det Dw\,(0) \neq 0$. We find that the total variation of the Jacobian determinant of u in Ω (an open set of $\mathbb{R}^n$ containing the origin) is equal to the measure ω_n of the unit ball.

The aim of this paper is to give an explicit characterization of the total variation $TV\,(u, \Omega)$ of the Jacobian determinant $\det Du\,(x)$, defined in (2), for some classes of functions $u \in L^{\infty}_{\text{loc}}\,(\Omega; \mathbb{R}^n) \cap W^{1,p}\,(\Omega; \mathbb{R}^n)$ with $p > n - 1$, in particular for those u locally Lipschitz-continuous away from a given point $x_0 \in \Omega$ (and thus with the Jacobian determinant $\det Du$ possibly singular only at x_0).

Statements of the main results are given in the following Section 2. In Section 3 we relate the notion of total variation of the Jacobian determinant to the *topological degree*. A relevant *geometrical interpretation* is given by Corollary 15 of Section 3. In particular, denoting by B_1 the unit ball of $\mathbb{R}^n$ and by $S^{n-1} := \partial B_1$ its boundary, we prove that, if $v : S^{n-1} \to S^{n-1}$ is a map of class C^1 onto S^{n-1}, locally invertible with local inverse of class C^1 at any point of S^{n-1}, and if $u : B_1 \backslash \{0\} \to S^{n-1}$ is defined by $u\,(x) := v\left(\frac{x}{|x|}\right)$, then the total variation $TV\,(u, B_1)$ of the Jacobian determinant of u may be expressed in terms of the *topological degree* of the maps v and $\widetilde{v}$, where $\widetilde{v} : B_1 \to B_1$ is any *Lipschitz-continuous extension* of v to the unit ball B_1. Precisely,

$$TV\,(u, B_1) = \omega_n\,|\deg v| = \omega_n\,|\deg \widetilde{v}|\ . \tag{4}$$

Note that formula (4) *does not hold*, in general, if the map $v : S^{n-1} \to \mathbb{R}^n$ takes values on a set $v\left(S^{n-1}\right)$ not diffeomorphic to S^{n-1} (see Theorem 4 and the examples of Section 10). A generalization of this resuly holds if we assume that u is in $W^{1,p}\left(\Omega; \mathbb{R}^N\right)$ for some $p \in (1, N)$ and is locally Lypschitz outside a finite number of points $a_i \in \Omega$, $i = 1, \ldots, k$, provided that u satisfies in a neighborhood of each a_i the hypotheses of Theorem 1 for suitable functions v_i. In this case the total variation of the Jacobian of u is given by

$$TV(u, \Omega) = \int_{\Omega} |\det Du(x)|\,dx + \sum_{i=1}^{k} \pi |\deg v_i|\,.$$

For possible extensions of this formula to more general spaces we refer to [12], [13].

Section 4 is dedicated to explaining how the study of the total variation $TV\,(u, \Omega)$ fits squarely within the framework of *relaxation problems* with *nonstandard growth conditions*. In Section 5 we present a thorough study of the 2−d case, which plays a very special role. In fact, in two dimensions we are able to perform a deeper analysis and to find more general assumptions which allow us to characterize fully the total variation $TV\,(u, \Omega)$. In particular, it is possible to identify $TV\,(u, \Omega)$ of maps $u : B_1 \subset \mathbb{R}^2 \to \Gamma$, with values on a set Γ which is the *boundary* of a simply connected domain $D \subset \mathbb{R}^2$, starshaped with respect to a point ξ in the *interior* of D (for example, when $\Gamma = S^1$ is the boundary of the unit ball B_1). We emphasize Lemma 22, which we call "*the umbrella lemma*", and which plays a crucial role in our argument, as explained in Section 5. For the sake of completeness, we include the statement, without proof, of some 2-dimensional results that have been presented in [24].

In Section 8 we move on to the general $n-$dimensional framework, and in Section 8 we apply the results thus obtained to the study of relaxation of *polyconvex functionals*. Indeed, we provide an explicit representation formula for the related energy associated to the *polyconvex integral functional*

$$F\,(u, \Omega) := \int_{\Omega} g\,(M\,(Du))\;dx\,,$$

where $g : \mathbb{R}^N \to [0, +\infty)$ is a *convex function*, $M\,(Du)$ is the map with values in $\mathbb{R}^N$, $N = \sum_{j=1}^{n} \binom{n}{j}^2$, defined by

$$M\,(Du) := \left(Du, \text{adj}_2 Du, \ldots, \text{adj}_{n-1} Du, \det Du\right)\ ,$$

and where $\mathrm{adj}_j Du$ denotes, for every $j = 2, \ldots, n-1$, the matrix of all *minors* $j \times j$ of Du.

Finally, in Section 9 we study in detail $TV(u,\Omega)$ when u is as in (3). Additional $2-$dimensional and $3-$dimensional examples are proposed in Section 10. The special, but representative, case analyzed in Section 6 concerns maps $u : B_1 \subset \mathbb{R}^2 \to \gamma = \gamma^+ \cup \gamma^-$, were γ is the *"eight" curve*, i.e., the union of the two tangent circles $\gamma^{\pm} = \left\{ (x_1, x_2) \in \mathbb{R}^2 : (x_1 \mp 1)^2 + (x_2)^2 = 1 \right\}$ in $\mathbb{R}^2$. In particular, we show (see Theorem 4 and Section 10) that in general formula (4), which relates the total variation $TV(u,\Omega)$ of the Jacobian determinant with the topological degree, does not hold if the map $u : B_1 \subset \mathbb{R}^2 \to \gamma$ takes values on the *"eight" curve* γ.

2 Statement of the main results

In this section we state several representation formulas for the total variation $TV(u,\Omega)$ of the Jacobian determinant, defined in (2). We consider first the $2-$d case in detail, where the assumptions needed are more general than in the case $n \geq 2$, and in the second part of this section we describe the general $n-$dimensional case.

In order to fix the notation, here we consider $x_0 = 0$ and $\Omega \subset \mathbb{R}^2$ is an open set containing the origin. With an obvious abuse of notation, we write $u(x) = u(x_1, x_2) = u(\varrho, \vartheta)$, where (ρ, ϑ), $\varrho \geq 0$, $0 \leq \vartheta \leq 2\pi$, are the *polar coordinates* in $\mathbb{R}^2$. We also denote by $D_\tau u$ the *tangential derivative* of u (in the $\tau = (-\sin\vartheta, \cos\vartheta)$ direction), which is related to the (vector-valued) derivative u_ϑ by the formula

$$u_\vartheta =: \frac{\partial u(\varrho\cos\vartheta, \varrho\sin\vartheta)}{\partial\vartheta} = \varrho\left[-u_{x_1}\sin\vartheta + u_{x_2}\cos\vartheta\right] = \varrho D_\tau u\,.$$

We denote by $v : [0, 2\pi] \to \Gamma \subset \mathbb{R}^2$ a Lipschitz-continuous map, with $v(0) = v(2\pi)$, with components $v(\vartheta) = \left(v^1(\vartheta), v^2(\vartheta)\right)$, and with values on a curve $\Gamma \supseteq v([0, 2\pi])$. We assume that Γ can be parametrized in the following way

$$\Gamma = \{\xi + r(\vartheta)(\cos\vartheta, \sin\vartheta) : \vartheta \in [0, 2\pi]\}\,, \tag{5}$$

where $r(\vartheta)$ is a piecewise C^1-function such that $r(0) = r(2\pi)$, and $r(\vartheta) \geq r_0$ for every $\vartheta \in [0, 2\pi]$ and for some $r_0 > 0$. Condition (5) reduces to saying that Γ is the boundary of a domain

$$D := \{\xi + \varrho(\cos\vartheta, \sin\vartheta) : \vartheta \in [0, 2\pi],\ 0 \leq \varrho \leq r(\vartheta)\}\,, \tag{6}$$

starshaped with respect to a point ξ in the *interior* of D. The following theorem was proved in [24].

Theorem 1 (General result in 2−d) *Let u be a function of class $W^{1,p}\left(\Omega;\mathbb{R}^2\right) \cap W^{1,\infty}_{\mathrm{loc}}\left(\Omega\setminus\{0\};\mathbb{R}^2\right)$ for some $p \in (1,2)$. Let $v : [0, 2\pi] \to \Gamma$, $v(\vartheta) = \left(v^1(\vartheta), v^2(\vartheta)\right)$, $\vartheta \in [0, 2\pi]$, be a Lipschitz-continuous map, with $v(0) = v(2\pi)$ and Γ as in (5), and such that*

$$\lim_{\varrho\to 0} \|u(\varrho,\cdot) - v(\cdot)\|_{L^\infty((0,2\pi);\mathbb{R}^2)} = 0\,. \tag{7}$$

If the tangential derivative $D_\tau u$ of u satisfies the bound

$$\sup_{\varrho>0} \frac{1}{\varrho^{2-p}} \int_{B_\varrho} |D_\tau u|^p \, dx = \sup_{\varrho>0} \frac{1}{\varrho^{2-p}} \int_0^\varrho r^{1-p}\, dr \int_0^{2\pi} |u_\vartheta(r,\vartheta)|^p \, d\vartheta \leq M_0$$

for a positive constant M_0, then the total variation of u is given by

$$TV(u,\Omega) = \int_\Omega |\det Du(x)| \, dx + \frac{1}{2}\left|\int_0^{2\pi} \left\{v^1(\vartheta)\, v^2_\vartheta(\vartheta) - v^2(\vartheta)\, v^1_\vartheta(\vartheta)\right\} d\vartheta\right|\,.$$

Note that, by (7), there exists $r>0$ such that $B_r \subset \Omega$ and $u \in L^\infty\left(B_r;\mathbb{R}^2\right)$. Therefore in the statement of Theorem 1 we have in fact $u \in L^\infty_{\mathrm{loc}}\left(\Omega;\mathbb{R}^2\right)\cap W^{1,p}\left(\Omega;\mathbb{R}^2\right)\cap W^{1,\infty}_{\mathrm{loc}}\left(\Omega\backslash\{0\};\mathbb{R}^2\right)$ for some $p\in(1,2)$. Moreover, the assumption of Lipschitz-continuity of v may be replaced by the weaker assumption that $v\in W^{1,p}\left((0,2\pi);\mathbb{R}^2\right)$.

Consider the particular case in which the map $u=u(\varrho,\vartheta)$ does not depend on ϱ, that is $u=u(\vartheta)$. Then, as a function of ϑ, $u=u(\vartheta):[0,2\pi]\to\mathbb{R}^2$ is a Lipschitz-continuous map and $u(0)=u(2\pi)$. Looked upon as a function of two variables, i.e., $u:\Omega=B_1\to\mathbb{R}^2$ constant with respect to $\varrho\in(0,1]$, it turns out that $u\in L^\infty\left(\Omega;\mathbb{R}^2\right)\cap W^{1,p}\left(\Omega;\mathbb{R}^2\right)\cap W^{1,\infty}_{\mathrm{loc}}\left(\Omega\backslash\{0\};\mathbb{R}^2\right)$ for every $p\in[1,2)$, but $u\notin W^{1,2}\left(\Omega;\mathbb{R}^2\right)$ unless $u(\vartheta)$ is constant.

From the previous result, with $u=v$, we immediately obtain the following consequence (see [24]).

Corollary 2 (Radially independent maps in $2-$d) *Let Γ be as in (5), and let $u=v:[0,2\pi]\to\Gamma$ be a Lipschitz-continuous map such that $v(0)=v(2\pi)$. Then $\det Du(x)=0$ for almost every $x\in\mathbb{R}^2$ and the total variation of the Jacobian determinant is given by*

$$TV(u,\Omega)=\frac{1}{2}\left|\int_0^{2\pi}\left\{v^1(\vartheta)\,v^2_\vartheta(\vartheta)-v^2(\vartheta)\,v^1_\vartheta(\vartheta)\right\}\,d\vartheta\right|. \tag{8}$$

We observe that formula (8) has a relevant geometrical meaning because the right hand side represents the *"winding number"* of the curve $v=\left(v^1,v^2\right)$. See Section 3 for a further discussion on the geometric interpretation of (8).

With the aim to compare the previous result with the $n-$dimensional results given below, in Section 5 we present the following equivalent formulation of Corollary 2.

Corollary 3 (Analytic interpretation in $2-$d) *Let Γ be as in (5), and let $v:[0,2\pi]\to\Gamma$ be a Lipschitz-continuous map such that $v(0)=v(2\pi)$. Then the total variation $TV(u,\Omega)$ is given by*

$$TV(u,\Omega)=\left|\int_{B_1}\det D\widetilde{u}(x)\,dx\right|, \tag{9}$$

where $\widetilde{u}:B_1\to\mathbb{R}^2$ is any Lipschitz-continuous extension of v to B_1.

Note the surprising fact that the integral in the right hand side of (9) (and in (8) as well) appears with the absolute value *outside* the integral sign, and not *inside*!

Another relevant $2-$dimensional result is related to the *"eight" curve* in $\mathbb{R}^2$, i.e., to the union γ of the two circles γ^+,γ^-, of radius 1 with centers at $(1,0)$ and at $(-1,0)$ respectively. Some explicit examples related to the "eight" curve are given in Section 10. Below we present two *estimates*, proven in [24], an upper bound and a lower bound, which will allow us to study these examples.

Theorem 4 (The "eight" curve) *Let $\gamma=\gamma^+\cup\gamma^-\subset\mathbb{R}^2$ be the union of the two circles of radius 1 with centers at $(1,0)$ and at $(-1,0)$. Let $v:[0,2\pi]\to\gamma$ be a Lipschitz-continuous curve, with parametric representation $v(\vartheta)=\left(v^1(\vartheta),v^2(\vartheta)\right)$, $\vartheta\in[0,2\pi]$, such that $v(0)=v(2\pi)$. Let $(I_j)_{j\in\mathbb{N}}$ be a sequence of disjoint open intervals (possibly empty) of $[0,2\pi]$ such that the image $v(I_j)$ is contained either in γ^+ or in γ^-, and $v(\vartheta)=(0,0)$ when $\vartheta\notin\cup_{j\in\mathbb{N}}I_j$. Then, with $u(x):=v(x/|x|)$, the following upper estimate holds*

$$TV(u,B_1)\le\frac{1}{2}\sum_{j\in\mathbb{N}}\left|\int_{I_j}\left\{v^1(\vartheta)\,v^2_\vartheta(\vartheta)-v^2(\vartheta)\,v^1_\vartheta(\vartheta)\right\}\,d\vartheta\right|.$$

For the lower estimate, we denote by I_j^+*, with the* $+$ *sign, any previous interval* I_j *such that* $v(I_j) \subset \gamma^+$*, and by* I_k^- *any previous interval* I_k *such that* $v(I_k) \subset \gamma^-$*. Then we also have*

$$TV(u, B_1) \geq \frac{1}{2}\left\{\left|\sum_{j\in\mathbb{N}} \int_{I_j^+} \{v^1 v_\vartheta^2 - v^2 v_\vartheta^1\}\, d\vartheta\right| + \left|\sum_{k\in\mathbb{N}} \int_{I_k^-} \{v^1 v_\vartheta^2 - v^2 v_\vartheta^1\}\, d\vartheta\right|\right\}.$$

Moving on to the $n-$dimensional case, we first establish in Theorem 5 a general inequality between the total variation of the distributional determinant $\operatorname{Det} Du$ (see (1)), that we denote by $|\operatorname{Det} Du|(\Omega)$, and the total variation $TV(u,\Omega)$ if the Jacobian, defined in (2). Note that in the first half of the statement of the next theorem we *do not* assume that $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega\setminus\{0\};\mathbb{R}^n)$, while in the second half we require that $u \in W^{1,n}_{\mathrm{loc}}(\Omega\setminus\{0\};\mathbb{R}^n)$.

Theorem 5 (Comparison between $|\operatorname{Det} Du|(\Omega)$ and $TV(u,\Omega)$) *Let* $p > n-1$ *and assume that* $u \in L^\infty(\Omega;\mathbb{R}^n) \cap W^{1,p}(\Omega;\mathbb{R}^n)$*. If* $TV(u,\Omega) < +\infty$*, then* $TV(u,\cdot)$ *and* $\operatorname{Det} Du$ *are finite Radon measures,* $\det Du \in L^1(\Omega)$*, and*

$$TV(u, A) = \int_A |\det Du(x)|\, dx + \lambda_s(A), \tag{10}$$

$$\operatorname{Det} Du(A) = \int_A \det Du(x)\, dx + \mu_s(A), \tag{11}$$

for every open set $A \subset \Omega$*, where* λ_s*,* μ_s*, are finite Radon measures, singular with respect to the Lebesgue measure* $\mathcal{L}^n$*, and* $|\mu_s| \leq \lambda_s$*, i.e., for every open set* $A \subset \Omega$*,*

$$|\operatorname{Det} Du|(A) \leq TV(u, A). \tag{12}$$

If, in addition, $u \in W^{1,n}_{\mathrm{loc}}(\Omega\setminus\{0\};\mathbb{R}^n)$*, then* $\lambda_s = \lambda\delta_0$*,* $\mu_s = \mu\delta_0$*, for some constants* $\lambda \geq 0$*,* $\mu \in \mathbb{R}$*, with* $|\mu| \leq \lambda$*, where* δ_0 *is the Dirac mass at the origin.*

Examples given in Section 10 show that in general the equality between $|\operatorname{Det} Du|(A)$ and $TV(u,A)$ should not be expected. In particular, this equality fails for maps valued on the *"eight" curve*. The proof of Theorem 5 is presented at the end of Section 4. We note that one of the main contributions of this paper is the identification of the *defect constants* $\lambda \geq 0$, $\mu \in \mathbb{R}$.

Let us denote by B_r the ball in $\mathbb{R}^n$, $n \geq 2$, with center in 0 and radius $r > 0$. In particular, B_1 is the ball of radius $r = 1$ and $\partial B_1 = S^{n-1}$ is its boundary.

We call the attention of the reader to the fact that, in dealing with the general $n-$dimensional case, we denote by v a map from S^{n-1} into $\mathbb{R}^n$, while in $2-$d $v = v(\vartheta)$ does not denote a map from S^1 into $\mathbb{R}^2$, but instead a periodic function from $[0, 2\pi]$ into $\mathbb{R}^2$. Therefore, if $\overline{v}$ is the corresponding map from S^1 into $\mathbb{R}^2$, then we have $v(\vartheta) = \overline{v}(\cos\vartheta, \sin\vartheta)$.

Let $\omega_0 \in S^{n-1}$ be fixed. For every $j \in \{1, 2, \ldots, n-1\}$ let $\tau_j : S^{n-1}\setminus\{\omega_0\} \to \partial B_1$ by a vector field of class C^1 such that, for every $x \in S^{n-1}\setminus\{\omega_0\}$, the set of vectors $\{\tau_1(\omega), \tau_2(\omega), \ldots, \tau_{n-1}(\omega)\}$ is an *orthonormal basis* for the tangent plane to the surface ∂B_1 at the point ω.

The following theorem provides a general *representation formula* for the total variation of the distributional determinant $|\operatorname{Det} Du|(\Omega)$. Note that, under the assumption $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega\setminus\{0\};\mathbb{R}^n)$, by formula (15) we give a representation of the total variation of the singular measure μ_s in (10).

Theorem 6 (Total variation of the distributional determinant) *Let* $n \geq 2$ *and let* Ω *be an open set containing the origin. Let* $u \in W^{1,p}(\Omega;\mathbb{R}^n) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega\setminus\{0\};\mathbb{R}^n)$ *for some* $p \in (n-1, n)$*.*

Let $v : \partial B_1 = S^{n-1} \to \mathbb{R}^n$, $v \in W^{1,\infty}\left(S^{n-1};\mathbb{R}^n\right)$, $v = \left(v^1, v^2, \ldots, v^n\right)$, be a Lipschitz-continuous map such that

$$\lim_{\varrho \to 0^+} \max\left\{\left|u\left(\varrho\omega\right) - v\left(\omega\right)\right| : \ \omega \in S^{n-1}\right\} = 0\,. \tag{13}$$

Let us assume that

$$\sup_{\varrho > 0} \frac{1}{\varrho^{n-p}} \int_{B_\varrho} \left|D_\tau u\right|^p \, dx \, \leq M_0 \tag{14}$$

for a positive constant M_0. If $\det Du \in L^1(\Omega)$ then $\operatorname{Det} Du$ is a Radon measure and its the total variation $\left|\operatorname{Det} Du\right|$ is given by

$$\begin{aligned} \left|\operatorname{Det} Du\right|(\Omega) = & \int_\Omega \left|\det Du\left(x\right)\right| \, dx \\ & + \frac{1}{n}\left|\int_{\partial B_1} \sum_{i=1}^n \left(-1\right)^{i+1} v^i\left(\omega\right) \frac{\partial\left(v^1, \ldots, v^{i-1}, v^{i+1}, \ldots, v^n\right)}{\partial\left(\tau_1, \tau_2, \ldots, \tau_{n-1}\right)}\left(\omega\right) \, dH^{n-1}\right|. \end{aligned} \tag{15}$$

Moreover, if we denote by $\widetilde{u} : B_1 \to \mathbb{R}^n$ any Lipschitz-continuous extension of v to B_1, then

$$\left|\operatorname{Det} Du\right|(\Omega) = \int_\Omega \left|\det Du\left(x\right)\right| \, dx \, + \left|\int_{B_1} \det D\widetilde{u}\left(x\right) \, dx\right|. \tag{16}$$

By assumption (13) there exists $r > 0$ such that $u \in L^\infty\left(B_r;\mathbb{R}^2\right)$. Thus, in the statement of Theorem 6 (and in Theorem 9 below), we actually have that u is a function of class $L^\infty_{\text{loc}}\left(\Omega;\mathbb{R}^n\right) \cap W^{1,p}\left(\Omega;\mathbb{R}^n\right) \cap W^{1,\infty}_{\text{loc}}\left(\Omega \backslash \{0\};\mathbb{R}^n\right)$ for some $p \in (n-1, n)$.

Remark 7 *A simple calculation shows that in $2-d$ the last term on the right hand side of (15) reduces to*

$$\frac{1}{2}\left|\int_0^{2\pi} \left\{v^1\left(\vartheta\right) v^2_\vartheta\left(\vartheta\right) - v^2\left(\vartheta\right) v^1_\vartheta\left(\vartheta\right)\right\} \, d\vartheta\right|,$$

where $v : [0, 2\pi] \to \mathbb{R}^2$ is the asymptotic limit map in (7). Indeed, denoting by $\overline{v} : S^1 \to \mathbb{R}^2$ the map related to v through the condition $v\left(\vartheta\right) := \overline{v}\left(\cos\vartheta, \sin\vartheta\right)$, we have

$$\frac{dv^i}{d\vartheta} = \frac{\partial \overline{v}^i}{\partial x_1}\left(-\sin\vartheta\right) + \frac{\partial \overline{v}^i}{\partial x_2}\cos\vartheta\,, \qquad i = 1, 2,$$

and, since the unit tangent vector $\tau : [0, 2\pi] \to \partial B_1$ can be represented by $\tau\left(\theta\right) = \left(-\sin\vartheta, \cos\vartheta\right)$, we obtain

$$\frac{dv^i}{d\vartheta} = \frac{\partial \overline{v}^i}{\partial \tau}\,, \qquad i = 1, 2.$$

With the notation $\omega = \left(\frac{x_1}{|x|}, \frac{x_2}{|x|}\right) = \left(\cos\vartheta, \sin\vartheta\right) \in \partial B_1 = S^1$, we finally have

$$\begin{aligned} \int_0^{2\pi} \left\{v^1\left(\vartheta\right) v^2_\vartheta\left(\vartheta\right) - v^2\left(\vartheta\right) v^1_\vartheta\left(\vartheta\right)\right\} \, d\vartheta & = \int_0^{2\pi} \left\{\overline{v}^1 \frac{\partial \overline{v}^2}{\partial \tau} - \overline{v}^2 \frac{\partial \overline{v}^1}{\partial \tau}\right\} \, d\vartheta \\ & = \int_{\partial B_1} \sum_{i=1}^2 \left(-1\right)^{i+1} \overline{v}^i\left(\omega\right) \frac{d\overline{v}^i}{d\tau}\left(\omega\right) \, dH^1\,. \end{aligned}$$

Therefore (15) in $2-d$ becomes

$$\left|\operatorname{Det} Du\right|(\Omega) = \int_\Omega \left|\det Du\left(x\right)\right| \, dx + \frac{1}{2}\left|\int_0^{2\pi} \left\{v^1 v^2_\vartheta - v^2 v^1_\vartheta\right\} \, d\vartheta\right|,$$

and the conclusion of Theorem 1 now can be restated in the form

$$TV\left(u, \Omega\right) = \left|\operatorname{Det} Du\right|(\Omega)\,.$$

Remark 8 *In the case of the "eight" curve studied by Theorem 4, with* $v:[0,2\pi]\to\gamma=\gamma^{+}\cup\gamma^{-}\subset\mathbb{R}^2$ *and* $u(x)=v(x/|x|)$*, we have*

$$\begin{aligned} TV(u,B_1) &\geq \frac{1}{2}\left\{\left|\sum_{j\in\mathbb{N}}\int_{I_j^{+}}\left\{v^1v^2_\vartheta-v^2v^1_\vartheta\right\}\,d\vartheta\right|+\left|\sum_{k\in\mathbb{N}}\int_{I_k^{-}}\left\{v^1v^2_\vartheta-v^2v^1_\vartheta\right\}\,d\vartheta\right|\right\} \\ &\geq \frac{1}{2}\left|\int_0^{2\pi}\left\{v^1(\vartheta)\,v^2_\vartheta(\vartheta)-v^2(\vartheta)\,v^1_\vartheta(\vartheta)\right\}\,d\vartheta\right| = |\mathrm{Det}\,Du|\,(B_1)\ . \end{aligned} \tag{17}$$

Therefore, as in the general case (see Theorem 5 and (12) in particular), $TV(u,B_1)\geq|\mathrm{Det}\,Du|\,(B_1)$*. Moreover, in view of the inequalities in (17), we can easily find an example such that the strict inequality* $TV(u,B_1)>|\mathrm{Det}\,Du|\,(B_1)$ *holds. See Section 10.*

Next we state the main result for the $n-$d case, analogous to Theorem 1. The proof of the theorem may be found in Section 7.

Theorem 9 (General result in $n-$d) *Let* $n\geq2$ *and let* Ω *be an open set containing the origin. Let* $u\in W^{1,p}(\Omega;\mathbb{R}^n)\cap W^{1,\infty}_{\mathrm{loc}}(\Omega\setminus\{0\};\mathbb{R}^n)$ *for some* $p\in(n-1,n)$ *and let* $v\in W^{1,\infty}\left(S^{n-1};\mathbb{R}^n\right)$ *satisfying (13) and (14). If* $\det Du\notin L^1(\Omega)$ *then* $TV(u,\Omega)=+\infty$*. If* $\det Du\in L^1(\Omega)$*, then the total variation of the distributional determinant* $|\mathrm{Det}\,Du|\,(\Omega)$ *is given by (15) and* $TV(u,\Omega)\geq|\mathrm{Det}\,Du|\,(\Omega)$*. Moreover, if the quantity*

$$\sum_{i=1}^{n}(-1)^{i+1}\,v^i\frac{\partial\left(v^1,\dots,v^{i-1},v^{i+1},\dots,v^n\right)}{\partial\left(\tau_1,\tau_2,\dots,\tau_{n-1}\right)} \tag{18}$$

has constant sign $H^{n-1}-$*almost everywhere on* ∂B_1*, then*

$$TV(u,\Omega)=|\mathrm{Det}\,Du|\,(\Omega)\ . \tag{19}$$

In Section 9 we apply Theorem 9 to calculate explicitly the total variation of the singular map $u:\Omega\setminus\{0\}\to\mathbb{R}^n$, $u(x)=\frac{w(x)-w(0)}{|w(x)-w(0)|}$, where w is a map differentiable at $x=0$, with $\det Dw(0)\neq0$, to obtain $TV(u,\Omega)=|B_1|=\omega_n$.

Remark 10 *We conjecture that formula (19) holds independently of the sign condition (18) for a certain of subclass of mappings* u *with asymptotic limit* v *at* $x=0$*, in particular if* $v:S^{n-1}\to\mathbb{R}^n$ *takes values on* S^{n-1}*. Theorem 1 above asserts that this conjecture is true in the* $2-$*dimensional case, and when* $v\left(S^1\right)$ *is the set* Γ *in (5), boundary of a starshaped set. With the Example 42 we propose a* $3-d$ *case where the conjecture is also true. However, if* $v\left(S^{n-1}\right)$ *is not diffeomorphic to* S^{n-1}*, as in the case of the "eight" curve considered in Theorem 4 (see also the examples of Section 10), then the representation formula for* $TV(v,B_1)$ *should take into account the topology of* $v\left(S^{n-1}\right)$*.*

As further applications of Theorem 9, now we consider radially independent maps $u:\Omega\to\mathbb{R}^n$, defined through a Lipschitz-continuous map $v:S^{n-1}\to\mathbb{R}^n$ by the position

$$u(x):=v\left(\frac{x}{|x|}\right),\qquad\forall\,x\in B_1\setminus\{0\}\ .$$

Clearly $u\in W^{1,p}(\Omega;\mathbb{R}^n)\cap W^{1,\infty}_{\mathrm{loc}}(\Omega\setminus\{0\};\mathbb{R}^n)$ for every $p\in[1,n)$, but $u\notin W^{1,n}(\Omega;\mathbb{R}^n)$ unless v is a constant function. We obtain immediately from Theorem 9 the following result.

Corollary 11 (Radially independent maps) *Let* $v : \partial B_1 = S^{n-1} \to \mathbb{R}^n$, $v = \left(v^1, v^2, \ldots, v^n\right)$, *be a Lipschitz-continuous map. For every open set* Ω *containing the origin we consider the map* $u : \Omega \to \mathbb{R}^n$, *defined by* $u(x) := v(x/|x|)$ *for* $x \in \Omega \setminus \{0\}$. *For every* $p \in (n-1, n)$ *the total variation of the Jacobian of* u *is given by*

$$TV(u, \Omega) = \frac{1}{n} \left| \int_{\partial B_1} \sum_{i=1}^{n} (-1)^{i+1} v^i(\omega) \frac{\partial \left(v^1, \ldots, v^{i-1}, v^{i+1}, \ldots, v^n\right)}{\partial (\tau_1, \tau_2, \ldots, \tau_{n-1})} (\omega)\, dH^{n-1} \right|, \tag{20}$$

provided the quantity (18) has constant sign $H^{n-1}-$*almost everywhere on* ∂B_1.

The following result is similarly to Corollary 3, valid in the 2−d case.

Corollary 12 (Analytic interpretation in $n-$**d)** *Let* $v : S^{n-1} \to \mathbb{R}^n$ *be a Lipschitz-continuous map, let* Ω *be an open set containing the origin, and let* $u : \Omega \to \mathbb{R}^n$ *be defined by* $u(x) := v(x/|x|)$ *for* $x \in \Omega \setminus \{0\}$. *Denote by* $\widetilde{u} : B_1 \to \mathbb{R}^n$ *the Lipschitz-continuous extension of* v *to* B_1 *given by* $\widetilde{u}(0) = 0$ *and*

$$\widetilde{u}(x) := |x| \cdot v\left(\frac{x}{|x|}\right), \qquad \forall\, x \in B_1 \setminus \{0\}.$$

If the Jacobian $\det D\widetilde{u}(x)$ *has constant sign* $H^{n-1}-$*almost everywhere on* B_1, *then*

$$TV(u, \Omega) = \left| \int_{B_1} \det D\widetilde{u}(x)\, dx \right|. \tag{21}$$

Remark 13 *Let us assume that* $v : S^{n-1} \to S^{n-1}$ *is a map of class* C^1 *onto* S^{n-1}, *locally invertible with* C^1 *local inverse at any point of* S^{n-1}. *If* $\widetilde{u}$ *is defined as before by* $\widetilde{u}(x) = |x| \cdot v(x/|x|)$, *then also* $\widetilde{u} : B_1 \to B_1$ *is a map of class* C^1 *and it is locally invertible with* C^1 *local inverse at any point of* $B_1 \setminus \{0\}$. *Then the assumption of Corollary 12 is satisfied. Indeed,* $\det D\widetilde{u}(x) \neq 0$ *for every* $x \in B_1 \setminus \{0\}$ *and, by continuity,* $\det D\widetilde{u}(x)$ *has constant sign in* $B_1 \setminus \{0\}$. *We also notice that, by (65) of Lemma 35, when* $\eta(t) = t$ *we have*

$$\det D\widetilde{u}(x) = \sum_{i=1}^{n} (-1)^{i+1} v^i\left(\frac{x}{|x|}\right) \frac{\partial \left(v^1, \ldots, v^{i-1}, v^{i+1}, \ldots, v^n\right)}{\partial (\tau_1, \tau_2, \ldots, \tau_{n-1})} \left(\frac{x}{|x|}\right),$$

therefore the sign assumption in Corollary 12 is equivalent to the sign assumption of Theorem 9.

A final remark about the definition (2) of the *total variation* $TV(u, \Omega)$ of the Jacobian determinant $\det Du(x)$: as before, consider $u \in L^\infty(\Omega; \mathbb{R}^n) \cap W^{1,p}(\Omega; \mathbb{R}^n)$ for some $p > n-1$. The definition in (2) of $TV(u, \Omega)$ is based on the convergence of a generic sequence $\{u_h\}_{h \in \mathbb{N}} \subset W^{1,n}(\Omega; \mathbb{R}^n)$ to u in the *weak topology* of $W^{1,p}(\Omega; \mathbb{R}^n)$. Instead, we could consider the *strong norm topology* and give the following definition of $TV^s(u, \Omega)$:

$$\begin{aligned} TV^s(u, \Omega) = \inf \Big\{ \liminf_{h \to +\infty} \int_\Omega |\det Du_h(x)|\, dx \ : \\ u_h \to u \text{ strongly in } W^{1,p}(\Omega; \mathbb{R}^n),\ u_h \in W^{1,n}(\Omega; \mathbb{R}^n) \Big\}. \end{aligned} \tag{22}$$

Clearly we have

$$TV(u, \Omega) \leq TV^s(u, \Omega), \quad \forall\, u \in L^\infty(\Omega; \mathbb{R}^n) \cap W^{1,p}(\Omega; \mathbb{R}^n).$$

However it is interesting, and somewhat surprising, to observe that Theorems 1, 4 and 9 (as well as Corollaries 2 and 11) still hold if we replace $TV\,(u,\Omega)$ by $TV^s\,(u,\Omega)$. In particular, under the assumptions of Theorems 1 and 9 we have indeed

$$TV\,(u,\Omega) = TV^s\,(u,\Omega)\ ,$$

for every open set $\Omega \subset \mathbb{R}^n$, and for every $u \in L^\infty\,(\Omega;\mathbb{R}^n) \cap W^{1,p}\,(\Omega;\mathbb{R}^n)$ with $p > n-1$.

Using the argument of Lemma 32, for every $u \in L^\infty\left(\Omega;\mathbb{R}^2\right) \cap W^{1,p}\left(\Omega;\mathbb{R}^2\right) \cap W^{1,\infty}_{\mathrm{loc}}\left(\Omega\backslash\left\{0\right\};\mathbb{R}^2\right)$ with $p > 1$, it can be shown that admissible sequences for $TV^s\,(u,\Omega)$ may be required to assume prescribed boundary values, precisely

$$\begin{aligned} TV^s\,(u,\Omega) = \inf \Bigg\{ & \liminf_{h\to+\infty} \int_\Omega \left|\det Du_h\,(x)\right|\,dx\ : \\ & u_h \to u \text{ strongly in } W^{1,p}\left(\Omega;\mathbb{R}^2\right),\ u_h \in u + W_0^{1,\infty}\left(\Omega;\mathbb{R}^2\right) \Bigg\}. \end{aligned}$$

3 Geometrical interpretation

In this section we give a geometrical interpretation of the results stated in Section 2, by means of the notion of topological degree of maps between manifolds.

We recall that if $w : \Omega \to \mathbb{R}^n$ is a Lipschitz-continuous map, then the *topological degree of the map w at a point $y \in \mathbb{R}^n$* is

$$\deg\,(w,\Omega,y) := \sum_{x\in w^{-1}(y)\cap A(w)} \operatorname{sign}\left(\det Dw\,(x)\right)\ ,$$

where $A\,(w) = \{x \in \Omega :\ w \text{ is differentiable at } x\}$. The *degree of the map w in the set* Ω, denoted by $\deg w$, is

$$\begin{aligned} \deg w &:= \frac{1}{|w\,(\Omega)|} \int_{w(\Omega)} \sum_{x\in w^{-1}(y)\cap A(w)} \operatorname{sign}\left(\det Dw\,(x)\right)\,dy\,, \qquad (23) \\ &= \frac{1}{|w\,(\Omega)|} \int_{w(\Omega)} \sum_{x\in w^{-1}(y)} \operatorname{sign}\left(\det Dw\,(x)\right)\,dy \end{aligned}$$

(above we used the fact that, since w is a Lipschitz-continuous map, then the measure of the sets $\Omega\backslash A\,(w)$ and of its image $w\,(\Omega\backslash A\,(w))$ are equal to zero). See the books by Giaquinta, Modica and Souček [38] and by Fonseca and Gangbo [26] for more details.

It is well known that

$$\int_\Omega \det Dw\,(x)\,dx = \int_{w(\Omega)} \deg\,(w,\Omega,y)\,dy\,,$$

and thus

$$\deg w = \frac{1}{|w\,(\Omega)|} \int_\Omega \det Dw\,(x)\,dx\,. \qquad (24)$$

Using of the symbol # to denote the *cardinality* of the set, we have

$$\int_\Omega \left|\det Dw\,(x)\right|\,dx = \int_{w(\Omega)} \#\left\{x\in\Omega : w\,(x) = y\right\}\,dy\,. \qquad (25)$$

For our purpose it is also useful to recall the definition of *degree of a map* $v : S^{n-1} \to S^{n-1}$, v onto S^{n-1}. To this aim let us denote by T_ω the tangential plane to S^{n-1} at the point $\omega \in S^{n-1}$. If v is

Lipschitz-continuous, then for $H^{n-1}-$a.e. $\omega \in S^{n-1}$ the differential $dv_\omega : T_\omega \to T_{v(\omega)}$ exists. Similarly to the Euclidean case (23), the degree of v is defined by (see Chapter 5 of the book by Milnor [50])

$$\deg v := \frac{1}{n\omega_n} \int_{S^{n-1}} \sum_{\omega \in v^{-1}(\sigma)} \operatorname{sign}\left(\det dv_\omega\right) \, dH^{n-1}_\sigma \, ,$$

where, with an obvious abuse of notation, we denote by dv_ω also the $(n-1) \times (n-1)$ matrix representing the differential with respect to two fixed bases in T_ω and $T_{v(\omega)}$. Using again the area formula for maps between manifolds, as in (24) we get (see also [12], BN2)

$$\deg v = \frac{1}{n\omega_n} \int_{S^{n-1}} \det dv_\omega \, dH^{n-1}_\omega \, .$$

Fix $\omega_0 \in \partial B_1$ and denote by $\tau_j : S^{n-1} \backslash \{\omega_0\} \to \mathbb{R}^n$, for $j \in \{1, 2, \dots, n-1\}$, a vector field of class C^1 such that, for every $x \in S^{n-1} \backslash \{\omega_0\}$, the set of vectors $\{\tau_1(x), \tau_2(x), \dots, \tau_{n-1}(x)\}$ is an orthonormal basis for the tangent plane to the surface S^{n-1} at the point x. The following representation formula (26) for $\deg v$ holds.

Theorem 14 *Let $v : S^{n-1} \to S^{n-1}$ be a Lipschitz-continuous map onto S^{n-1}. Then, for $H^{n-1}-$a.e. $\omega \in S^{n-1}$, we have*

$$\det dv_\omega = \sum_{i=1}^{n} (-1)^{i+1} v^i(\omega) \frac{\partial\left(v^1, \dots, v^{i-1}, v^{i+1}, \dots, v^n\right)}{\partial\left(\tau_1, \tau_2, \dots, \tau_{n-1}\right)}(\omega) \, . \tag{26}$$

Theorem 14 is proved below in this section. We deduce from Theorem 14 and Corollary 11 the following consequence.

Corollary 15 (Geometric interpretation) *Let $v : S^{n-1} \to S^{n-1}$ be a map of class C^1 and onto, and let $u : B_1 \backslash \{0\} \to S^{n-1}$ be defined by $u(x) := v(x/|x|)$. If dv_ω is not singular at any $\omega \in S^{n-1}$, i.e., if v is locally invertible with C^1 local inverse at any point of S^{n-1}, then*

$$TV(u, B_1) = \omega_n \left|\deg v\right| = \omega_n \left|\deg \widetilde{v}\right| \, , \tag{27}$$

where $\widetilde{v} : B_1 \to \mathbb{R}^n$ is any Lipschitz-continuous extension of v to B_1.

Remark 16 *In two dimensions the total variation $TV(u, B_1)$ can be expressed in terms of the degree as in (27) under the sole assumption that v maps S^1 into a simple curve enclosing a starshaped domain (see Corollary 2). However, as shown in Section 10, this is not true anymore if v maps S^1 into a non-simple curve, such as the "eight" curve.*

Proof of Theorem 14. Fix $\omega \in S^{n-1}$ and denote by $\{\tau_1, \tau_2, \dots, \tau_{n-1}\}$, $\{\sigma_1, \sigma_2, \dots, \sigma_{n-1}\}$, two orthonormal bases for the tangent planes to T_ω and $T_{v(\omega)}$, respectively. With respect to these two bases the linear map $dv_\omega : T_\omega \to T_{v(\omega)}$ is represented by the $(n-1) \times (n-1)$ matrix with coefficients

$$\left(dv_\omega\right)_{ij} = \left\langle \sigma_j, \frac{\partial v}{\partial \tau_i}(\omega) \right\rangle \, .$$

Therefore the matrix dv_ω is the product of A and B, where A is the $(n-1) \times n$ matrix whose rows are $\frac{\partial v}{\partial \tau_i}$ and B is the $n \times (n-1)$ matrix whose columns are σ_j. By (iii) of Lemma 37 we have

$$\det dv_\omega = \sum_{i=1}^{n} \det X_{,i}(A) \cdot \det X_{i,}(B) \, ,$$

where

$$\det X_{,i}(A) = \frac{\partial\left(v^1, \ldots, v^{i-1}, v^{i+1}, \ldots, v^n\right)}{\partial\left(\tau_1, \tau_2, \ldots, \tau_{n-1}\right)}(\omega) \ ,$$

and in view of (78)

$$\det X_{i,}(B) = (-1)^{i+1}\,\nu_i\left(v(\omega)\right) \ ,$$

where $\nu_i(v(\omega))$ is the i-th component of the outward unit normal S^{n-1} at $v(\omega)$, i.e., $\nu_i(v(\omega)) = v^i(\omega)$. This concludes the proof of (26). ■

Proof of Corollary 15. From the previous theorem we deduce that, if dv_ω is not singular at $\omega \in S^{n-1}$, then $\det dv_\omega \neq 0$. Therefore, the right hand side of (26) is different from zero and has constant sign, since by assumption the map v is of class C^1. The first equality in (27) follows from Theorem 14 and (20). The second equality is consequence of (21) and of (24). ■

We conclude by giving a geometrical interpretation of some of the estimates given in this paper. In the following statement we use again of the symbol # to denote the *cardinality* of a set.

Theorem 17 *Let $v : S^{n-1} \to S^{n-1}$ be a Lipschitz-continuous map and let $u : B_1\setminus\{0\} \to S^{n-1}$ be defined by $u(x) := v(x/|x|)$. The total variation $TV(u, B_1)$ of the Jacobian of u can be estimated by*

$$TV(u, B_1) \geq \omega_n\,|\deg v| \ , \tag{28}$$

$$TV(u, B_1) \leq \frac{1}{n}\int_{\partial B_1} \#\left\{x \in S^{n-1} :\ v(x) = \omega\right\}\, dH^{n-1}_\omega \,. \tag{29}$$

Proof. Inequality (28) follows from inequality $TV(u, B_1) \geq |\mathrm{Det}\, Du|(B_1)$, equality (15) of Theorem 6 on the representation of $|\mathrm{Det}\, Du|(B_1)$, and formula (26) of Theorem 14.

To prove (29), we apply the estimate (72) and formula (25). Precisely, we denote by $\widetilde{v} : B_1 \to \mathbb{R}^n$ the extension of v defined by $\widetilde{v}(0) = 0$ and

$$\widetilde{v}(x) := |x| \cdot v\left(\frac{x}{|x|}\right), \qquad \forall\, x \in B_1\setminus\{0\} \,.$$

Let $\varrho_h \to 0^+$ and define

$$u_h(x) := \begin{cases} \frac{1}{\varrho_h}\widetilde{v}(x) & \text{if } x \in B_{\varrho_h}, \\ u(x) := v(x/|x|) & \text{if } x \in B_1\setminus B_{\varrho_h}. \end{cases} .$$

Clearly $u_h \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^n)$ and, by (25),

$$\begin{aligned} TV(u, B_1) &\leq \liminf_{h\to+\infty} \int_{B_1} |\det Du_h(x)|\, dx = \liminf_{h\to+\infty} \int_{B_{\varrho_h}} \left|\det D\frac{1}{\varrho_h}\widetilde{v}(x)\right| dx \\ &= \int_{B_1} |\det D\widetilde{v}(x)|\, dx = \int_{\widetilde{v}(B_1)} \#\left\{x \in B_1 : \widetilde{v}(x) = y\right\} dy \,, \end{aligned}$$

and, since $\widetilde{v}(B_1) \subseteq B_1$,

$$TV(u, B_1) \leq \int_{B_1} \#\left\{x \in S^{n-1} : v(x) = \frac{y}{|y|}\right\} dy$$

$$= \int_0^1 \varrho^{n-1}\, d\varrho \int_{\partial B_1} \#\left\{x \in S^{n-1} : v(x) = \omega\right\}\, dH^{n-1}_\omega = \frac{1}{n}\int_{\partial B_1} \#\left\{x \in S^{n-1} : v(x) = \omega\right\}\, dH^{n-1}_\omega \,.$$

■

4 $\operatorname{Det} Du$ versus $\det Du$

In this section we give a brief overview of relations between $\operatorname{Det} Du$, $\det Du$ and $TV(u, \Omega)$. We recall that the *Jacobian* $\det Du$ is given by

$$\det Du(x) := \frac{\partial\left(u^1, u^2, \ldots, u^n\right)}{\partial\left(x_1, x_2, \ldots, x_n\right)} = \sum_{i=1}^{n} \frac{\partial u^1}{\partial x_i} (\operatorname{adj} Du)_1^i, \tag{30}$$

where $\operatorname{adj} Du$ stands for the *adjugate* of Du, i.e., the transpose of the matrix of cofactors of Du. It is clear that when $u \in W^{1,n}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ then $\det Du \in L^1_{\mathrm{loc}}(\Omega)$. However, it is well known that, within some ranges of lower regularity for u, it is still possible to introduce a new concept of determinant which agrees with $\det Du$ when $u \in W^{1,n}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$.

Consider the *distributional Jacobian determinant*, which, as usual, is denoted by $\operatorname{Det} Du$ capitalized, and is given by

$$\operatorname{Det} Du := \sum_{i=1}^{n} (-1)^{i+1} \frac{\partial}{\partial x_i}\left(u^1 \frac{\partial\left(u^2, \ldots, u^n\right)}{\partial\left(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\right)}\right) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(u^1 (\operatorname{adj} Du)_1^i\right). \tag{31}$$

Note that $\operatorname{Det} Du$ is a *distribution* when $u \in W^{1,p}(\Omega; \mathbb{R}^n)$, $\operatorname{adj} Du \in L^q(\Omega; \mathbb{R}^{n\times n})$, with $1/p + 1/q \leq 1 + 1/n$ (in particular, when $u \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ for some $p > n^2/(n+1)$), or when $u \in L^{\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^n) \cap W^{1,n-1}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ (actually, it suffices to require that $u^1 \in L^{\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$, and that the vector field of derivatives $\left(Du^2, Du^3, \ldots, Du^n\right) \in L^{n-1}_{\mathrm{loc}}\left(\Omega; \mathbb{R}^{(n-1)\times n}\right)$). In the latter case, it is clear that the products in (31) are in $L^1_{\mathrm{loc}}(\Omega)$. Also, if $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ and $\operatorname{adj} Du \in L^q(\Omega; \mathbb{R}^{n\times n})$ with $1/p + 1/q \leq 1 + 1/n$, then this integrability property still holds by virtue of Hölder's inequality together with the fact that $1/q + 1/(p^*) \leq 1$ and, due to the Sobolev Embedding Theorem, $u \in L^{p^*}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$.

For smooth functions the Jacobian determinant $\det Du(x)$ and the distributional Jacobian determinant $\operatorname{Det} Du$ coincide. In fact, if $u \in W^{1,n}(\Omega; \mathbb{R}^n)$ then using the fact that the adjugate is divergence free, it is easy to see that (30) reduces to (31). Also, Müller, Tang and Yan proved in [57] that if $u \in W^{1,n-1}(\Omega; \mathbb{R}^n)$ and if $\operatorname{adj} Du \in L^{n/(n-1)}(\Omega; \mathbb{R}^{n\times n})$ then $\operatorname{Det} Du = \det Du$ and it belongs to $L^1(\Omega)$. This relation may fail if u is not sufficiently regular. As an example, consider (see [38])

$$u(x) := \sqrt[n]{a^n + |x|^n}\, \frac{x}{|x|}, \qquad \Omega := B_1,$$

where B_1, as in the previous sections, stands for the open ball in $\mathbb{R}^n$ centered at zero and with radius one. Then $u \in W^{1,p}(B_1; \mathbb{R}^n)$ for all $p < n$, $\det Du = 1$ a.e. in B_1, but

$$\operatorname{Det} Du = \mathcal{L}^n \lfloor B_1 + \omega_n\, a^n\, \delta_0,$$

where $\mathcal{L}^n$ denotes the Lebesgue measure in $\mathbb{R}^n$ and ω_n is the volume of the unit ball B_1. Similarly, as shown in [29], if $u(x) := x/|x|$ then $\det Du = 0$ a.e. in B_1 and $\operatorname{Det} Du = \omega_n\, \delta_0$.

These examples suggest that, at least for some ranges of p, when $\operatorname{Det} Du$ is a Radon measure then its absolutely continuous part with respect to the $n-$dimensional Lebesgue measure reduces to $\det Du$. Indeed, this holds when $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ and $\operatorname{adj} Du \in L^q(\Omega; \mathbb{R}^{n\times n})$ with $1/p + 1/q \leq 1 + 1/n$ (see [53]); see also Theorem 5.

The presence of singular measures in $\operatorname{Det} Du$ is in perfect agreement with recent experiments, which suggest that, in addition to bulk energy, surface contributions and singular measures may also be energetically relevant, thus disfavoring the creation of extremely small cavities (see [17], [32] and [33]). These considerations have motivated the search for a characterization of the singular measures which may appear in the description of the distributional Jacobian determinant. If we do not impose any geometrical or analytical restrictions on the function u, then it is possible to attain Radon measures with support of arbitrary Hausdorff dimension. Precisely, it was proven by Müller [55] (see also [53]) that, given $\alpha \in (0, n)$, there exists a compact set $K \subset B_1$ with Hausdorff dimension α, and there exists $u \in W^{1,p}(B_1; \mathbb{R}^n) \cap C^0(\overline{B_1})$ for all $p < n$, such that

$$\operatorname{Det} Du = \det Du \, \mathcal{L}^n \lfloor B_1 + \mu_s, \tag{32}$$

where μ_s is a positive Radon measure, singular with respect to $\mathcal{L}^n$, and such that $\operatorname{supp} \mu_s = K$. The situation is dramatically different if $u \in W^{n-1}(\Omega, S^{n-1})$, as it can be shown that if $\operatorname{Det} Du$ is a finite, signed, Radon measure then $\operatorname{Det} Du$ is a finite integer combination of Dirac masses (see Brezis and Nirenberg [12], [13]). The use of BMO and Hardy spaces allows one to obtain higher integrability results along the lines of Müller [52], [54], and Coifman, Lions, Meyers and Semmes [18]. As an example, it can be shown that if $u \in W^{1,n}(\Omega; \mathbb{R}^n)$ is such that $\det Du \geq 0$, then (see also Brezis, Fusco and Sbordone [11] and Iwaniec and Sbordone [41]) $\det Du \log(2 + \det Du) \in L^1_{\text{loc}}(\Omega)$.

As mentioned before, in this paper we assume that u is a function of class

$$W^{1,p}(\Omega; \mathbb{R}^n) \cap W^{1,\infty}_{\text{loc}}(\Omega \backslash \{0\}; \mathbb{R}^n)$$

for some $p \in (n-1, n)$ and for an open set $\Omega \subset \mathbb{R}^n$ containing the origin. The definition of the total variation $TV(u, \Omega)$ introduced in (2) follows the approach commonly used for variational problems with non-standard growth and coercivity conditions (see [1], [2], [9], [15], [27], [29], [46], [38], [48], [49]). The aim of this paper is to characterize $TV(u, \Omega)$. In [29] Fonseca and Marcellini accomplished this for $u(x) = x/|x|$. Fonseca and Malý [27], and Bouchitté, Fonseca and Malý [9] set up the problem into a broader context. Precisely, if $f : \Omega \times \mathbb{R}^{n \times n} \to \mathbb{R}$ is a *Carathéodory function*, then the *effective* (or *relaxed*) energy is defined as

$$\mathcal{F}_{p,q}(u, \Omega) := \inf \left\{ \liminf_{h \to \infty} \int_\Omega f(x, Du_h) \, dx : \; u_h \in W^{1,q}_{\text{loc}}, \;\; u_h \rightharpoonup u \;\text{ in } W^{1,p} \right\}. \tag{33}$$

In the case, where $f(x, \xi) := g(\det \xi)$ and $g :\to [0, +\infty)$ is a convex function, then (see [15], [22], [27])

$$\mathcal{F}_{p,n}(u, \Omega) \geq \int_\Omega g(\det Du(x)) \, dx \qquad \text{if } \; p \geq n-1,$$

and if $p > n-1$ then (see [9])

$$\mathcal{F}_{p,n}(u, \Omega) = \int_\Omega g(\det Du(x)) \, dx + \mu_s(\Omega),$$

for some Radon measure μ_s, singular with respect to the Lebesgue measure $\mathcal{L}^n$. For a general integrand f, and under the growth condition $0 \leq f(x, \xi) \leq C(1 + |\xi|^q)$, with $p > \frac{n-1}{n} q$, we have

$$\mathcal{F}_{p,q}(u, \Omega) = h_u \, \mathcal{L}^n \lfloor \Omega \, + \, \lambda_s, \tag{34}$$

where (see [1]) $h_u \leq Qf(x, Du)$, and λ_s is a singular measure. If $f = f(\xi)$ then it can be shown that (see [9], [27])

$$h_u = Qf(Du), \tag{35}$$

where Qf stands for the *quasiconvexification* of f, precisely (see [19], [51])

$$Qf(\xi) := \inf \left\{ \int_{(0,1)^n} f(\xi + D\varphi(x)) \, dx : \varphi \in C^1_0(\Omega; \mathbb{R}^n) \right\}.$$

This may no longer be true when f depends also on x and $p < q$ (although it is still valid if $f(x, \cdot)$ is convex, see [1]). Indeed, Gangbo [31] constructed an example where $f(x, \xi) = \chi_K(x) \, |\det \xi|$, and

$h_u = f$ if and only if $\mathcal{L}^N(\partial K) = 0$. Hence, in general, (35) fails and $f^{**}(x, \nabla u) \leq h_u$ is the only known lower bound (see also [1], [9], [27], [29], [46], [47]).

Further understanding of the total variation $TV(u, \Omega)$ asks for mastery of weak convergence of minors for $p < n$. Works by Ball [4], Dacorogna and Murat [21], Giaquinta, Modica and Souček [38], and Reshetnyak [59], established that

$$u_h \rightharpoonup u \;\text{ in }\; W^{1,n}(\Omega; \mathbb{R}^n) \quad \Longrightarrow \quad \det Du \rightharpoonup \det Du$$

in the sense of measures, where we recall that a sequence $\{\mu_h\}$ of Radon measures is said to *converge in the sense of measures* to a Radon measure μ in Ω if for every $\varphi \in C_c(\Omega; \mathbb{R})$ we have

$$\int_\Omega \varphi \, d\mu_h \to \int_\Omega \varphi \, d\mu \, .$$

Müller [52] has shown that, if in addition $\det Du_h \geq 0$, then $\det Du_h \rightharpoonup \det Du$ weakly in $L^1(\Omega)$. Moreover, if $u_h \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^n)$ and $\{\operatorname{adj} Du_h\}$ is bounded in $L^q\left(\Omega; \mathbb{R}^{d\times n}\right)$ with $p \geq n-1$, $q \geq n/(n-1)$, one of these two inequalities being strict, then

$$\det Du_h \rightharpoonup \det Du \qquad \text{in the sense of measures.}$$

Also, if $u_h \in W^{1,n}(\Omega; \mathbb{R}^n)$, $u_h \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^n)$ and $p > n-1$, then

$$\operatorname{adj} Du_h \rightharpoonup \operatorname{adj} Du \qquad \text{in} \;\; L^{p/(n-1)} \;\; \forall\, p > n-1. \tag{36}$$

A complete characterization of weak convergence of the determinant has been obtained by Fonseca, Leoni and Malý in [28], where it was shown that, if the sequence $\{u_h\} \subset W^{1,n}(\Omega; \mathbb{R}^n)$ converges to a function u in $L^1(\Omega; \mathbb{R}^n)$, if $\{u_h\}$ is bounded in $W^{1,n-1}(\Omega; \mathbb{R}^n)$, and if $\det Du_h \rightharpoonup \mu$ for some Radon measure μ, then

$$\frac{d\mu}{d\mathcal{L}^n} = \det Du, \quad \text{ a.e. } x \in \Omega. \tag{37}$$

For related works we refer to [2], [4], [15], [19], [20], [22], [30], [31], [34], [38], [45], [51], [52], [57].

What can we then say about the singular measure μ_s in (32), its significance and interpretation, and what are the relations, if any, between the total variation of $\operatorname{Det} Du$, i.e. $|\operatorname{Det} Du|(\Omega)$, and $TV(u, \Omega)$? An answer is given by Theorem 5, which contemplates a general framework where only integrability assumptions are considered, and no structural properties of the function u are prescribed. Next we present the proof of this result.

Proof of Theorem 5. Since $TV(u, \Omega) < +\infty$, by (34) and (35) $TV(u, \cdot)$ is a finite Radon measure, and it admits the Radon-Nikodym decomposition (10). In particular, it follows that $\det Du \in L^1(\Omega)$.

Let $\delta > 0$ be fixed and consider a sequence $\{u_h\}_{h\in\mathbb{N}} \subset C^1(\Omega; \mathbb{R}^n)$ such that $u_h \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^n)$, with $p > n-1$, and

$$TV(u, \Omega) + \delta \geq \lim_{h\to+\infty} \int_\Omega |\det Du_h| \, dx \, . \tag{38}$$

We first observe that, without loss of generality, we may assume that the sequence of the first components $\{u_h^1\}_{h\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$. Indeed, under the notation $M := \left\|u^1\right\|_\infty$, it suffices to consider the truncation

$$w_h^1(x) := \begin{cases} -M & \text{if } \; u_h^j(x) \leq -M, \\ u_h^j(x) & \text{if } \; -M \leq u_h^j(x) \leq M, \\ M & \text{if } \; u_h^j(x) \geq M, \end{cases}$$

and to set $w_h := \left(w_h^1, u_h^2, \ldots, u_h^n\right)$, for every $h \in \mathbb{N}$. It is easy to verify that, as $h \to +\infty$, w_h converges to u in the weak topology of $W^{1,p}(\Omega; \mathbb{R}^n)$ and, since $|\det Dw_h| \leq |\det Du_h|$, for almost every $x \in \Omega$, inequality (38) still holds with $\{u_h\}_{h\in\mathbb{N}}$ replaced by $\{w_h\}_{h\in\mathbb{N}}$.

Since $TV(u,\Omega)<+\infty$, by (38) the sequence $\{\det Du_h\}_{h\in\mathbb{N}}$ is bounded in $L^1(\Omega)$, therefore, up to a subsequence (not relabeled) $\det Du_h \rightharpoonup^* \widetilde{\mu}$ as $h\to+\infty$, where $\widetilde{\mu}$ is a *finite* Radon measure. By (37) we have

$$\frac{d\widetilde{\mu}}{d\mathcal{L}^n}=\det Du, \quad \text{a.e. } x\in\Omega\,. \tag{39}$$

Next we prove that the distribution $\operatorname{Det} Du$ coincides with $\widetilde{\mu}$ on $C_0^1(\Omega)$ and hence, by regularization and density, on $C_0^0(\Omega)$. To prove this, for fixed $\varphi\in C_0^1(\Omega)$ we have

$$\begin{aligned}\langle \operatorname{Det} Du,\varphi\rangle &= -\int_\Omega \sum_{i=1}^n u^1(\operatorname{adj} Du)_1^i\,\frac{\partial\varphi}{\partial x_i}\,dx\\ &= -\lim_{h\to\infty}\int_\Omega \sum_{i=1}^n u_h^1\,(\operatorname{adj} Du_h)_1^i\,\frac{\partial\varphi}{\partial x_i}\,dx=\lim_{h\to\infty}\int_\Omega \det Du_h\,\varphi\,dx=\langle\widetilde{\mu},\varphi\rangle\ .\end{aligned}$$

Here we have used the facts that, since $u_h\rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^n)$ for $p>n-1$, then $\operatorname{adj} Du_h$ weakly converge to $\operatorname{adj} Du$ in $L^{p/(n-1)}$, and since the sequence $\{u_h^1\}_{h\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$ and converges as $h\to+\infty$ to u^1 in $L^p(\Omega)$, it also converges to u^1 in $L^q(\Omega)$, for every $q<+\infty$, in particular for $q=\frac{p}{p-(n-1)}$, the conjugate exponent of $\frac{p}{n-1}$. Therefore, in view of (39), we deduce the Radon-Nikodym decomposition for $\operatorname{Det} Du$ as asserted in (11).

Let A be an open subset of Ω and let $\varphi\in C_0^1(A;\mathbb{R})$ be such that $\|\varphi\|_\infty\le 1$. By (38), a similar argument yields

$$\begin{aligned}|\langle \operatorname{Det} Du,\varphi\rangle| &= \left|\int_A \sum_{i=1}^n u^1(\operatorname{adj} Du)_1^i\,\frac{\partial\varphi}{\partial x_i}\,dx\right| = \lim_{h\to\infty}\left|\int_A \sum_{i=1}^n u_h^1\,(\operatorname{adj} Du_h)_1^i\,\frac{\partial\varphi}{\partial x_i}\,dx\right|\\ &= \lim_{h\to\infty}\left|\int_A \det Du_h\,\varphi\,dx\right| \le \limsup_{k\to\infty}\|\varphi\|_\infty\int_A |\det Du_h|\,dx\le TV(u,A)+\delta\,.\end{aligned}$$

It suffices to let $\delta\to0^+$, and to take the supremum over all such functions φ, to conclude (12), i.e., $|\operatorname{Det} Du|(A)\le TV(u,A)$.

Suppose now, in addition, that $u\in W^{1,n}_{\mathrm{loc}}(\Omega\backslash\{0\};\mathbb{R}^n)$. Let A be an open subset of Ω such that $0\notin A$. We recall that for every sequence u_h which converges to u in the weak topology of $W^{1,p}(A;\mathbb{R}^n)$ for some $p>n-1$, with $u,u_h\in W^{1,n}_{\mathrm{loc}}(A;\mathbb{R}^n)$ for every $h\in\mathbb{N}$, we have (see [20])

$$\liminf_{h\to+\infty}\int_A |\det Du_h|\,dx\ \ge\ \int_A |\det Du|\,dx\,.$$

Hence

$$TV(u,A)=\int_A |\det Du|\,dx\,,$$

whenever A is an open subset of Ω and $0\notin A$. Therefore we conclude that $\operatorname{supp}\lambda_s\subset\{0\}$, and thus $\lambda_s=\lambda\delta_0$ for some constant $\lambda\ge0$, where δ_0 is the Dirac measure at the origin.

On the other hand, in view of the inequality $|\operatorname{Det} Du|(A)\le TV(u,A)$ in (12), it follows that $\operatorname{supp}\mu_s\subset\operatorname{supp}\lambda_s\subset\{0\}$, therefore $\mu_s=\mu\delta_0$, for some constant $\mu\in\mathbb{R}$, with $|\mu|\le\lambda$, where we have used (12).once more. ■

Remark 18 *The result stated in Theorem 5 holds also under the assumption that $u\in W^{1,p}(\Omega;\mathbb{R}^n)$ for some $p>n^2/(n+1)$. Indeed, in this case, instead of truncating the sequence $\{u_h^1\}_{h\in\mathbb{N}}$ we use the fact that, by Kondrachov's Compact Embedding Theorem, $u_h\to u$ strongly in L^{n^2}, with n^2 being the conjugate exponent of $n^2/(n^2-1)$. Again, $\{Du_h\}_{h\in\mathbb{N}}$ weakly converges in $L^p(\Omega;\mathbb{R}^{n\times n})$ and the sequence $\{\operatorname{adj} Du_h\}_{h\in\mathbb{N}}$ weakly converges in $L^{n^2/(n^2-1)}$.*

5 The 2−dimensional case

Let $n = 2$. For every $\xi = (\xi^1, \xi^2) \in \mathbb{R}^2$, $\xi \neq 0$, we denote by $\operatorname{Arg}\xi$ the unique angle in $[-\pi, \pi)$ such that

$$\cos \operatorname{Arg}\xi = \frac{\xi^1}{|\xi|}, \qquad \sin \operatorname{Arg}\xi = \frac{\xi^2}{|\xi|}.$$

As before, we denote by B_r the circle in $\mathbb{R}^2$ with center in 0 and radius $r > 0$. Then B_1 is the circle of radius $r = 1$ and $\partial B_1 = S^1$ is its boundary. If $\alpha, \beta \in [0, 2\pi]$, $\alpha < \beta$, then $S(\alpha, \beta)$ stands for the *polar sector* given by

$$S(\alpha, \beta) := \left\{ \xi = \varrho (\cos\vartheta, \sin\vartheta) \in \mathbb{R}^2 : \ \ \varrho \leq 1, \ \ \vartheta \in [\alpha, \beta] \right\}.$$

In the sequel $v : [0, 2\pi] \to \mathbb{R}^2$ is a Lipschitz-continuous *closed* curve, i.e., $v(0) = v(2\pi)$, that we represent as $v = (v^1, v^2) = (v^1(\vartheta), v^2(\vartheta))$, with $\vartheta \in [0, 2\pi]$. We shall denote by $v_\vartheta := (v^1_\vartheta, v^2_\vartheta)$ the gradient of v, which exists for almost every $\vartheta \in [0, 2\pi]$. If $v(\vartheta) \neq 0$ for every $\vartheta \in [0, 2\pi]$, then we denote by $A_v(\vartheta)$ the quantity

$$A_v(\vartheta) := \operatorname{Arg} v(0) + \int_0^\vartheta \frac{v^1(t)\, v^2_\vartheta(t) - v^2(t)\, v^1_\vartheta(t)}{|v(t)|^2}\, dt.$$

There exists a simple relation between A_v and $\operatorname{Arg} v$, which is inferred from the next lemma.

Lemma 19 *If $v : [0, 2\pi] \to \mathbb{R}^2$ is a Lipschitz-continuous curve such that $v(\vartheta) \neq 0$ for every $\vartheta \in [0, 2\pi]$, then, for every $\alpha, \beta \in [0, 2\pi]$ with $\alpha < \beta$, there exists $k \in \mathbb{Z}$ such that*

$$A_v(\beta) - A_v(\alpha) = \operatorname{Arg} v(\beta) - \operatorname{Arg} v(\alpha) + 2k\pi. \tag{40}$$

Proof. Assume first that $v \in C^1([0, 2\pi]; \mathbb{R}^2)$ and that there exist at most a finite number of angles $\vartheta_i \in [0, 2\pi)$ such that either $v^1(\vartheta_i) = 0$ or $v^2(\vartheta_i) = 0$. Then, for every $\vartheta \neq \vartheta_i$ (since $v^1(\vartheta_i) \neq 0$) we have

$$\frac{d}{d\vartheta} \operatorname{Arg} v(\vartheta) = \frac{d}{d\vartheta} \arctan \frac{v^2(\vartheta)}{v^1(\vartheta)} = \frac{v^1(\vartheta)\, v^2_\vartheta(\vartheta) - v^2(\vartheta)\, v^1_\vartheta(\vartheta)}{|v(\vartheta)|^2}.$$

The result then follows by integrating this equality and recalling that, each time that $v(\vartheta)$ crosses the half line $\left\{(x, y) \in \mathbb{R}^2 : \ x < 0, \ y = 0\right\}$, and this may happen at most a finite number of times (necessarily for ϑ equal to some ϑ_i, where $v^2(\vartheta_i) = 0$), the function $\operatorname{Arg} v(\vartheta)$ has a jump of $\pm 2\pi$.

In the general case, we approximate v by a sequence $\{v_j\}_{j\in\mathbb{N}}$ of curves of class $C^1([0, 2\pi]; \mathbb{R}^2)$ such that $\{v_j\}_{j\in\mathbb{N}}$ uniformly converges to v and $\{dv_j/d\vartheta\}_{j\in\mathbb{N}}$ converges to $dv/d\vartheta$ in $L^p([0, 2\pi])$ for every $p \in [1, +\infty)$. We may construct the curves v_j so that $v_j(\vartheta) \neq 0$ for all $\vartheta \in [0, 2\pi]$ and either $v^1(\vartheta_i) = 0$ or $v^2(\vartheta_i) = 0$ only for finitely many i. Moreover, if $\operatorname{Arg} v(\vartheta) \neq -\pi$, then $\operatorname{Arg} v_j(\vartheta) \to \operatorname{Arg} v(\vartheta)$, while, if $\operatorname{Arg} v(\vartheta) = -\pi$, then, up to a subsequence, $\operatorname{Arg} v_j(\vartheta) \to \operatorname{Arg} v(\vartheta) = -\pi$ or $\operatorname{Arg} v_j(\vartheta) \to \pi$. Finally, the quantity

$$A_{v_j}(\beta) - A_{v_j}(\alpha) = \int_\alpha^\beta \frac{v^1_j v^2_{j,\vartheta} - v^2_j v^1_{j,\vartheta}}{|v_j|^2}\, dt$$

converges, as $j \to +\infty$, to $A_v(\beta) - A_v(\alpha)$. From the relation

$$A_{v_j}(\beta) - A_{v_j}(\alpha) = \operatorname{Arg} v_j(\beta) - \operatorname{Arg} v_j(\alpha) + 2k_j\pi,$$

valid for every $j \in \mathbb{N}$ and for some $k_j \in \mathbb{Z}$, we see that the sequence k_j is bounded, since $\operatorname{Arg} v_j(\beta)$,$\operatorname{Arg} v_j(\alpha) \in [-\pi, \pi)$. Then, up to a subsequence, we obtain the conclusion (40) as $j \to +\infty$. ■

As in Section 2, we denote by Γ a curve in $\mathbb{R}^2$ parametrized in the following way

$$\Gamma := \{\xi + r(\vartheta)(\cos\vartheta, \sin\vartheta) : \ \vartheta \in [0, 2\pi]\}, \tag{41}$$

where $r(\vartheta)$ is a piecewise C^1 function such that $r(0) = r(2\pi)$, and $r(\vartheta) \geq r_0$ for every $\vartheta \in [0, 2\pi]$ and for some $r_0 > 0$. Condition (41) means that Γ is the Lipschitz-continuous boundary of a domain

$$D := \{\xi + \varrho(\cos\vartheta, \sin\vartheta) : \vartheta \in [0, 2\pi], \ 0 \leq \varrho \leq r(\vartheta)\},$$

starshaped with respect to a point ξ in the *interior* of D. In the sequel it is understood that the function $r(\vartheta)$ is extended to $\mathbb{R}$ by periodicity.

Lemma 20 *Let Γ be as in (41) and let $v : [0, 2\pi] \to \Gamma$ be a Lipschitz-continuous map such that* $\operatorname{Arg}(v(0) - \xi) = 0$. *Then the curve v may be represented in the form*

$$v(\vartheta) = \xi + r(A_{v-\xi}(\vartheta))(\cos A_{v-\xi}(\vartheta), \sin A_{v-\xi}(\vartheta)) \tag{42}$$

for all $\vartheta \in [0, 2\pi]$.

The proof of this lemma may be found in [24].

Remark 21 *Under the assumptions of Lemma 20, from the representation formula (42) for $v(\vartheta)$ it follows that, if $A_{v-\xi}(\alpha) = A_{v-\xi}(\beta)$, then $v(\alpha) = v(\beta)$. Conversely, if $v(\alpha) = v(\beta)$ then there exists $k \in \mathbb{Z}$ such that $A_{v-\xi}(\alpha) = A_{v-\xi}(\beta) + 2k\pi$. However, notice that if Γ is the boundary of a simply connected domain which is not starshaped with respect to ξ, then the conclusion of Lemma 20 may not be true. In particular, the condition $A_{v-\xi}(\alpha) = A_{v-\xi}(\beta)$ may not imply that $v(\alpha) = v(\beta)$.*

The next Lemma 22, found in [24], plays a central role in the study of the 2-dimensional case. For the convenience of the reader, we include its proof below.

Lemma 22 (The "umbrella" lemma) *Let $\Gamma = \{\xi + r(\vartheta)(\cos\vartheta, \sin\vartheta)\}$ and let $v : [0, 2\pi] \to \Gamma$ be a Lipschitz-continuous map. If $\alpha, \beta \in [0, 2\pi]$, $\alpha < \beta$, are such that $A_{v-\xi}(\alpha) = A_{v-\xi}(\beta)$, then for every $\varepsilon > 0$ there exists a Lipschitz-continuous map $w : S(\alpha, \beta) \to \mathbb{R}^2$ satisfying the boundary conditions*

$$\begin{cases} w(1, \vartheta) = v(\vartheta) & \forall\, \vartheta \in [\alpha, \beta], \\ w(\varrho, \alpha) = w(\varrho, \beta) = \xi + \varrho(v(\alpha) - \xi) & \forall\, \varrho \in [0, 1], \end{cases} \tag{43}$$

and such that

$$\int_{S(\alpha,\beta)} |\det Dw(x)| \ dx < \varepsilon . \tag{44}$$

Proof. Without loss of generality we can assume that $\operatorname{Arg}(v(0) - \xi) = 0$. Fix $h \in \mathbb{N}$ and set

$$w_h(\varrho, \vartheta) := \xi + \varrho r(\varphi_h(\varrho, \vartheta))(\cos\varphi_h(\varrho, \vartheta), \sin\varphi_h(\varrho, \vartheta)), \tag{45}$$

where, for every $\varrho \in [0, 1]$ and for every $\vartheta \in [\alpha, \beta]$,

$$\varphi_h(\varrho, \vartheta) := \varrho^h A_{v-\xi}(\vartheta) + \left(1 - \varrho^h\right) A_{v-\xi}(\alpha) .$$

Since $\varphi_h(1, \vartheta) = A_{v-\xi}(\vartheta)$, $\varphi_h(\varrho, \alpha) = \varphi_h(\varrho, \beta) = A_{v-\xi}(\alpha)$, by the representation formula (42) of Lemma 20 we obtain the validity of the boundary conditions (43).

Now we evaluate the left hand side in (44). We observe that, if $u(x) = \left(u^1(\varrho, \vartheta), u^2(\varrho, \vartheta)\right)$, and using the notation $\frac{\partial u^i}{\partial \varrho} = u^i_\varrho$, $\frac{\partial u^i}{\partial \vartheta} = u^i_\vartheta$ $(i = 1, 2)$, we have

$$\det Du(x) = \frac{1}{\varrho} \begin{vmatrix} u^1_\varrho(\varrho, \vartheta) & u^1_\vartheta(\varrho, \vartheta) \\ u^2_\varrho(\varrho, \vartheta) & u^2_\vartheta(\varrho, \vartheta) \end{vmatrix} . \tag{46}$$

For the function w_h we obtain

$$\int_{S(\alpha,\beta)} |\det Dw_h(x)| \, dx = \int_0^1 d\varrho \int_\alpha^\beta \left| \frac{\partial \left(w_h^1, w_h^2\right)}{\partial (\varrho, \vartheta)} \right| d\vartheta \, .$$

Now the Jacobian determinant of w_h is

$$\frac{\partial \left(w_h^1, w_h^2\right)}{\partial (\varrho, \vartheta)} = \varrho r^2 (\varphi_h) \frac{\partial \varphi_h}{\partial \vartheta} = \varrho^{h+1} r^2 (\varphi_h) A'_{v-\xi} (\vartheta) \, ,$$

and we conclude that

$$\int_{S(\alpha,\beta)} |\det Dw_h(x)| \, dx = \int_0^1 \varrho^{h+1} \, d\varrho \int_\alpha^\beta r^2 (\varphi_h) \left| A'_{v-\xi} (\vartheta) \right| d\vartheta = \frac{c}{h+2} \, ,$$

where we denote by c a suitable constant. The conclusion follows by choosing $h \in \mathbb{N}$ sufficiently large.

■

Remark 23 *We call Lemma 22 the "the umbrella lemma"due to the fact that the geometric representation of the graph of the map $w : \mathbb{R}^2 \to \mathbb{R}^2$ considered in Lemma 22 is some sort of "umbrella" (under some mathematical tolerance and human imagination!). In fact, let us consider for simplicity the case where the image Γ of the map v is the unit circle $[0, 2\pi] \subset \mathbb{R}^2$ centered around $\xi = 0$. Then the graph of w is a subset of S^1: it "starts" from the center $\xi = 0$ (the starting point of the "umbrella-stick", in correspondence to $\varrho = 0$) and it "ends" for $\varrho = 1$, at the surface $\{w(1, \vartheta) = v(\vartheta) : \ \vartheta \in [\alpha, \beta]\} \subset S^1$, which can be interpreted as the upper surface of the open umbrella, to protect one from the rain. Moreover, by (44), like an umbrella, the total volume of the image of w is small (large upper surface, small volume! In our $2-d$ case, we have a $2-$dimensional "picture" of an umbrella, with large upper length and small area).*

We refer to Figures 1, 2 and 3, where we represented the image of the map $w_h(\varrho, \vartheta)$ in (45) under three particular choices of the parameters. Precisely, for fixed $h \in \mathbb{N}$ we considered $w_h : S(\alpha, \beta) \to B_1$ (i.e., $r(\varphi_h(\varrho, \vartheta))$ in (45) identically equal to 1 and $\xi = 0$) given by

$$\begin{cases} w_h(\varrho, \vartheta) = \varrho (\cos \varphi_h(\varrho, \vartheta), \sin \varphi_h(\varrho, \vartheta)) \\ \varphi_h(\varrho, \vartheta) = \varrho^h A_v(\vartheta) + \left(1 - \varrho^h\right) A_v(\alpha) \end{cases} , \tag{47}$$

where $A_v : [\alpha, \beta] \to \mathbb{R}$ is a function such that $A_v(\alpha) = A_v(\beta)$. The common value of A_v at $\vartheta = \alpha$ and $\vartheta = \beta$ is the asymptotic value of the angle $\varphi_h(\varrho, \vartheta)$ as $\varrho \to 0^+$ and it represents the angle which the umbrella-stick forms with the $x-$axis. At $\varrho = 1$ the angle $\varphi_h(1, \vartheta)$ holds $A_v(\vartheta)$; therefore the maximum M and the minimum m of $A_v(\vartheta)$ represent the bounds for the angle $\varphi_h(1, \vartheta)$ of the image $w(1, \vartheta)$ at the surface S^1 of the ball B_1. These pictures has been made by Emanuele Paolini, starting from the analytic expression of w in (47). We thank him for the beautiful job.

□

Figure 1: A $2-d$ image of the map w defined in (47), with $h = 4$, for a particular (piecewise linear) function $A_u(\vartheta)$. The angle which the *umbrella-stick* forms with the $x-$axis is given by $A_u(\alpha) = A_u(\beta) = \pi/2$. The *maximun* M and the *minimum* m of $A_u(\vartheta) = \varphi(1, \vartheta)$, which give the bounds for the *angles* of the image $w(1, \vartheta)$ at the surface S^1 of the ball B_1, in this case are equal to $m = \pi/6$, $M = 5\pi/6$, respectively. Note that the map is radially linear when $\vartheta = \alpha$ and $\vartheta = \beta$, where the angle of the image is equal to $\pi/2$.

An abbreviated proof of the result below may be found in [24].

□

Figure 2: Another $2-d$ image of the map w defined in (47), with $h=4$, for a different choise of the piecewise linear function $A_u(\vartheta)$. In this case we obtain an *asymmetric* umbrella. Again $A_u(\alpha)=A_u(\beta)=\pi/2$, while in this case $m=-\pi/6$, $M=5\pi/6$. The map is not one-to-one: only the image points with angles $-\pi/6$ and $5\pi/6$ may be assumed once; all the other points are hit at least twice; the points with angle $\pi/2$ are hit at least three times.

□

Figure 3: Map $w:S(\alpha,\beta)\to B_1$ in (47). Here we fixed $h=3$, while the angle which the *umbrella-stick* forms with the $x-$axis is equal to $A_u(\alpha)=A_u(\beta)=0$. The bounds for the *angle* of the image $w(1,\vartheta)$, at the surface S^1 of the unit ball B_1, in this case are equal to $m=0$, $M=2\pi+\pi/2$. The map is radially linear when the angle $A_u(\vartheta)$ of the image is 0 (and this happens only if $\vartheta=\alpha=\beta$, when $\varphi(1,\vartheta)=A_u(\vartheta)=0$). The map w is not one-to-one: due to the overlaping phenomenon, some points with ϱ close to 1 and $0\le\varphi<\pi/2$ are assumed at least four times.

Lemma 24 *Let $v:[0,2\pi]\to\Gamma$ be a Lipschitz-continuous map. Let $\alpha,\beta\in[0,2\pi]$, $\alpha<\beta$, be such that $A_{v-\xi}(\alpha)=A_{v-\xi}(\beta)$. If $A_{v-\xi}(\vartheta)$ is piecewise strictly monotone in $[\alpha,\beta]$ (with a finite number of monotonicity intervals) then*

$$\int_\alpha^\beta \left\{\left(v^1(\vartheta)-\xi^1\right)v^2_\vartheta(\vartheta)-\left(v^2(\vartheta)-\xi^2\right)v^1_\vartheta(\vartheta)\right\}\,d\vartheta\;=\;0\,.$$

Proof. Without loss of generality we assume that $\xi=(0,0)$. Since $A_v(\vartheta)$ is piecewise strictly monotone in $[\alpha,\beta]$ and $A_v(\alpha)=A_v(\beta)$, there exists a partition of the interval $[\alpha,\beta]$, $\alpha=\vartheta_0<\vartheta_1<\ldots<\vartheta_N=\beta$, $N\ge 2$, such that, for every $i=1,2,\ldots N$, the real function $A_v(\vartheta)$ is strictly increasing in $[\vartheta_{i-1},\vartheta_i]$ and is strictly decreasing in $[\vartheta_i,\vartheta_{i+1}]$ (or viceversa). We will prove the lemma by an induction argument based on the number N of these maximal intervals of monotonicity.

Let us first assume that $N=2$. Hence there exists $\vartheta_1\in(\alpha,\beta)$ such that $A_v(\vartheta)$ is strictly increasing in $[\alpha,\vartheta_1]$ and is strictly decreasing in $[\vartheta_1,\beta]$, or conversely. To fix the ideas, let us assume that $A_v(\vartheta)$ is strictly increasing in $[\alpha,\vartheta_1]$. For every $(\varrho,\vartheta)\in S(\alpha,\beta)$ let us define $\widetilde{v}(\varrho,\vartheta):=\varrho\, v(\vartheta)$. If $A_v(\vartheta_1)-A_v(\alpha)\le 2\pi$, then $\widetilde{v}$ restricted to the interior of $S(\alpha,\vartheta_1)$ and $S(\vartheta_1,\beta)$ is one-to-one. Moreover the images $\widetilde{v}(S(\alpha,\vartheta_1))$ and $\widetilde{v}(S(\vartheta_1,\beta))$ are equal. Therefore, by the area formula,

$$\int_{S(\alpha,\vartheta_1)}|\det D\widetilde{v}(x)|\;dx=\operatorname{area}(\widetilde{v}(S(\alpha,\vartheta_1)))=\operatorname{area}(\widetilde{v}(S(\vartheta_1,\beta)))=\int_{S(\vartheta_1,\beta)}|\det D\widetilde{v}(x)|\;dx\,.$$

Since $\det D\widetilde{v}\ge 0$ in $S(\alpha,\vartheta_1)$ and $\det D\widetilde{v}\le 0$ in $S(\vartheta_1,\beta)$, we obtain

$$\int_{S(\alpha,\vartheta_1)}\det D\widetilde{v}(x)\;dx=\operatorname{area}(\widetilde{v}(S(\alpha,\vartheta_1)))=\operatorname{area}(\widetilde{v}(S(\vartheta_1,\beta)))=-\int_{S(\vartheta_1,\beta)}\det D\widetilde{v}(x)\;dx\,.$$

By using again (46), we have

$$\det D\widetilde{v}(\varrho,\vartheta)=\frac{1}{\varrho}\begin{vmatrix} v^1(\vartheta) & \varrho v^1_\vartheta(\vartheta)\\ v^2(\vartheta) & \varrho v^2_\vartheta(\vartheta)\end{vmatrix}=v^1(\vartheta)\,v^2_\vartheta(\vartheta)-v^2(\vartheta)\,v^1_\vartheta(\vartheta)=A_v(\vartheta)\,|v(\vartheta)|^2\,.$$

Therefore, as claimed,

$$\begin{aligned}0&=\int_{S(\alpha,\beta)}\det D\widetilde{v}(x)\;dx=\int_0^1\varrho\,d\varrho\int_\alpha^\beta\left\{v^1(\vartheta)\,v^2_\vartheta(\vartheta)-v^2(\vartheta)\,v^1_\vartheta(\vartheta)\right\}\;d\vartheta\\&=\frac{1}{2}\int_\alpha^\beta\left\{v^1(\vartheta)\,v^2_\vartheta(\vartheta)-v^2(\vartheta)\,v^1_\vartheta(\vartheta)\right\}\;d\vartheta\,.\end{aligned}$$

If $2k\pi < A_v(\vartheta_1) - A_v(\alpha) \le 2\pi(k+1)$ for some $k \ge 1$, then we denote by $\vartheta' \in (\alpha, \vartheta_1)$, $\vartheta'' \in (\vartheta_1, \beta)$ the points such that $A_v(\vartheta') = A_v(\vartheta'') = 2k\pi$. Again, using the area formula, we have

$$\int_{S(\alpha,\vartheta_1)} |\det D\widetilde{v}(x)| \; dx = \int_{S(\alpha,\vartheta')} |\det D\widetilde{v}(x)| \; dx + \int_{S(\vartheta',\vartheta_1)} |\det D\widetilde{v}(x)| \; dx = k \,\mathrm{area}\, D + \mathrm{area}\, E\,,$$

where D is the domain in (6) enclosed by Γ and E is the domain represented in polar coordinates by

$$\begin{aligned} E &= \left\{ \varrho\left(\cos A_v(\vartheta), \sin A_v(\vartheta)\right) : \; \vartheta \in \left[\vartheta', \vartheta_1\right], \; 0 \le \varrho \le r(\vartheta) \right\} \\ &= \left\{ \varrho\left(\cos A_v(\vartheta), \sin A_v(\vartheta)\right) : \; \vartheta \in \left[\vartheta_1, \vartheta''\right], \; 0 \le \varrho \le r(\vartheta) \right\} . \end{aligned}$$

Therefore, we also have

$$\int_{S(\vartheta_1,\beta)} |\det D\widetilde{v}(x)| \; dx = \int_{S(\vartheta_1,\vartheta'')} |\det D\widetilde{v}(x)| \; dx + \int_{S(\vartheta'',\beta)} |\det D\widetilde{v}(x)| \; dx = \mathrm{area}\, E + k \,\mathrm{area}\, D\,.$$

Arguing as before we get the thesis (with $N = 2$)

$$\begin{aligned} \frac{1}{2} \int_\alpha^\beta \left\{ v^1(\vartheta)\, v^2_\vartheta(\vartheta) - v^2(\vartheta)\, v^1_\vartheta(\vartheta) \right\} \; d\vartheta &= \int_{S(\alpha,\beta)} \det D\widetilde{v}(x) \; dx \\ &= \int_{S(\alpha,\vartheta_1)} |\det D\widetilde{v}(x)| \; dx - \int_{S(\vartheta_1,\beta)} |\det D\widetilde{v}(x)| \; dx = 0. \end{aligned}$$

By induction, we assume that the result is true if there are $N-1$ maximal intervals of monotonicity for the function $A_v(\vartheta)$. Then we consider the case where there are N of such intervals, with endpoints $\alpha = \vartheta_0 < \vartheta_1 < \ldots < \vartheta_N = \beta$. Without loss of generality, we can assume that $A_v(\vartheta)$ is strictly increasing in $[\alpha, \vartheta_1]$ and is strictly decreasing in $[\vartheta_1, \vartheta_2]$. If $A_v(\alpha) > A_v(\vartheta_2)$, then there exists $\gamma \in (\vartheta_1, \vartheta_2)$ such that $A_v(\gamma) = A_v(\alpha)$; since the thesis holds for the case of two intervals $[\alpha, \vartheta_1]$, $[\vartheta_1, \gamma]$, we obtain

$$\int_\alpha^\gamma \left\{ v^1 v^2_\vartheta - v^2 v^1_\vartheta \right\} \; d\vartheta = 0. \tag{48}$$

The thesis also holds for the $N-1$ intervals $[\gamma, \vartheta_2], [\vartheta_2, \vartheta_3], \ldots, [\vartheta_{N-1}, \beta]$, and so we have

$$\int_\gamma^\beta \left\{ v^1 v^2_\vartheta - v^2 v^1_\vartheta \right\} \; d\vartheta = 0\,,$$

which, together with (48), yields the conclusion if $A_v(\alpha) > A_v(\vartheta_2)$.

If $A_v(\alpha) = A_v(\vartheta_2)$, then the same argument works with $\gamma = \vartheta_2$. If $A_v(\alpha) < A_v(\vartheta_2)$ then there exists $\delta \in (\alpha, \vartheta_1)$ such that $A_v(\delta) = A_v(\vartheta_2)$ and, as before, by considering the two intervals $[\delta, \vartheta_1]$, $[\vartheta_1, \vartheta_2]$, we have

$$\int_\delta^{\vartheta_2} \left\{ v^1 v^2_\vartheta - v^2 v^1_\vartheta \right\} \; d\vartheta = 0\,. \tag{49}$$

Then we "modify" the function $v(\vartheta)$ by "cutting out" the interval (δ, ϑ_2) from $[\alpha, \beta]$. Precisely, we define in the interval $[\alpha + [\vartheta_2 - \delta], \beta]$

$$w(\vartheta) := \begin{cases} v(\vartheta - [\vartheta_2 - \delta]) & \text{if } \alpha + [\vartheta_2 - \delta] \le \vartheta \le \vartheta_2, \\ v(\vartheta) & \text{if } \vartheta_2 \le \vartheta \le \beta. \end{cases}$$

Then $A_w(\vartheta)$ is piecewise strictly monotone in $[\alpha + [\vartheta_2 - \delta], \beta]$, with $N-1$ monotonicity intervals. By the induction assumption we have

$$0 = \int_{\alpha+[\vartheta_2-\delta]}^\beta \left\{ w^1 w^2_\vartheta - w^2 w^1_\vartheta \right\} \; d\vartheta = \int_\alpha^\delta \left\{ v^1 v^2_\vartheta - v^2 v^1_\vartheta \right\} \; d\vartheta + \int_{\vartheta_2}^\beta \left\{ v^1 v^2_\vartheta - v^2 v^1_\vartheta \right\} \; d\vartheta\,,$$

which, together with (49), yields the conclusion. ∎

The lemma below is proven in [24].

Lemma 25 *Let $v : [0, 2\pi] \to \Gamma$ be a Lipschitz-continuous map. Let $A_{v-\xi}(\vartheta)$ be piecewise strictly monotone in $[a,b]$ (with a finite number of monotonicity intervals). For every $\varepsilon > 0$ there exists a Lipschitz-continuous map $w : B_1 \to \mathbb{R}^2$ such that $w(1,\vartheta) = v(\vartheta)$ for every $\vartheta \in [0, 2\pi]$, and*

$$\int_{B_1} |\det Dw(x)| \ dx < \varepsilon + \frac{1}{2} \left| \int_0^{2\pi} \left\{ v^1(\vartheta) v^2_\vartheta(\vartheta) - v^2(\vartheta) v^1_\vartheta(\vartheta) \right\} \ d\vartheta \right| .$$

Next we consider maps $u = u(\varrho, \vartheta)$ depending explicitly on ϱ as well. We assume first that u is a smooth map in the unit ball $B_1 \subset \mathbb{R}^2$.

Lemma 26 (The integral of the Jacobian for smooth maps) *Let $u \in W^{1,\infty}(B_1; \mathbb{R}^2)$. For every $r \in (0, 1]$ we have*

$$\int_{B_r} \det Du(x) \ dx = \frac{1}{2} \int_0^{2\pi} \left\{ u^1(r,\vartheta) \frac{\partial u^2}{\partial \vartheta}(r,\vartheta) - u^2(r,\vartheta) \frac{\partial u^1}{\partial \vartheta}(r,\vartheta) \right\} d\vartheta . \tag{50}$$

Proof. If first $u \in C^2(B_1; \mathbb{R}^2)$, then by the divergence theorem, we have for every $r \in (0,1)$

$$\int_{B_r} \det Du(x) \ dx = \int_{B_r} \operatorname{div} \left(u^1 \frac{\partial u^2}{\partial x_2}, -u^1 \frac{\partial u^2}{\partial x_1} \right) dx = \int_{\partial B_r} \left\{ u^1 \frac{\partial u^2}{\partial x_2} \nu^1 - u^1 \frac{\partial u^2}{\partial x_1} \nu^2 \right\} dH^1 , \tag{51}$$

where $\nu = (\nu^1, \nu^2)$ is the exterior normal to ∂B_r and $dH^1 = ds = r d\vartheta$ is the element of archlenght. A standard approximation argument yields formula (51) for every $u \in W^{1,\infty}(B_1; \mathbb{R}^2)$ and for every $r \in (0,1)$ (since $u \in W^{1,\infty}(\partial B_r; \mathbb{R}^2)$ for every $r \in (0,1)$ too).

With an obvious abuse of notation, we write u in polar coordinates (ϱ, ϑ), i.e., $u(x_1, x_2) = u(\varrho, \vartheta)$. We have

$$\frac{\partial u}{\partial x_1} = \frac{\partial u}{\partial \rho} \frac{\partial \varrho}{\partial x_1} + \frac{\partial u}{\partial \vartheta} \frac{\partial \vartheta}{\partial x_1} = \frac{\partial u}{\partial \rho} \cos\vartheta - \frac{\partial u}{\partial \vartheta} \frac{\sin\vartheta}{\varrho} ,$$
$$\frac{\partial u}{\partial x_2} = \frac{\partial u}{\partial \rho} \frac{\partial \varrho}{\partial x_2} + \frac{\partial u}{\partial \vartheta} \frac{\partial \vartheta}{\partial x_2} = \frac{\partial u}{\partial \rho} \sin\vartheta + \frac{\partial u}{\partial \vartheta} \frac{\cos\vartheta}{\varrho} .$$

Since on ∂B_r the exterior normal reduces to $\nu = (\nu^1, \nu^2) = (\cos\vartheta, \sin\vartheta)$, from (51) we obtain

$$\begin{aligned} \frac{\partial u^2}{\partial x_2} \nu^1 - \frac{\partial u^2}{\partial x_1} \nu^2 &= \left(\frac{\partial u^2}{\partial \rho} \sin\vartheta + \frac{\partial u^2}{\partial \vartheta} \frac{\cos\vartheta}{\varrho} \right) \cos\vartheta - \left(\frac{\partial u^2}{\partial \rho} \cos\vartheta - \frac{\partial u^2}{\partial \vartheta} \frac{\sin\vartheta}{\varrho} \right) \sin\vartheta \\ &= \frac{\partial u^2}{\partial \rho} \sin\vartheta\cos\vartheta + \frac{1}{\varrho} \frac{\partial u^2}{\partial \vartheta} \cos^2\vartheta - \frac{\partial u^2}{\partial \rho} \sin\vartheta\cos\vartheta + \frac{1}{\varrho} \frac{\partial u^2}{\partial \vartheta} \sin^2\vartheta = \frac{1}{\varrho} \frac{\partial u^2}{\partial \vartheta} . \end{aligned}$$

Thus on ∂B_r, since $dH^1 = r d\vartheta$, we get

$$\int_{B_r} \det Du(x) \ dx = \int_{\partial B_r} \left\{ u^1 \frac{\partial u^2}{\partial x_2} \nu^1 - u^1 \frac{\partial u^2}{\partial x_1} \nu^2 \right\} dH^1 = \int_0^{2\pi} u^1 \frac{\partial u^2}{\partial \vartheta} \, d\vartheta .$$

For symmetric reasons, starting now from $\det Du(x) = -\left(u^2 \cdot u^1_{x_2}\right)_{x_1} + \left(u^2 \cdot u^1_{x_1}\right)_{x_2}$, we also obtain

$$\int_{B_r} \det Du(x) \ dx = \int_0^{2\pi} -u^2 \frac{\partial u^1}{\partial \vartheta} \, d\vartheta ,$$

and thus, for every value of a real parameter λ,

$$\int_{B_r} \det Du(x) \ dx = \lambda \int_0^{2\pi} u^1 \frac{\partial u^2}{\partial \vartheta} \, d\vartheta + (1 - \lambda) \int_0^{2\pi} -u^2 \frac{\partial u^1}{\partial \vartheta} \, d\vartheta$$

and, in particular, for $\lambda = 1/2$ we reach the conclusion in (50). ■

6 The "eight" curve

Let us denote by γ the image of the *"eight" curve*, i.e., the union of the two circles γ^+ and γ^- of radius 1, respectively of center at $(1,0)$ and at $(-1,0)$. Below we will use some elementary representation formulas for γ^+ and γ^-. Precisely, for γ^+ we will use the representation formulas

$$\gamma^+ := \left\{(\xi_1,\xi_2)\in\mathbb{R}^2 : \xi_1^2+\xi_2^2-2\xi_1=0\right\},$$

$$\xi\in\gamma^+\setminus(0,0) \iff \left\{\begin{array}{l}\xi_1 = 2\cos^2\operatorname{Arg}\xi\\ \xi_2 = 2\cos\operatorname{Arg}\xi\cdot\sin\operatorname{Arg}\xi\end{array}\right. . \tag{52}$$

With the aim to prove Theorem 4, we start with some preliminary results concerning a map w with values in the circle γ^+.

Lemma 27 *Let $w:[0,2\pi]\to\gamma^+$ be a Lipschitz-continuous curve such that $w(0)=(2,0)$. The real function $R(\vartheta)$ defined by $R(\vartheta):=0$ if $w(\vartheta)=(0,0)$ and by*

$$R(\vartheta) := \frac{w^1(\vartheta)\,w^2_\vartheta(\vartheta)-w^2(\vartheta)\,w^1_\vartheta(\vartheta)}{|w(\vartheta)|^2},\quad \text{if } w(\vartheta)\neq(0,0)\ , \tag{53}$$

is bounded in $[0,2\pi]$ by a constant depending only on the Lipschitz constant of w. Moreover, if

$$A_w(\vartheta) = \int_0^\vartheta \frac{w^1(t)\,w^2_\vartheta(t)-w^2(t)\,w^1_\vartheta(t)}{|w(t)|^2}\,dt$$

then, for every $\alpha,\beta\in[0,2\pi]$ such that $w(\alpha)\neq(0,0)$ and $w(\beta)\neq(0,0)$, there exists $k\in\mathbb{Z}$ such that

$$A_w(\beta)-A_w(\alpha) = \operatorname{Arg}\,w(\beta)-\operatorname{Arg}\,w(\alpha)+k\pi\,. \tag{54}$$

Proof. Step 1 (boundedness of $R(\vartheta)$). Let L be the Lipschitz constant of w. If $|w(\vartheta)|\geq\frac{1}{2}$, then there exists a constant c such that

$$|R(\vartheta)|\leq cL\,. \tag{55}$$

On the other hand, if $|w(\vartheta)|<\frac{1}{2}$ then, since $\left[w^1(\vartheta)\right]^2+\left[w^2(\vartheta)\right]^2-2w^1(\vartheta)=0$, we deduce that $|w(\vartheta)|^2=2w^1(\vartheta)$ and $w^1(\vartheta)=1-\sqrt{1-[w^2(\vartheta)]^2}$. Taking the derivative of both sides we obtain

$$w^1_\vartheta(\vartheta) = \frac{w^2(\vartheta)\,w^2_\vartheta(\vartheta)}{\sqrt{1-[w^2(\vartheta)]^2}}\,.$$

Therefore, if $w(\vartheta)\neq(0,0)$, for almost every ϑ we also have

$$\begin{aligned} R(\vartheta) &= \frac{w^1(\vartheta)\,w^2_\vartheta(\vartheta)-w^2(\vartheta)\,w^1_\vartheta(\vartheta)}{|w(\vartheta)|^2} = \frac{w^2_\vartheta(\vartheta)}{2}-\frac{w^2(\vartheta)\,w^1_\vartheta(\vartheta)}{2w^1(\vartheta)}\\ &= \frac{w^2_\vartheta(\vartheta)}{2}\left(1-\frac{\left[w^2(\vartheta)\right]^2}{\left(1-\sqrt{1-[w^2(\vartheta)]^2}\right)\cdot\sqrt{1-[w^2(\vartheta)]^2}}\right). \end{aligned}$$

The derivative of the real function $g(t)=1-\sqrt{1-t}$ satisfies the condition $g'(t)\geq 1/2$ for every $t\in[0,1)$; thus we have

$$1-\sqrt{1-[w^2(\vartheta)]^2}\geq\frac{1}{2}\left[w^2(\vartheta)\right]^2\,.$$

We deduce that

$$|R(\vartheta)| \leq \frac{1}{2}\left|w_\vartheta^2(\vartheta)\right|\left(1+\frac{2}{\sqrt{1-\left[w^2(\vartheta)\right]^2}}\right),$$

and again (55) holds for an appropriate constant c since $\left|w^2(\vartheta)\right| < \frac{1}{2}$. This proves the first assertion of the lemma.

Step 2 (proof of (54) under special assumptions). To prove assertion (54) we first make the further assumption that there exist N disjoint open intervals (α_i, β_i) such that

$$0 = \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \ldots \leq \alpha_N < \beta_N = 2\pi,$$

and $w(\vartheta) = (0,0)$ if and only if $\vartheta \in [0, 2\pi] \setminus \cup_{i=1}^N (\alpha_i, \beta_i)$. Fix $\alpha, \beta \in (0, 2\pi)$ such that $w(\alpha) \neq (0,0)$ and $w(\beta) \neq (0,0)$. If $\alpha, \beta \in (\alpha_i, \beta_i)$ for some $i \in \{1, 2, \ldots, N\}$, then, using an argument similar to that of the first part of Lemma 19, we have

$$A_w(\beta) - A_w(\alpha) = \text{Arg } w(\beta) - \text{Arg } w(\alpha) . \tag{56}$$

Otherwise, if there exists $i \in \{1, 2, \ldots, N\}$ such that

$$\alpha_i < \alpha < \beta_i \leq \alpha_{i+1} < \beta < \beta_{i+1}, \tag{57}$$

then we apply (56) to the interval $(\alpha, \beta_i - \varepsilon)$ to obtain

$$A_w(\beta_i - \varepsilon) - A_w(\alpha) = \text{Arg } w(\beta_i - \varepsilon) - \text{Arg } w(\alpha) .$$

In the limit as $\varepsilon \to 0^+$, since when $w(\vartheta) \in \gamma^+ \setminus \{(0,0)\}$ then $\text{Arg } w(\vartheta) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we obtain

$$A_w(\beta_i) - A_w(\alpha) = \pm\frac{\pi}{2} - \text{Arg } w(\alpha) , \tag{58}$$

where the sign $+$ holds if $w^2(\vartheta) > 0$ as $\vartheta \to \beta_i^-$, and the sign $-$ holds otherwise. Similarly, we have

$$A_w(\beta) - A_w(\alpha_{i+1}) = \text{Arg } w(\beta) - \left(\pm\frac{\pi}{2}\right) \tag{59}$$

and, adding side by side (58) and (59), yields

$$A_w(\beta) - A_w(\alpha) = \text{Arg } w(\beta) - \text{Arg } w(\alpha) + k\pi,$$

where $k \in \{-1, 0, 1\}$. The general case, when (57) is not necessarily satisfied, follows from the previous case by iteration.

Step 3 (proof of (54)). Let $w : [0, 2\pi] \to \gamma^+$ be a Lipschitz-continuous map. Let $\{I_j\}_{j\in\mathbb{N}}$ be a sequence of disjoint open intervals (possibly empty) such that $w(\vartheta) \neq (0,0)$ if and only if $\vartheta \in \cup_{j\in\mathbb{N}} I_j$. For every $h \in \mathbb{N}$ we define

$$w_h(\vartheta) := \begin{cases} w(\vartheta) & \text{if } \vartheta \in \cup_{j=1}^h I_j, \\ (0,0) & \text{if } \vartheta \notin \cup_{j=1}^h I_j. \end{cases}$$

Then the sequence of Lipschitz constants L_h of w_h is bounded. Moreover w_h converges uniformly to w in $[0, 2\pi]$, as $h \to +\infty$, and the corresponding sequence $\{R_h(\vartheta)\}_{h\in\mathbb{N}}$ converge to $R(\vartheta)$ almost everywhere in $[0, 2\pi]$. Therefore, integrating (53), we deduce that $\{A_{w_h}(\vartheta)\}_{h\in\mathbb{N}}$ converges to $A_w(\vartheta)$ uniformly in $[0, 2\pi]$.

Let $\alpha, \beta \in [0, 2\pi]$ be such that $w(\alpha) \neq (0,0)$ and $w(\beta) \neq (0,0)$. For h large enough we also have $w_h(\alpha) = w(\alpha) \neq (0,0)$ and $w_h(\beta) = w(\beta) \neq (0,0)$ and, by the previous step,

$$A_{w_h}(\beta) - A_{w_h}(\alpha) = \text{Arg } w(\beta) - \text{Arg } w(\alpha) + k_h\pi .$$

Since the sequence $\{k_h\}$ is bounded, we can pass to the limit in a subsequence and we arrive at the conclusion (54). ∎

Lemma 28 *Under the same assumptions of the previous Lemma 27, for every $\vartheta \in [0, 2\pi]$ we have*

$$w(\vartheta) = 2\cos A_w(\vartheta)(\cos A_w(\vartheta), \sin A_w(\vartheta)) . \tag{60}$$

Proof. Recall that $w(0) = (2, 0)$ and so $\operatorname{Arg} w(\vartheta) = 0$. By Lemma 27, if $w(\vartheta) \neq (0, 0)$, then there exists $k_\vartheta \in \mathbb{Z}$ such that $\operatorname{Arg} w(\vartheta) = A_w(\vartheta) + k_\vartheta \pi$. By (52) we deduce the conclusion

$$\begin{cases} w^1(\vartheta) = 2\cos^2 \operatorname{Arg} w(\vartheta) = 2\cos^2 A_w(\vartheta) \\ w^2(\vartheta) = \sin 2 \operatorname{Arg} w(\vartheta) = \sin 2 A_w(\vartheta) = 2\sin A_w(\vartheta) \cdot \cos A_w(\vartheta) \end{cases} .$$

If $w(\vartheta_0) = (0, 0)$ and there exists a sequence $\vartheta_i \to \vartheta_0$ such that $w(\vartheta_i) \neq (0, 0)$ for every $i \in \mathbb{N}$, then (60) holds for $\vartheta = \vartheta_i$. Since $A_w(\vartheta)$ is a continuous function, (60) holds for $\vartheta = \vartheta_0$ as well.

If $w(\vartheta_0) = (0, 0)$ and a sequence $\vartheta_i \to \vartheta_0$ such that $w(\vartheta_i) \neq (0, 0)$ for every $i \in \mathbb{N}$ does not exist, then there exists an interval $(\vartheta_0 - \delta, \vartheta_0 + \delta)$, with $\delta > 0$, such that $w(\vartheta)$ is identically equal to $(0, 0)$ in $(\vartheta_0 - \delta, \vartheta_0 + \delta)$. In this case let us denote by (α, β) the largest interval containing ϑ_0 with this property; since $R(\vartheta) = 0$ in (α, β) we have $A_w(\alpha) = A_w(\vartheta_0)$. On the other hand (60) holds for $\vartheta = \alpha$ since (α, β) is an extremal interval; hence

$$\begin{aligned} w(\vartheta_0) = (0, 0) = w(\alpha) &= 2\cos A_w(\alpha)(\cos A_w(\alpha), \sin A_w(\alpha)) \\ &= 2\cos A_w(\vartheta_0)(\cos A_w(\vartheta_0), \sin A_w(\vartheta_0)) . \end{aligned}$$

■

The next lemma is similar to the *"umbrella" Lemma 22*, with the main difference that here the starting point of the "*umbrella-stick*" is placed at a boundary point of the circle γ^+.

Lemma 29 (The "umbrella" lemma for the "eight" curve) *Let $w : [0, 2\pi] \to \gamma^+$ be a Lipschitz continuous curve. Assume that there exist $\alpha, \beta \in [0, 2\pi]$, $\alpha < \beta$, such that $A_w(\alpha) = A_w(\beta)$. Then, for every $\varepsilon > 0$, there exists a Lipschitz-continuous map $\widetilde{w} : S(\alpha, \beta) \to \mathbb{R}^2$ satisfying the boundary conditions*

$$\begin{cases} \widetilde{w}(1, \vartheta) = w(\vartheta) & \forall\, \vartheta \in [\alpha, \beta] , \\ \widetilde{w}(\varrho, \alpha) = \varrho w(\alpha) & \forall\, \varrho \in [0, 1] , \\ \widetilde{w}(\varrho, \beta) = \varrho w(\beta) & \forall\, \varrho \in [0, 1] , \end{cases}$$

(note that $w(\alpha) = w(\beta)$) and such that

$$\int_{S(\alpha,\beta)} |\det D\widetilde{w}(x)| \; dx < \varepsilon .$$

Proof. For fixed $h \in \mathbb{N}$ we set

$$\widetilde{w}_h(\varrho, \vartheta) := 2\varrho \cos\varphi_h(\varrho, \vartheta)(\cos\varphi_h(\varrho, \vartheta), \sin\varphi_h(\varrho, \vartheta)) ,$$

where

$$\varphi_h(\varrho, \vartheta) := \varrho^h A_w(\vartheta) + \left(1 - \varrho^h\right) A_w(\alpha) .$$

Let us test the boundary conditions of $\widetilde{w}(\varrho, \vartheta)$. By Lemma 28, for every $\vartheta \in [\alpha, \beta]$ we have

$$\widetilde{w}_h(1, \vartheta) = 2\cos A_w(\vartheta)(\cos A_w(\vartheta), \sin A_w(\vartheta)) = \left(w_h^1(\vartheta), w_h^2(\vartheta)\right) = w(\vartheta) ,$$

and, for every $\varrho \in [0, 1]$,

$$\widetilde{w}_h(\varrho, \alpha) = 2\varrho \cos A_w(\alpha)(\cos A_w(\alpha), \sin A_w(\alpha)) = \varrho\left(w_h^1((\alpha)), w_h^2((\alpha))\right) = \varrho w(\alpha) .$$

Similarly $\widetilde{w}_h(\varrho,\beta)=\varrho w(\beta)$ for every $\varrho\in[0,1]$. Using an argument similar to the one used in Lemma 22, we can see that (we do not denote in the matrix the dependence on h)

$$\det D\widetilde{w}_h(x)=\frac{1}{\varrho}\begin{vmatrix}\widetilde{w}_\varrho^1(\varrho,\vartheta) & \widetilde{w}_\vartheta^1(\varrho,\vartheta)\\ \widetilde{w}_\varrho^2(\varrho,\vartheta) & \widetilde{w}_\vartheta^2(\varrho,\vartheta)\end{vmatrix}=4\varrho^h\cos^2\varphi_h(\varrho,\vartheta)\,A_w'(\vartheta)\ .$$

By Lemma 27 the function

$$A_w'(\vartheta)=\frac{w^1(\vartheta)\,w_\vartheta^2(\vartheta)-w^2(\vartheta)\,w_\vartheta^1(\vartheta)}{|w(\vartheta)|^2}$$

is bounded; thus there exists a constant c such that

$$\int_{S(\alpha,\beta)}|\det D\widetilde{w}_h(x)|\ dx\le c\int_0^1\varrho^{h+1}\,d\varrho=\frac{c}{h+2}\,,$$

and this concludes the proof of our lemma. ■

Lemma 30 *Let $w:[0,2\pi]\to\gamma^+$ be a Lipschitz-continuous map. If $\alpha,\beta\in[0,2\pi]$, $\alpha<\beta$, are such that $A_w(\alpha)=A_w(\beta)$, and if the function $A_w(\vartheta)$ is piecewise strictly monotone in $[\alpha,\beta]$ (with a finite number of monotonicity intervals), then*

$$\int_\alpha^\beta\left\{w^1(\vartheta)\,w_\vartheta^2(\vartheta)-w^2(\vartheta)\,w_\vartheta^1(\vartheta)\right\}\,d\vartheta\ =\ 0\,.$$

Proof. This result can be proved just as in Lemma 24. ■

The lemma below was established in [24].

Lemma 31 *Let $u:[0,2\pi]\to\gamma=\gamma^+\cup\gamma^-$ be a Lipschitz-continuous map. Assume that there exist N disjoint open intervals $I_j\subset[0,2\pi]$ such that $u(I_j)$ is contained either in γ^+ or in γ^- for every $j=1,2,\ldots,N$, and $u(\vartheta)=(0,0)$ when $\vartheta\notin\cup_{j=1}^N I_j$. Assume, in addition, that the function*

$$\vartheta\ \longrightarrow\ u^1(\vartheta)\,u_\vartheta^2(\vartheta)-u^2(\vartheta)\,u_\vartheta^1(\vartheta)$$

has piecewise constant sign in $[0,2\pi]$. Then, for every $\varepsilon>0$, there exists a Lipschitz-continuous map $\widetilde{w}:B_1\to\mathbb{R}^2$ satisfying the boundary condition $\widetilde{w}(1,\vartheta)=u(\vartheta)$ for every $\vartheta\in[0,2\pi]$, and such that

$$\int_{B_1}|\det D\widetilde{w}(x)|\ dx<\varepsilon+\frac{1}{2}\sum_{j=1}^N\left|\int_{I_j}\left\{u^1(\vartheta)\,u_\vartheta^2(\vartheta)-u^2(\vartheta)\,u_\vartheta^1(\vartheta)\right\}\,d\vartheta\right|.$$

7 The $n-$dimensional case

In this section we prove Theorem 9. We first recall a lower bound and an upper bound estimates for $TV(u,\Omega)$ that have been obtained in [24].We note that Lemma 32 is a variant of Lemma 5.1 (see also Lemma 2.3) by Marcellini [48], who considered the general *quasiconvex case* with the exponent p below the critical growth exponent n, precisely $n^2/(n+1)<p<n$.

Lemma 32 (Lower bound - first estimate) *Let $u\in L^\infty(\Omega;\mathbb{R}^n)\cap W^{1,p}(\Omega;\mathbb{R}^n)\cap W^{1,\infty}_{\text{loc}}(\Omega\setminus\{0\};\mathbb{R}^n)$ for some $p\in(n-1,n)$. The following estimate holds*

$$TV(u,\Omega)\ge\left|\int_\Omega\det D\widetilde{u}(x)\ dx\right|\,,$$

whenever $\widetilde{u}:\Omega\to\mathbb{R}^n$ is a Lipschitz-continuous map which agrees with u on the boundary of Ω, i.e., $\widetilde{u}(x)=u(x)$ on $\partial\Omega$.

Lemma 33 (Lower bound - second estimate) *Let $u \in L^{\infty}(\Omega;\mathbb{R}^n)\cap W^{1,p}(\Omega;\mathbb{R}^n)\cap W^{1,\infty}_{\mathrm{loc}}(\Omega\backslash\{0\};\mathbb{R}^n)$ for some $p \in (n-1,n)$. For every $r>0$ such that $B_r \subset \Omega$ the following estimate holds*

$$TV(u,\Omega) \geq \int_{\Omega\backslash B_r} |\det Du(x)|\ dx + \left| \int_{B_r} \det D\widetilde{u}(x)\ dx \right| , \tag{61}$$

where $\widetilde{u}: B_r \to \mathbb{R}^n$ is any Lipschitz-continuous map which coincides with u on the boundary of B_r, i.e., $\widetilde{u}(x) = u(x)$ on ∂B_r.

Lemma 34 *Let $u \in W^{1,p}(B_1;\mathbb{R}^n)\cap W^{1,\infty}_{\mathrm{loc}}(B_1\backslash\{0\};\mathbb{R}^n)$ for some $p \in [1,n)$. If*

$$\frac{1}{\varrho^{n-p}} \int_{B_\varrho} |D_\tau u|^p\ dx \leq M_0$$

for every $\varrho \in (0,1)$ and for some positive constant M_0, then there exists a constant $c(n,p)$ and a sequence $\varrho_j \to 0$ such that

$$\frac{1}{\varrho_j^{n-p-1}} \int_{\partial B_{\varrho_j}} |D_\tau u|^p\ dH^{n-1} \leq c(n,p)\, M_0 .$$

Proof. For every $j \geq 2$ we have

$$\int_{1/(2j)}^{1/j} d\varrho \int_{\partial B_\varrho} |D_\tau u|^p\ dH^{n-1} \leq \int_{B_{1/j}} |D_\tau u|^p\ dx \leq \frac{M_0}{j^{n-p}} . \tag{62}$$

Therefore there exist $\varrho_j \in \left(\frac{1}{2j}, \frac{1}{j}\right)$ such that

$$\int_{\partial B_{\varrho_j}} |D_\tau u|^p\ dH^{n-1} \leq \frac{3M_0}{j^{n-p-1}} ; \tag{63}$$

in fact, if (63) does not hold, then for every $\varrho \in \left(\frac{1}{2j}, \frac{1}{j}\right)$ we should have

$$\int_{\partial B_\varrho} |D_\tau u|^p\ dH^{n-1} \geq \frac{3M_0}{j^{n-p-1}}$$

and thus

$$\int_{1/(2j)}^{1/j} d\varrho \int_{\partial B_\varrho} |D_\tau u|^p\ dH^{n-1} \geq \frac{3M_0}{j^{n-p-1}} \cdot \frac{1}{2j} > \frac{M_0}{j^{n-p}} ,$$

which is in contradiction with (62). Since $\frac{1}{2j} < \varrho_j < \frac{1}{j}$, we deduce that $\varrho_j \to 0$, and that $\left(\frac{1}{j}\right)^{n-p-1} \leq \varrho_j^{n-p-1}$ if $p \geq n-1$, while $\left(\frac{1}{j}\right)^{n-p-1} \leq (2\varrho_j)^{n-p-1}$ if $p < n-1$. From (63) we finally have

$$\int_{\partial B_{\varrho_j}} |D_\tau u|^p\ dH^{n-1} \leq \frac{3M_0}{j^{n-p-1}} \leq c(n,p)\, M_0\, \varrho_j^{n-p-1} ,$$

where $c(n,p) = 3$ if $p \in [n-1,n)$, $c(n,p) = 3 \cdot 2^{n-p-1}$ if $p \in [1,n-1)$. ■

We denote a generic element of the surface of the unit ball $\partial B_1 = S^{n-1}$ by ω. Let $\omega_0 \in S^{n-1}$ be fixed. For every $j \in \{1,2,\ldots,n-1\}$ let $\tau_j : S^{n-1} - \{\omega_0\} \to S^{n-1}$ by a vector field of class C^1 such that, for every $x \in S^{n-1} - \{\omega_0\}$, the set of vectors $\{\tau_1(\omega), \tau_2(\omega), \ldots, \tau_{n-1}(\omega)\}$ is an orthonormal basis for the tangent plane to the surface S^{n-1} at the point ω. Without loss of generality (up to a change of sign to one of the vectors) we can assume that $\tau_1(\omega), \tau_2(\omega), \ldots, \tau_{n-1}(\omega)$ have the property that, if we denote by $\nu(\omega)$ the *exterior* normal unit vector to S^{n-1} at ω, then the system of vectors $\{\nu(\omega), \tau_1(\omega), \ldots, \tau_{n-1}(\omega)\}$ is a positively oriented basis of $\mathbb{R}^n$. I.e.,

$$\nu(\omega) \wedge \tau_1(\omega) \wedge \ldots \wedge \tau_{n-1}(\omega) = e_1 \wedge e_2 \wedge \ldots \wedge e_n$$

or, equivalently, that the determinant of the matrix whose column vectors are the components of $\nu(\omega), \tau_1(\omega), \ldots, \tau_{n-1}(\omega)$ with respect to $e_1, e_2, \ldots, e_n$, is equal to 1.

If $v : S^{n-1} \to \mathbb{R}^n$, $v \in W^{1,\infty}(S^{n-1}; \mathbb{R}^n)$, $v = (v^1, v^2, \ldots, v^n)$, is a Lipschitz-continuous map, we denote by $D_\tau v$ the vector of $\mathbb{R}^{n-1}$ whose components are $D_{\tau_1} v, D_{\tau_2} v, \ldots, D_{\tau_{n-1}} v$.

Lemma 35 *Let $v \in W^{1,\infty}(S^{n-1}; \mathbb{R}^n)$, $\eta \in C^1([0,1])$, with $\eta(0) = 0$ and let $w(x) = \eta(|x|)\, v\left(\frac{x}{|x|}\right)$. For almost every $x \in B_1$ we have*

$$|Dw(x)|^2 = \left|\eta'(|x|)\, v\left(\frac{x}{|x|}\right)\right|^2 + \frac{\eta^2(|x|)}{|x|^2}\left|D_\tau v\left(\frac{x}{|x|}\right)\right|^2; \tag{64}$$

$$\det Dw(x) = \frac{\eta'(|x|)\,\eta^{n-1}(|x|)}{|x|^{n-1}} \sum_{i=1}^{n} (-1)^{i-1} v^i\left(\frac{x}{|x|}\right) \frac{\partial(v^1, \ldots, v^{i-1}, v^{i+1}, \ldots, v^n)}{\partial(\tau_1, \tau_2, \ldots, \tau_{n-1})}\left(\frac{x}{|x|}\right). \tag{65}$$

Moreover, if $\eta(t) = t$ for every $t \in [0,1]$, then

$$\int_{B_1} \det Dw(x)\, dx = \frac{1}{n} \int_{\partial B_1} \sum_{i=1}^{n} (-1)^{i-1} v^i(\omega) \frac{\partial(v^1, \ldots, v^{i-1}, v^{i+1}, \ldots, v^n)}{\partial(\tau_1, \tau_2, \ldots, \tau_{n-1})}(\omega)\, dH^{n-1}. \tag{66}$$

Proof. Since $v : S^{n-1} \to \mathbb{R}^n$ is a Lipschitz-continuous map, then $D_\tau v(\omega)$ exists (in the classical sense) H^{n-1} almost everywhere on S^{n-1} and the map $x \to v(x/|x|)$ is classically differentiable for almost every $x \in B_1$. Let $x \neq 0$ be a point of B_1 where $v(x/|x|)$ is differentiable; since the vectors

$$\nu = \nu\left(\frac{x}{|x|}\right), \ \tau_1 = \tau_1\left(\frac{x}{|x|}\right), \ldots, \ \tau_{n-1} = \tau_{n-1}\left(\frac{x}{|x|}\right),$$

form a basis of $\mathbb{R}^n$, for every $i = 1, 2, \ldots, n$ we have

$$Dw^i(x) = \frac{\partial w^i(x)}{\partial \nu}\nu + \sum_{j=1}^{n-1} \frac{\partial w^i(x)}{\partial \tau_j}\tau_j = \eta'(|x|)\, v^i\left(\frac{x}{|x|}\right)\nu + \frac{\eta(|x|)}{|x|} \sum_{j=1}^{n-1} \frac{\partial v^i(x/|x|)}{\partial \tau_j}\tau_j,$$

and thus we obtain (64). Moreover, $Dw(x)$ is equal to the matrix $\{Dw^1(x), Dw^2(x), \ldots, Dw^n(x)\}$. If we express each column of $Dw(x)$ as linear combination of the elements of the basis $\{\nu, \tau_1, \ldots, \tau_{n-1}\}$, since $w(x) = \eta(|x|)\, v\left(\frac{x}{|x|}\right)$, we obtain the matrix

$$Dw(x) = \begin{pmatrix} \eta'(|x|)\, v^1\left(\frac{x}{|x|}\right) & \eta'(|x|)\, v^2\left(\frac{x}{|x|}\right) & \ldots & \eta'(|x|)\, v^n\left(\frac{x}{|x|}\right) \\ \frac{\eta(|x|)}{|x|}\frac{\partial v^1}{\partial \tau_1}\left(\frac{x}{|x|}\right) & \frac{\eta(|x|)}{|x|}\frac{\partial v^2}{\partial \tau_1}\left(\frac{x}{|x|}\right) & \ldots & \frac{\eta(|x|)}{|x|}\frac{\partial v^n}{\partial \tau_1}\left(\frac{x}{|x|}\right) \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\eta(|x|)}{|x|}\frac{\partial v^1}{\partial \tau_{n-1}}\left(\frac{x}{|x|}\right) & \frac{\eta(|x|)}{|x|}\frac{\partial v^2}{\partial \tau_{n-1}}\left(\frac{x}{|x|}\right) & \ldots & \frac{\eta(|x|)}{|x|}\frac{\partial v^n}{\partial \tau_{n-1}}\left(\frac{x}{|x|}\right) \end{pmatrix}.$$

Thus the determinant of the matrix $Dw(x)$, computed by developing the first row, is given by (65). By integrating over B_1 both sides of (65), with $\eta(t)=t$ for every $t\in[0,1]$, since $\frac{\eta'(|x|)\eta^{n-1}(|x|)}{|x|^{n-1}}=1$, we obtain

$$\begin{aligned}\int_{B_1}\det Dw(x)\,dx &= \int_{B_1}\sum_{i=1}^{n}(-1)^{i-1}v^i\left(\frac{x}{|x|}\right)\frac{\partial\left(v^1,\dots,v^{i-1},v^{i+1},\dots,v^n\right)}{\partial\left(\tau_1,\tau_2,\dots,\tau_{n-1}\right)}\left(\frac{x}{|x|}\right)dx\\ &= \int_0^1 d\varrho\int_{\partial B_\varrho}\sum_{i=1}^{n}(-1)^{i-1}v^i\left(\frac{x}{|x|}\right)\frac{\partial\left(v^1,\dots,v^{i-1},v^{i+1},\dots,v^n\right)}{\partial\left(\tau_1,\tau_2,\dots,\tau_{n-1}\right)}\left(\frac{x}{|x|}\right)dH^{n-1}\\ &= \int_0^1\varrho^{n-1}d\varrho\int_{\partial B_1}\sum_{i=1}^{n}(-1)^{i-1}v^i(\omega)\frac{\partial\left(v^1,\dots,v^{i-1},v^{i+1},\dots,v^n\right)}{\partial\left(\tau_1,\tau_2,\dots,\tau_{n-1}\right)}(\omega)\,dH^{n-1}\\ &= \frac{1}{n}\int_{\partial B_1}\sum_{i=1}^{n}(-1)^{i-1}v^i(\omega)\frac{\partial\left(v^1,\dots,v^{i-1},v^{i+1},\dots,v^n\right)}{\partial\left(\tau_1,\tau_2,\dots,\tau_{n-1}\right)}(\omega)\,dH^{n-1}.\end{aligned}$$

■

Lemma 36 *Let Ω be an open set containing the origin. Assume that, for some $p\in(n-1,n)$, $u\in W^{1,p}(\Omega;\mathbb{R}^n)\cap W^{1,\infty}_{\mathrm{loc}}(\Omega\setminus\{0\};\mathbb{R}^n)$ satisfies*

$$\sup_{\varrho>0}\frac{1}{\varrho^{n-p}}\int_{B_\varrho}|D_\tau u|^p\,dx\le M_0$$

for a positive constant M_0. Let $v\in W^{1,\infty}\left(S^{n-1};\mathbb{R}^n\right)$ be such that

$$\lim_{\varrho\to0^+}\max\{|u(\varrho\omega)-v(\omega)|:\ \omega\in\partial B_1\}=0\,.$$

Then there exists a sequence $\varrho_j\to0$ such that

$$\lim_{j\to+\infty}\int_{B_1}\det Dw_j(x)\,dx=\int_{B_1}\det Dw(x)\,dx\,, \tag{67}$$

where $w_j(x):=|x|\ u\left(\varrho_j\frac{x}{|x|}\right)$ and $w(x):=|x|\ v\left(\frac{x}{|x|}\right)$.

Proof. Let ϱ_j be the real sequence converging to zero of Lemma 34. By assumption $w_j(x):=|x|\ u\left(\varrho_j\frac{x}{|x|}\right)$ converges to $w(x):=|x|\ v\left(\frac{x}{|x|}\right)$ uniformly in B_1. Let us prove that w_j weakly converge in $W^{1,p}(B_1;\mathbb{R}^n)$ to w. In fact, by (64) of Lemma 35 we have

$$|Dw_j(x)|^2=\left|u\left(\varrho_j\frac{x}{|x|}\right)\right|^2+\varrho_j^2\left|D_\tau u\left(\varrho_j\frac{x}{|x|}\right)\right|^2$$

and thus the L^p norm of Dw_j remains bounded. In fact, by Lemma 34,

$$\begin{aligned}\int_{B_1}|Dw_j(x)|^p\,dx &\le c_1+c_2\varrho_j^p\int_{B_1}\left|D_\tau u\left(\varrho_j\frac{x}{|x|}\right)\right|^p dx\\ &= c_1+c_2\varrho_j^p\int_0^1 dr\int_{\partial B_r}\left|D_\tau u\left(\varrho_j\frac{x}{|x|}\right)\right|^p dH^{n-1}\\ &= c_1+c_2\varrho_j^p\int_0^1\frac{r^{n-1}}{\varrho_j^{n-1}}\,dr\int_{\partial B_{\varrho_j}}|D_\tau u(y)|^p\,dH_y^{n-1}\\ &= c_1+\frac{c_2}{n\varrho_j^{n-p-1}}\int_{\partial B_{\varrho_j}}|D_\tau u(y)|^p\,dH_y^{n-1}=c_1+\frac{c_2}{n}\,c(n,p)\,M_0\,.\end{aligned}$$

By (65) we also have, with $\alpha = p/(n-1)$,

$$\int_{B_1} |\det Dw_j(x)|^\alpha \, dx \le c_3 \varrho_j^{\alpha(n-1)} \int_{B_1} \left| u\left(\varrho_j \frac{x}{|x|}\right)\right|^\alpha \left| D_\tau u\left(\varrho_j \frac{x}{|x|}\right)\right|^{\alpha(n-1)} dx \tag{68}$$

$$\le c_4 \varrho_j^{\alpha(n-1)} \varrho_j^{1-n} \int_{\partial B_{\varrho_j}} |D_\tau u(y)|^p \, dH_y^{n-1} = c_4 \frac{1}{\varrho_j^{n-1-p}} \int_{\partial B_{\varrho_j}} |D_\tau u(y)|^p \, dH_y^{n-1},$$

which is bounded, again by Lemma 34. Therefore, since $\alpha > 1$, to obtain the conclusion (67) it is sufficient to prove that

$$\lim_{j\to+\infty} \int_{B_1} \varphi \det Dw_j(x)\, dx = \int_{B_1} \varphi \det Dw(x)\, dx\,, \qquad \forall\, \varphi \in C_0^1(B_1)\,. \tag{69}$$

Since $p > n-1$, we apply Reshetnyak's [59] weak continuity result on the matrix $\mathrm{adj}_{n-1} Dw_j$ of minors $(n-1)\times(n-1)$ of Dw_j, which weakly converge in $L^{\frac{p}{n-1}}$ to the corresponding matrix $\mathrm{adj}_{n-1} Dw$ of minors of Dw (see)(36)). By the uniform convergence of w_j to w, for every $\varphi \in C_0^1(B_1)$ we get the conclusion

$$\lim_{j\to+\infty} \int_{B_1} \varphi \det Dw_j \, dx = \lim_{j\to+\infty} -\int_{B_1} w_j^1 \frac{\partial\left(\varphi, w_j^2, \dots, w_j^n\right)}{\partial(x_1, x_2, \dots, x_n)}\, dx$$

$$= -\int_{B_1} w^1 \frac{\partial\left(\varphi, w^2, \dots, w^n\right)}{\partial(x_1, x_2, \dots, x_n)}\, dx = \int_{B_1} \varphi \det Dw\, dx\,.$$

■

Proof of Theorem 6. Let $u \in L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^2) \cap W^{1,p}(\Omega;\mathbb{R}^n) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega\setminus\{0\};\mathbb{R}^n)$ for some $p \in (n-1, n)$. Let $\{\varepsilon_h\}_{h\in\mathbb{N}}$ be a sequence converging to zero and consider the *convolution* $u_h := u * \eta_{\varepsilon_h}$ of u with a smooth *mollifier* η_{ε_h}. For every $h \in \mathbb{N}$, $u_h \in C^1(\Omega_h;\mathbb{R}^n)$, where we set $\Omega_h := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \varepsilon_h\}$. Moreover, for every $\Omega' \subset\subset \Omega\setminus\{0\}$, $u_h \to u$ uniformly in Ω', $Du_h(x) \to Du(x)$ for every $x \in \Omega \setminus E$, where E is a Borel set of zero measure, and the sequence $\{u_h\}_{h\in\mathbb{N}}$ is Lipschitz-continuos in Ω', with a Lipschitz constant independent of h. Denote by N_0 the set of real numbers given by

$$N_0 := \left\{\varrho > 0 : \; H^{n-1}(\partial B_\varrho \cap E) > 0\right\}\,.$$

If $B_r \subset\subset \Omega$ then we have

$$0 = |E \cap B_r| = \int_0^r H^{n-1}(\partial B_\varrho \cap E)\, d\varrho\,,$$

and thus the one-dimensional Lebesgue measure of N_0 is equal to zero. We can repeat the proof of Lemma 36 to reach the same conclusion for a sequence $\{\varrho_j\}_{j\in\mathbb{N}} \subset (0, r)$, $\{\varrho_j\}_{j\in\mathbb{N}} \cap N_0 = \varnothing$, $\varrho_j \to 0$. Since $u_h \to u$ uniformly on B_{ϱ_j}, $D_\tau u_h(x) \to D_\tau u(x)$ H^{n-1}−almost everywhere on B_{ϱ_j}, and the sequence $\{u_h\}_{h\in\mathbb{N}}$ is Lipschitz-continuos on B_{ϱ_j} with a Lipschitz constant independent of h, then $D_\tau u_h \to D_\tau u$ in $L^q\left(\partial B_{\varrho_j}\right)$ for every $q \ge 1$. Fixed $\varphi \in C_0^1(\Omega)$ and denoting by $\nu = \nu(x) =$

$(\nu^1, \nu^2, \ldots, \nu^n)$ the *exterior* normal unit vector to ∂B_{ϱ_j}, we have

$$\begin{aligned}
\int_{\Omega\setminus B_{\varrho_j}} u^1 \frac{\partial\left(\varphi, u^2, \ldots, u^n\right)}{\partial\left(x_1, x_2, \ldots, x_n\right)}\, dx &= \lim_{h\to+\infty} \int_{\Omega\setminus B_{\varrho_j}} u_h^1 \frac{\partial\left(\varphi, u_h^2, \ldots, u_h^n\right)}{\partial\left(x_1, x_2, \ldots, x_n\right)}\, dx \\
&= -\lim_{h\to+\infty} \left\{ \int_{\Omega\setminus B_{\varrho_j}} \varphi \frac{\partial\left(u_h^1, u_h^2, \ldots, u_h^n\right)}{\partial\left(x_1, x_2, \ldots, x_n\right)}\, dx \right. \\
&\quad \left. + \int_{\partial B_{\varrho_j}} u_h^1 \sum_{i=1}^n (-1)^{i-1} \varphi \frac{\partial\left(u_h^2, u_h^3, \ldots, u_h^n\right)\, \nu^i}{\partial\left(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\right)}\, dH^{n-1} \right\} \\
&= -\int_{\Omega\setminus B_{\varrho_j}} \varphi \det Du\, dx \\
&\quad - \int_{\partial B_{\varrho_j}} u^1 \sum_{i=1}^n (-1)^{i-1} \varphi \frac{\partial\left(u^2, u^3, \ldots, u^n\right)\, \nu^i}{\partial\left(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\right)}\, dH^{n-1}\,.
\end{aligned}$$

By the analytic expression (78) of ν, together with (iii) of Lemma 37, with the notation $w_j(x) := |x|\; u\left(\varrho_j \frac{x}{|x|}\right)$, we obtain

$$\begin{aligned}
&\int_{\Omega\setminus B_{\varrho_j}} u^1 \frac{\partial\left(\varphi, u^2, \ldots, u^n\right)}{\partial\left(x_1, x_2, \ldots, x_n\right)}\, dx = -\int_{\Omega\setminus B_{\varrho_j}} \varphi \det Du\, dx - \int_{\partial B_{\varrho_j}} u^1 \varphi \frac{\partial\left(u^2, u^3, \ldots, u^n\right)}{\partial\left(\tau_1, \tau_2, \ldots,, \tau_{n-1}\right)}\, dH^{n-1} \\
&= -\int_{\Omega\setminus B_{\varrho_j}} \varphi \det Du\, dx - \int_{\partial B_1} w_j^1 \varphi\left(\varrho_j \omega\right) \frac{\partial\left(w_j^2, w_j^3, \ldots, w_j^n\right)}{\partial\left(\tau_1, \tau_2, \ldots,, \tau_{n-1}\right)}\, dH^{n-1} \\
&= -\int_{\Omega\setminus B_{\varrho_j}} \varphi \det Du\, dx - \int_{\partial B_1} w_j^1 \varphi\left(\varrho_j \omega\right) \sum_{i=1}^n (-1)^{i-1} \frac{\partial\left(w_j^2, w_j^3, \ldots, w_j^n\right)\, \nu^i}{\partial\left(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\right)}\, dH^{n-1} \\
&= -\int_{\Omega\setminus B_{\varrho_j}} \varphi \det Du\, dx - \int_{B_1} \frac{\partial\left(w_j^1 \varphi\left(\varrho_j \frac{x}{|x|}\right), w_j^2, \ldots, w_j^n\right)}{\partial\left(x_1, x_2, \ldots, x_n\right)}\, dx \\
&= -\int_{\Omega\setminus B_{\varrho_j}} \varphi \det Du\, dx - \int_{B_1} \varphi\left(\varrho_j \frac{x}{|x|}\right) \det Dw_j\, dx - \int_{B_1} w_j^1 \frac{\partial\left(\varphi\left(\varrho_j \frac{x}{|x|}\right), w_j^2, \ldots, w_j^n\right)}{\partial\left(x_1, x_2, \ldots, x_n\right)}\, dx\,. \quad (70)
\end{aligned}$$

As $j \to +\infty$ the quantity $\varphi\left(\varrho_j \frac{x}{|x|}\right)$ converges to $\varphi(0)$ uniformly in B_1. Then, by the bound (68) and by (69), we obtain

$$\lim_{j\to+\infty} \int_{B_1} \varphi\left(\varrho_j \frac{x}{|x|}\right) \det Dw_j(x)\; dx = \int_{B_1} \varphi(0) \det Dw(x)\; dx\,.$$

Moreover, as in the proof of Lemma 36, the sequence $\{|Dw_j|\}_{j\in\mathbb{N}}$ is bounded in $L^p(B_1)$ and

$$\left| \int_{B_1} w_j^1 \frac{\partial\left(\varphi\left(\varrho_j \frac{x}{|x|}\right), w_j^2, \ldots, w_j^n\right)}{\partial\left(x_1, x_2, \ldots, x_n\right)}\, dx \right| \le c_1 \varrho_j \int_{B_1} \frac{|w_j|\; |Dw_j|^{n-1}}{|x|}\, dx \le c_2 \varrho_j\,,$$

which converges to zero as $j \to +\infty$. Therefore, since $\det Du \in L^1(\Omega)$, letting $j \to +\infty$ in (70) we obtain

$$\int_{\Omega} u^1 \frac{\partial\left(\varphi, u^2, \ldots, u^n\right)}{\partial\left(x_1, x_2, \ldots, x_n\right)}\, dx = -\int_{\Omega} \varphi \det Du\, dx - \varphi(0) \int_{B_1} \det Dw\, dx\,,$$

with $w(x) := |x|\ v\left(\frac{x}{|x|}\right)$; i.e.,

$$\operatorname{Det} Du = \det Du + m_0\delta_0\,, \qquad \text{where } m_0 = \int_{B_1} \det Dw\, dx\,.$$

Then, the total variation $|\operatorname{Det} Du|(\Omega)$ of $\operatorname{Det} Du$ is equal to

$$|\operatorname{Det} Du|(\Omega) = \int_\Omega |\det Du|\ dx + |m_0|\ ,$$

which agrees with the conclusion (16). ■

Proof of Theorem 9. Step 1 (lower bound). We first notice that, by virtue of (13), there exists $r > 0$ such that $u \in L^\infty(B_r;\mathbb{R}^n)$. Let $p \in (n-1, n)$. Let $\varrho_j \to 0$ be the sequence of the Lemma 34, and consider $j \in \mathbb{N}$ sufficiently large so that $B_{\varrho_j} \subset B_r \subset \Omega$. By the estimate (61) of Lemma 33 we have

$$TV(u,\Omega) \geq \int_{\Omega\setminus B_{\varrho_j}} |\det Du(x)|\ dx\ + \left|\int_{B_{\varrho_j}} \det D\widetilde{u}(x)\ dx\right|\ , \tag{71}$$

where $\widetilde{u} : B_{\varrho_j} \to \mathbb{R}^n$ is any Lipschitz-continuous map which assumes the boundary value $\widetilde{u}(x) = u(x)$ on ∂B_{ϱ_j}. In particular, we consider the extension $\widetilde{u} = \widetilde{w}_j$ given by $\widetilde{w}_j(x) := \frac{|x|}{\varrho_j}\ u\left(\varrho_j \frac{x}{|x|}\right)$, and, using a change of variables, we have

$$\int_{B_{\varrho_j}} \det D\widetilde{w}_j(x)\ dx = \int_{B_1} \det Dw_j(x)\ dx\,,$$

where $w_j(x) := |x|\ u\left(\varrho_j \frac{x}{|x|}\right)$. Letting $j \to +\infty$ in (71), by Lemma 36 we get

$$TV(u,\Omega) \geq \liminf_{j\to+\infty} \int_{\Omega\setminus B_{\varrho_j}} |\det Du(x)|\ dx\ + \lim_{j\to+\infty} \left|\int_{B_{\varrho_j}} \det D\widetilde{w}_j(x)\ dx\right|$$

$$= \int_\Omega |\det Du(x)|\ dx\ + \lim_{j\to+\infty} \left|\int_{B_1} \det Dw_j(x)\ dx\right| = \int_\Omega |\det Du(x)|\ dx\ + \left|\int_{B_1} \det Dw(x)\ dx\right|\ ,$$

where $w(x) := |x|\ v\left(\frac{x}{|x|}\right)$. We represent $\det Dw(x)$ using (66) of Lemma 35, and we obtain the lower bound

$$\begin{aligned} TV(u,\Omega) \geq \int_\Omega |\det Du(x)|\ dx\ + \left|\int_{B_1} \det Dw(x)\ dx\right| = \int_\Omega |\det Du(x)|\ dx \\ + \frac{1}{n}\left|\int_{\partial B_1} \sum_{i=1}^n (-1)^{i-1}\, v^i(\omega)\, \frac{\partial\left(v^1,\ldots,v^{i-1},v^{i+1},\ldots,v^n\right)}{\partial(\tau_1,\tau_2,\ldots,\tau_{n-1})}(\omega)\ dH^{n-1}\right|\,. \end{aligned}$$

Step 2 (upper bound in the radially symmetric case). Here we assume that $u(x) := v(x/|x|)$. Let ϱ_h be a sequence of positive numbers converging to zero as $h \to +\infty$ and let $h \in \mathbb{N}$ be sufficiently large so that $B_{\varrho_h} \subset \Omega$. As before, we use the notation $w(x) := |x|\, v(x/|x|)$, and we define

$$u_h(x) := \begin{cases} \frac{|x|}{\varrho_h}\, v\left(\frac{x}{|x|}\right) = \frac{1}{\varrho_h} w(x) = w\left(\frac{x}{\varrho_h}\right) & \text{if } x \in B_{\varrho_h}, \\ u(x) = v\left(\frac{x}{|x|}\right) & \text{if } x \in \Omega\setminus B_{\varrho_h} \end{cases}\ .$$

Then $\{u_h\}_{h\in\mathbb{N}}$ converges to u in in the *strong* norm topology of $W^{1,p}(\Omega;\mathbb{R}^n)$. Therefore we can use the definition (22) of $TV^s(u,\Omega)$ and, since $\det Du(x)=0$ in $\Omega\backslash B_{\varrho_h}$ we have

$$TV^s(u,\Omega)\le\liminf_{h\to+\infty}\int_{B_{\varrho_h}}\left|\frac{1}{\varrho_h^n}\det Dw\left(\frac{x}{\varrho_h}\right)\right|\,dx=\int_{B_1}\left|\det Dw(x)\right|\,dx=\left|\int_{B_1}\det Dw(x)\,dx\right|,\tag{72}$$

where the last equality follows from the fact that, by assumption, $\det Dw(x)$ has constant sign in B_1. In fact, by (65) of Lemma 35, with $\eta(|x|)=|x|$, we have

$$\det Dw(x)=\sum_{i=1}^{n}(-1)^{i-1}v^i\left(\frac{x}{|x|}\right)\frac{\partial\left(v^1,\ldots,v^{i-1},v^{i+1},\ldots,v^n\right)}{\partial\left(\tau_1,\tau_2,\ldots,\tau_{n-1}\right)}\left(\frac{x}{|x|}\right),$$

and thus, by the *sign condition* (18), the left hand side has constant sign as well as the right hand side. Therefore, from Step 1 and from (72), when $u(x):=v(x/|x|)$ we get

$$TV(u,\Omega)=TV^s(u,\Omega)=TV(v,B_1)$$
$$=\frac{1}{n}\left|\int_{\partial B_1}\sum_{i=1}^{n}(-1)^{i+1}v^i(\omega)\frac{\partial\left(v^1,\ldots,v^{i-1},v^{i+1},\ldots,v^n\right)}{\partial\left(\tau_1,\tau_2,\ldots,\tau_{n-1}\right)}(\omega)\,dH^{n-1}\right|.$$

We explicitly observe that, as a consequence of what we have shown in Steps 1 and 2, we have achieved the proof of Theorem 11 in the radially symmetric case; moreover, the representation formula for $TV(v,\Omega)$ is independent of the open set Ω containing the origin.

Step 3 (upper bound in the general case). By Lemma 34 there exists a sequence $(\varrho_h)_{h\in\mathbb{N}}$, converging to zero as $h\to+\infty$, and such that

$$\frac{1}{\varrho_h^{n-p-1}}\int_{\partial B_{\varrho_h}}|D_\tau u|^p\,dH^{n-1}\le c(n,p)\,M_0\,.\tag{73}$$

For every $h\in\mathbb{N}$, we denote by σ_h a real sequence in $(0,1)$ to be chosen later (see (76)). For every $h=1,2,\ldots$, let $\eta_h(\varrho)$ be a *cut-off* function such that $\eta_h(\varrho)=1$ if $0\le\varrho\le\varrho_h(1-\sigma_h)$, $\eta_h(\varrho)=0$ if $\varrho_h\le\varrho\le1$, $\eta_h(\varrho)$ is linear in the interval $[\varrho_h(1-\sigma_h),\varrho_h]$. Fix $\varepsilon>0$. From Step 2 there exists a Lipschitz-continuous map $w:B_1\to\mathbb{R}^n$ such that $w(x):=v\left(\frac{x}{|x|}\right)$ on a neighborhood of ∂B_1 and

$$\int_{B_1}|\det Dw(x)|\,dx<TV(v,B_1)+\varepsilon\,.\tag{74}$$

Then, with the notation $\omega:=x/|x|$, we define

$$u_h(x):=\begin{cases}w\left(\frac{x}{\varrho_h(1-\sigma_h)}\right) & \text{if } 0\le|x|\le\varrho_h(1-\sigma_h)\,,\\ \eta_h(|x|)\,v(\omega)+[1-\eta_h(|x|)]\,u(\varrho_h\omega) & \text{if } \varrho_h(1-\sigma_h)<|x|<\varrho_h,\\ u(x) & \text{if } x\in\Omega\backslash B_{\varrho_h}\end{cases}.\tag{75}$$

We first prove that $\{u_h\}_{h\in\mathbb{N}}$ converges to u in the *strong topology* of $W^{1,p}(\Omega;\mathbb{R}^n)$. In fact

$$\int_\Omega|u_h-u|^p\,dx=\int_{B_{\varrho_h}}|u_h-u|^p\,dx\le c\int_{B_{\varrho_h(1-\sigma_h)}}\left|w\left(\frac{x}{\varrho_h(1-\sigma_h)}\right)\right|^p\,dx$$
$$+c\int_{B_{\varrho_h}\backslash B_{\varrho_h(1-\sigma_h)}}\left\{\left|v\left(\frac{x}{|x|}\right)\right|^p+\left|u\left(\varrho_h\frac{x}{|x|}\right)\right|^p\right\}\,dx+c\int_{B_{\varrho_h}}|u(x)|^p\,dx$$
$$\le c\varrho_h^n\left\{\|w\|^p_{L^\infty(B_1)}+\|v\|^p_{L^\infty(\partial B_1)}+\|u(\varrho_h\omega)-v(\omega)\|^p_{L^\infty(\partial B_1)}\right\}+c\int_{B_{\varrho_h}}|u(x)|^p\,dx,$$

which goes to zero as $h \to +\infty$, since $\varrho_h \to 0$ and $\|u(\varrho_h \omega) - v(\omega)\|^p_{L^\infty(\partial B_1)} \to 0$. Moreover, by (64) of Lemma 35, we have

$$\begin{aligned}
\int_\Omega |Du_h - Du|^p \, dx &\le c_1 \int_{B_{\varrho_h(1-\sigma_h)}} \left| Dw\left(\frac{x}{\varrho_h(1-\sigma_h)}\right)\right|^p dx + c_1 \int_{B_{\varrho_h} \setminus B_{\varrho_h(1-\sigma_h)}} \frac{|D_\tau v|^p}{|x|^p} \, dx \\
&+ c_1 \int_{B_{\varrho_h} \setminus B_{\varrho_h(1-\sigma_h)}} \left| D_\tau u\left(\varrho_h \frac{x}{|x|}\right)\right|^p dx + \frac{c_1}{\varrho_h^p \sigma_h^p} \int_{B_{\varrho_h} \setminus B_{\varrho_h(1-\sigma_h)}} \left| u\left(\varrho_h \frac{x}{|x|}\right) - v\left(\frac{x}{|x|}\right)\right|^p dx \\
&+ c_1 \int_{B_{\varrho_h}} |Du|^p \, dx \\
&\le c_2 \varrho_h^{n-p} (1-\sigma_h)^{n-p} \int_{B_1} |Dw(x)|^p \, dx + c_2 \int_{B_{\varrho_h} \setminus B_{\varrho_h(1-\sigma_h)}} \frac{1}{|x|^p} \, dx \\
&+ c_2 \int_{\varrho_h(1-\sigma_h)}^{\varrho_h} dr \int_{\partial B_r} \left| D_\tau u\left(\varrho_h \frac{x}{|x|}\right)\right|^p dH^{n-1} + c_2 \frac{\varrho_h^{n-p}}{\sigma_h^{p-1}} \|u(\varrho_h \omega) - v(\omega)\|^p_{L^\infty(\partial B_1)} \\
&+ c_1 \int_{B_{\varrho_h}} |Du|^p \, dx \\
&\le c_3 \varrho_h^{n-p} + \frac{c_3}{\varrho_h^{n-1}} \varrho_h^n \sigma_h \int_{\partial B_{\varrho_h}} |D_\tau u(y)|^p \, dH_y^{n-1} + c_3 \frac{\varrho_h^{n-p}}{\sigma_h^{p-1}} \|u(\varrho_h \omega) - v(\omega)\|^p_{L^\infty(\partial B_1)} \\
&+ c_1 \int_{B_{\varrho_h}} |Du|^p \, dx \, .
\end{aligned}$$

By the bound (73) we obtain

$$\int_\Omega |Du_h - Du|^p \, dx \le c(w, v, M_0) \, \varrho_h^{n-p} + c \frac{\varrho_h^{n-p}}{\sigma_h^{p-1}} \|u(\varrho_h \omega) - v(\omega)\|^p_{L^\infty(\partial B_1)} + c_1 \int_{B_{\varrho_h}} |Du|^p \, dx$$

and this quantity goes to zero as $h \to +\infty$ if we assume that

$$\sigma_h := \varrho_h^{\frac{n-p}{p-1}} \tag{76}$$

(we use here the fact that $p < n$). Therefore, as $h \to +\infty$, u_h converges to u in the strong norm topology of $W^{1,p}(\Omega; \mathbb{R}^n)$. Thus, by (74) and by the lower semicontinuity of $TV^s(u, \Omega)$ with respect to the strong convergence in $W^{1,p}(\Omega; \mathbb{R}^n)$, we have

$$\begin{aligned}
TV(u, \Omega) \le TV^s(u, \Omega) &\le \liminf_{h \to +\infty} \int_\Omega |\det Du_h(x)| \, dx \\
&\le \int_{B_1} |\det Dw(x)| \, dx + \int_\Omega |\det Du(x)| \, dx + \liminf_{h \to +\infty} \int_{B_{\varrho_h} \setminus B_{\varrho_h(1-\sigma_h)}} |\det Du_h(x)| \, dx \\
&\le TV(v, B_1) + \varepsilon + \int_\Omega |\det Du(x)| \, dx + \liminf_{h \to +\infty} \int_{B_{\varrho_h} \setminus B_{\varrho_h(1-\sigma_h)}} |\det Du_h(x)| \, dx \, .
\end{aligned}$$

If we prove that

$$\lim_{h \to +\infty} \int_{B_{\varrho_h} \setminus B_{\varrho_h(1-\sigma_h)}} |\det Du_h(x)| \, dx = 0 \, , \tag{77}$$

then, letting $\varepsilon \to 0^+$ we reach the upper bound

$$TV(u, \Omega) \le TV^s(u, \Omega) \le TV(v, B_1) + \int_\Omega |\det Du(x)| \, dx$$

which, together to the lower bound in Step 1, yields the conclusion

$$TV(u,\Omega)=TV^{s}(u,\Omega)=TV(v,B_1)+\int_{\Omega}|\det Du(x)|\;dx\,.$$

Therefore it remains to prove (77). To this aim, arguing as in the proof of (65), we can evaluate $\det Du_h(x)$ by taking first the derivative of u_h with respect to the radial direction, and then the tangential derivatives. We get

$$\begin{aligned}\int_{B_{\varrho_h}\setminus B_{\varrho_h(1-\sigma_h)}}|\det Du_h(x)|\;dx &\le \frac{c_1}{\varrho_h\sigma_h}\int_{B_{\varrho_h}\setminus B_{\varrho_h(1-\sigma_h)}}\left\{\left|u\left(\varrho_h\frac{x}{|x|}\right)-v\left(\frac{x}{|x|}\right)\right|\cdot\right.\\ &\left.\cdot\left[\left|D_\tau v\left(\frac{x}{|x|}\right)\right|^{n-1}+\left|D_\tau u\left(\varrho_h\frac{x}{|x|}\right)\right|^{n-1}\right]\right\}dx\\ \le\frac{c_1}{\varrho_h\sigma_h}\left\|u(\varrho_h\omega)-v(\omega)\right\|_{L^\infty(\partial B_1)}&\cdot\left\{c_2\int_{B_{\varrho_h}\setminus B_{\varrho_h(1-\sigma_h)}}\frac{1}{|x|^{n-1}}\,dx+\frac{\varrho_h^n\sigma_h}{\varrho_h^{n-1}}\int_{\partial B_{\varrho_h}}|D_\tau u|^{n-1}\;dH^{n-1}\right\}\\ &\le c_3\left\|u(\varrho_h\omega)-v(\omega)\right\|_{L^\infty(\partial B_1)}\left\{c_2+\int_{\partial B_{\varrho_h}}|D_\tau u|^{n-1}\;dH^{n-1}\right\}.\end{aligned}$$

Finally, since by (73) we also have

$$\int_{\partial B_{\varrho_h}}|D_\tau u|^{n-1}\;dH^{n-1}\le c_4\left\{\frac{1}{\varrho_h^{n-p-1}}\int_{\partial B_{\varrho_h}}|D_\tau u|^{p}\;dH^{n-1}\right\}^{\frac{n-1}{p}}\le c_5\,,$$

then, from the above inequality, we deduce that

$$\int_{B_{\varrho_h}\setminus B_{\varrho_h(1-\sigma_h)}}|\det Du_h(x)|\;dx\le c_6\left\|u(\varrho_h\omega)-v(\omega)\right\|_{L^\infty(\partial B_1)}\,,$$

which converges to zero as $h\to+\infty$. Thus (77) is proved. ■

We conclude this section with some algebraic results used in the paper. We introduce some notations. Denote $M^{m\times n}$ by the family of $m\times n$ matrices. If A is an $n\times n$ matrix ($A\in M^{n\times n}$), $X_{i,j}(A)$ is the matrix obtained from A by deleting the $i-$th row and the $j-$th column of A. If S is an $(n-1)\times n$ matrix ($S\in M^{(n-1)\times n}$), $X_{,j}(S)$ stands for the matrix obtained from S by deleting the $j-$th column of S. If T is an $n\times(n-1)$ matrix ($T\in M^{n\times(n-1)}$), then $X_{i,}(T)$ is the matrix obtained from T by deleting the $i-$th row of T.

The properties stated in the next two lemmas are known and we do not give their proofs. We refer the reader for instance to the book by Cartan [14].

Lemma 37 (Algebraic lemma) *The following properties hold:*

(i) Let $\xi,\eta\in\mathbb{R}^n$ and let $B\in M^{n\times n}$. If $A_{ij}=\xi_i\eta_j$ and $A=(A_{ij})\in M^{n\times n}$, then

$$\det(A+B)=\sum_{i,j=1}^{n}(-1)^{i+j}\,\xi_i\eta_j\det(X_{i,j}(B))+\det(B)\;.$$

(ii) Let $T\in M^{n\times(n-1)}$ be a matrix whose column vectors $\{\tau_1,\tau_2,\ldots,\tau_{n-1}\}$ form an orthonormal basis of $\mathbb{R}^n$. Then

$$\sum_{i=1}^{n}\left[\det(X_{i,}(T))\right]^2=1\,.$$

(iii) Let $S \in M^{(n-1)\times n}$ and $T \in M^{n\times(n-1)}$. Then

$$\det(S \cdot T) = \sum_{i=1}^{n} \det(X_{,i}(S)) \cdot \det(X_{i,}(T)) \, .$$

As in Section 2, fixed $\omega_0 \in \partial B_1$, for every $j \in \{1, 2, \ldots, n-1\}$ we consider a vector field $\tau_j : \partial B_1 \setminus \{\omega_0\} \to \mathbb{R}^n$ of class C^1 such that the set of vectors $\{\tau_1(x), \tau_2(x), \ldots, \tau_{n-1}(x)\}$ is an orthonormal basis for the tangent plane to the surface ∂B_1 at the point x, for every $x \in \partial B_1 \setminus \{\omega_0\}$. For every $x \in \partial B_1 \setminus \{\omega_0\}$ we denote by $T(x)$ the $n \times (n-1)$ matrix whose columns are given by the vectors $\{\tau_1(x), \tau_2(x), \ldots, \tau_{n-1}(x)\}$. Consider the vector

$$\nu(x) := \sum_{i=1}^{n} (-1)^{i+1} \det(X_{i,}(T(x))) \, e_i \, . \tag{78}$$

Up to a change of sign to one of the vectors $\tau_1(x), \tau_2(x), \ldots, \tau_{n-1}(x)$, we can assume that, at every $x \in \partial B_1 \setminus \{\omega_0\}$, $\nu(x)$ represent the *exterior* normal unit vector to ∂B_1. That $\nu(x)$ is a *normal unit vector* to the surface ∂B_1 follows from the following result.

Lemma 38 (On the normal unit vector) *For every $x \in \partial B_1 \setminus \{\omega_0\}$ the vector $\nu(x)$ has norm equal to 1 and it is orthogonal to the vectors $\tau_1(x), \tau_2(x), \ldots, \tau_{n-1}(x)$; i.e.,*

$$\begin{cases} |\nu(x)| = 1 & \forall\, x \in \partial B_1 \setminus \{\omega_0\} \, , \\ \langle \nu(x), \tau_i(x) \rangle = 0 & \forall\, x \in \partial B_1 \setminus \{\omega_0\} \, , \ \ \forall\, i = 1, 2, \ldots, n-1. \end{cases}$$

8 Relaxation in the general polyconvex case

As mentioned in Section 4, the characterization of $TV(u, \Omega)$ may be viewed within a broader context, namely as part of a program to search for the description and identification of the defect measure obtained through relaxation of energies when there is a gap between the space of coercivity and the space guaranteeing apriori continuity. Indeed, $TV(u, \Omega)$ is a particular case of a functional of the type $\mathcal{F}_{p,q}(u, \Omega)$ in (33).

Here formally we may consider

$$\overline{F}(u, \Omega) := \inf \Big\{ \liminf_{h \to +\infty} \int_\Omega g(M(Du_h(x))) \, dx \ : \tag{79}$$

$$u_h \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega; \mathbb{R}^n) \, , \ u_h \in W^{1,n}(\Omega; \mathbb{R}^n) \Big\} \, .$$

Then $\overline{F}(u, \Omega)$ is the *relaxed functional* of the integral functional

$$F(u, \Omega) := \int_\Omega g(M(Du)) \, dx \, ,$$

where $u : \Omega \to \mathbb{R}^n$. The vector-valued map $M(Du)$ of minors of Du is given by

$$M(Du) := \big(Du, \mathrm{adj}_2 Du, \ldots, \mathrm{adj}_{n-1} Du, \det Du \big) \in \mathbb{R}^N,$$

where, for $j = 2, \ldots, n-1$, $\mathrm{adj}_j Du$ denotes the matrix of all minors $j \times j$ of Du and $N = \sum_{j=1}^{n} \binom{n}{j}^2$ (in particular $N = 5$ if $n = 2$). Finally $g : \mathbb{R}^N \to [0, +\infty)$ is a *convex function* satisfying the growth conditions

$$g_\infty |\det \xi| \leq g(M(\xi)) \leq L(1 + |\xi|^p) + g_\infty |\det \xi| \, , \tag{80}$$

for some constants $L \geq 0$, $g_\infty > 0$, for all matrices $\xi \in \mathbb{R}^{n\times n}$ and for some exponent $p \in [1,n)$.

A particularly important case of $F(u,\Omega)$ is the *area integral*

$$A(u,\Omega) := \int_\Omega \sqrt{1+|M(Du)|^2}\, dx\,,$$

which in the $2-d$ setting reduces to

$$A(u,\Omega) = \int_\Omega \sqrt{1+|Du(x)|^2+|\det Du(x)|^2}\, dx\,.$$

Theorem 39 below has been proved by Marcellini [47], [48] for $p > n^2/(n+1)$ and by Dacorogna and Marcellini [20] for $p > n-1$ ($p \geq 1$ if $n=2$). A limiting case, with $p = n-1$, has been considered under different assumptions by Acerbi and Dal Maso [2], Celada and Dal Maso [15], Dal Maso and Sbordone [22] and by Fusco and Hutchinson [30]. The *relaxation* in this context has been first considered by Fonseca and Marcellini [29].

Theorem 39 (Lower semicontinuity below the critical exponent) *Let Ω be an open set of $\mathbb{R}^n$. Let $g:\mathbb{R}^N \to \mathbb{R}$ be a nonnegative convex function. Then*

$$\liminf_{h\to+\infty} \int_\Omega g(M(Du_h))\, dx \geq \int_\Omega g(M(Du))\, dx\,,$$

for every sequence u_h which converge to u in the weak topology of $W^{1,p}(\Omega;\mathbb{R}^n)$ for some $p > n-1$, with $u, u_h \in W^{1,n}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ for every $h \in \mathbb{N}$.

It has been shown in [9] that, if $p > n-1$, then $\overline{F}(u,\cdot)$ is a Radon measure and, for every open set $A \subset \Omega$,

$$\overline{F}(u,A) = g(M(Du))\,\mathcal{L}^n\lfloor A + \mu_s(A)\,,$$

where μ_s is a finite Radon measure, singular with respect to the Lebesgue measure $\mathcal{L}^n$. A longtime question has been to identify the singular measure μ_s. In Theorem 40 we achieve this for the class of maps $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega\backslash\{0\};\mathbb{R}^n)$ considered in Section 2. Precisely, using Theorem 1 in $2-$d and Theorem 9 for the general $n-$d case, we can prove the following relaxation result.

Theorem 40 (Relaxation in $n-$d) *Let Ω be an open set of $\mathbb{R}^n$, $n \geq 2$, containing the origin. Let $u \in W^{1,p}(\Omega;\mathbb{R}^n) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega\backslash\{0\};\mathbb{R}^n)$ for some $p \in (n-1,n)$, such that, for a positive constant M_0,*

$$\sup_{\varrho>0} \frac{1}{\varrho^{n-p}} \int_{B_\varrho} |Du|^p\, dx \leq M_0\,.$$

Let $v:\partial B_1 = S^{n-1} \to \mathbb{R}^n$, $v \in W^{1,\infty}\left(S^{n-1};\mathbb{R}^n\right)$, be a Lipschitz-continuous map such that

$$\lim_{\varrho\to 0^+} \max\left\{\left|u\left(\varrho\frac{x}{|x|}\right) - v\left(\frac{x}{|x|}\right)\right| : \, x \in B_1\backslash\{0\}\right\} = 0\,.$$

Moreover, if $n=2$ we assume that the map v has values in the set Γ defined in (5); while, if $n \geq 3$, then we assume that the quantity

$$\sum_{i=1}^n (-1)^{i+1} v^i \frac{\partial\left(v^1,\ldots,v^{i-1},v^{i+1},\ldots,v^n\right)}{\partial(\tau_1,\tau_2,\ldots,\tau_{n-1})}$$

has constant sign $H^{n-1}-$almost everywhere on ∂B_1. Then the relaxed functional $\overline{F}(u,\Omega)$, defined in (79) with $g:\mathbb{R}^N \to [0,+\infty)$ satisfying (80), is given by

$$\overline{F}(u,\Omega) = \int_\Omega g(M(Du(x)))\, dx + g_\infty TV(v,B_1)\,,$$

where the total variation $TV(v,B_1)$ of v is given in (20).

Proof. Step 1 (lower bound). Consider a sequence $\{u_h\}_{h\in\mathbb{N}}$ of class $W^{1,n}(\Omega;\mathbb{R}^n)$ converging to u in the weak topology of $W^{1,p}(\Omega;\mathbb{R}^n)$, as $h\to+\infty$. Let $\varrho\in(0,1)$ be fixed. By Theorem 39, on the lower semicontinuity below the critical exponent, using the bound on the left hand side of (80), we have

$$\begin{aligned}\liminf_{h\to+\infty}\ F(u_h,\Omega) &\geq \liminf_{h\to+\infty}\int_{\Omega\setminus B_\varrho} g\left(M\left(Du_h(x)\right)\right)dx+\liminf_{h\to+\infty}\ g_\infty\int_{B_\varrho}\left|\det Du_h(x)\right|dx\\ &\geq \int_{\Omega\setminus B_\varrho} g\left(M\left(Du(x)\right)\right)dx+g_\infty TV\left(v,B_1\right).\end{aligned}$$

Letting $\varrho\to 0$ we deduce the lower bound

$$\overline{F}(u,\Omega)\geq\int_\Omega g\left(M\left(Du(x)\right)\right)dx+g_\infty TV\left(v,B_1\right).$$

Step 2 (upper bound). For every $\varepsilon>0$ there exists a Lipschitz-continuous map $w:B_1\to\mathbb{R}^n$ satisfying

$$\int_{B_1}\left|\det Dw(x)\right|dx<\varepsilon\ +TV\left(v,B_1\right)\tag{81}$$

and such that $w=v$ on ∂B_1. Indeed, if $n=2$ we use (**??**), while if $n\geq 3$ we use (74). By Lemma 34 there exists a sequence $(\varrho_h)_{h\in\mathbb{N}}$, converging to zero as $h\to+\infty$, and such that

$$\frac{1}{\varrho_h^{n-p-1}}\int_{\partial B_{\varrho_h}}\left|D_\tau u\right|^p\ dH^{n-1}\ \leq c(n,p)\,M_0\,.$$

For every $h\in\mathbb{N}$ we set $\sigma_h:=\varrho_h^{\frac{n-p}{p-1}}$, and we define $u_h(x)$ as in (75). As in Step 3 of the proof of Theorem 9, we can show that

$$\lim_{h\to+\infty}\int_{B_{\varrho_h}\setminus B_{\varrho_h(1-\sigma_h)}}\left|Du_h\right|^p\,dx=\lim_{h\to+\infty}\int_{B_{\varrho_h}\setminus B_{\varrho_h(1-\sigma_h)}}\left|\det Du_h(x)\right|dx=0\tag{82}$$

and, by also using the inequality on the right hand side of (80), we can prove the upper bound

$$\begin{aligned}\overline{F}(u,\Omega)\leq\liminf_{h\to+\infty}\ F(u_h,\Omega)&\leq\int_\Omega g\left(M\left(Du(x)\right)\right)dx\\ &+\liminf_{h\to+\infty}\int_{B_{\varrho_h}\setminus B_{\varrho_h(1-\sigma_h)}}\left\{L\left(1+\left|Du_h\right|^p\right)+g_\infty\left|\det Du_h\right|\right\}dx\\ &+g_\infty\int_{B_1}\left|\det Dw(x)\right|dx\,.\end{aligned}$$

By (81) and (82), letting ε go to zero, we conclude that

$$\overline{F}(u,\Omega)\leq\int_\Omega g\left(M\left(Du(x)\right)\right)dx+g_\infty TV\left(v,B_1\right).$$

■

9 A relevant $n-$dimensional class of maps

The singular map $u : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n$, defined for $x \neq 0$ by

$$u(x) = \frac{x}{|x|}, \tag{83}$$

belongs to the class $W^{1,p}(B_1;\mathbb{R}^n) \cap W^{1,\infty}_{\text{loc}}(\Omega\backslash\{0\};\mathbb{R}^n)$ for every $p \in [1,n)$, but $u \notin W^{1,n}(B_1;\mathbb{R}^n)$. In this case a formula for the total variation $TV(u,\Omega)$ was already known. Indeed, (84) below has been first given in 1986 by Marcellini [48] (see also Fonseca and Marcellini [29]). In this section we generalize the formula to more general maps.

To deduce (84) using the tools developed in this work, write $u(x) = v(x/|x|)$, where the map $v : \partial B_1 \to \mathbb{R}^n$ is the identity on $\partial B_1 = S^{n-1}$. The map $\widetilde{v}(x) = |x| \cdot v(x/|x|) = x$ is the smooth extension of u according with Corollary 12. Clearly $D\widetilde{v}(x) = Id$ is the *identity matrix* and $\det D\widetilde{v}(x) = 1$. Therefore, if Ω is any open set of $\mathbb{R}^n$ containing the origin, Corollary 12 gives

$$TV\left(\frac{x}{|x|},\Omega\right) = \left|\int_{B_1} \det D\widetilde{v}(x)\, dx\right| = \int_{B_1} dx = |B_1| = \omega_n. \tag{84}$$

Next we generalize the structure (83) and we consider a class of maps recently studied by Jerrard and Soner [43]. Consider a function $w \in C^1(\Omega;\mathbb{R}^n)$ (or, more generally, a locally Lipschitz-continuous map $w : \Omega \to \mathbb{R}^n$ classically differentiable at $x = 0$) such that $\det Dw(0) \neq 0$. Let Ω be an open set containing the origin and define $u : \Omega\backslash\{0\} \to \mathbb{R}^n$ by

$$u(x) := \frac{w(x) - w(0)}{|w(x) - w(0)|}. \tag{85}$$

Note that the condition $\det Dw(0) \neq 0$ ensures the existence of $r > 0$ such that $w(x) \neq w(0)$ for every $x \in B_r\backslash\{0\}$, and in the sequel we limit ourselves to open sets $\Omega \subset B_r$ containing the origin.

First we show that, without loss of generality, we may assume that $Dw(0) = Id$ is the *identity matrix*. Indeed, by assumption, the gradient $Dw(0)$ of w at $x = 0$ is a nonsingular matrix $n \times n$; let us denote by $A := Dw(0)$ this matrix, and by A^{-1} its inverse matrix. Define on $\Omega\backslash\{0\}$

$$z(x) := u\left(A^{-1}x\right) = \frac{w\left(A^{-1}x\right) - w(0)}{|w(A^{-1}x) - w(0)|}, \quad \forall\, x \in \Omega\backslash\{0\}.$$

Let $\{u_h\}_{h\in\mathbb{N}}$ be a sequence in $W^{1,n}(\Omega;\mathbb{R}^n)$ which converges, as $h \to +\infty$, to u weakly in $W^{1,p}(\Omega;\mathbb{R}^n)$. Then $z_h(x) := u_h\left(A^{-1}x\right)$ converges weakly in $W^{1,p}(\Omega;\mathbb{R}^n)$ to $z(x) = u\left(A^{-1}x\right)$. Since

$$\int_{A(\Omega)} |\det Dz_h(x)|\, dx = \int_{A(\Omega)} \left|\det Du_h(A^{-1}x)\right| \cdot \left|\det A^{-1}\right|\, dx = \int_{\Omega} |\det Du_h(x)|\, dx,$$

we deduce that $TV(z, A(\Omega)) = TV(u,\Omega)$. We also have $\left[Dw\left(A^{-1}x\right)\right]_{x=0} = Dw(0) \cdot A^{-1} = Id$, where Id is the identity matrix. Therefore, the above computations show that, without loss of generality, to evaluate the total variation $TV(u,\Omega)$ of the Jacobian determinant we may assume that

$$A = Dw(0) = Id. \tag{86}$$

Under (86), with u given in (85), we define $v : \partial B_1 \to \mathbb{R}^n$ by $v(y) := y$, for every $y \in \partial B_1$. We have

$$\lim_{\varrho\to 0} \max\{|u(\varrho y) - v(y)| : \ x \in B_1\backslash\{0\}\} = 0. \tag{87}$$

Indeed, since w is differentiable at $x=0$, we obtain

$$u(\varrho y)-v(y)=\frac{w(\varrho y)-w(0)}{|w(\varrho y)-w(0)|}-y=\frac{\varrho y+o(\varrho)}{|\varrho y+o(\varrho)|}-y=\frac{y+\frac{o(\varrho)}{\varrho}}{\left|y+\frac{o(\varrho)}{\varrho}\right|}-y\,,$$

which converges to zero as $\varrho\to 0$. Thus assertion (87) is proved.

Moreover, for every $x\in B_\varrho\setminus\{0\}$ with B_ϱ compactly contained in Ω, if we denote by L the Lipschitz constant of w in B_ϱ, we have

$$|Du(x)|\le c_1\frac{|Dw(x)|}{|w(x)-w(0)|}\le c_1\frac{L}{|w(x)-w(0)|}\,,$$

for a constant c_1. Since $A=Dw(0)=Id$, then

$$|w(x)-w(0)|=|Dw(0)\cdot x+o(|x|)|=|x+o(|x|)|\ge\frac{1}{2}|x|$$

for every $x\in B_{\varrho_0}$ with ϱ_0 sufficiently small; thus

$$|Du(x)|\le c_1\frac{L}{|w(x)-w(0)|}\le\frac{2c_1L}{|x|}\,.$$

Also, for every $p<n$, we have

$$\begin{aligned}\sup_{0<\varrho\le\varrho_0}\frac{1}{\varrho^{n-p}}\int_{B_\varrho}|Du|^p\,dx &\le \sup_{0<\varrho\le\varrho_0}\frac{c_2}{\varrho^{n-p}}\int_{B_\varrho}\frac{1}{|x|^p}\,dx\\ &\le \sup_{0<\varrho\le\varrho_0}\frac{c_2\cdot\omega_n}{\varrho^{n-p}}\int_0^\varrho r^{n-1-p}\,dr\;=\;\frac{c_2\cdot\omega_n}{n-p}.\end{aligned}$$

Therefore the assumptions (13), (14) are satisfied, and we can apply Theorem 9, when $v:S^{n-1}\to S^{n-1}$ is the identity map. Since $|u(x)|=1$ for every $x\in\Omega\setminus\{0\}$, then $\det Du(x)=0$ in $\Omega\setminus\{0\}$, and hence, by (84) we finally get

$$TV(u,\Omega)=TV\left(\frac{x}{|x|},\Omega\right)=\omega_n\,,\qquad\text{with }\;u(x):=\frac{w(x)-w(0)}{|w(x)-w(0)|}\,.$$

10 Some 2− and 3−dimensional examples

We start with a simple application of the general 2−d result of Theorem 1.

Example 41 *Let $u(x):=v(x/|x|)$, where $v:[0,2\pi]\to S^1$ is the map $v(\vartheta)=(\cos g(\vartheta),\sin g(\vartheta))$, with $g:[0,2\pi]\to\mathbb{R}$ Lipschitz-continuous function such that $g(2\pi)=g(0)+2k\pi$, for some $k\in\mathbb{Z}$. Since $v^1(\vartheta)\,v^2_\vartheta(\vartheta)-v^2(\vartheta)\,v^1_\vartheta(\vartheta)=g'(\vartheta)$, by Theorem 1 we obtain*

$$TV(u,B_1)=\frac{1}{2}\left|\int_0^{2\pi}g'(\vartheta)\,d\vartheta\right|=|k|\,\pi\,.$$

Note that here g is not necessarily a monotone function and that $TV(u,B_1)=\frac{1}{2}\left|\int_0^{2\pi}g'(\vartheta)\,d\vartheta\right|$, with the absolute value sign outside the integral sign, and not inside as could have been expected. On the other hand, if $w(x)=|x|\,u(x)$ is the radially linear Lipschitz-continuous extension of v, we have instead $TV(w,B_1)=\frac{1}{2}\int_0^{2\pi}|g'(\vartheta)|\,d\vartheta$.

Consider a Lipschitz-continuous closed curve $v : [0, 2\pi] \to \gamma$, with parametric representation $v(\vartheta) = \left(v^1(\vartheta), v^2(\vartheta)\right)$ and with $v(0) = v(2\pi)$. As in Section 2, we denote by $\left\{I_j^+\right\}_j$ and by $\left\{I_k^-\right\}_k$ sequences of disjoint open intervals of $[0, 2\pi]$ such that $v(I_j) \subset \gamma^+$ and $v(I_k) \subset \gamma^-$ (and $v(\vartheta) = (0, 0)$ when $\vartheta \notin \left(\cup_j I_j^+\right) \cup \left(\cup_k I_k^-\right)$). With $u(x) := v(x/|x|)$, we stated in Theorem 4 the following upper and lower estimates

$$TV(u, B_1) \leq \frac{1}{2} \sum_{j\in\mathbb{N}} \left| \int_{I_j} \left\{v^1(\vartheta) v_\vartheta^2(\vartheta) - v^2(\vartheta) v_\vartheta^1(\vartheta)\right\} \, d\vartheta \right| ; \tag{88}$$

$$TV(u, B_1) \geq \frac{1}{2} \left\{ \left| \sum_{j\in\mathbb{N}} \int_{I_j^+} \left\{v^1 v_\vartheta^2 - v^2 v_\vartheta^1\right\} \, d\vartheta \right| + \left| \sum_{k\in\mathbb{N}} \int_{I_k^-} \left\{v^1 v_\vartheta^2 - v^2 v_\vartheta^1\right\} \, d\vartheta \right| \right\} . \tag{89}$$

We notice that, if the curve $v : [0, 2\pi] \to \gamma = \gamma^+ \cup \gamma^-$ admits only two intervals I_1^+ and I_2^- where $v\left(I_1^+\right) \subset \gamma^+$, $v\left(I_2^-\right) \subset \gamma^-$ respectively, then the above estimates for $TV(u, B_1)$ are in fact equalities, and

$$TV(u, B_1) = \frac{1}{2} \left\{ \left| \int_{I_1^+} \left\{v^1 v_\vartheta^2 - v^2 v_\vartheta^1\right\} \, d\vartheta \right| + \left| \int_{I_2^-} \left\{v^1 v_\vartheta^2 - v^2 v_\vartheta^1\right\} \, d\vartheta \right| \right\} . \tag{90}$$

Moreover, the total variation of the distributional determinant $|\mathrm{Det}\, Du|(B_1)$ is given by

$$|\mathrm{Det}\, Du|(B_1) = \frac{1}{2} \left| \int_0^{2\pi} \left\{v^1(\vartheta) v_\vartheta^2(\vartheta) - v^2(\vartheta) v_\vartheta^1(\vartheta)\right\} \, d\vartheta \right| .$$

In [24] we presented 2-dimensional examples illustrating situations where $TV(u, \Omega) > |\mathrm{Det} Du|(\Omega)$ and where there is a gap between (88) and (89).

Finally we consider a $3-$dimensional example.

Example 42 *Let us consider the map $v : S^2 \to S^2 \subset \mathbb{R}^3$ defined, in spherical coordinates, by*

$$v(\vartheta, \psi) := \begin{cases} v^1 = \cos g(\vartheta) \sin\psi \\ v^2 = \sin g(\vartheta) \sin\psi \\ v^3 = \cos\psi \end{cases} ,$$

for $\vartheta \in [0, 2\pi]$, $\psi \in [0, \pi]$, where $g : [0, 2\pi] \to [0, 2\pi]$ is a Lipschitz-continuous function such that $g(2\pi) - g(0) = 2k\pi$ for some $k \in \mathbb{Z}$. By formula (65) we can see that, if ω is a generic point of S^2, represented in the form $\omega = (\cos\vartheta \sin\psi, \sin\vartheta \sin\psi, \cos\psi)$, then we have

$$v^1(\omega) \frac{\partial\left(v^2, v^3\right)}{\partial(\tau_1, \tau_2)}(\omega) - v^2(\omega) \frac{\partial\left(v^1, v^3\right)}{\partial(\tau_1, \tau_2)}(\omega) + v^3(\omega) \frac{\partial\left(v^2, v^1\right)}{\partial(\tau_1, \tau_2)}(\omega) = g'(\vartheta) .$$

Thus, if the function g is monotone, then the sign assumption (18) is satisfied and, by Theorem 9, we obtain

$$TV(v, B_1) = \frac{2}{3} |g(2\pi) - g(0)| = \frac{4}{3}\pi |k| , \tag{91}$$

which, as expected, is equal to the absolute value $|k|$ of the topological degree of the map times the volume $\omega_3 = \frac{4}{3}\pi$ of the unit ball in $\mathbb{R}^3$.

However, formula (91) also holds if the function g is not monotone, i.e., if the sign assumption (18) is not satisfied. To assert this fact (that we do not want to prove in all details), we can follow the argument used in Section 5 to prove Theorem 1. In particular, if for some α, β, with $0 \leq \alpha < \beta \leq 2\pi$, *we have $g(\alpha) = g(\beta)$, then for every $\varepsilon > 0$ we can construct a Lipschitz-continuous map $w : S_{\alpha,\beta} \to \mathbb{R}^3$ such that $w(x) := |x|\, v\left(\frac{x}{|x|}\right)$ if $x \in \partial S_{\alpha,\beta}$ and*

$$\int_{S_{\alpha,\beta}} |\det Dw(x)| \; dx < \varepsilon \; ,$$

where $S_{\alpha,\beta}$ is the subset of B_1 of points $x = (\varrho\cos\vartheta\sin\psi, \varrho\sin\vartheta\sin\psi, \varrho\cos\psi)$, with $0 \leq \varrho \leq 1$, $\alpha \leq \vartheta \leq \beta$, $0 \leq \psi \leq \pi$. The map w can be defined similarly to the one used in the proof of the "umbrella" Lemma 22, setting $w(\varrho,\vartheta,\psi) := \varrho(\cos\varphi(\varrho,\vartheta)\sin\psi, \sin\varphi(\varrho,\vartheta)\sin\psi, \cos\psi)$, where $\varphi(\varrho,\vartheta) := \varrho^h g(\vartheta) + \left(1-\varrho^h\right) g(\alpha)$, with h sufficiently large.

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