Homotopical Algebraic Geometry II: geometric stacks and applications

is given by α_u. By definition of the model structure on $A-QCoh$, the functor

$$f^* : A-QCoh \longrightarrow A'-QCoh$$

is clearly a left Quillen functor, as the natural diagram

$$\begin{array}{ccc} A-QCoh & \xrightarrow{f^*} & A'-QCoh \\ \downarrow & & \downarrow \\ A-Mod & \xrightarrow[f^*]{} & A'-Mod \end{array}$$

commutes up to a natural isomorphism. Furthermore, for any pair of morphisms

$$A \xrightarrow{f} A' \xrightarrow{g} A''$$

in $Comm(\mathcal{C})$, there is an equality $(g\circ f)^* = g^* \circ f^*$. In other words, the rule

$$A \mapsto A-QCoh \qquad (f : A \to A') \mapsto f^*$$

defines a U-cofibrantly generated left Quilllen presheaf on $Aff_{\mathcal{C}} = Comm(\mathcal{C})^{op}$ in the sense of Appendix B.

We now consider for any $A \in Comm(\mathcal{C})$, the subcategory $A - QCoh^c_W$ of $A - QCoh$, consisting of equivalences between cofibrant objects. As these are preserved by the pullback functors f^*, we obtain this way a new presheaf of $\mathbb{V}$-small categories

$$\begin{array}{rccc} QCoh^c_W : & Comm(\mathcal{C}) = Aff^{op}_{\mathcal{C}} & \longrightarrow & Cat_{\mathbb{V}} \\ & A & \mapsto & A-QCoh^c_W. \end{array}$$

Composing with the nerve functor

$$N : Cat_{\mathbb{V}} \longrightarrow SSet_{\mathbb{V}}$$

we get a simplicial presheaf

$$\begin{array}{rccc} N(QCoh^c_W) : & Comm(\mathcal{C}) = Aff^{op}_{\mathcal{C}} & \longrightarrow & SSet_{\mathbb{V}} \\ & A & \mapsto & N(A-QCoh^c_W). \end{array}$$

DEFINITION 1.3.7.1. *The simplicial presheaf of* quasi-coherent modules *is* $N(QCoh^c_W)$ *defined above. It is denoted by* **QCoh**, *and is considered as an object in* $Aff^{\sim,\tau}_{\mathcal{C}}$.

It is important to note that for any $A \in Comm(\mathcal{C})$, the simplicial set $\mathbf{QCoh}(A)$ is naturally equivalent to the nerve of $A - Mod^c_W$, the subcategory of equivalences between cofibrant objects in $A-Mod$, and therefore also to the nerve of $A-Mod_W$, the subcategory of equivalences in $A - Mod$. In particular, $\pi_0(\mathbf{QCoh}(A))$ is in bijection with isomorphisms classes of objects in $\mathrm{Ho}(A - Mod)$ (i.e. equivalence classes of objects in $A - Mod$). Furthermore, by [**D-K**3] (see also Appendix A), for any $x \in \mathbf{QCoh}(A)$, corresponding to an equivalence class of $M \in A - Mod$, the connected component of $\mathbf{QCoh}(A)$ containing x is naturally equivalent to $BAut(M)$, where $Aut(M)$ is the simplicial monoid of self equivalences of M in $A - Mod$. In particular, we have

$$\pi_1(\mathbf{QCoh}(A), x) \simeq Aut_{\mathrm{Ho}(A-Mod)}(M)$$

$$\pi_{i+1}(\mathbf{QCoh}(A), x) \simeq [S^i M, M]_{A-Mod} \; \forall\, i > 1.$$

The main result of this section is the following.

THEOREM 1.3.7.2. *The simplicial presheaf* **QCoh** *is a stack.*

PROOF. This is a direct application of the strictification theorem B.0.7 recalled in Appendix B.

More precisely, we use our lemma 1.3.2.3 (2). Concerning finite direct sums, we have already seen (during the proof of lemma 1.3.2.3) that the natural functor

$$(-\otimes_{A\times^h B} A)\times(-\otimes_{A\times^h B} B) : (A\times^h B)-Mod \longrightarrow A-Mod\times B-Mod$$

is a Quillen equivalence. This implies that for any two objects $X, Y \in Aff_{\mathcal{C}}$ the natural morphism

$$\mathbf{QCoh}(X\coprod^{\mathbb{L}} Y)\longrightarrow \mathbf{QCoh}(X)\times\mathbf{QCoh}(Y)$$

is an equivalence. It only remains to show that **QCoh** has the descent property with respect to hypercovers of the type described in lemma 1.3.2.3 (2). But this is nothing else than our assumption 1.3.2.2 (3) together with Cor. B.0.8. □

An important consequence of theorem 1.3.7.2 is the following.

COROLLARY 1.3.7.3. *Let $A \in Comm(\mathcal{C})$ be a commutative monoid, $A-Mod_W$ the subcategory of equivalences in $A-Mod$, and $N(A-Mod_W)$ be its nerve. Then, there exists natural isomorphisms in* $\mathrm{Ho}(SSet)$

$$\mathbb{R}\mathbf{QCoh}(A)\simeq \mathbb{R}_{\tau}\underline{Hom}(\mathbb{R}\underline{Spec}\,A,\mathbf{QCoh})\simeq N(A-Mod_W).$$

To finish this section, we will describe two important sub-stacks of **QCoh**, namely the stack of *perfect modules* and the stack of *vector bundles*.

For any commutative monoid A in $\mathcal{C}$, we let $\mathbf{Perf}(A)$ be the sub-simplicial set of $\mathbf{QCoh}(A)$ consisting of all connected components corresponding to perfect objects in $\mathrm{Ho}(A-Mod)$ (in the sense of Def. 1.2.3.6). More precisely, if $Iso(D)$ denotes the set of isomorphisms classes of a category D, the simplicial set $\mathbf{Perf}(A)$ is defined as the pullback

$$\begin{array}{ccc}\mathbf{Perf}(A) & \longrightarrow & Iso(\mathrm{Ho}(A-Mod)^{perf})\\ \downarrow & & \downarrow\\ \mathbf{QCoh}(A) & \longrightarrow & \pi_0\mathbf{QCoh}(A)\simeq Iso(\mathrm{Ho}(A-Mod))\end{array}$$

where $\mathrm{Ho}(A-Mod)^{perf}$ is the full subcategory of $\mathrm{Ho}(A-Mod)$ consisting of perfect A-modules in the sense of Def. 1.2.3.6.

We say that an A-module $M\in\mathrm{Ho}(A-Mod)$ is a *rank n vector bundle*, if there exists a covering family $A\longrightarrow A'$ such that $M\otimes^{\mathbb{L}}_A A'$ is isomorphic in $\mathrm{Ho}(A'-Mod)$ to $(A')^n$. As we have defined the sub simplicial set $\mathbf{Perf}(A)$ of $\mathbf{QCoh}(A)$ we define $\mathbf{Vect}_n(A)$ to be the sub simplicial set of $\mathbf{QCoh}(A)$ consisting of connected components corresponding to rank n vector bundles.

For any morphism of commutative monoids $u: A\longrightarrow A'$, the base change functor

$$\mathbb{L}u^*:\mathrm{Ho}(A-Mod)\longrightarrow\mathrm{Ho}(A'-Mod)$$

preserves perfect modules as well as rank n vector bundles. Therefore, the sub simplicial sets $\mathbf{Vect}_n(A)$ and **Perf** form in fact full sub simplicial presheaves

$$\mathbf{Vect}_n\subset\mathbf{QCoh}\qquad \mathbf{Perf}\subset\mathbf{QCoh}.$$

The simplicial presheaves $\mathbf{Vect}_n$ and **Perf** then define objects in $Aff_{\mathcal{C}}^{\sim,\tau}$.

COROLLARY 1.3.7.4. *The simplicial presheaves* **Perf** *and* $\mathbf{Vect}_n$ *are stacks.*

PROOF. Indeed, as they are full sub-simplicial presheaves of the stack **QCoh**, it is clearly enough to show that being a perfect module and being a vector bundle or rank n is a local condition for the topology τ. For vector bundles this is obvious from the definition.

Let $A \in Comm(\mathcal{C})$ be a commutative monoid, and P be an A-module, such that there exists a τ-covering $A \longrightarrow B$ such that $P \otimes^{\mathbb{L}}_A B$ is a perfect B-module. Assume that $A \to B$ is a cofibration, and let B_* be its co-nerve, considered as a co-simplicial object in $A - Comm(\mathcal{C})$. Let Q be any A-module, and define two objects in $\mathrm{Ho}(csB_* - Mod)$ by

$$\mathbb{R}\underline{Hom}_A(P,Q)_* := \mathbb{R}\underline{Hom}_A(P,Q) \otimes^{\mathbb{L}}_A B_*$$
$$\mathbb{R}\underline{Hom}_A(P,Q_*) := \mathbb{R}\underline{Hom}_A(P,Q \otimes^{\mathbb{L}}_A B_*).$$

There is a natural morphism

$$\mathbb{R}\underline{Hom}_A(P,Q)_* \longrightarrow \mathbb{R}\underline{Hom}_A(P,Q_*).$$

These co-simplicial objects are both cartesian, and by assumption 1.3.2.2 (3) applied to the A-modules Q and $\mathbb{R}\underline{Hom}_A(P,Q)$ the induced morphism in $\mathrm{Ho}(A - Mod)$

$$\int \mathbb{R}\underline{Hom}_A(P,Q)_* \simeq \mathbb{R}\underline{Hom}_A(P,Q) \simeq$$
$$\simeq Holim_{n\in\Delta}\mathbb{R}\underline{Hom}_A(P,Q \otimes^{\mathbb{L}}_A B_n) \simeq \int \mathbb{R}\underline{Hom}_A(P,Q \otimes^{\mathbb{L}}_A B_*)$$

is an isomorphism. Therefore, assumption 1.3.2.2 (3) implies that the natural morphism

$$\mathbb{R}\underline{Hom}_A(P,Q)_0 \longrightarrow \mathbb{R}\underline{Hom}_A(P,Q_0)$$

is an isomorphism. By definition this implies that

$$\mathbb{R}\underline{Hom}_A(P,Q) \otimes^{\mathbb{L}}_A B \longrightarrow \mathbb{R}\underline{Hom}_A(P,Q \otimes^{\mathbb{L}}_A B)$$

is an isomorphism in $\mathrm{Ho}(A - Mod)$. In particular, when $Q = A$ we find that the natural morphism

$$P^{\vee} \otimes^{\mathbb{L}}_A B \longrightarrow \mathbb{R}\underline{Hom}_B(P \otimes^{\mathbb{L}}_A B, B)$$

is an isomorphism in $\mathrm{Ho}(B - Mod)$. As $P \otimes^{\mathbb{L}}_A B$ is by assumption a perfect B-module, we find that the natural morphism

$$P^{\vee} \otimes^{\mathbb{L}}_A Q \longrightarrow \mathbb{R}\underline{Hom}(P,Q)$$

becomes an isomorphism after base changing to B, and this for any A-module Q. As $A \longrightarrow B$ is a τ-covering, we see that this implies that

$$P^{\vee} \otimes^{\mathbb{L}}_A Q \longrightarrow \mathbb{R}\underline{Hom}(P,Q)$$

is always an isomorphism in $\mathrm{Ho}(A - Mod)$, for any Q, and thus that P is a perfect A-module. □

DEFINITION 1.3.7.5. *The stack of vector bundles of rank n is $\mathbf{Vect}_n$. The stack of perfect modules is* **Perf**.

The same construction can also been done in the stable context. For a commutative monoid A in $\mathcal{C}$, we define a category $A - QCoh^{Sp}$, of stable quasi-coherent modules on A (or equivalently on $Spec\,A$) in the following way. Its objects are the data of a stable B-module $M_B \in Sp(B - Mod)$ for any commutative A-algebra $B \in A - Comm(\mathcal{C})$, together with an isomorphism

$$\alpha_u : M_B \otimes_B C \longrightarrow M_{B'}$$

for any morphism $u : B \longrightarrow B'$ in $A-Comm(\mathcal{C})$, such that one has $\alpha_v \circ (\alpha_u \otimes_{B'} B'') = \alpha_{v\circ u}$ for any pair of morphisms

$$B \xrightarrow{u} B' \xrightarrow{v} B''$$

in $A-Comm(\mathcal{C})$. Such data will be denoted by (M,α). A morphism in $A-QCoh^{Sp}$, from (M,α) to (M',α') is given by a family of morphisms of stable B-modules $f_B : M_B \longrightarrow M'_B$, for any $B \in A-Comm(\mathcal{C})$, such that for any $u : B \to B'$ in $A-Comm(\mathcal{C})$ the diagram

$$\begin{array}{ccc} M_B \otimes_B B' & \xrightarrow{\alpha_u} & M_{B'} \\ \downarrow{\scriptstyle f_B \otimes_B B'} & & \downarrow{\scriptstyle f_{B'}} \\ (M'_B) \otimes_B B' & \xrightarrow[\alpha'_u]{} & M'_{B'} \end{array}$$

commutes. As the categories $Sp(A-Mod)$ and $Comm(\mathcal{C})$ are all $\mathbb{V}$-small, so are the categories $A-QCoh^{Sp}$.

There exists a natural projection $A-QCoh^{Sp} \longrightarrow Sp(A-Mod)$, sending (M,α) to M_A, and it is straightforward to check that it is an equivalence of categories. In particular, the model structure on $Sp(A-Mod)$ will be transported naturally on $A-QCoh^{Sp}$ through this equivalence.

Let now $f : A \longrightarrow A'$ be a morphism of commutative monoids in $\mathcal{C}$. There exists a pullback functor

$$f^* : A-QCoh^{Sp} \longrightarrow A'-QCoh^{Sp}$$

defined by $f(M,\alpha)_B := M_B$ for any $B \in A-Comm(\mathcal{C})$, and for $u : B \longrightarrow B'$ in $A-Comm(\mathcal{C})$ the transition morphism

$$f(M,\alpha)_B \otimes_B B' = M_B \otimes_B B' \longrightarrow f(M,\alpha)_{B'} = M_{B'}$$

is given by α_u. By definition of the model structure on $A-QCoh^{Sp}$, the functor

$$f^* : A-QCoh^{Sp} \longrightarrow A'-QCoh^{Sp}$$

is clearly a left Quillen functor. Furthermore, for any pair of morphisms

$$A \xrightarrow{f} A' \xrightarrow{g} A''$$

in $Comm(\mathcal{C})$, there is an equality $(g\circ f)^* = g^* \circ f^*$. In other words, the rule

$$A \mapsto A-QCoh^{Sp} \qquad (f : A \to A') \mapsto f^*$$

defines a $\mathbb{U}$-cofibrantly generated left Quilllen presheaf on $Aff_{\mathcal{C}} = Comm(\mathcal{C})^{op}$ in the sense of Appendix B.

We now consider for any $A \in Comm(\mathcal{C})$, the subcategory $A-QCoh^{Sp,c}_W$ of $A-QCoh^{Sp}$, consisting of equivalences between cofibrant objects. As these are preserved by the pullback functors f^*, one gets this way a new presheaf of $\mathbb{V}$-small categories

$$\begin{array}{cccc} QCoh^{Sp,c}_W : & Comm(\mathcal{C}) = Aff_{\mathcal{C}}^{op} & \longrightarrow & Cat_{\mathbb{V}} \\ & A & \mapsto & A-QCoh^{Sp,c}_W. \end{array}$$

Composing with the nerve functor

$$N : Cat_{\mathbb{V}} \longrightarrow SSet_{\mathbb{V}}$$

one gets a simplicial presheaf

$$\begin{array}{cccc} N(QCoh^{Sp,c}_W) : & Comm(\mathcal{C}) = Aff_{\mathcal{C}}^{op} & \longrightarrow & SSet_{\mathbb{V}} \\ & A & \mapsto & N(A-QCoh^{Sp,c}_W). \end{array}$$

DEFINITION 1.3.7.6. *The simplicial presheaf of* stable quasi-coherent modules *is* $N(QCoh_W^{Sp,c})$ *defined above. It is denoted by* $\mathbf{QCoh}^{Sp}$*, and is considered as an object in* $Aff_{\mathcal{C}}^{\sim,\tau}$.

It is important to note that for any $A \in Comm(\mathcal{C})$, the simplicial set $\mathbf{QCoh}^{Sp}(A)$ is naturally equivalent to the nerve of $A - Mod_W^{Sp,c}$, the subcategory of equivalences between cofibrant objects in $Sp(A - Mod)$, and therefore also to the nerve of $Sp(A - Mod)_W$, the subcategory of equivalences in $Sp(A - Mod)$. In particular, $\pi_0(\mathbf{QCoh}^{Sp}(A))$ is in bijection with isomorphisms classes of objects in $\mathrm{Ho}(Sp(A - Mod))$ (i.e. equivalence classes of objects in $Sp(A - Mod)$). Furthermore, by [**D-K3**] (see also Appendix A), for any $x \in \mathbf{QCoh}^{Sp}(A)$, corresponding to an equivalence class of $M \in Sp(A - Mod)$, the connected component of $\mathbf{QCoh}^{Sp}(A)$ containing x is naturally equivalent to $BAut(M)$, where $Aut(M)$ is the simplicial monoid of self equivalences of M in $Sp(A - Mod)$. In particular, we have

$$\pi_1(\mathbf{QCoh}^{Sp}(A), x) \simeq Aut_{\mathrm{Ho}(Sp(A-Mod))}(M)$$

$$\pi_{i+1}(\mathbf{QCoh}^{Sp}(A), x) \simeq [S^i M, M]_{Sp(A-Mod)}\ \forall\, i > 1.$$

The same proof as theorem 1.3.7.2, but based on Proposition 1.2.12.5 gives the following stable version.

THEOREM 1.3.7.7. *Assume that the two conditions are satisfied.*

1. *The suspension functor* $S : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{C})$ *is fully faithful.*
2. *For all* τ*-covering family* $\{U_i \to X\}$ *in* $Aff_{\mathcal{C}}$*, each morphism* $U_i \to X$ *is flat in the sense of Def. 1.2.4.1.*

Then, the simplicial presheaf $\mathbf{QCoh}^{Sp}$ *is a stack.*

For any commutative monoid A in $\mathcal{C}$, we let $\mathbf{Perf}^{Sp}(A)$ be the sub-simplicial set of $\mathbf{QCoh}^{Sp}(A)$ consisting of all connected components corresponding to perfect objects in $\mathrm{Ho}(Sp(A - Mod))$ (in the sense of Def. 1.2.3.6). This defines a full sub-prestack $\mathbf{Perf}^{Sp}$ of $\mathbf{QCoh}^{Sp}$. Then, the same argument as for Cor. 1.3.7.4 gives the following corollary.

COROLLARY 1.3.7.8. *Assume that the two conditions are satisfied.*

1. *The suspension functor* $S : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{C})$ *is fully faithful.*
2. *For all* τ*-covering family* $\{U_i \to X\}$ *in* $Aff_{\mathcal{C}}$*, each morphism* $U_i \to X$ *is flat in the sense of Def. 1.2.4.1.*

The simplicial presheaf $\mathbf{Perf}^{Sp}$ *is a stack.*

This justifies the following definition.

DEFINITION 1.3.7.9. *Under the condition of Cor. 1.3.7.8, the* stack of stable perfect modules *is* $\mathbf{Perf}^{Sp}$.

We finish this section by the standard description of $\mathbf{Vect}_n$ as the classifying stack of the group stack $\mathbf{Gl}_n$, of invertible n by n matrices. For this, we notice that the natural morphism of stacks $* \longrightarrow \mathbf{Vect}_n$, pointing the trivial rank n vector bundle, induces an isomorphism of sheaves of sets $* \simeq \pi_0(\mathbf{Vect}_n)$. Therefore, the stack $\mathbf{Vect}_n$ can be written as the geometric realization of the homotopy nerve of the morphism $* \longrightarrow \mathbf{Vect}_n$. In other words, we can find a Segal groupoid object X_*, such that $X_0 = *$, and with $|X_*| \simeq \mathbf{Vect}_n$. Furthermore, the object X_1 is naturally equivalent to the loop stack $\Omega_*\mathbf{Vect}_n := * \times^h_{\mathbf{Vect}_n} *$. By construction and by Prop. A.0.6, this loop stack can be described as the simplicial presheaf

$$\begin{array}{rccc} \Omega_*\mathbf{Vect}_n : & Comm(\mathcal{C}) & \longrightarrow & SSet \\ & A & \mapsto & Map'_{\mathcal{C}}(1^n, A^n), \end{array}$$

where $Map'_{\mathcal{C}}(1^n, A^n)$ is the sub simplicial set of the mapping space $Map_{\mathcal{C}}(1^n, A^n)$ consisting of all connected components corresponding to automorphisms in

$$\pi_0 Map_{\mathcal{C}}(1^n, A^n) \simeq \pi_0 Map_{A-Mod}(A^n, A^n) \simeq [A^n, A^n]_{A-Mod}.$$

The important fact concerning the stack $\Omega_*\mathbf{Vect}_n$ is the following result.

PROPOSITION 1.3.7.10. (1) *The stack* $\Omega_*\mathbf{Vect}_n$ *is representable and the morphism* $\Omega_*\mathbf{Vect}_n \longrightarrow *$ *is formally smooth.*

(2) *If Moreover* $\mathbf{1}$ *is finitely presented in* $\mathcal{C}$*, then the morphism* $\Omega_*\mathbf{Vect}_n \longrightarrow *$ *is finitely presented (and thus smooth by (1)).*

PROOF. (1) We start by defining a larger stack $\mathbf{M}_n$, of n by n matrices. We set

$$\begin{array}{cccc} \mathbf{M}_n: & Comm(\mathcal{C}) & \longrightarrow & SSet \\ & A & \mapsto & Map_{\mathcal{C}}(1^n, A^n). \end{array}$$

This stack is representable, as it is isomorphic in $\mathrm{St}(\mathcal{C}, \tau)$ to $\mathbb{R}\underline{Spec}\, B$, where $B = \mathbb{L}F(\mathbf{1}^{n^2})$ is the free commutative monoid generated by the object $\mathbf{1}^{n^2} \in \mathcal{C}$. We claim that the natural inclusion morphism

$$\Omega_*\mathbf{Vect}_n \longrightarrow \mathbf{M}_n$$

is (-1)-representable and a formally étale morphism. Indeed, let A be any commutative monoid,

$$x : X := \mathbb{R}\underline{Spec}\, A \longrightarrow \mathbf{M}_n$$

be a morphism of stacks, and let us consider the stack

$$F := \Omega_*\mathbf{Vect}_n \times^h_{\mathbf{M}_n} X \longrightarrow X.$$

The point x corresponds via the Yoneda lemma to a morphism $u : A^n \longrightarrow A^n$ in $\mathrm{Ho}(A - Mod)$. Now, for any commutative monoid A', the natural morphism

$$\mathbb{R}F(A') \longrightarrow (\mathbb{R}\underline{Spec}\, A)(A')$$

identifies $\mathbb{R}F(A')$ with the sub simplicial set of $(\mathbb{R}\underline{Spec}\, A)(A') \simeq Map_{Comm(\mathcal{C})}(A, A')$ consisting of all connected components corresponding to morphisms $A \longrightarrow A'$ in $\mathrm{Ho}(Comm(\mathcal{C}))$ such that

$$u \otimes^{\mathbb{L}}_A A' : (A')^n \longrightarrow (A')^n$$

is an isomorphism in $\mathrm{Ho}(A' - Mod)$. Considering u as an element of $[A^n, A^n] \simeq M_n(\pi_0(A))$, we can consider its determinant $d(u) \in \pi_0(A)$. Then, using notations from Def. 1.2.9.2, we clearly have an isomorphism of stacks

$$F \simeq \mathbb{R}\underline{Spec}\,(A[d(u)^{-1}]).$$

By Prop. 1.2.9.5 this shows that the morphism

$$F \longrightarrow \mathbb{R}\underline{Spec}\, A$$

is a formally étale morphism between representable stacks. As $\mathbf{M}_n$ is representable this implies that $\Omega_*\mathbf{Vect}_n$ is a representable stack and that the morphism

$$\Omega_*\mathbf{Vect}_n \longrightarrow \mathbf{M}_n$$

is formally étale (it is also a flat monomorphism by 1.2.9.4).

Moreover, we have $\mathbf{M}_n \simeq \mathbb{R}\underline{Spec}\, B$, where $B := \mathbb{L}F(\mathbf{1}^{n^2})$ is the derived free commutative monoid over $\mathbf{1}^{n^2}$. This implies that $\mathbb{L}_B$ is a free B-module of rank n^2, and therefore that the morphism $\mathbf{1} \longrightarrow B$ is formally smooth in the sense of Def. 1.2.7.1. By composition, we find that $\Omega_*\mathbf{Vect}_n \longrightarrow *$ is a formally smooth morphism as required.

(2) This follows from (1), Prop. 1.2.9.4 (2) and the fact that $B = \mathbb{L}F(\mathbf{1}^{n^2})$ is a finitely presented object in $Comm(\mathcal{C})$. □

DEFINITION 1.3.7.11. (1) *The stack* $\Omega_*\mathbf{Vect}_n$ *is denoted by* $\mathbf{Gl}_n$, *and is called the* linear group stack of rank n. *The stack* $\mathbf{Gl}_1$ *is denoted by* $\mathbb{G}_m$, *and is called the* multiplicative group stack.

(2) *The stack* $\mathbf{M}_n$ *defined during the proof of 1.3.7.10* (1) *is called the* stack of $n \times n$ matrices. *The stack* $\mathbf{M}_1$ *is denoted by* $\mathbb{G}_a$, *and is called the* additive group stack.

Being a stack of loops, the stack $\mathbf{Gl}_n = \Omega_*\mathbf{Vect}_n$ has a natural group structure, encoded in the fact that it is the X_1 of a Segal groupoid object X_* with $X_0 = *$. Symbolically, we will simply write

$$B\mathbf{Gl}_n := |X_*|.$$

Our conclusion is that the stack $\mathbf{Vect}_n$ can be written as $B\mathbf{Gl}_n$, where $\mathbf{Gl}_n$ is a *formally smooth representable group stack*. Furthermore this group stack is smooth when the unit $\mathbf{1}$ is finitely presented. As a corollary we get the following geometricity result on $\mathbf{Vect}_n$.

COROLLARY 1.3.7.12. *Assume that the unit* 1 *is a finitely presented object in* $\mathcal{C}$. *Assume that all smooth morphisms in* $Comm(\mathcal{C})$ *belong to* $\mathbf{P}$. *Then, the stack* $\mathbf{Vect}_n$ *is* 1*-geometric, the morphism* $\mathbf{Vect}_n \longrightarrow *$ *is in* $\mathbf{P}$ *and finitely presented, and furthermore its diagonal is a* (-1)*-representable morphism.*

PROOF. The 1-geometricity statement is a consequence of Prop. 1.3.4.2 and the fact that the natural morphism $* \longrightarrow \mathbf{Vect}_n$ is a 1-$\mathbf{P}$-atlas. That $*$ is a 1-$\mathbf{P}$-atlas also implies that $\mathbf{Vect}_n \longrightarrow *$ is in $\mathbf{P}$ and finitely presented. The statement concerning the diagonal follows from the fact that $\mathbf{Gl}_n$ is a representable stack and the locality of representable objects Prop. 1.3.2.8. □

We finish with an analogous situation for perfect modules. We let K be a perfect object in $\mathcal{C}$, and we define a stack $\mathbb{R}\underline{End}(K)$ in the following way. We chose a cofibrant replacement QK of K, and Γ_* a simplicial resolution functor on $\mathcal{C}$. One sets

$$\begin{array}{cccc} \mathbb{R}\underline{End}(K): & Comm(\mathcal{C}) & \longrightarrow & SSet_{\mathbb{V}} \\ & A & \mapsto & Hom(QK, \Gamma_*(QK \otimes A)). \end{array}$$

Note that for any A the simplicial set $\mathbb{R}\underline{End}(K)(A)$ is naturally equivalent to

$$Map_{A-Mod}(K \otimes^{\mathbb{L}} A, K \otimes^{\mathbb{L}} A).$$

LEMMA 1.3.7.13. *The simplicial presheaf* $\mathbb{R}\underline{End}(K)(A) \in Aff_{\mathcal{C}}^{\sim,\tau}$ *is a stack. It is furthermore representable.*

PROOF. This is clear as K being perfect one sees that there exists an isomorphism in $\mathrm{Ho}(SPr(Aff_{\mathcal{C}}))$

$$\mathbb{R}\underline{End}(K) \simeq \mathbb{R}\underline{Spec}\, B,$$

where $B := \mathbb{L}F(K \otimes^{\mathbb{L}} K^{\vee})$ is the derived free commutative monoid on the object $K \otimes^{\mathbb{L}} K^{\vee}$. □

We now define a sub-stack $\mathbb{R}\underline{Aut}(K)$ of $\mathbb{R}\underline{End}(K)$. For a commutative monoid $A \in Comm(\mathcal{C})$, we define $\mathbb{R}\underline{Aut}(K)(A)$ to be the union of connected components of $\mathbb{R}\underline{End}(K)(A)$ corresponding to isomorphisms in

$$\pi_0(\mathbb{R}\underline{End}(K)(A)) \simeq [K \otimes^{\mathbb{L}} A, K \otimes^{\mathbb{L}} A]_{A-Mod}.$$

This clearly defines a full sub-simplicial presheaf $\mathbb{R}\underline{Aut}(K)$ of $\mathbb{R}\underline{End}(K)$, which is a sub-stack as one can see easily using Cor. 1.3.2.7.

PROPOSITION 1.3.7.14. *Assume that C is a stable model category. Then, the stack $\mathbb{R}\underline{Aut}(K)$ is representable. Furthermore the morphism $\mathbb{R}\underline{Aut}(K) \longrightarrow \mathbb{R}\underline{End}(K)$ is a formal Zariski open immersion, and $\mathbb{R}\underline{Aut}(K) \longrightarrow *$ is fp. If furthermore 1 is finitely presented in C then $\mathbb{R}\underline{Aut}(K) \longrightarrow *$ is a perfect morphism.*

PROOF. It is the same as 1.3.7.10 but using the construction A_K and Prop. 1.2.10.1, instead of the standard localization $A[a^{-1}]$. More precisely, for a representable stack $X := \mathbb{R}\underline{Spec}\,A$ and a morphism $x : X \longrightarrow \mathbb{R}\underline{End}(K)$, the homotopy pullback

$$\mathbb{R}\underline{Aut}(K) \times^{h}_{\mathbb{R}\underline{End}(K)} X \longrightarrow X$$

is isomorphic in $\mathrm{Ho}(Aff_C^{\sim,\tau}/X)$ to

$$\mathbb{R}\underline{Spec}\,A_E \longrightarrow \mathbb{R}\underline{Spec}\,A,$$

where E is the homotopy cofiber of the endomorphism $x : K \otimes^{\mathbb{L}} A \longrightarrow K \otimes^{\mathbb{L}} A$ corresponding to the point x. □

CHAPTER 1.4

Geometric stacks: Infinitesimal theory

As in the previous chapter, we fix once for all a HAG context $(\mathcal{C}, \mathcal{C}_0, \mathcal{A}, \tau, \mathbf{P})$.

In this chapter we will assume furthermore that the suspension functor

$$S : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

is fully faithful. In particular, for any commutative monoid $A \in Comm(\mathcal{C})$, the stabilization functor

$$S_A : \mathrm{Ho}(A-Mod) \longrightarrow \mathrm{Ho}(Sp(A-Mod))$$

is fully faithful (see 1.2.11.2). We will therefore forget to mention the functor S_A and simply consider A-modules as objects in $\mathrm{Ho}(Sp(A-Mod))$, corresponding to 0-connective objects.

1.4.1. Tangent stacks and cotangent complexes

We consider the initial commutative monoid $\mathbf{1} \in Comm(\mathcal{C})$. It can be seen as a module over itself, and gives rise to a trivial square zero extension $\mathbf{1} \oplus \mathbf{1}$ (see §1.2.1).

DEFINITION 1.4.1.1. *The* dual numbers over $\mathcal{C}$ *is the commutative monoid* $\mathbf{1} \oplus \mathbf{1}$, *and is denoted by* $\mathbf{1}[\epsilon]$. *The corresponding representable stack is denoted by*

$$\mathbb{D}_\epsilon := \mathbb{R}\underline{Spec}\,(\mathbf{1}[\epsilon])$$

and is called the infinitesimal disk.

Of course, as every trivial square zero extension the natural morphism $\mathbf{1} \longrightarrow \mathbf{1}[\epsilon]$ possesses a natural retraction $\mathbf{1}[\epsilon] \longrightarrow \mathbf{1}$. On the level of representable stacks this defines a natural global point

$$* \longrightarrow \mathbb{D}_\epsilon.$$

We recal that $\mathrm{Ho}(Aff_{\mathcal{C}}^{\sim,\tau})$ being the homotopy category of a model topos has internal *Hom*'s objects $\mathbb{R}_\tau\underline{\mathcal{H}om}(-,-)$. They satisfy the usual adjunction isomorphisms

$$Map_{Aff_{\mathcal{C}}^{\sim,\tau}}(F, \mathbb{R}_\tau\underline{\mathcal{H}om}(G,H)) \simeq Map_{Aff_{\mathcal{C}}^{\sim,\tau}}(F \times^h G, H).$$

DEFINITION 1.4.1.2. *Let* $F \in \mathrm{St}(\mathcal{C},\tau)$ *be a stack. The* tangent stack of F *is defined to be*

$$TF := \mathbb{R}_\tau\underline{\mathcal{H}om}(\mathbb{D}_\epsilon, F).$$

The natural morphism $* \longrightarrow \mathbb{D}_\epsilon$ *induces a well defined projection*

$$\pi : TF \longrightarrow F,$$

and the projection $\mathbb{D}_\epsilon \longrightarrow *$ *induces a natural section of* π

$$e : F \longrightarrow TF.$$

For any commutative monoid $A \in Comm(\mathcal{C})$, it is clear that there is a natural equivalence

$$A \otimes^{\mathbb{L}} (\mathbf{1}[\epsilon]) \simeq A \oplus A,$$

where $A \oplus A$ is the trivial square zero extension of A by itself. We will simply denote $A \oplus A$ by $A[\epsilon]$, and

$$\mathbb{D}_{\epsilon}^{A} := \mathbb{R}\underline{Spec}(A[\epsilon]) \simeq \mathbb{R}\underline{Spec}(A) \times \mathbb{D}_{\epsilon}$$

the infinitesimal disk over A.

With these notations, and for any fibrant object $F \in Aff_{\mathcal{C}}^{\sim,\tau}$, the stack TF can be described as the following simplicial presheaf

$$\begin{array}{rccc} TF: & Comm(\mathcal{C}) = Aff_{\mathcal{C}}^{op} & \longrightarrow & SSet_{\mathbb{V}} \\ & A & \mapsto & F(A[\epsilon]). \end{array}$$

Note that if F is fibrant, then so is TF as defined above. In other words for any $A \in Comm(\mathcal{C})$ there exists a natural equivalence of simplicial sets

$$\mathbb{R}TF(A) \simeq \mathbb{R}F(A[\epsilon]).$$

PROPOSITION 1.4.1.3. *The functor $F \mapsto TF$ commutes with $\mathbb{V}$-small homotopy limits.*

PROOF. This is clear as $\mathbb{R}_{\tau}\underline{\mathcal{H}om}(H, -)$ always commutes with homotopy limits for any H. □

Let $F \in \mathrm{St}(\mathcal{C}, \tau)$ be a stack, $A \in Comm(\mathcal{C})$ a commutative monoid and

$$x : \mathbb{R}\underline{Spec}\, A \longrightarrow F$$

be a A-point. Let M be an A-module, and let $A \oplus M$ be the trivial square zero extension of A by M. Let us fix the following notations

$$X := \mathbb{R}\underline{Spec}\, A \qquad X[M] := \mathbb{R}\underline{Spec}\,(A \oplus M).$$

The natural augmentation $A \oplus M \longrightarrow A$ gives rise to a natural morphism of stacks $X \longrightarrow X[M]$.

DEFINITION 1.4.1.4. *Let $x : X \longrightarrow F$ be as above. The simplicial set of* derivations from F to M at the point x *is defined by*

$$\mathbb{D}er_F(X, M) := Map_{X/Aff_{\mathcal{C}}^{\sim,\tau}}(X[M], F) \in \mathrm{Ho}(SSet_{\mathbb{V}}).$$

It will also denoted by $\mathbb{D}er_F(x, M)$.

As the construction $M \mapsto X[M]$ is functorial in M, we get this way a well defined functor

$$\mathbb{D}er_F(X, -) : \mathrm{Ho}(A - Mod) \longrightarrow \mathrm{Ho}(SSet_{\mathbb{V}}).$$

This functor is furthermore naturally compatible with the $\mathrm{Ho}(SSet)$-enrichment, in the sense that there exists natural morphisms

$$Map_{A-Mod}(M, N) \longrightarrow Map_{SSet}(\mathbb{D}er_F(X, M), \mathbb{D}er_F(X, N))$$

which are compatible with compositions. Note that the morphism $X \longrightarrow X[M]$ has a natural retraction $X[M] \longrightarrow X$, and therefore that the simplicial set $\mathbb{D}er_F(X, M)$ has a distinguished base point, the trivial derivation. The functor $\mathbb{D}er_F(X, -)$ takes its values in the homotopy category of pointed simplicial sets.

We can also describe the functor $\mathbb{D}er_F(X, M)$ using functors of points in the following way. Let $F \in Aff_{\mathcal{C}}^{\sim,\tau}$ be a fibrant object. For any commutative monoid A, any A-module M and any point $x \in F(A)$, we consider the standard homotopy fiber[1] of

$$F(A \oplus M) \longrightarrow F(A)$$

at the point x. This clearly defines a functor

$$\begin{array}{ccc} A - Mod & \longrightarrow & SSet_{\mathbb{V}} \\ M & \mapsto & Hofiber\,(F(A \oplus M) \to F(A)) \end{array}$$

which is a lift of the functor considered above

$$\mathbb{D}er_F(X, -) : \mathrm{Ho}(A - Mod) \longrightarrow \mathrm{Ho}(SSet_{\mathbb{V}}),$$

where $x \in F(A)$ corresponds via the Yoneda lemma to a morphism $X = \mathbb{R}\underline{Spec}\,A \longrightarrow F$. The fact that the functor $\mathbb{D}er_F(X, -)$ has a natural lift as above is important, as it then makes sense to say that it commutes with homotopy limits or homotopy colimits. The functor $\mathbb{D}er_F(X, -)$ can be considered as an object in $\mathrm{Ho}((A - Mod^{op})^\wedge)$, the homotopy category of pre-stacks on the model category $A - Mod^{op}$, as defined in [**HAGI**, §4.1] (see also §1.3.1). Restricting to the subcategory $A - Mod_0$ we get an object

$$\mathbb{D}er_F(X, -) \in \mathrm{Ho}((A - Mod_0^{op})^\wedge).$$

In the sequel the functor $\mathbb{D}er_F(X, -)$ will always be considered as an object in $\mathrm{Ho}((A - Mod_0^{op})^\wedge)$.

DEFINITION 1.4.1.5. *Let F be any stack and let $A \in \mathcal{A}$.*

(1) *Let $x : X := \mathbb{R}\underline{Spec}\,A \longrightarrow F$ be an A-point. We say that F* has a cotangent complex at x *if there exists an integer n, an $(-n)$-connective stable A-module $\mathbb{L}_{F,x} \in \mathrm{Ho}(Sp(A - Mod))$, and an isomorphism in $\mathrm{Ho}((A - Mod_0^{op})^\wedge)$*

$$\mathbb{D}er_F(X, -) \simeq \mathbb{R}\underline{h}_s^{\mathbb{L}_{F,x}}.$$

(2) *If F has a cotangent complex at x, the stable A-module $\mathbb{L}_{F,x}$ is then called the* cotangent complex of F at x.

(3) *If F has a cotangent complex at x, the* tangent complex of F at x *is then the stable A-module*

$$\mathbb{T}_{F,x} := \mathbb{R}\underline{Hom}_A^{Sp}(\mathbb{L}_{F,x}, A) \in \mathrm{Ho}(Sp(A - Mod)).$$

In other words, the existence of a cotangent complex of F at $x : X := \mathbb{R}\underline{Spec}\,A \longrightarrow F$ is equivalent to the co-representability of the functor

$$\mathbb{D}er_F(X, -) : A - Mod_0 \longrightarrow SSet_{\mathbb{V}},$$

by some $(-n)$-connective object $\mathbb{L}_{F,x} \in \mathrm{Ho}(Sp(A - Mod))$. The fact that $\mathbb{L}_{F,x}$ is well defined is justified by our Prop. 1.2.11.3.

The first relation between cotangent complexes and the tangent stack is given by the following proposition.

PROPOSITION 1.4.1.6. *Let F be a stack and $x : X := \mathbb{R}\underline{Spec}\,A \longrightarrow F$ be an A-point with $A \in \mathcal{A}$. If F has a cotangent complex $\mathbb{L}_{F,x}$ at the point x then there exists a natural isomorphism in $\mathrm{Ho}(SSet_{\mathbb{V}})$*

$$\mathbb{R}\underline{Hom}_{Aff_{\mathcal{C}}^{\sim,\tau}/F}(X, TF) \simeq Map_{Sp(A-Mod)}(A, \mathbb{T}_{F,x}) \simeq Map_{Sp(A-Mod)}(\mathbb{L}_{F,x}, A).$$

[1] The standard homotopy fiber product of a diagram $X \longrightarrow Z \longleftarrow Y$ of fibrant simplicial sets is defined for example by

$$X \times_Z^h Y := (X \times Y) \times_{Z \times Z} Z^{\Delta^1}.$$

PROOF. By definition of TF, there is a natural isomorphism

$$\mathbb{R}\underline{Hom}_{Aff_{\mathcal{C}}^{\sim,\tau}/F}(X,TF) \simeq \mathbb{R}\underline{Hom}_{Aff_{\mathcal{C}}^{\sim,\tau}/X}(X[A],TF) \simeq Map_{Sp(A-Mod)}(\mathbb{L}_{F,x},A).$$

Moreover, by definition of the tangent complex we have

$$Map_{Sp(A-Mod)}(\mathbb{L}_{F,x},A) \simeq Map_{Sp(A-Mod)}(A,\mathbb{T}_{F,x}).$$

□

Now, let F be a stack, and u a morphism in $\mathrm{Ho}(Aff_{\mathcal{C}}^{\sim,\tau}/F)$

$$\begin{array}{ccc} Y & \xrightarrow{u} & X \\ & {\scriptstyle y}\searrow \quad \swarrow{\scriptstyle y} & \\ & F, & \end{array}$$

with $X = \mathbb{R}\underline{Spec}\,A$ and $Y = \mathbb{R}\underline{Spec}\,B$ belonging to $\mathcal{A}$. Let $M \in B-Mod_0$, which is also an A-module by the forgetful functor $A-Mod_0 \longrightarrow B-Mod_0$. There is a commutative diagram of commutative monoids

$$\begin{array}{ccc} A \oplus M & \longrightarrow & B \oplus M \\ \downarrow & & \downarrow \\ A & \longrightarrow & B, \end{array}$$

inducing a commutative square of representable stacks

$$\begin{array}{ccc} X[M] & \longleftarrow & Y[M] \\ \uparrow & & \uparrow \\ X & \longleftarrow & Y. \end{array}$$

This implies the existence of a natural morphism in $\mathrm{Ho}(SSet_{\mathbb{V}})$

$$\mathbb{D}er_F(Y,M) \longrightarrow \mathbb{D}er_F(X,M).$$

If the stack F has cotangent complexes at both points x and y, Prop. 1.2.11.3 induces a well defined morphism in $\mathrm{Ho}(Sp(B-Mod))$

$$u^* : \mathbb{L}_{F,x} \otimes_A^{\mathbb{L}} B \longrightarrow \mathbb{L}_{F,y}.$$

Of course, we have $(u \circ v)^* = v^* \circ u^*$ whenever this formula makes sense.

In the same way, the construction of $\mathbb{L}_{F,x}$ is functorial in F. Let $f : F \longrightarrow F'$ be a morphism of stacks, $A \in \mathcal{A}$, and $x : \mathbb{R}\underline{Spec}\,A \longrightarrow F$ be an A-point with image $f(x) : \mathbb{R}\underline{Spec}\,A \longrightarrow F'$. Then, for any A-module M, there is a natural morphism

$$\mathbb{D}er_F(X,M) \longrightarrow \mathbb{D}er_{F'}(X,M).$$

Therefore, if F has a cotangent complex at x and F' has a cotangent complex at x', we get a natural morphism in $\mathrm{Ho}(Sp(A-Mod))$

$$df_x : \mathbb{L}_{F',f(x)} \longrightarrow \mathbb{L}_{F,x},$$

called the *differential of f at x*. Once again, we have $d(f \circ g)_x = dg_x \circ df_x$ each time this formula makes sense (this is the chain rule). Dually, we also get by duality the *derivative of f at x*

$$Tf_x : \mathbb{T}_{F,x} \longrightarrow \mathbb{T}_{F',f(x)}.$$

DEFINITION 1.4.1.7. *A stack F has a global cotangent complex relative to the HA context $(\mathcal{C},\mathcal{C}_0,\mathcal{A})$ (or simply* has a cotangent complex *when there is no ambiguity on the context) if the following two conditions are satisfied.*

(1) *For any $A \in \mathcal{A}$, and any point $x : \mathbb{R}\underline{Spec}\,A \longrightarrow F$, the stack F has a cotangent complex $\mathbb{L}_{F,x}$ at x.*

(2) *For any morphism $u : A \longrightarrow B$ in $\mathcal{A}$, and any morphism in* $\mathrm{Ho}(Aff_{\mathcal{C}}^{\sim,\tau}/F)$

$$\begin{array}{ccc} Y := \mathbb{R}\underline{Spec}\,B & \xrightarrow{\quad u \quad} & X := \mathbb{R}\underline{Spec}\,A \\ & {\scriptstyle y}\searrow \quad \swarrow{\scriptstyle x} & \\ & F, & \end{array}$$

the induced morphism

$$u^* : \mathbb{L}_{F,x} \otimes^{\mathbb{L}}_A B \longrightarrow \mathbb{L}_{F,y}$$

is an isomorphism in $\mathrm{Ho}(Sp(B-Mod))$.

As a corollary of Prop. 1.2.1.2 and the standard properties of derivations any representable stack has a cotangent complex.

PROPOSITION 1.4.1.8. *Any representable stack $F = \mathbb{R}\underline{Spec}A$ has a global cotangent complex.*

PROOF. This is nothing else than the existence of a universal derivation as proved in Prop. 1.2.1.2. □

If $X = \mathbb{R}\underline{Spec}\,A$ is a representable stack in $\mathcal{A}$, and $x : X \to X$ is the identity, then the stable A-module $\mathbb{L}_{X,x}$ is naturally isomorphic in $\mathrm{Ho}(Sp(A-Mod))$ to $\mathbb{L}_A$. More generally, for any morphism $A \longrightarrow B$ with $B \in \mathcal{A}$, corresponding to $y : \mathbb{R}\underline{Spec}\,B \longrightarrow X$, the B-module $\mathbb{L}_{X,y}$ is naturally isomorphic in $\mathrm{Ho}(Sp(B-Mod))$ to $\mathbb{L}_A \otimes^{\mathbb{L}}_A B$.

The next proposition explains the relation between the tangent stack and the global cotangent complex when it exists. It is a globalization of Prop.1.4.1.6.

PROPOSITION 1.4.1.9. *Let F be a stack having a cotangent complex. Let $x : X = \mathbb{R}\underline{Spec}\,A \longrightarrow F$ be any morphism, and*

$$TF_x := TF \times^h_F X \longrightarrow X$$

the natural projection. Let $A \longrightarrow B$ be a morphism with $B \in \mathcal{A}$, corresponding to a morphism of representable stacks $X = \mathbb{R}\underline{Spec}\,A \longrightarrow Y = \mathbb{R}\underline{Spec}\,B$. Then, there exists a natural isomorphism in $\mathrm{Ho}(SSet)$

$$\mathbb{R}TF_x(B) \simeq Map_{Aff_{\mathcal{C}}^{\sim,\tau}/F}(Y, TF) \simeq Map_{Sp(A-Mod)}(\mathbb{L}_{F,x}, B).$$

PROOF. This is a reformulation of Prop. 1.4.1.6, and the fact that

$$Map_{Sp(A-Mod)}(\mathbb{L}_{F,x}, B) \simeq Map_{Sp(B-Mod)}(\mathbb{L}_{F,x} \otimes^{\mathbb{L}}_A B, B).$$

□

REMARK 1.4.1.10. Of course, the isomorphism of proposition Prop. 1.4.1.9 is functorial in F.

PROPOSITION 1.4.1.11. *Let F be an n-geometric stack. We assume that for any $A \in \mathcal{A}$, and any point $x : X := \mathbb{R}\underline{Spec}\,A \longrightarrow F$, and any A-module $M \in A - Mod_0$, the natural morphism*

$$\mathbb{D}er_F(X, M) \simeq \mathbb{D}er_F(X, \Omega S(M)) \longrightarrow \Omega\mathbb{D}er_F(X, S(M))$$

is an isomorphism in $\mathrm{Ho}(SSet)$. Then F has a global cotangent complex, which is furthermore $(-n)$-connective.

PROOF. The proof is by induction on n. For $n = -1$ this is Prop. 1.4.1.8 and does not use our exactness condition on the functor $\mathbb{D}er_F(X, -)$.

Let $n \geq 0$ be an integer and F be an n-geometric stack. Let $X = \mathbb{R}\underline{Spec}\,A$ be a representable stack in $\mathcal{A}$ and $x : X \longrightarrow F$ be any morphism in $\mathrm{St}(\mathcal{C}, \tau)$. We consider the natural morphisms

$$d : X \longrightarrow X \times^h X \qquad d_F : X \longrightarrow X \times^h_F X.$$

By induction on n, we see that the stacks $X \times^h X$ and $X \times^h_F X$ both have cotangent complexes at the point d and d_F, denoted respectively by $\mathbb{L}$ and $\mathbb{L}'$. There is moreover a natural morphism in $\mathrm{Ho}(Sp(A - Mod))$

$$f : \mathbb{L}' \longrightarrow \mathbb{L},$$

induced by

$$X \longrightarrow X \times^h_F X \longrightarrow X \times^h X.$$

We set $\mathbb{L}''$ as the homotopy cofiber of f in $Sp(A - Mod)$. By construction, for any A-module $M \in A - Mod_0$, the simplicial set $Map_{Sp(A-Mod)}(\mathbb{L}'', M)$ is naturally equivalent to the homotopy fiber of

$$\mathbb{D}er_X(X, M) \times^h_{\mathbb{D}er_F(X,M)} \mathbb{D}er_X(X, M) \longrightarrow \mathbb{D}er_X(X, M) \times \mathbb{D}er_X(X, M),$$

and thus is naturally equivalent to $\Omega_* \mathbb{D}er_F(X, M)$. By assumption we have natural isomorphisms in $\mathrm{Ho}(SSet)$

$$Map_{Sp(A-Mod)}(\Omega(\mathbb{L}''), M) \simeq Map_{Sp(A-Mod)}(\mathbb{L}'', S(M))$$
$$\simeq \Omega_* \mathbb{D}er_F(X, S(M)) \simeq \mathbb{D}er_F(X, \Omega_* S(M)) \simeq \mathbb{D}er_F(X, M).$$

This implies that $\Omega_* \mathbb{L}'' \in \mathrm{Ho}(Sp(A - Mod))$ is a cotangent complex of F at the point x. By induction on n and by construction we also see that this cotangent complex is $(-n)$-connective.

Now, let

$$\begin{array}{ccc} Y = \mathbb{R}\underline{Spec}\,B & \xrightarrow{u} & X = \mathbb{R}\underline{Spec}\,A \\ & {\scriptstyle y}\searrow \quad \swarrow{\scriptstyle x} & \\ & F, & \end{array}$$

be a morphism in $\mathrm{Ho}(Aff_{\mathcal{C}}^{\sim,\tau}/F)$, with A and B in $\mathcal{A}$. We consider the commutative diagram with homotopy cartesian squares

$$\begin{array}{ccccccc} & & X & \longrightarrow & X \times^h_F X & \longrightarrow & X \times^h X \\ & \nearrow & \uparrow & & \uparrow & & \uparrow \\ Y & \longrightarrow & Y \times^h_X Y & \longrightarrow & Y \times^h_F Y & \longrightarrow & Y \times^h Y. \end{array}$$

By the above explicit construction and an induction on n, the fact that the natural morphism

$$u^* : \mathbb{L}_{F,x} \otimes^{\mathbb{L}}_A B \longrightarrow \mathbb{L}_{F,y}$$

is an isomorphism simply follows from the next lemma.

LEMMA 1.4.1.12. *Let*

$$\begin{array}{ccccc} X = \mathbb{R}\underline{Spec}\,A & \xrightarrow{x} & F & \longrightarrow & G \\ {\scriptstyle u}\uparrow & & \uparrow & & \uparrow \\ Y = \mathbb{R}\underline{Spec}\,B & \xrightarrow[y]{} & F' & \longrightarrow & G' \end{array}$$

be a commutative diagram with the right hand square being homotopy cartesian in $Aff_{\mathcal{C}}^{\sim,\tau}$. We assume that A and B are in $\mathcal{A}$ and that F and G have global cotangent complexes. Then the natural square

$$\begin{array}{ccc} \mathbb{L}_{G',y} & \longrightarrow & \mathbb{L}_{F',y} \\ \uparrow & & \uparrow \\ \mathbb{L}_{G,x} \otimes^{\mathbb{L}}_A B & \longrightarrow & \mathbb{L}_{F,x} \otimes^{\mathbb{L}}_A B \end{array}$$

is homotopy cartesian in $Sp(B-Mod)$.

PROOF. This is immediate from the definition and the homotopy cartesian square

$$\begin{array}{ccc} \mathbb{D}er_F(Y,M) \simeq \mathbb{D}er_F(X,M) & \longrightarrow & \mathbb{D}er_G(Y,M) \simeq \mathbb{D}er_G(X,M) \\ \uparrow & & \uparrow \\ \mathbb{D}er_{F'}(Y,M) & \longrightarrow & \mathbb{D}er_{G'}(Y,M) \end{array}$$

for any A-module M. □

This finishes the proof of Prop. 1.4.1.11. □

In fact, the proof of Proposition 1.4.1.11 also proves the following

PROPOSITION 1.4.1.13. *Let F be a stack such that the diagonal $F \longrightarrow F \times^h F$ is $(n-1)$-representable. We suppose that for an $A \in \mathcal{A}$, any point $x : X := \mathbb{R}\underline{Spec}\, A \longrightarrow F$, and any A-module $M \in A - Mod_0$ the natural morphism*

$$\mathbb{D}er_F(X,M) \simeq \mathbb{D}er_F(X,\Omega S(M)) \longrightarrow \Omega\mathbb{D}er_F(X,S(M))$$

is an isomorphism in Ho$(SSet)$. *Then F has a cotangent complex, which is furthermore $(-n)$-connective.*

We finish this section by the notion of relative cotangent complex and its relation with the absolute notion. Let $f : F \longrightarrow G$ be a morphism of stacks, $A \in \mathcal{A}$, and $X := \mathbb{R}\underline{Spec}\, A \longrightarrow F$ be a morphism. We define an object $\mathbb{D}er_{F/G}(X,-) \in \mathrm{Ho}((A-Mod_0^{op})^{\wedge})$, to be the standard homotopy fiber of the morphism of the natural morphism

$$df : \mathbb{D}er_F(X,-) \longrightarrow \mathbb{D}er_G(X,-).$$

In terms of functors the object $\mathbb{D}er_{F/G}(X,-)$ sends an A-module $M \in A - Mod_0$ to the simplicial set

$$\mathbb{D}er_{F/G}(X,M) = Map_{X/Aff_{\mathcal{C}}^{\sim,\tau}/G}(X[M],F).$$

DEFINITION 1.4.1.14. *Let $f : F \longrightarrow G$ be a morphism of stacks.*

(1) *Let $A \in \mathcal{A}$, and $x : X := \mathbb{R}\underline{Spec}\, A \longrightarrow F$ be an A-point. We say that f* has a (relative) cotangent complex at x relative to $\mathcal{A}$ *(or simply f* has a (relative) cotangent complex at x *when $\mathcal{A}$ is unambiguous) if there exists an integer n, and an $(-n)$-connective stable A-module $\mathbb{L}_{F/G,x} \in$* Ho$(Sp(A-Mod))$, *and an isomorphism in* Ho$(A-Mod_0^{op})^{\wedge}$

$$\mathbb{D}er_{F/G}(X,-) \simeq \mathbb{R}\underline{h}_s^{\mathbb{L}_{F/G,x}}.$$

(2) *If f has a cotangent complex at x, the stable A-module $\mathbb{L}_{F,x}$ is then called the* (relative) cotangent complex of f at x.

(3) *If f has a cotangent complex at x, the* (relative) tangent complex of f at x *is then the stable A-module*

$$\mathbb{T}_{F/G,x} := \mathbb{R}\underline{Hom}_A^{Sp}(\mathbb{L}_{F/G,x}, A) \in \mathrm{Ho}(Sp(A-Mod)).$$

Let now be a morphism of stacks $f : F \longrightarrow G$, and a commutative diagram in $Aff_{\mathcal{C}}^{\sim,\tau}$

$$\begin{array}{ccc} Y & \xrightarrow{u} & X \\ & {\scriptstyle y}\searrow \quad \swarrow{\scriptstyle y} & \\ & F, & \end{array}$$

with $X = \mathbb{R}\underline{Spec}\,A$ and $Y = \mathbb{R}\underline{Spec}\,B$ belonging to $\mathcal{A}$. We have a natural morphism in $\mathrm{Ho}((A-Mod_0^{op})^{\wedge})$

$$\mathbb{D}er_{F/G}(Y,) \longrightarrow \mathbb{D}er_{F/G}(X,-).$$

If the morphism $f : F \longrightarrow G$ has cotangent complexes at both points x and y, Prop. 1.2.11.3 induces a well defined morphism in $\mathrm{Ho}(Sp(B-Mod))$

$$u^* : \mathbb{L}_{F/G,x} \otimes_A^{\mathbb{L}} B \longrightarrow \mathbb{L}_{F/G,y}.$$

Of course, we have $(u \circ v)^* = v^* \circ u^*$ when this formula makes sense.

DEFINITION 1.4.1.15. *A morphism of stacks $f : F \longrightarrow G$* has a relative cotangent complex relative to $(\mathcal{C}, \mathcal{C}_0, \mathcal{A})$ *(or simply* has a cotangent complex *when the HA context is clear) if the following two conditions are satisfied.*

(1) *For any $A \in \mathcal{A}$, and any point $x : \mathbb{R}\underline{Spec}\,A \longrightarrow F$, the morphism f has a cotangent complex $\mathbb{L}_{F/G,x}$ at x.*

(2) *For any morphism $u : A \longrightarrow B$ in $\mathcal{A}$, and any morphism in* $\mathrm{Ho}(Aff_{\mathcal{C}}^{\sim,\tau}/F)$

$$\begin{array}{ccc} Y := \mathbb{R}\underline{Spec}\,B & \xrightarrow{u} & X := \mathbb{R}\underline{Spec}\,A \\ & {\scriptstyle y}\searrow \quad \swarrow{\scriptstyle x} & \\ & F, & \end{array}$$

the induced morphism

$$u^* : \mathbb{L}_{F/G,x} \otimes_A^{\mathbb{L}} B \longrightarrow \mathbb{L}_{F/G,y}$$

is an isomorphism in $\mathrm{Ho}(Sp(B-Mod))$.

The important remark is the following, relating absolute and relative notions of cotangent complexes.

LEMMA 1.4.1.16. *Let $f : F \longrightarrow G$ be a morphism of stacks.*

(1) *If both stacks F and G have cotangent complexes then the morphism f has a cotangent complex. Furthermore, for any $A \in \mathcal{A}$, and any morphism of stacks $X = \mathbb{R}\underline{Spec}\,A \longrightarrow F$, there is a natural homotopy cofiber sequence of stable A-modules*

$$\mathbb{L}_{G,x} \longrightarrow \mathbb{L}_{F,x} \longrightarrow \mathbb{L}_{F/G,x}.$$

(2) *If the morphism f has a cotangent complex then for any stack H and any morphism $H \longrightarrow G$, the morphism $F \times_G^h H \longrightarrow H$ has a relative cotangent complex and furthermore we have*

$$\mathbb{L}_{F/G,x} \simeq \mathbb{L}_{F \times_G^h H/H,x}$$

for any $A \in \mathcal{A}$, and any morphism of stacks $X = \mathbb{R}\underline{Spec}\,A \longrightarrow F \times_G^h H$.

(3) *If for any $A \in \mathcal{A}$ and any morphism of stacks $x : X := \mathbb{R}\underline{Spec}\, A \longrightarrow F$, the morphism $F \times^h_G X \longrightarrow X$ has a relative cotangent complex, then the morphism f has a relative cotangent complex. Furthermore, we have*

$$\mathbb{L}_{F/G,x} \simeq \mathbb{L}_{F\times^h_G X/X,x}.$$

(4) *If for any $A \in \mathcal{A}$ and any morphism of stacks $x : X := \mathbb{R}\underline{Spec}\, A \longrightarrow F$, the stack $F \times^h_G X$ has a cotangent complex, then the morphism f has a relative cotangent complex. Furthermore we have a natural homotopy cofiber sequence*

$$\mathbb{L}_A \longrightarrow \mathbb{L}_{F\times^h_G X,x} \longrightarrow \mathbb{L}_{F/G,x}.$$

PROOF. (1) and (2) follow easily from the definition. Point (3) follows from (2). Finally, point (4) follows from (3), (1) and Prop. 1.4.1.8. □

1.4.2. Obstruction theory

Recall from 1.2.1.7 that for any commutative monoid A, any A-module M, and any derivation $d : A \longrightarrow A \oplus M$, we can form the square zero extension of A by ΩM, denoted by $A \oplus_d \Omega M$, as the homotopy cartesian square in $Comm(\mathcal{C})$

$$\begin{array}{ccc} A \oplus_d \Omega M & \xrightarrow{p} & A \\ \downarrow & & \downarrow d \\ A & \xrightarrow{s} & A \oplus M \end{array}$$

where $s : A \longrightarrow A \oplus M$ is the trivial derivation. In the sequel, the morphism $p : A \oplus_d \Omega M \longrightarrow A$ will be called the natural projection.

DEFINITION 1.4.2.1. (1) *A stack F is* infinitesimally cartesian relative to the HA context $(\mathcal{C}, \mathcal{C}_0, \mathcal{A})$ *(or simply* inf-cartesian *when the HA context is unambiguous) if for any commutative monoid $A \in \mathcal{A}$, any $M \in A - Mod_1$, and any derivation $d \in \pi_0(\mathbb{D}er(A, M))$, corresponding to a morphism $d : A \longrightarrow A \oplus M$ in $\mathrm{Ho}(Comm(\mathcal{C})/A)$, the square*

$$\begin{array}{ccc} \mathbb{R}F(A \oplus_d \Omega M) & \longrightarrow & \mathbb{R}F(A) \\ \downarrow & & \downarrow d \\ \mathbb{R}F(A) & \xrightarrow{s} & \mathbb{R}F(A \oplus M) \end{array}$$

is homotopy cartesian.

(2) *A stack F* has an obstruction theory *(relative to $(\mathcal{C}, \mathcal{C}_0, \mathcal{A})$) if it has a (global) cotangent complex and if it is infinitesimally cartesian (relative to $(\mathcal{C}, \mathcal{C}_0, \mathcal{A})$).*

One also has a relative version.

DEFINITION 1.4.2.2. (1) *A morphism of stacks $F \longrightarrow G$ is* infinitesimally cartesian relative to $(\mathcal{C}, \mathcal{C}_0, \mathcal{A})$ *(or simply* inf-cartesian *if the HA context is clear) if for any commutative monoid $A \in \mathcal{A}$, any A-module $M \in A - Mod_1$, and any derivation $d \in \pi_0(\mathbb{D}er(A, M))$, corresponding to a morphism d :*

$A \longrightarrow A \oplus M$ *in* $\mathrm{Ho}(Comm(\mathcal{C})/A)$*, the square*

$$\begin{array}{ccc} \mathbb{R}F(A \oplus_d \Omega M) & \longrightarrow & \mathbb{R}G(A \oplus_d \Omega M) \\ \downarrow & & \downarrow \\ \mathbb{R}F(A) \times^h_{\mathbb{R}F(A\oplus M)} \mathbb{R}F(A) & \longrightarrow & \mathbb{R}G(A) \times^h_{\mathbb{R}G(A\oplus M)} \mathbb{R}G(A) \end{array}$$

is homotopy cartesian.

(2) *A morphism of stacks* $f : F \longrightarrow G$ has an obstruction theory relative to $(\mathcal{C}, \mathcal{C}_0, \mathcal{A})$ *(or simply* has an obstruction theory *if the HA context is clear) if it has a (global) cotangent complex and if it is infinitesimally cartesian relative* $(\mathcal{C}, \mathcal{C}_0, \mathcal{A})$.

As our HA context $(\mathcal{C}, \mathcal{C}_0, \mathcal{A})$ is fixed once for all we will from now avoid to mention the expression *relative to* $(\mathcal{C}, \mathcal{C}_0, \mathcal{A})$ when referring to the property of having an obstruction theory. The more precise terminology will only be used when two different HA contexts are involved (this will only happen in §2.3).

We have the following generalization of lemma 1.4.1.16.

Lemma 1.4.2.3. *Let* $f : F \longrightarrow G$ *be a morphism of stacks.*

(1) *If both stacks* F *and* G *have an obstruction theory then the morphism* f *has an obstruction theory.*
(2) *If the morphism* f *has an obstruction theory then for any stack* H *and any morphism* $H \longrightarrow G$*, the morphism* $F \times^h_G H \longrightarrow H$ *has a relative obstruction theory.*
(3) *If for any* $B \in Comm(\mathcal{C})$ *and any morphism of stacks* $y : Y := \mathbb{R}\underline{Spec}\, B \longrightarrow G$*, the stack* $F \times^h_G B$ *has an obstruction theory , then the morphism* f *has a relative obstruction theory.*

Proof. The existence of the cotangent complexes is done in Lem. 1.4.1.16, and it only remains to deal with the inf-cartesian property. The points (1) and (2) are clear by definition.

(3) Let $A \in \mathcal{A}$, $M \in A - Mod_1$ and $d \in \pi_0(\mathbb{D}er(A, M))$. We need to show that the square

$$\begin{array}{ccc} \mathbb{R}F(A \oplus_d \Omega M) & \longrightarrow & \mathbb{R}G(A \oplus_d \Omega M) \\ \downarrow & & \downarrow \\ \mathbb{R}F(A) \times^h_{\mathbb{R}F(A\oplus M)} \mathbb{R}F(A) & \longrightarrow & \mathbb{R}G(A) \times^h_{\mathbb{R}G(A\oplus M)} \mathbb{R}G(A) \end{array}$$

is homotopy cartesian. Let z be a point in $\mathbb{R}G(A \oplus_d \Omega M)$. We need to prove that the morphism induced on the homotopy fibers of the two horizontal morphisms taken at z is an equivalence. But this easily follows from the fact that the pullback of f by the morphism corresponding to z,

$$F \times^h_G X_d[\Omega M] \longrightarrow X_d[\Omega M],$$

has an obstruction theory. $\square$

Proposition 1.4.2.4. (1) *Any representable stack has an obstruction theory.*
(2) *Any representable morphism has an obstruction theory.*

PROOF. (1) By Prop. 1.4.1.8 we already know that representable stacks have cotangent complexes. Using the Yoneda lemma, it is obvious to check that any representable stack is inf-cartesian. Indeed, for $F = \mathbb{R}\underline{Spec}\, B$ we have

$$\mathbb{R}F(A \oplus_d \Omega M) \simeq Map_{Comm(\mathcal{C})}(B, A \oplus_d \Omega M) \simeq$$

$$Map_{Comm(\mathcal{C})}(B, A) \times^h_{Map_{Comm(\mathcal{C})}(B, A \oplus M)} Map_{Comm(\mathcal{C})}(B, A) \simeq \mathbb{R}F(A) \times^h_{\mathbb{R}F(A \oplus M)} \mathbb{R}F(A).$$

(2) Follows from (1) and Lem. 1.4.2.3 (3). □

In general, the expression *has an obstruction theory relative to* $(\mathcal{C}, \mathcal{C}_0, \mathcal{A})$ is justified by the following proposition.

PROPOSITION 1.4.2.5. *Let F be a stack which has an obstruction theory. Let $A \in \mathcal{A}$, $M \in A - Mod_1$ and let $d \in \pi_0(\mathbb{D}er(A, M))$ be a derivation with $A \oplus_d \Omega M$ the corresponding square zero extension. Let us denote by*

$$X := \mathbb{R}\underline{Spec}\, A \longrightarrow X_d[\Omega M] := \mathbb{R}\underline{Spec}\,(A \oplus_d \Omega M)$$

the morphism of representable stacks corresponding to the natural projection $A \oplus_d \Omega M \longrightarrow A$. Finally, let $x : X \longrightarrow F$ be an A-point of F.

(1) *There exists a natural obstruction*

$$\alpha(x) \in \pi_0(Map_{Sp(A-Mod)}(\mathbb{L}_{F,x}, M)) = [\mathbb{L}_{F,x}, M]_{Sp(A-Mod)}$$

vanishing if and only x extends to a morphism x' in $\mathrm{Ho}(X/Aff_{\mathcal{C}}^{\sim,\tau})$

$$\begin{array}{ccc} X & \longrightarrow & X_d[\Omega M] \\ & {\scriptstyle x}\searrow \quad \swarrow{\scriptstyle x'} & \\ & F. & \end{array}$$

(2) *Let us suppose that $\alpha(x) = 0$. Then, the simplicial set of lifts of x*

$$\mathbb{R}\underline{Hom}_{X/Aff_{\mathcal{C}}^{\sim,\tau}}(X_d[\Omega M], F)$$

is (non canonically) isomorphic in $\mathrm{Ho}(SSet)$ *to the simplicial set*

$$Map_{Sp(A-Mod)}(\mathbb{L}_{F,x}, \Omega M) \simeq \Omega Map_{Sp(A-Mod)}(\mathbb{L}_{F,x}, M).$$

More precisely, it is a simplicial torsor over the simplicial group

$$\Omega Map_{Sp(A-Mod)}(\mathbb{L}_{F,x}, M).$$

PROOF. First of all, the space of lifts x' is by definition $\mathbb{R}\underline{Hom}_{X/Aff_{\mathcal{C}}^{\sim,\tau}}(X_d[\Omega M], F)$, which is naturally equivalent to the homotopy fiber at x of the natural morphism

$$\mathbb{R}_\tau\underline{Hom}(X_d[\Omega M], F) \longrightarrow \mathbb{R}_\tau\underline{Hom}(X, F).$$

Using that F is inf-cartesian, we see that there exists a homotopy cartesian square

$$\begin{array}{ccc} \mathbb{R}_\tau\underline{Hom}(X_d[\Omega M], F) & \longrightarrow & \mathbb{R}_\tau\underline{Hom}(X, F) \\ \downarrow & & \downarrow{\scriptstyle d} \\ \mathbb{R}_\tau\underline{Hom}(X, F) & \xrightarrow[s]{} & \mathbb{R}_\tau\underline{Hom}(X[M], F). \end{array}$$

Therefore, the simplicial set $\mathbb{R}\underline{Hom}_{X/Aff_{\mathcal{C}}^{\sim,\tau}}(X_d[\Omega M], F)$ fits into a homotopy cartesian square

$$\begin{array}{ccc} \mathbb{R}\underline{Hom}_{X/Aff_{\mathcal{C}}^{\sim,\tau}}(X_d[\Omega M], F) & \longrightarrow & \bullet \\ \downarrow & & \downarrow{\scriptstyle d} \\ \bullet & \xrightarrow{\ 0\ } & \mathbb{R}\underline{Hom}_{X/Aff_{\mathcal{C}}^{\sim,\tau}}(X[M], F). \end{array}$$

As F has a cotangent complex we have

$$\mathbb{R}\underline{Hom}_{X/Aff_{\mathcal{C}}^{\sim,\tau}}(X[M], F) \simeq Map_{Sp(A-Mod)}(\mathbb{L}_{F,x}, M),$$

and we see that the image of the right vertical arrow in the last diagram provides the element $\alpha(x) \in \pi_0(Map_{Sp(A-Mod)}(\mathbb{L}_{F,x}, M))$, which clearly vanishes if and only if $\mathbb{R}\underline{Hom}_{X/Aff_{\mathcal{C}}^{\sim,\tau}}(X_d[\Omega M], F)$ is non-empty. Furthermore, this last homotopy cartesian diagram also shows that if $\alpha(x) = 0$, then one has an isomorphism in $\mathrm{Ho}(SSet)$

$$\mathbb{R}\underline{Hom}_{X/Aff_{\mathcal{C}}^{\sim,\tau}}(X_d[\Omega M], F) \simeq \Omega Map_{Sp(A-Mod)}(\mathbb{L}_{F,x}, M).$$

□

One checks immediately that the obstruction of Prop. 1.4.2.5 is functorial in F and X in the following sense. If $f : F \longrightarrow F'$ be a morphism of stacks, then clearly

$$df(\alpha(x)) = \alpha(f(x)) \in \pi_0(\mathbb{D}er_{F'}(X, M)).$$

In the same way, if $A \longrightarrow B$ is a morphism in $\mathcal{A}$, corresponding to a morphism of representable stacks $Y \longrightarrow X$, and $y : Y \longrightarrow F$ be the composition, then we have

$$y^*(\alpha(x)) = \alpha_y \in \pi_0(\mathbb{D}er_F(Y, M \otimes_A^{\mathbb{L}} B)).$$

Proposition 1.4.2.5 also has a relative form, whose proof is essentially the same. We will also express it in a more precise way.

PROPOSITION 1.4.2.6. *Let $f : F \longrightarrow G$ be a morphism of stacks which has an obstruction theory. Let $A \in \mathcal{A}$, $M \in A - Mod_1$, $d \in \pi_0(\mathbb{D}er(A, M))$ a derivation and $A \oplus_d \Omega M$ the corresponding square zero extension. Let x be a point in $\mathbb{R}F(A) \times^h_{\mathbb{R}G(A \oplus_d \Omega M)} \mathbb{R}G(A)$ with projection $y \in \mathbb{R}F(A)$, and let $L(x)$ be the homotopy fiber, taken at x, of the morphism*

$$\mathbb{R}F(A \oplus_d \Omega M) \longrightarrow \mathbb{R}F(A) \times^h_{\mathbb{R}G(A \oplus_d \Omega M)} \mathbb{R}G(A).$$

Then there exists a natural point $\alpha(x)$ in $Map_{A-Mod}(\mathbb{L}_{F/G,x}, M)$, and a natural isomorphism in $\mathrm{Ho}(SSet)$

$$L(z) \simeq \Omega_{\alpha(x),0} Map_{A-Mod}(\mathbb{L}_{F/G,y}, M),$$

where $\Omega_{\alpha(x),0} Map_{A-Mod}(\mathbb{L}_{F/G,y}, M)$ is the simplicial set of paths from $\alpha(x)$ to 0.

PROOF. Essentially the same as for Prop. 1.4.2.5. The point x corresponds to a commutative diagram in $\mathrm{Ho}(Aff_{\mathcal{C}}^{\sim,\tau}/G)$

$$\begin{array}{ccc} X & \longrightarrow & F\,, \\ \downarrow & & \downarrow \\ X_d[\Omega M] & \longrightarrow & G \end{array}$$

where $X := \mathbb{R}\underline{Spec}\, A$ and $X_d[\Omega M] := \mathbb{R}\underline{Spec}\,(A \oplus_d \Omega M)$. Composing with the natural commutative diagram

$$\begin{array}{ccc} X[M] & \xrightarrow{d} & X \\ {\scriptstyle s}\downarrow & & \downarrow \\ X & \longrightarrow & X_d[\Omega M] \end{array}$$

we get a well defined commutative diagram in $\mathrm{Ho}(Aff_{\mathcal{C}}^{\sim,\tau}/G)$

$$\begin{array}{ccccc} X[M] & \xrightarrow{d} & X & \longrightarrow & F \\ {\scriptstyle s}\downarrow & & & & \downarrow \\ X & & \longrightarrow & & G, \end{array}$$

giving rise to a well defined point

$$\alpha(x) \in \mathbb{R}\underline{Hom}_{X/Aff_{\mathcal{C}}^{\sim,\tau}/G}(X[M], F) = \mathbb{D}er_{F/G}(X, M).$$

Using that the morphism f is inf-cartesian, we easily see that the simplicial set $\Omega_{\alpha(x),0}\mathbb{D}er_{F/G}(X, M)$ is naturally equivalent to the space of lifts

$$L(x) = \mathbb{R}\underline{Hom}_{X/Aff_{\mathcal{C}}^{\sim,\tau}/G}(X_d[\Omega M], F) \simeq \Omega_{\alpha(x),0}\mathbb{D}er_{F/G}(X, M).$$

□

PROPOSITION 1.4.2.7. *Let F be a stack whose diagonal $F \longrightarrow F \times^h F$ is n-representable for some n. Then F has an obstruction theory if and only if it is inf-cartesian.*

PROOF. It is enough to show that a stack F that is inf-cartesian satisfies the condition of proposition 1.4.1.11. But this follows easily from the following homotopy cartesian square

$$\begin{array}{ccc} A \oplus M & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle s} \\ A & \xrightarrow[s]{} & A \oplus S(M) \end{array}$$

for any commutative monoid $A \in \mathcal{A}$ and any A-module $M \in A - Mod_1$. □

1.4.3. Artin conditions

In this section we will give conditions on the topology τ and on the class of morphisms **P** ensuring that any stack which is geometric for such a τ and **P** will have an obstruction theory. We call these conditions *Artin's conditions*, though we warn the reader that these are not the rather famous conditions for a functor to be representable by an algebraic space. We refer instead to the fact that an algebraic stack in the sense of Artin (i.e. with a smooth atlas) has a good infinitesimal and obstruction theory. We think that this has been first noticed by M. Artin, since this is precisely one part of the easy direction of his representability criterion ([**Ar**, I, 1.6] or [**La-Mo**, Thm. 10.10]).

DEFINITION 1.4.3.1. *We will say that* τ and **P** satisfy Artin's conditions relative to $(\mathcal{C}, \mathcal{C}_0, \mathcal{A})$ *(or simply* satisfy Artin's conditions *if the HA context is clear) if there exists a class* **E** *of morphisms in* $Aff_{\mathcal{C}}$ *such that the following conditions are satisfied.*

(1) *Any morphism in* **P** *is formally i-smooth in the sense of Def. 1.2.8.1.*

(2) *Morphisms in* $\boldsymbol{E}$ *are formally étale, are stable by equivalences, homotopy pullbacks and composition.*
(3) *For any morphism* $A \longrightarrow B$ *in* $\boldsymbol{E}$ *with* $A \in \mathcal{A}$, *we have* $B \in \mathcal{A}$.
(4) *For any epimorphism of representable stacks* $Y \longrightarrow X$, *which is a* $\boldsymbol{P}$-*morphism, there exists an epimorphism of representable stacks* $X' \longrightarrow X$, *which is in* $\boldsymbol{E}$, *and a commutative diagram in* $\mathrm{St}(\mathcal{C}, \tau)$

$$\begin{array}{ccc} & \nearrow & Y \\ & & \downarrow \\ X' & \longrightarrow & X. \end{array}$$

(5) *Let* $A \in \mathcal{A}$, $M \in A - Mod_1$, $d \in \pi_0(\mathbb{D}er(A, M))$ *be a derivation, and*

$$X := \mathbb{R}\underline{Spec}\,A \longrightarrow X_d[\Omega M] := \mathbb{R}\underline{Spec}_{,}(A \oplus_d \Omega M)$$

be the natural morphism in $\mathrm{St}(\mathcal{C}, \tau)$. *Then, a formally étale morphism of representable stacks* $p : U \longrightarrow X_d[\Omega M]$ *is in* $\boldsymbol{E}$ *if and only if* $U \times^h_{X_d[\Omega M]} X \longrightarrow X$ *is so. Furthermore, if* p *is in* $\boldsymbol{E}$, *then* p *is an epimorphism of stacks if and only if* $U \times^h_{X_d[\Omega M]} X \longrightarrow X$ *is so.*

The above definition might seem a bit technical and somehow hard to follow. In order to fix his intuition, we suggest the reader to think in terms of standard algebraic geometry with τ being the etale topology, $\mathbf{P}$ the class of smooth morphisms, and $\mathbf{E}$ the class of étale morphisms. This is only meant to convey some classical geometric intuition because this classical situation in algebraic geometry does not really fit into the above definition; in fact in this case the base category is the category of k-modules and thus the suspension functor is not fully faithful.

In order to simplify notations we will say that a morphism $A \longrightarrow B$ in $Comm(\mathcal{C})$ is an ***E**-covering* if it is in $\mathbf{E}$ and if the corresponding morphism of stacks

$$\mathbb{R}\underline{Spec}\,B \longrightarrow \mathbb{R}\underline{Spec}\,A$$

is an epimorphism of stacks.

THEOREM 1.4.3.2. *Assume* τ *and* $\boldsymbol{P}$ *satisfy Artin's conditions.*
(1) *Any* n-*representable morphism of stacks has an obstruction theory.*
(2) *Let* $f : F \longrightarrow G$ *be an* n-*representable morphism of stacks. If* f *is in* $\boldsymbol{P}$ *then for any* $A \in \mathcal{A}$, *and any morphism* $x : X := \mathbb{R}\underline{Spec}\,A \longrightarrow F$ *there exists an* E-*covering*

$$x' : X' := \mathbb{R}\underline{Spec}\,A' \longrightarrow X$$

such that for any $M \in A' - Mod_1$ *the natural morphism*

$$[\mathbb{L}_{X'/G,x'}, M] \longrightarrow [\mathbb{L}_{F/G,x}, M]_{A-Mod}$$

is zero.

PROOF. Before starting the proof we will need the following general fact, that will be used all along the proof of the theorem.

LEMMA 1.4.3.3. *Let* D *be a pointed model category for which the suspension functor*

$$S : \mathrm{Ho}(D) \longrightarrow \mathrm{Ho}(D)$$

is fully faithful. Then, a homotopy co-cartesian square in D *is also homotopy cartesian.*

PROOF. When D is a stable model category this is well known since homotopy fiber sequences are also homotopy cofiber sequences (see [**Ho1**]). The general case is proved in the same way. When D is furthermore $\mathbb{U}$-cellular (which will be our case), one can even deduce the result from the stable case by using the left Quillen functor $D \longrightarrow Sp(D)$, from D to the model category of spectra in D as defined in [**Ho2**], and using the fact that it is homotopically fully faithful. □

The previous lemma can be applied to $\mathcal{C}$, but also to the model categories $B-Mod$ of modules over some commutative monoid B. In particular, homotopy cartesian square of B-modules which are also homotopy co-cartesian will remain homotopy cartesian after a derived tensor product by any B-module.

Let us now start the proof of theorem 1.4.3.2.

Some topological invariance statements

We start with several results concerning topological invariance of formally étale morphisms and morphisms in $\mathbf{E}$.

LEMMA 1.4.3.4. *Let A be a commutative monoid, M an A-module and $A\oplus M \longrightarrow A$ the natural augmentation. Then, the homotopy push out functor*

$$A \otimes^{\mathbb{L}}_{A\oplus M} - : \mathrm{Ho}((A \oplus M) - Comm(\mathcal{C})) \longrightarrow \mathrm{Ho}(A - Comm(\mathcal{C}))$$

induces an equivalence between the full sub-categories consisting of formally étale commutative $A \oplus M$-algebras and formally étale commutative A-algebras.

PROOF. We see that the functor is essentially surjective as the morphism $A \oplus M \longrightarrow A$ possesses a section $A \longrightarrow A\oplus M$. Next, we show that the functor $A\otimes^{\mathbb{L}}_{A\oplus M} -$ is conservative. For this, let us consider the commutative square

$$\begin{array}{ccc} A \oplus M & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \oplus S(M), \end{array}$$

which, as a commutative square in $\mathcal{C}$, is homotopy cocartesian and homotopy cartesian. Therefore, this square is homotopy cocartesian in $(A \oplus M) - Mod$, and thus lemma 1.4.3.3 implies that for any commutative $A \oplus M$-algebra B, the natural morphism

$$B \longrightarrow (A \otimes^{\mathbb{L}}_{A\oplus M} B) \times^h_{(A\oplus S(M))\otimes^{\mathbb{L}}_{A\oplus M}B} (A \otimes^{\mathbb{L}}_{A\oplus M} B) \simeq \left(A \times^h_{A\oplus S(M)} A\right) \otimes^{\mathbb{L}}_{A\oplus M} B$$

is then an isomorphism in $\mathrm{Ho}(Comm(\mathcal{C}))$. This clearly implies that the functor $A \otimes^{\mathbb{L}}_{A\oplus M} -$ is conservative.

Now, let $A \oplus M \longrightarrow B$ be a formally étale morphism of commutative monoids. The diagonal morphism $M \longrightarrow M \times^h M \simeq M \oplus M$ in $\mathrm{Ho}(A - Mod)$, induces a well defined morphism in $\mathrm{Ho}(Comm(\mathcal{C})/A \oplus M)$

$$\begin{array}{ccc} & & (A \oplus M) \oplus M \\ & \nearrow & \downarrow \\ A \oplus M & \xrightarrow{Id} & A \oplus M, \end{array}$$

and therefore a natural element in $\pi_0\mathbb{D}er(A\oplus M, M)$. Composing with $M \longrightarrow M\otimes^{\mathbb{L}}_A B$, we get a well defined element in $\pi_0\mathbb{D}er(A\oplus M, M\otimes^{\mathbb{L}}_A B)$. Using that $A\oplus M \longrightarrow B$ is formally étale, this element extends uniquely to an element in $\pi_0\mathbb{D}er(B, M\otimes^{\mathbb{L}}_A B)$. This last derivation gives rise to a well defined morphism in $\mathrm{Ho}((A\oplus M)-Comm(\mathcal{C}))$

$$u : B \longrightarrow (A\otimes^{\mathbb{L}}_{A\oplus M} B)\oplus M\otimes^{\mathbb{L}}_A B.$$

Furthermore, by construction, this morphism is sent to the identity of $A\otimes^{\mathbb{L}}_{A\oplus M} B$ by the functor $A\otimes^{\mathbb{L}}_{A\oplus M} -$, and as we have seen this implies that u is an isomorphism in $\mathrm{Ho}((A\oplus M)-Comm(\mathcal{C}))$.

We now finish the proof of the lemma by showing that the functor $A\otimes^{\mathbb{L}}_{A\oplus M} -$ is fully faithful. For this, let $A\oplus M \longrightarrow B$ and $A\oplus M \longrightarrow B'$ be two formally étale morphisms of commutative monoids. As we have seen, B' can be written as

$$B' \simeq (A\oplus M)\otimes^{\mathbb{L}}_A A' \simeq A'\oplus M'$$

where $A \longrightarrow A'$ is formally étale, and $M' := M\otimes^{\mathbb{L}}_A A'$. We consider the natural morphism

$$Map_{(A\oplus M)-Comm(\mathcal{C})}(B, B') \simeq Map_{(A\oplus M)-Comm(\mathcal{C})}(B, A'\oplus M') \longrightarrow$$

$$Map_{A-Comm(\mathcal{C})}(A\otimes^{\mathbb{L}}_{A\oplus M} B, A\otimes^{\mathbb{L}}_{A\oplus M} B') \simeq Map_{A-Comm(\mathcal{C})}(A\otimes^{\mathbb{L}}_{A\oplus M} B, A').$$

The homotopy fiber of this morphism at a point $B \to A'$ is identified with $\mathbb{D}er_{A\oplus M}(B, M')$, which is contractible as B is formally étale over $A\oplus M$. This shows that the morphism

$$Map_{(A\oplus M)-Comm(\mathcal{C})}(B, B') \longrightarrow Map_{A-Comm(\mathcal{C})}(A\otimes^{\mathbb{L}}_{A\oplus M} B, A\otimes^{\mathbb{L}}_{A\oplus M} B')$$

has contractible homotopy fibers and therefore is an isomorphism in $\mathrm{Ho}(SSet)$, and finishes the proof of the lemma. □

LEMMA 1.4.3.5. *Let A be a commutative monoid and M an A-module. Then the homotopy push out functor*

$$A\otimes^{\mathbb{L}}_{A\oplus M} - : \mathrm{Ho}((A\oplus M)-Comm(\mathcal{C})) \longrightarrow \mathrm{Ho}(A-Comm(\mathcal{C}))$$

induces an equivalence between the full sub-categories consisting of $\mathbf{E}$-coverings of $A\oplus M$ and of $\mathbf{E}$-coverings of A.

PROOF. Using lemma 1.4.3.4 it is enough to show that a formally étale morphism $f : A\oplus M \longrightarrow B$ is in $\mathbf{E}$ (respectively an $\mathbf{E}$-covering) if and only if $A \longrightarrow A' := A\otimes^{\mathbb{L}}_{A\oplus M} B$ is in $\mathbf{E}$ (respectively an $\mathbf{E}$-covering). But, as f is formally étale, lemma 1.4.3.4 implies that it can written as

$$A\oplus M \longrightarrow A'\oplus M' \simeq (A\oplus M)\otimes^{\mathbb{L}}_A A',$$

with $M' := M\otimes^{\mathbb{L}}_A A'$. Therefore the lemma simply follows from the stability of epimorphisms and morphisms in $\mathbf{E}$ by homotopy pullbacks. □

LEMMA 1.4.3.6. *Let $A \in Comm(\mathcal{C})$, $M \in A-Mod_1$, and $d \in \pi_0(\mathbb{D}er(A, M))$ be a derivation. Let $B := A\oplus_d \Omega M \longrightarrow A$ be the natural augmentation, and let us consider the base change functor*

$$A\otimes^{\mathbb{L}}_B - : \mathrm{Ho}(B-Comm(\mathcal{C})) \longrightarrow \mathrm{Ho}(A-Comm(\mathcal{C})).$$

Then, $A\otimes^{\mathbb{L}}_B -$ restricted to the full subcategory consisting of formally étale commutative B-algebras is fully faithful.

PROOF. Let

$$\begin{array}{ccc} B & \longrightarrow & A \\ \downarrow & & \downarrow d \\ A & \xrightarrow[s]{} & A \oplus M \end{array}$$

be the standard homotopy cartesian square of commutative monoids, which is also homotopy co-cartesian in $\mathcal{C}$ as $M \in A-Mod_1$. We represent it as a fibered square in $Comm(\mathcal{C})$

$$\begin{array}{ccc} B & \longrightarrow & A \\ \downarrow & & \downarrow d \\ A' & \xrightarrow[s']{} & A \oplus M, \end{array}$$

where $s' : A' \longrightarrow A \oplus M$ is fibrant replacement of the trivial section $s : A \longrightarrow A \oplus M$.

We define a model category D whose objects are 5-plets (B_1, B_2, B_3, a, b), where $B_1 \in A - Comm(\mathcal{C})$, $B_2 \in A' - Comm(\mathcal{C})$, $B_3 \in (A \oplus M) - Comm(\mathcal{C})$, and a and b are morphisms of commutative $(A \oplus M)$-algebras

$$(A \oplus M) \otimes_A B_1 \xrightarrow{a} B_3 \xleftarrow{b} (A \oplus M) \otimes_{A'} B_2$$

(where the co-base change on the left is taken with respect of the morphism $s' : A' \to A \oplus M$ and the one on the right with respect to $d : A \to A \oplus M$). For an object (B_1, B_2, B_3, a, b) in D, the morphisms a and b can also be understood as $B_1 \longrightarrow B_3$ in $A - Comm(\mathcal{C})$ and $B_2 \longrightarrow B_3$ in $A' - Comm(\mathcal{C})$. The morphisms

$$(B_1, B_2, B_3, a, b) \longrightarrow (B'_1, B'_2, B'_3, a', b')$$

in D are defined in the obvious way, as families of morphisms $\{B_i \to B'_i\}$ commuting with the a's and b's. A morphism in D is defined to be an equivalence or a cofibration if each morphism $B_i \to B'_i$ is so. A morphism in D is defined to be a fibration if each morphism $B_i \to B'_i$ is a fibration in $\mathcal{C}$, and if the natural morphisms

$$B_1 \longrightarrow B'_1 \times_{B'_3} B_3 \qquad B_2 \longrightarrow B'_2 \times_{B'_3} B_3$$

are fibrations in $\mathcal{C}$. This defines a model category structure on D which is a Reedy type model structure. An important fact concerning D is the description of its mapping spaces as the following homotopy cartesian square

$$\begin{array}{ccc} Map_D((\underline{B}, a, b), (\underline{B'}, a', b')) & \longrightarrow & Map_{A-Comm(\mathcal{C})}(B_1, B'_1) \times Map_{A'-Comm(\mathcal{C})}(B_2, B'_2) \\ \downarrow & & \downarrow \\ Map_{(A\oplus M)-Comm(\mathcal{C})}(B_3, B'_3) & \longrightarrow & Map_{A-Comm(\mathcal{C})}(B_1, B'_3) \times Map_{A'-Comm(\mathcal{C})}(B_2, B'_3) \end{array}$$

where we have denoted $\underline{B} := (B_1, B_2, B_3)$ and $\underline{B'} := (B'_1, B'_2, B'_3)$. There exists a natural functor

$$F : B - Comm(\mathcal{C}) \longrightarrow D$$

sending a commutative B-algebra B' to the object

$$F(B') := (A \otimes_B B', A' \otimes_B B', (A \oplus M) \otimes_B B', a, b)$$

where

$$a : (A \oplus M) \otimes_A (A \otimes_B B') \simeq (A \oplus M) \otimes_B B' \qquad b : (A \oplus M) \otimes_{A'} (A' \otimes_B B') \simeq (A \oplus M) \otimes_B B'$$

are the two natural isomorphisms in $(A\oplus M)-Comm(\mathcal{C})$. The functor F has a right adjoint G, sending an object (B_1, B_2, B_3, a, b) to the pullback in $B-Comm(\mathcal{C})$

$$\begin{array}{ccc} G(B_1,B_2,B_3,a,b) & \longrightarrow & B_1 \\ \downarrow & & \downarrow a \\ B_2 & \xrightarrow[b]{} & B_3. \end{array}$$

Clearly the adjunction (F, G) is a Quillen adjunction. Furthermore, lemma 1.4.3.3 implies that for any commutative B-algebra $B \to B'$ the adjunction morphism

$$B' \longrightarrow \mathbb{R}G\mathbb{L}F(B') = A\otimes^{\mathbb{L}}_B B' \times^h_{(A\oplus M)\otimes^{\mathbb{L}}_B B'} A'\otimes^{\mathbb{L}}_B B' \simeq (A\times^h_{A\oplus M} A')\otimes^{\mathbb{L}}_B B'$$

is an isomorphism in $\mathrm{Ho}(B-Comm(\mathcal{C}))$. This implies in particular that

$$\mathbb{L}F : \mathrm{Ho}(B-Comm(\mathcal{C})) \longrightarrow \mathrm{Ho}(D)$$

is fully faithful.

We now consider the functor

$$D \longrightarrow A-Comm(\mathcal{C})$$

sending (B_1, B_2, B_3, a, b) to B_1. Using our lemma 1.4.3.4, and the description of the mapping spaces in D, it is not hard to see that the induced functor

$$\mathrm{Ho}(D) \longrightarrow \mathrm{Ho}(A-Comm(\mathcal{C}))$$

becomes fully faithful when restricted to the full subcategory of $\mathrm{Ho}(D)$ consisting of objects (B_1, B_2, B_3, a, b) such that $A \to B_1$ and $A' \to B_2$ are formally étale and the induced morphism

$$a : (A\oplus M)\otimes^{\mathbb{L}}_A B_1 \longrightarrow B_3 \qquad b : (A\oplus M)\otimes^{\mathbb{L}}_{A'} B_2 \longrightarrow B_3$$

are isomorphisms in $\mathrm{Ho}((A\oplus M)-Comm(\mathcal{C}))$. Putting all of this together we deduce that the functor

$$A\otimes^{\mathbb{L}}_B - : \mathrm{Ho}(B-Comm(\mathcal{C})) \longrightarrow \mathrm{Ho}(A-Comm(\mathcal{C}))$$

is fully faithful when restricted to the full subcategory of formally étale morphisms. □

LEMMA 1.4.3.7. *Let A be a commutative monoid, M an A-module and $d : \mathbb{D}er(A, M)$ be a derivation. Let $B := A\oplus_d \Omega M \longrightarrow A$ be the natural augmentation. Then, there exists a natural homotopy cofiber sequence of A-modules*

$$\mathbb{L}_B\otimes^{\mathbb{L}}_B A \longrightarrow \mathbb{L}_A \xrightarrow{d} \mathbb{L}QZ(M),$$

where

$$Q : A-Comm_{nu}(\mathcal{C}) \longrightarrow A-Mod \qquad A-Comm_{nu}(\mathcal{C}) \longleftarrow A-Mod : Z$$

is the Quillen adjunction described during the proof of 1.2.1.2.

PROOF. When d is the trivial derivation, we know the lemma is correct as by Prop. 1.2.1.6 (4) we have

$$\mathbb{L}_{A\oplus M}\otimes^{\mathbb{L}}_{A\oplus M} A \simeq \mathbb{L}_A \coprod \mathbb{L}QZ(M).$$

For the general situation, we use our left Quillen functor

$$F : B-Comm(\mathcal{C}) \longrightarrow D$$

defined during the proof of lemma 1.4.3.6. The commutative monoid A is naturally an $(A \oplus M)$-algebra, and can be considered as a natural object (A, A, A) in D with the obvious transition morphisms

$$A \otimes_A (A \oplus M) \simeq A \oplus M \to A,$$

$$A \otimes_{A'} (A \oplus M) \to A \otimes_A (A \oplus M) \simeq A \oplus M \to A.$$

For any A-module N, one can consider $A \oplus N$ as a commutative A-algebra, and therefore as as an object $(A \oplus N, A \oplus N, A \oplus N)$ in D (with the obvious transition morphisms). We will simply denote by A the object $(A, A, A) \in D$, and by $A \oplus N$ the object $(A \oplus N, A \oplus N, A \oplus N) \in D$. The left Quillen property of F implies that

$$\mathbb{D}er(B, N) \simeq Map_{D/A}(F(B), A \oplus N).$$

This shows that the morphism

$$\mathbb{D}er(A, N) \longrightarrow \mathbb{D}er(B, N)$$

is equivalent to the morphism

$$\mathbb{D}er(A, N) \longrightarrow \mathbb{D}er(A, N) \times^h_{\mathbb{D}er(A \oplus M, N)} \mathbb{D}er(A, N).$$

This implies the the morphism

$$\mathbb{L}_B \otimes^{\mathbb{L}}_B A \longrightarrow \mathbb{L}_A$$

is naturally equivalent to the morphism of A-modules

$$\mathbb{L}_A \coprod^{\mathbb{L}}_{\mathbb{L}_{A \oplus M} \otimes^{\mathbb{L}}_{A \oplus M} A} \mathbb{L}_A \longrightarrow \mathbb{L}_A.$$

Using the already known result for the trivial extension $A \oplus M$ we get the required natural cofiber sequence

$$\mathbb{L}_B \otimes^{\mathbb{L}}_B A \longrightarrow \mathbb{L}_A \longrightarrow \mathbb{L}QZ(M).$$

□

LEMMA 1.4.3.8. *Let A be a commutative monoid, $M \in A - Mod_1$ and $d \in \pi_0(\mathbb{D}er(A, M))$ be a derivation. Let $B := A \oplus_d \Omega M \longrightarrow A$ be the natural augmentation, and let us consider the base change functor*

$$A \otimes^{\mathbb{L}}_B - : \mathrm{Ho}(B - Comm(\mathcal{C})) \longrightarrow \mathrm{Ho}(A - Comm(\mathcal{C})).$$

Then, $A \otimes^{\mathbb{L}}_B -$ induces an equivalence between the full sub-categories consisting of formally étale commutative B-algebras and of formally étale commutative A-algebras.

PROOF. By lemma 1.4.3.6 we already know that the functor is fully faithful, and it only remains to show that any formally étale A-algebra $A \to A'$ is of the form $A \otimes^{\mathbb{L}}_B B'$ for some formally étale morphism $B \to B'$.

Let $A \longrightarrow A'$ be a formally étale morphism. The derivation $d \in \pi_0\mathbb{D}er(A, M)$ lifts uniquely to a derivation $d' \in \pi_0\mathbb{D}er(A', M')$ where $M' := M \otimes^{\mathbb{L}}_A A'$. We form the corresponding square zero extension

$$\begin{array}{ccc} B' := A' \oplus_d \Omega M' & \longrightarrow & A' \\ \downarrow & & \downarrow{\scriptstyle d'} \\ A' & \longrightarrow & A' \oplus M' \end{array}$$

which comes equipped with a natural morphism $B \longrightarrow B'$ fitting in a homotopy commutative square

$$\begin{array}{ccc} B' & \longrightarrow & A' \\ \uparrow & & \uparrow \\ B & \longrightarrow & A. \end{array}$$

We claim that $B \longrightarrow B'$ is formally étale and that the natural morphism $A \otimes^{\mathbb{L}}_B B' \longrightarrow A'$ is an isomorphism in $\mathrm{Ho}(A - Comm(\mathcal{C}))$.

There are natural cofiber sequences in $\mathcal{C}$

$$B' \longrightarrow A' \longrightarrow M'$$

$$B \longrightarrow A \longrightarrow M,$$

as well as a natural morphism of cofiber sequences in $\mathcal{C}$

$$\begin{array}{ccccc} A \otimes^{\mathbb{L}}_B B' & \longrightarrow & A \otimes^{\mathbb{L}}_B A' & \longrightarrow & A \otimes^{\mathbb{L}}_B M' \\ \downarrow & & \downarrow & & \downarrow \\ B \otimes^{\mathbb{L}}_B A' \simeq A' & \longrightarrow & A \otimes^{\mathbb{L}}_B A' & \longrightarrow & M \otimes^{\mathbb{L}}_B A', \end{array}$$

which by our lemma 1.4.3.3 is also a morphism of fiber sequences. The two right vertical morphisms are equivalences and thus so is the arrow on the left. This shows that $A \otimes^{\mathbb{L}}_B B' \simeq A'$.

Finally, lemma 1.4.3.7 implies the existence of a natural morphism of cofiber sequences of A'-modules

$$\begin{array}{ccccc} \mathbb{L}_B \otimes^{\mathbb{L}}_B A & \longrightarrow & \mathbb{L}_A & \longrightarrow & \mathbb{L}QZ(M) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{L}_{B'} \otimes^{\mathbb{L}}_{B'} A' & \longrightarrow & \mathbb{L}_{A'} & \longrightarrow & \mathbb{L}QZ(M') \simeq \mathbb{L}QZ(M) \otimes^{\mathbb{L}}_A A'. \end{array}$$

As $A \to A'$ is étale, this implies that the natural morphism

$$(\mathbb{L}_B \otimes^{\mathbb{L}}_B B') \otimes^{\mathbb{L}}_{B'} A' \simeq (\mathbb{L}_B \otimes^{\mathbb{L}}_B A) \otimes^{\mathbb{L}}_A A' \longrightarrow \mathbb{L}_{B'} \otimes^{\mathbb{L}}_{B'} A'$$

is an isomorphism in $\mathrm{Ho}(A' - Mod)$. This would show that $B \longrightarrow B'$ is formally étale if one knew that the base change functor

$$A' \otimes^{\mathbb{L}}_{B'} - : \mathrm{Ho}(B' - Mod) \longrightarrow \mathrm{Ho}(A' - Mod)$$

were conservative. However, this is the case as lemma 1.4.3.3 implies that for any B'-module N we have a natural isomorphism in $\mathrm{Ho}(B' - Mod)$

$$N \simeq (A' \otimes^{\mathbb{L}}_{B'} N) \times^h_{(A' \oplus M') \otimes^{\mathbb{L}}_{B'} N} (A' \otimes^{\mathbb{L}}_{B'} N).$$

□

LEMMA 1.4.3.9. *Let $A \in \mathcal{A}$, $M \in A - Mod_1$ and $d \in \pi_0(\mathbb{D}er(A, M))$ be a derivation. Let*

$$A \oplus_d \Omega M \longrightarrow A$$

be the natural morphism and let us consider the homotopy push-out functor

$$(A \oplus_d \Omega M) \otimes^{\mathbb{L}}_A - : \mathrm{Ho}((A \oplus_d \Omega M) - Comm(\mathcal{C})) \longrightarrow \mathrm{Ho}(A - Comm(\mathcal{C})).$$

*Then, $(A \oplus_d \Omega M) \otimes^{\mathbb{L}}_A -$ induces an equivalence between the full sub-categories consisting of **E**-covers of $A \oplus_d \Omega M$ and of **E**-covers of A.*

PROOF. This is immediate from Lem. 1.4.3.8 and condition (4) of Artin's conditions 1.4.3.1. □

Proof of Theorem 1.4.3.2

We are now ready to prove that F has an obstruction theory. To simplify notations we assume that F is fibrant in $Aff_{\mathcal{C}}^{\sim,\tau}$, so $F(A) \simeq \mathbb{R}F(A)$ for any $A \in Comm(\mathcal{C})$. We then argue by induction on the integer n. For $n = -1$ Theorem 1.4.3.2 follows from Prop. 1.4.2.4, hypothesis Def. 1.4.3.1 (5) and Prop. 1.2.8.3. We now assume that $n \geq 0$ and that both statement of theorem 1.4.3.2 are true for all $m < n$.

We start by proving Thm. 1.4.3.2 (1) for rank n. For this, we use Lem. 1.4.2.3 and Prop. 1.4.2.7, which show that we only need to prove that any n-geometric stack is inf-cartesian.

LEMMA 1.4.3.10. *Let F be an n-geometric stack. Then F is inf-cartesian.*

PROOF. Let $A \in \mathcal{A}$, $M \in A - Mod_1$, and $d \in \pi_0(\mathbb{D}er(A, M))$. Let x be a point in $\pi_0(F(A) \times^h_{F(A \oplus M)} F(A))$, whith projection $x_1 \in \pi_0(F(A))$ on the first factor. We need to show that the homotopy fiber, taken at x, of the morphism

$$F(A \oplus_d \Omega M) \longrightarrow F(A) \times^h_{F(A\oplus M)} F(A)$$

is contractible. For this, we replace the homotopy cartesian diagram

$$\begin{array}{ccc} A \oplus_d \Omega M & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \oplus M \end{array}$$

be an equivalent commutative diagram in $Comm(\mathcal{C})$

$$\begin{array}{ccc} B & \longrightarrow & B_1 \\ \downarrow & & \downarrow \\ B_2 & \longrightarrow & B_3, \end{array}$$

in such a way that each morphism is a cofibration in $Comm(\mathcal{C})$. The point x can be represented as a point in the standard homotopy pullback $F(B_1) \times^h_{F(B_3)} F(B_2)$. We then define a functor

$$S : B - Comm(\mathcal{C}) = (Aff_{\mathcal{C}}/Spec\, B)^{op} \longrightarrow SSet_{\mathbb{V}},$$

in the following way. For any morphism of commutative monoids $B \longrightarrow B'$, the simplicial set $S(B')$ is defined to be the standard homotopy fiber, taken at x, of the natural morphism

$$F(B') \longrightarrow F(B_1 \otimes_B B') \times^h_{F(B_3 \otimes_B B')} F(B_2 \otimes_B B').$$

Because of our choices on the B_i's, it is clear that the simplicial presheaf S is a stack on the comma model site $Aff_{\mathcal{C}}/Spec\, B$. Therefore, in order to show that $S(B)$ is contractible it is enough to show that $S(B')$ is contractible for some morphism $B \longrightarrow B'$ such that $\mathbb{R}\underline{Spec}\, B' \longrightarrow \mathbb{R}\underline{Spec}\, B$ is an epimorphism of stacks. In particular, we are allowed to homotopy base change by some $\mathbf{E}$-covering of B. Also, using our lemma 1.4.3.8 (or rather its proof), we see that for any **E**-covering $B \to B'$, the homotopy cartesian square

$$\begin{array}{ccc} B' & \longrightarrow & B_1 \otimes_B B' \\ \downarrow & & \downarrow \\ B_2 \otimes_B B' & \longrightarrow & B_3 \otimes_B B', \end{array}$$

is in fact equivalent to some

$$\begin{array}{ccc} A' \oplus_{d'} \Omega M' & \longrightarrow & A' \\ \downarrow & & \downarrow{\scriptstyle d'} \\ A' & \longrightarrow & A' \oplus M', \end{array}$$

for some **E**-covering $A \longrightarrow A'$ (and with $M' \simeq M \otimes^{\mathbb{L}}_A A'$, and where d' is the unique derivation $d' \in \pi_0\mathbb{D}er(A', M')$ extending d). This shows that we can always replace A by A', d by d' and M by M'. In particular, for an $(n-1)$-atlas $\{U_i \longrightarrow F\}$ F, we can assume that the point $x_1 \in \pi_0(F(A))$, image of the point x, lifts to a point in $y_1 \in \pi_0(U_j(A))$ for some j. We will denote $U := U_j$.

SUB-LEMMA 1.4.3.11. *The point $x \in \pi_0(F(A) \times^h_{F(A\oplus M)} F(A))$ lifts to point $y \in \pi_0(U(A) \times^h_{U(A\oplus M)} U(A))$*

PROOF. We consider the commutative diagram of simplicial sets

$$\begin{array}{ccc} U(A) \times^h_{U(A\oplus M)} U(A) & \stackrel{f}{\longrightarrow} & F(A) \times^h_{F(A\oplus M)} F(A) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ U(A) & \longrightarrow & F(A) \end{array}$$

induced by the natural projection $A \oplus M \to A$. Let $F(p)$ and $F(q)$ be the homotopy fibers of the morphisms p and q taken at y_1 and x_1. We have a natural morphism $g : F(p) \longrightarrow F(q)$. Moreover, the homotopy fiber of the morphism f, taken at the point x, receives a natural morphism from the homotopy fiber of the morphism g. It is therefore enough to show that the homotopy fiber of g is not empty. But, by definition of derivations, the morphism g is equivalent to the morphism

$$\Omega_{d,0}\mathbb{D}er_U(X, M) \longrightarrow \Omega_{d,0}\mathbb{D}er_F(X, M),$$

where the derivation d is given by the image of the point y_1 by $d : A \to A \oplus M$. Therefore, the homotopy fiber of the morphism g is equivalent to

$$\Omega_{d,0}\mathbb{D}er_{U/F}(X, M) \simeq \Omega_{d,0}Map(\mathbb{L}_{U/F,y_1}, M).$$

But, using Thm. 1.4.3.2 (2) at rank $(n-1)$ and for the morphism $U \longrightarrow F$ we obtain that $\Omega_{d,0}\mathbb{D}er_{U/F}(X, M)$ is non empty. This finishes the proof of the sub-lemma. $\square$

We now consider the commutative diagram

$$\begin{array}{ccc} U(A \oplus_d \Omega M) & \stackrel{a}{\longrightarrow} & U(A) \times^h_{U(A\oplus M)} U(A) \\ {\scriptstyle b'}\downarrow & & \downarrow{\scriptstyle b} \\ F(A \oplus_d \Omega M) & \underset{a'}{\longrightarrow} & F(A) \times^h_{F(A\oplus M)} F(A). \end{array}$$

The morphism a is an equivalence because U is representable. Furthermore, by our inductive assumption the above square is homotopy cartesian. This implies that the homotopy fiber of a' at x is either contractible or empty. But, by the above sub-lemma the point x lifts, up to homotopy, to a point in $U(A)\times^h_{U(A\oplus M)}U(A)$, showing that this homotopy fiber is non empty. □

We have finished the proof of lemma 1.4.3.10 which implies that any n-geometric stack is inf-cartesian, and thus that any n-representable morphism has an obstruction theory. It only remain to show part (2) of Thm. 1.4.3.2 at rank n. For this, we use Lem. 1.4.1.16 (3) which implies that we can assume that $G=*$. Let $U\longrightarrow F$ be an n-atlas, $A\in\mathcal{A}$ and $x: X:=\mathbb{R}\underline{Spec}\,A\longrightarrow F$ be a point. By passing to an epimorphism of representable stacks $X'\longrightarrow X$ which is in $\mathbf{E}$, we can suppose that the point x factors through a point $u: X\longrightarrow U$, where U is representable and $U\longrightarrow F$ is in $\mathbf{P}$. By composition and the hypothesis that morphisms in $\mathbf{P}$ are formally i-smooth, we see that $U\longrightarrow *$ is a formally i-smooth morphism. We then have a diagram

$$\mathbb{L}_{F,x}\longrightarrow\mathbb{L}_{U,u}\longrightarrow\mathbb{L}_{X,x},$$

which obviously implies that for any $M\in A-Mod_1$ the natural morphism

$$[\mathbb{L}_{X,x},M]\longrightarrow[\mathbb{L}_{F,x},M]$$

factors through the morphism

$$[\mathbb{L}_{X,x},M]\longrightarrow[\mathbb{L}_{U,u},M]$$

which is itself equal to zero by Prop. 1.2.8.3. □

We also extract from Lem. 1.4.3.8 and its proof the following important corollary.

Corollary 1.4.3.12. *Let $A\in Comm(\mathcal{C})$, $M\in A-Mod$ and $d\in\pi_0(\mathbb{D}er(A,M))$ be a derivation. Assume that the square*

$$\begin{array}{ccc} B=A\oplus_d\Omega M & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & A\oplus M \end{array}$$

is homotopy co-cartesian in $\mathcal{C}$, then the base change functor

$$A\otimes^{\mathbb{L}}_B - : \mathrm{Ho}(B-Comm(\mathcal{C}))\longrightarrow\mathrm{Ho}(A-Comm(\mathcal{C}))$$

induces an equivalence between the full sub-categories of formally étale commutative B-algebras and formally étale commutative A-algebras. The same statement holds with formally étale *replaced by* étale.

Proof. Only the assertion with *formally étale* replaced by *étale* requires an argument. For this, we only need to prove that if a formally étale morphism $f: B\longrightarrow B'$ is such that $A\longrightarrow A\otimes^{\mathbb{L}}_B B'=A'$ is finitely presented, then so is f. For this we use the fully faithful functor

$$\mathbb{L}F:\mathrm{Ho}(B-Comm(\mathcal{C}))\longrightarrow\mathrm{Ho}(D)$$

defined during the proof of Lem. 1.4.3.6. Using the description of mappping spaces in D in terms of a certain homotopy pullbacks, and using the fact that filtered homotopy colimits in $SSet$ commutes with homotopy pullbacks, we deduce the statement. □

The morphism a' is an equivalence because U is representable. Furthermore, by our inductive assumption the above square is homotopy cartesian. This implies that the homotopy fiber of a' at x is either contractible or empty. But, by the above sub-lemma the point x lifts, up to homotopy, to a point in $U(A) \times^h_{U(A \oplus M)} U'(A)$, showing that this homotopy fiber is non empty. □

We have finished the proof of lemma 1.4.3.10 which implies that any n-geometric stack is inf-cartesian, and thus that any n-representable morphism has an obstruction theory. It only remains to show part (2) of Thm. 1.4.3.2. For this, we use Lem. 1.4.1.16 (3) which implies that we can assume that $G = *$. Let $U \longrightarrow F$ be an n-atlas, $A \in \mathcal{A}$ and $x : X := \mathbb{R}\mathrm{Spec}\, A \longrightarrow F$ be a point. By passing to an epimorphism of representable stacks $X' \longrightarrow X$ which is in **E**, we can suppose that the point x factors through a point $y : X \longrightarrow U$, where U is representable and $U \longrightarrow F$ is in **P**. By composition and the hypothesis that morphisms in **P** are formally i-smooth, we see that $U \longrightarrow *$ is a formally i-smooth morphism. We then have a diagram

$$\mathbb{L}_{F,x} \longrightarrow \mathbb{L}_{U,y} \longrightarrow \mathbb{L}_{X}$$

which obviously implies that for any $M \in A - Mod_1$, the natural morphism

$$[\mathbb{L}_{X}, M] \longrightarrow [\mathbb{L}_{F,x}, M]$$

factors through the morphism

$$[\mathbb{L}_{U,y}, M] \longrightarrow [\mathbb{L}_{F,x}, M]$$

which is itself equal to zero by Prop. 1.2.8.3. □

We also extract from Lem. 1.4.3.8 and its proof the following important corollary

COROLLARY 1.4.3.12. *Let $A \in Comm(C)$, $M \in A-Mod$ and $d \in \pi_0(\mathrm{Der}(A, M))$ be a derivation. Assume that the square*

$$\begin{array}{ccc} B = A \oplus_d \Omega M & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle d} \\ A & \longrightarrow & A \oplus M \end{array}$$

is homotopy co-cartesian in C. Then, the base change functor

$$A \otimes^{\mathbb{L}}_B - : \mathrm{Ho}(B - Comm(C)) \longrightarrow \mathrm{Ho}(A - Comm(C))$$

induces an equivalence between the full sub-categories of formally étale commutative B-algebras and formally étale commutative A-algebras. The same statement holds with formally étale replaced by étale.

PROOF. Only the assertion with *formally étale* replaced by *étale* requires an argument. For this, we only need to prove that if a formally étale morphism $f : B \longrightarrow B'$ is such that $A \longrightarrow A \otimes^{\mathbb{L}}_B B'$ is finitely presented, then so is f. For this we use the fully faithful functor

$$\mathrm{Ho}(B - Comm(C)) \longrightarrow \mathrm{Ho}(D)$$

defined during the proof of Lem. 1.4.3.8. Using the description of mappping spaces in D in terms of a certain homotopy pullbacks, and using the fact that filtered homotopy colimits in SSet commutes with homotopy pullbacks, we deduce the statement. □

Part 2

Applications

Introduction to Part 2

In this second part we apply the theory developed in the first part to study the geometry of stacks in various HAG contexts (Def. 1.3.2.13). In particular we will specialize our base symmetric monoidal model category $\mathcal{C}$ to the following cases:

- $\mathcal{C} = \mathbb{Z} - Mod$, the category of $\mathbb{Z}$-modules to get a theory of *geometric stacks in* (classical) *Algebraic Geometry* (§2.1);
- $\mathcal{C} = sMod_k$, the category of simplicial modules over an arbitrary base commutative ring k to get a theory of *derived* or *D^--geometric stacks* (§2.2);
- $\mathcal{C} = C(k)$, the category of unbounded cochain complexes of modules over a characteristic zero base commutative ring k to get a theory of *geometric stacks in complicial algebraic geometry*, also called geometric D-stacks (§2.3);
- $\mathcal{C} = Sp^{\Sigma}$, the category of symmetric spectra ([**HSS, Shi**]) to get a theory of *geometric stacks in brave new algebraic geometry* (§2.4).

In §2.1 we are concerned with **classical algebraic geometry**, the base category $\mathcal{C} = \mathbb{Z} - Mod$ being endowed with the trivial model structure. We verify that if $k-Aff := Comm(\mathcal{C})^{\mathrm{op}} = (k-Alg)^{\mathrm{op}}$ is endowed with its étale Grothendieck topology and $\mathbf{P}$ is the class of smooth morphisms between (usual) commutative rings then

$$(\mathcal{C}, \mathcal{C}_0, \mathcal{A}, \tau, \mathbf{P}) := (k - Mod, k - Mod, k - Alg, \text{ét}, \mathbf{P})$$

is a HAG-context according to Def. 1.3.2.13. n-geometric stacks in this context will be called *Artin n-stacks*, and essentially coincide with geometric n-stacks as defined in [**S3**]; their model category will be denoted by $k - Aff^{\sim,\text{ét}}$ and its homotopy category by $\mathrm{St}(k)$.

After having established a coherent dictionary (Def. 2.1.1.4), we show in §2.1.2 how the theory of schemes, of algebraic spaces, and of Artin's algebraic stacks in groupoids ([**La-Mo**]) embeds in our theory of geometric stacks (Prop. 2.1.2.1). We also remark (Rmk. 2.1.2.2) that the general infinitesimal theory for geometric stacks developed in Part I does not apply to this context. The reason for this is that the category $\mathcal{C} = k - Mod$ (with its trivial model structure) is as unstable as it could possibly be: the suspension functor $S : \mathrm{Ho}(\mathcal{C}) = k - Mod \to k - Mod = \mathrm{Ho}(\mathcal{C})$ is trivial (i.e. sends each k-module to the zero k-module). The explanation for this is the following. Usual infinitesimal theory that applies to schemes, algebraic spaces or to (some classes of) Artin's algebraic stacks in groupoids is in fact (as made clear e.g. by the definition of cotangent complex of a scheme ([**Ill**, II.2]) which uses simplicial resolutions of objects $Comm(\mathcal{C})$), already conceptually part of *derived algebraic geometry* in the sense that its classical definition already requires to embed $Comm(\mathcal{C}) = k - Alg$ into the category of simplicial k-algebras. And in fact (as we will show in §2.2) when schemes, algebraic spaces and Artin algebraic stacks in groupoids are viewed as *derived stacks*, then their classical infinitesimal theory can be recovered (and generalized, see Cor. 2.2.4.5) and interpreted geometrically within our general formalism of Chapter 1.4 (in particular in Prop. 1.4.1.6).

Therefore we are naturally brought to §2.2 where we treat the case of **derived algebraic geometry**, i.e. the case where $\mathcal{C} := sk - Mod$, the category of simplicial modules over an arbitrary commutative ring k.

In §2.2.1 we describe the model categories $\mathcal{C} = sk-Mod$ and $Comm(\mathcal{C}) = sk-Alg$ whose opposite is denoted by $k-D^-Aff$, in particular finite cell and finitely presented objects, suspension and loop functors, Postnikov towers and stable modules. We also show that $(\mathcal{C}, \mathcal{C}_0, \mathcal{A}) := (sk - Mod, sk - Mod, sk - Alg)$ is a HA context in the sense of Def. 1.1.0.11.

In §2.2.2 we show how the general definitions of properties of modules (e.g. projective, flat, perfect) and of morphisms between commutative rings in $\mathcal{C}$ (e.g. finitely presented, flat, (formally) smooth, (formally) étale, Zariski open immersion) given in Chapter 1.2 translates concretely in the present context. The basic idea here is that of *strongness* which says that a module M over a simplicial k-algebra A (respectively, a morphism $A \to B$ in $sk - Alg$) has the property $\mathcal{P}$, defined in the abstract setting of Chapter 1.2, if and only if $\pi_0(M)$ has the corresponding classical property as a $\pi_0(A)$-module and $\pi_0(M) \otimes_{\pi_0(A)} \pi_*(A) \simeq \pi_*(M)$ (respectively, the induced morphism $\pi_0(A) \to \pi_0(B)$ has the corresponding classical property, and $\pi_0(B) \otimes_{\pi_0(A)} \pi_*(A) \simeq \pi_*(B)$). A straightforward extension of the étale topology to simplicial k-algebras, then provides us with an étale model site $(k - D^-Aff, \text{ét})$ satisfying assumption 1.3.2.2 (Def. 2.2.2.12 and Lemma 2.2.2.13) and with the corresponding model category $k - D^-Aff^{\sim,\text{ét}}$ of ét D^--stacks (Def. 2.2.2.14). The homotopy category $\mathrm{Ho}(k - D^-Aff^{\sim,\text{ét}})$ of ét D^--stacks will be simply denoted by $D^-\mathrm{St}(k)$. We conclude the section with two useful corollaries about topological invariance of étale and Zariski open immersions (Cor. 2.2.2.9 and 2.2.2.10), stating that ths small étale and Zariski site of a simplicial ring A is equivalent to the corresponding site of $\pi_0(A)$.

In §2.2.3 we describe our HAG context (Def. 1.3.2.13) for derived algebraic geometry by choosing the class $\mathbf{P}$ to be the class of smooth morphisms in $sk - Alg$ and the model topology to be the étale topology. This HAG context $(\mathcal{C}, \mathcal{C}_0, \mathcal{A}, \tau, \mathbf{P}) := (sk - Mod, sk - Mod, sk - Alg, \text{ét}, \text{smooth})$ is shown to satisfy Artin's conditions (Def. 2.2.3.2) relative to the HA context $(\mathcal{C}, \mathcal{C}_0, \mathcal{A}) := (sk - Mod, sk - Mod, sk - Alg)$ in Prop. 1.4.3.1; as a corollary of the general theory of Part I, this gives (Cor. 2.2.3.3) an obstruction theory (respectively, a relative obstruction theory) for any n-geometric D^--stack (resp., for any n-representable morphism between D^--stacks), and in particular a (relative) cotangent complex for any n-geometric D^--stack (resp., for any n-representable morphism). We finish the section showing that the properties of being flat, smooth, ètale and finitely presented can be extended to n-representable morphisms between D^--stacks (Lemma 2.2.3.4), and by the definition of open and closed immersion of D^--stacks (Def. 2.2.3.5).

In §2.2.4 we study *truncations* of derived stacks. The inclusion functor $j : k - Aff \to sk - Alg$, that sends a commutative k-algebra R to the constant simplicial k-algebra R, is Quillen right adjoint to the functor $\pi_0 : sk - Alg \to k - Alg$, and this Quillen adjunction induces an adjunction

$$i := \mathbb{L}j_! : \mathrm{St}(k) \longrightarrow D^-\mathrm{St}(k)$$

$$\mathrm{St}(k) \longleftarrow D^-\mathrm{St}(k) : \mathbb{R}i^* =: t_0$$

between the (homotopy) categories of derived and un-derived stacks. The functor i is fully faithful and commutes with homotopy colimits (Lemma 2.2.4.1) and embeds the theory of stacks into the theory of derived stacks, while the functor t_0, called the truncation functor, sends the affine stack corresponding to a simplicial k-algebra A to the affine scheme Spec $\pi_0(A)$, and commutes with homotopy limits and colimits (Lemma 2.2.4.2). Both the inclusion and the truncation functor preserve n-geometric stacks and flat, smooth, étale morphisms between them (Prop. 2.2.4.4). This gives a nice compatibility between the theories in §2.1 and §2.2, and therefore between moduli spaces and their derived analogs.

In particular, we get the that for *any* Artin algebraic stack in groupoids (actually, any Artin n-stack) $\mathcal{X}$, $i(\mathcal{X})$ has an obstruction theory, and therefore a cotangent complex; in other words viewing Artin stacks as derived stacks simplifies and clarifies a lot their infinitesimal theory, as already remarked in this introduction. We finish the section by showing (Prop. 2.2.4.7) that for any geometric D^--stack F, its truncation, viewed again as a derived stack (i.e. $it_0(F)$) sits inside F as a closed sub-stack, and one can reasonably think of F as behaving like a formal thickening of its truncation.

In §2.2.5 we give useful criteria for a n-representable morphism between D^--stacks being smooth (respectively, étale) in terms of locally finite presentation of the induced morphism on truncations, and infinitesimal lifting properties (Prop. 2.2.5.1, resp., Prop. 2.2.5.4) or properties of the cotangent complex (Cor. 2.2.5.3, resp., Cor. 2.2.5.6).

The final §2.2.6 of Chapter 2.2 contains applications of derived algebraic geometry to the construction of various derived versions of moduli spaces as D^--stacks. In 2.2.6.1 we first show (Lemma 2.2.6.1) that the stack of rank n vector bundles when viewed as a D^--stack using the inclusion $i : \mathrm{St}(k) \to D^-\mathrm{St}(k)$ is indeed isomorphic to the derived stack $\mathbf{Vect}_n$ of rank n vector bundles defined as in §1.3.7. Then, for any simplicial set K, we define the derived stack $\mathbb{R}\mathbf{Loc}_n(K)$ as the derived exponentiation of $\mathbf{Vect}_n$ with respect to K (Def. 2.2.6.2), show that when K is finite dimensional then $\mathbb{R}\mathbf{Loc}_n(K)$ is a finitely presented 1-geometric D^--stack (Lemma 2.2.6.3), and identify its truncation with the usual Artin stack of rank n local systems on K (Lemma 2.2.6.4). Finally we give a more concrete geometric interpretation of $\mathbb{R}\mathbf{Loc}_n(K)$ as a moduli space of derived geometric objects (*derived rank n local systems*) on the topological realization $|K|$ (Prop. 2.2.6.5) and show that the tangent space of $\mathbb{R}\mathbf{Loc}_n(K)$ at a global point correponding to a rank n local system E on K is the cohomology complex $C^*(K, E \otimes_k E^\vee)[1]$ (Prop. 2.2.6.6). The latter result shows in particular that the D^-- stack $\mathbb{R}\mathbf{Loc}_n(K)$ depends on strictly more than the fundamental groupoid of K (because its tangent spaces can be nontrivial even if K is simply connected) and therefore carries higher homotopical informations as opposed to the usual (i.e underived) Artin stack of rank n local systems. In subsection 2.2.6.2 we treat the case of algebras over an operad. If $\mathcal{O}$ is an operad in the category of k-modules, we consider a simplicial presheaf $\mathbf{Alg}_n^{\mathcal{O}}$ on $k - D^-Aff$ which associates to any simplicial k-algebra A the nerve of the subcategory $\mathcal{O} - Alg(A)$ of weak equivalences in the category of cofibrant algebras over the operad $\mathcal{O} \otimes_k A$ (which is an operad in simplicial A-modules) whose underlying A-module is a vector bundle of rank n. In Prop. 2.2.6.8 we show that $\mathbf{Alg}_n^{\mathcal{O}}$ is a 1-geometric quasi-compact D^--stack, and in Prop. 2.2.6.9 we identify its tangent space in terms of derived derivations. In subsection 2.2.6.3, using a special case of J.Lurie's representability criterion (see Appendix C), we give sufficient conditions for the D^--stack $\mathbf{Map}(\mathcal{X}, F)$ of morphisms of derived stacks $i(\mathcal{X}) \to F$ (for $\mathcal{X} \in \mathrm{St}(k)$) to be n-geometric (Thm. 2.2.6.11), and compute its tangent space in two particular cases (Cor. 2.2.6.14 and 2.2.6.15). The latter of these, i.e. the case $\mathbb{R}\mathcal{M}_{DR}(X) := \mathbf{Map}(i(X_{DR}), \mathbf{Vect}_n)$, where X is a complex smooth projective variety, is particularly important because it is the first step in the construction of a *derived* version of *non-abelian Hodge theory* which will be investigated in a future work.

In §2.3 we treat the case of **complicial** (or **unbounded**) **algebraic geometry**, i.e. homotopical algebraic geometry over the base category $C := C(k)$ of (unbounded) complexes of modules over a commutative $\mathbb{Q}$-algebra k [2].

Here $Comm(\mathcal{C})$ is the model category $k-cdga$ of commutative differential graded k-algebras, (called shortly cdga's) whose opposite model category will be denoted by $k-DAff$. A new feature of complicial algebraic geometry with respect to derived algebraic geometry is the existence of *two* interesting HA contexts (Lemma 2.3.1.1): the *weak* one $(\mathcal{C}, \mathcal{C}_0, \mathcal{A}) := (C(k), C(k), k-cdga)$, and the *connective* one $(\mathcal{C}, \mathcal{C}_0, \mathcal{A}) := (C(k), C(k)_{\leq 0}, k-cdga_0)$ where $C(k)_{\leq 0}$ is the full subcategory of $C(k)$ consisting of complexes which are cohomologically trivial in positive degrees and $k-cdga_0$ is the full subcategory of $k-cdga$ of connected algebras (i.e. cohomologically trivial in non-zero degrees). A related phenomenon is the fact that the notion of *strongness* that describes completely the properties of modules and of morphisms between between simplicial algebras in derived algebraic geometry, is less strictly connected with standard properties of modules and of morphisms between between cdga's. For example, morphisms between cdga's which have strongly the property $\mathcal{P}$ (e.g. $\mathcal{P}$= flat, étale etc.; see Def. 2.3.1.3) have the property $\mathcal{P}$ but the converse is true only if additional hypotheses are met (Prop. 2.3.1.4). Using the strong version of étale morphisms, we endow the category $k-DAff$ with the *strongly étale model topology* s-ét (Def. 2.3.1.6 and Lemma 2.3.1.7), and define the model category of *D-stacks* as $k-DAff^{\sim,\text{s-ét}}$ (Def. 2.3.1.8). The homotopy category of $k-DAff^{\sim,\text{s-ét}}$ will be simply denoted by $D\mathrm{St}(k)$.

While the notion of stack does not depend on the HA or HAG contexts chosen but only the base model site, the notion of geometric stack depends on the HA and HAG contexts. In §2.3.2 we complete the weak HA context above to the HAG context $(\mathcal{C}, \mathcal{C}_0, \mathcal{A}, \tau, \mathbf{P}) := (C(k), C(k), k-cdga, \text{s-ét}, \mathbf{P}_w)$, where $\mathbf{P}_w$ is the class of formally perfect morphisms (Def. 1.2.7.2), and call the corresponding geometric stacks *weakly geometric stacks* (Def. 2.3.2.2).

Section 2.3.3 contains some interesting examples of weakly geometric stacks. We first show (Prop. 2.3.3.1) that the stack **Perf** of perfect modules (defined in an abstract context in Def. 1.3.7.5) is weakly 1-geometric, categorically locally of finite presentation and its diagonal is (-1)-representable. In subsection 2.3.3.2 we first define the simplicial presheaf **Ass** sending a cdga A to the nerve of the subcategory of weak equivalences in the category of associative and unital (not necessarily commutative) cofibrant A-dg algebras whose underlying A-dg module is perfect, and show that this is a weakly 1-geometric D-stack (Prop. 2.3.3.2 and Cor. 2.3.3.4). Then we define (using the model structure on dg-categories of [**Tab**]) a dg-categorical variation of **Ass**, denoted by $\mathbf{Cat}_*$, by sending a cdga A to the nerve of the subcategory of weak equivalences of the model category of A-dg categories $\mathcal{D}$ which are connected and have perfect and cofibrant A-dg-modules of morphisms, and prove (Prop. 2.3.3.5) that the canonical classifying functor $\mathbf{Ass} \to \mathbf{Cat}_*$ is a weakly 1-representable, fp epimorphism of D-stacks; it follows that $\mathbf{Cat}_*$ is a weakly 2-geometric D-stack (Cor. 2.3.3.5).

In §2.3.4 we switch to the connective HA context $(\mathcal{C}, \mathcal{C}_0, \mathcal{A}) := (C(k), C(k)_{\leq 0}, k-cdga_0)$ and complete it to the connective HAG context

$$(C(k), C(k)_{\leq 0}, k-cdga_0, \text{s-ét}, \text{fip-smooth}),$$

[2] The case of k of positive characteristic can be treated as a special case of *brave new algebraic geometry* (§2.4) over the base Hk, H being the Eilenberg-Mac Lane functor.

where fip-smooth is the class of formally perfect (Def. 1.2.7.1) and formally i-smooth morphisms (Def. 1.2.8.1). Geometric stacks in this HAG context will be simply called *geometric D-stacks* (Def. 2.3.4.2); since fip-smooth morphisms are in $\mathbf{P}_w$, any geometric D-stack is weakly geometric. As opposed to the weak HAG context, this connective context indeed satisfies Artin's condition of Def. 1.4.3.1 (Prop. 2.3.4.3), and therefore any geometric D-stack has an obstruction theory (and a cotangent complex).

In §2.3.5 we give some examples of geometric D-stacks. We first observe (subsection 2.3.5.1) that the normalization functor $N : sk - Alg \to k - cdga$ induces a fully faithful functor $j := \mathbb{L}N_! : D^{-}\mathrm{St}(k) \hookrightarrow D\mathrm{St}(k)$. This provides us with lots of examples of (geometric) D-stacks. In subsection 2.3.5.2 we study the D-stack of CW-perfect modules. After having defined, for any cdga A, the notion of CW-A-dg-module of amplitude in $[a,b]$ (Def. 2.3.5.2) and proved some stability properties of this notion (Lemma 2.3.5.3), we define the sub-D-stack $\mathbf{Perf}^{CW}_{[a,b]} \subset \mathbf{Perf}$, consisting of all perfect modules locally equivalent to some CW-dg-modules of amplitude contained in $[a,b]$. We prove that $\mathbf{Perf}^{CW}_{[a,b]}$ is 1-geometric (Prop. 2.3.5.4), and that its tangent space at a point corresponding to a perfect CW-A-dg-module E is given by the complex $(E^{\vee} \otimes^{\mathbb{L}}_{A} E)[1]$ (Cor. 2.3.5.6). In subsection 2.3.5.3 we define the D-stack of CW-dg-algebras as the homotopy pullback of $\mathbf{Ass} \to \mathbf{Perf}$ along the inclusion $\mathbf{Perf}^{CW}_{[a,b]} \to \mathbf{Perf}$, prove that it is 1-geometric and compute its tangent space at a global point in terms of derived derivations (Cor. 2.3.5.9). Finally subsection 2.3.5.4 is devoted to the analysis of the D-stack $\mathbf{Cat}^{CW}_{*,[n,0]}$ of CW-dg-categories of perfect amplitude in $[n,0]$ (Def. 2.3.5.10). As opposite to the weakly geometric D-stack $\mathbf{Cat}_*$, that cannot have a reasonable infinitesimal theory, its full sub-D-stack $\mathbf{Cat}^{CW}_{*,[n,0]}$ is not only a 2-geometric D-stack but has a tangent space that can be computed in terms of Hochschild homology (Thm. 2.3.5.11). As corollaries of this important result we can prove a folklore statement (see e.g. [**Ko-So**, p. 266] in the case of A_∞-categories with one object) regarding the deformation theory of certain negative dg-categories being controlled by the Hochschild complex of dg-categories (Cor. 2.3.5.12), and a result showing that if one wishes to keep the existence of the cotangent complex, the restriction to non-positively graded dg-categories is unavoidable (Cor. 2.3.5.13).

In the last, short §2.4, we establish the basics of **brave new algebraic geometry**, i.e. of homotopical algebraic geometry over the base category $\mathcal{C} = Sp^{\Sigma}$ of symmetric spectra ([**HSS**, **Shi**]). We consider $\mathcal{C}$ endowed with the positive model structure of [**Shi**], which is better behaved than the usual one when dealing with commutative monoid objects an modules over them. We denote $Comm(\mathcal{C})$ by $S - Alg$ (and call its objects commutative S-algebras, S being the sphere spectrum, or sometimes bn-rings), and its opposite model category by $S - Aff$. Like in the case of complicial algebraic geometry, we consider two HA contexts here (Lemma 2.4.1.1): $(\mathcal{C}, \mathcal{C}_0, \mathcal{A}) := (Sp^{\Sigma}, Sp^{\Sigma}, S - Alg)$ and $(\mathcal{C}, \mathcal{C}_0, \mathcal{A}) := (Sp^{\Sigma}, Sp^{\Sigma}_c, S - Alg_0)$ where Sp^{Σ}_c is the subcategory of connective symmetric spectra, and $S - Alg_0$ the subcategory of S-algebras with homotopy groups concentrated in degree zero. After giving some examples of formally étale and formally *thh*-étale maps between bn-rings, we define (Def. 2.4.1.3) *strong versions* of flat, (formally) étale, (formally) smooth, and Zariski open immersions, exactly like in chapters 2.2 and 2.3, and give some results relating them to the corresponding non-strong notions (Prop. 2.4.1.4). An interesting exception to this relationship occurs in the case of smooth morphism: the Eilenberg-MacLane functor H from commutative rings to bn rings does not preserve (formal) smoothness in general, though it preserves (formal) strong smoothness, due to the presence of non-trivial Steenrod operations in characteristic $p > 0$ (Prop. 2.4.1.5). We endow the category $S - Aff$ with the *strong étale* model topology s-ét (Def. 2.4.1.7 and Lemma 2.4.1.8), and define the model category $S - Aff^{\sim, \text{s-ét}}$ of *S-stacks* (Def. 2.4.1.9). The homotopy category of $S - Aff^{\sim, \text{s-ét}}$ will be simply denoted by $\mathrm{St}(S)$. The two HA contexts defined above are completed (Cor. 2.4.1.11) to two different HAG contexts by choosing for both HA contexts the s-ét model topology, and the class $\mathbf{P}_{\text{s-ét}}$ of strongly étale morphsims for the first HA context (respectively, the class $\mathbf{P}$ of fip-morphisms for the second HA context). We call *geometric Deligne-Mumford S-stacks* (respectively, *geometric S-stacks*) the geometric stacks in the first (resp, second) HAG context, and observe that both contexts satisfy Artin's condition of Def. 1.4.3.1 (Prop. 2.4.1.13).

Finally in §2.4.2, we use the definition of topological modular forms of Hopkins-Miller to build a 1-geometric Deligne-Mumford stack $\overline{\mathcal{E}}_{\mathbf{S}}$ that is a "bn-derivation" of the usual stack $\overline{\mathcal{E}}$ of generalized elliptic curves [3] (i.e., the truncation of $\overline{\mathcal{E}}_{\mathbf{S}}$ is $\overline{\mathcal{E}}$), and such that the spectrum tmf coincide with the spectrum of *functions on* $\overline{\mathcal{E}}_{\mathbf{S}}$. We conclude by the remark that a moduli theoretic interpretation of $\overline{\mathcal{E}}_{\mathbf{S}}$ (or, most probably some variant of it), i.e. finding out which are the brave new objects that it classifies, could give not only interesting new geometry over bn rings but also new insights on classical objects of algebraic topology.

[3] See e.g. [**Del-Rap**, IV], where it is denoted by $\mathcal{M}_{(1)}$.

CHAPTER 2.1

Geometric n-stacks in algebraic geometry (after C. Simpson)

All along this chapter we fix an associative commutative ring $k \in \mathbb{U}$ with unit.

2.1.1. The general theory

We consider $\mathcal{C} = k - Mod$, the category of k-modules in the universe $\mathbb{U}$. We endow the category $\mathcal{C}$ with the trivial model structure for which equivalences are isomorphisms and all morphisms are cofibrations and fibrations. The category $\mathcal{C}$ is furthermore a symmetric monoidal model category for the monoidal structure given by the tensor product of k-modules. The assumptions 1.1.0.1, 1.1.0.2, 1.1.0.3 and 1.1.0.4 are all trivially satisfied. The category $Comm(\mathcal{C})$ is identified with the category $k - Alg$, of commutative (associative and unital) k-algebras in $\mathbb{U}$, endowed with the trivial model structure. Objects in $k - Alg$ will simply be called commutative k-algebras, without any reference to the universe $\mathbb{U}$. For any $A \in k - Alg$, the category $A - Mod$ is the usual symmetric monoidal category of A-modules in $\mathbb{U}$, also endowed with its trivial model structure. Furthermore, we have $\mathbb{R}\underline{Hom}_A(M,N) \simeq \underline{Hom}_A(M,N)$ for any two objects $A \in k - Alg$, and is the usual A-module of morphisms from M to N. We set $\mathcal{C}_0 := k - Mod$, and $\mathcal{A} := k - Alg$. The triplet $(k - Mod, k - Mod, k - Alg)$ is then a HA context in the sense of Def. 1.1.0.11.

The category $Aff_{\mathcal{C}}$ is identified with $k - Alg^{op}$, and therefore to the category of affine k-schemes in $\mathbb{U}$. It will simply be denoted by $k - Aff$ and its objects will simply be called affine k-schemes, without any reference to the universe $\mathbb{U}$. The model category of pre-stacks $k - Aff^{\wedge} = Aff_{\mathcal{C}}^{\wedge}$ is simply the model category of $\mathbb{V}$-simplicial presheaves on $k - Aff$, for which equivalences and fibrations are defined levelwise. The Yoneda functor

$$h : k - Aff \longrightarrow k - Aff^{\wedge}$$

is the usual one, and sends an affine k-scheme X to the presheaf of sets it represents h_X (considered as a presheaf of constant simplicial sets). Furthermore we have natural isomorphisms of functors

$$h \simeq \underline{h} \simeq \mathbb{R}\underline{h} : k - Aff \longrightarrow \mathrm{Ho}(k - Aff^{\wedge}),$$

which is nothing else than the natural composition

$$k - Aff \xrightarrow{h} k - Aff^{\wedge} \longrightarrow \mathrm{Ho}(k - Aff^{\wedge}).$$

We let $\tau = \text{ét}$, the usual étale pre-topology on $k - Aff$ (see e.g. [**Mil**]). Recall that a family of morphisms

$$\{X_i = Spec\, A_i \longrightarrow X = Spec\, A\}_{i \in I}$$

is an ét-covering family if and only if it contains a finite sub-family $\{X_i \longrightarrow X\}_{i\in J}$, $J \subset I$, such that the corresponding morphism of commutative k-algebras

$$A \longrightarrow \prod_{i\in J} A_i$$

is a faithfully flat and étale morphism of commutative rings.

LEMMA 2.1.1.1. *The étale topology on $k-Aff$ satisfies assumption 1.3.2.2.*

PROOF. Points (1) and (2) of 1.3.2.2 are clear. Point (3) is induced by the faithfully flat descent of quasi-coherent modules for affine $ffqc$-hypercovers (see e.g. [**SGA4-II**, Exp. V^{bis}])). □

The model category of stacks $k-Aff^{\sim,\text{ét}}$ is the projective model structure for simplicial presheaves on the Grothendieck site $(k-Aff, \text{ét})$, as defined for example in [**Bl**] (see also [**To1**, §1]). Its homotopy category, denoted simply as $\text{St}(k)$, is then identified with the full subcategory of $\text{Ho}(k-Aff^{\wedge})$ consisting of simplicial presheaves

$$F : k-Alg = k-Aff^{op} \longrightarrow SSet_{\mathbb{V}}$$

satisfying the following two conditions

- For any two commutative k-algebra A and B, the natural morphism

  $$F(A\times B) \longrightarrow F(A)\times F(B)$$

  is an isomorphism in $\text{Ho}(SSet)$.
- For any co-augmented co-simplicial commutative k-algebra, $A \longrightarrow B_*$, such that the augmented simplicial object

  $$Spec\, B_* \longrightarrow Spec\, A$$

  is an étale hypercover, the natural morphism

  $$F(A) \longrightarrow Holim_{n\in\Delta} F(B_n)$$

  is an isomorphism in $\text{Ho}(SSet)$.

It is well known (and also a consequence of Cor. 1.3.2.8) that the étale topology is sub-canonical; therefore there exists a fully faithful functor

$$h : k-Aff \longrightarrow \text{St}(k) \subset \text{Ho}(k-Aff^{\sim,\text{ét}}).$$

Furthermore, we have

$$Spec\, A \simeq \underline{Spec}\, A \simeq \mathbb{R}\underline{Spec}\, A$$

for any $A \in k-Alg$.

We set $\mathbf{P}$ to be the class of *smooth morphisms* in $k-Aff$ in the sense of [**EGAIV**, 17.3.1]. It is well known that our assumption 1.3.2.11 is satisfied (e.g. that smooth morphisms are étale-local in the source and target, see for example [**EGAIV**, 17.3.3, 17.3.4, 17.7.3]). In particular, we get that $(k-Mod, k-Mod, k-Alg, \text{ét}, \mathbf{P})$ is a HAG context in the sense of Def. 1.3.2.13. The general definition 1.3.3.1 can then be applied, and provides a notion of n-geometric stack in $\text{St}(k)$. A first important observation (the lemma below) is that n-geometric stacks are n-stacks in the sense of [**S3**]. This is a special feature of standard algebraic geometry, and the same would be true for any theory for which the model structure on C is trivial: the *geometric complexity* is a bound for the *stacky complexity.*

Recall that a stack $F \in \text{St}(k)$ is *n-truncated* if for any $X \in k-Aff$, any $s \in \pi_0(F(X))$ and any $i > n$, the sheaf $\pi_i(F, s)$ is trivial. By [**HAGI**, 3.7], this is equivalent to say that for any stack G the simplicial set $\mathbb{R}_\tau\underline{Hom}(G, F)$ is n-truncated.

LEMMA 2.1.1.2. *Let F be an n-geometric stack in* $\mathrm{St}(k)$. *Then F is $(n+1)$-truncated.*

PROOF. The proof is by induction on n. Representable stacks are nothing else than affine schemes, and therefore are 0-truncated. Suppose that the lemma is known for $m < n$. Let F be an n-geometric stack, $X \in k-Aff$ and $s \in \pi_0(F(X))$. We have a natural isomorphism of sheaves on $k-Aff/X$

$$\pi_i(F,s) \simeq \pi_{i-1}(X \times^h_F X, s),$$

where $X \longrightarrow F$ is the morphisms of stacks corresponding to s. As the diagonal of F is $(n-1)$-representable, $X \times^h_F X \simeq F \times_{F \times^h F} X \times^h X$ is $(n-1)$-geometric. By induction we find that $\pi_i(F,s) \simeq *$ for any $i > n$. □

Lemma 2.1.1.2 justifies the following terminology, closer to the usual terminology one can find in the literature.

DEFINITION 2.1.1.3. *An* Artin n-stack *is an n-truncated stack which is m-geometric for some integer m.*

The general theory of Artin n-stacks could then be pursued in a similar fashion as for Artin stacks in [**La-Mo**]. A part of this is done in [**S3**] and will not be reproduced here, as many of these statements will be settled down in the more general context of geometric D^--stacks (see §2.2). Let us mention however, that as explained in Def. 1.3.6.2, we can define the notions of flat, smooth, étale, unramified, regular, Zariski open immersion ... morphisms between Artin n-stacks. These kinds of morphisms are as usual stable by homotopy pullbacks, compositions and equivalences. In particular this allows the following definition.

DEFINITION 2.1.1.4.
- *An Artin n-stack is a* Deligne-Mumford n-stack *if there exists an n-atlas $\{U_i\}$ for F such that each morphism $U_i \longrightarrow F$ is an étale morphism.*
- *An Artin n-stack is an* algebraic space *if it is a Deligne-Mumford n-stack, and if furthermore the diagonal $F \longrightarrow F \times^h F$ is a monomorphism in the sense of Def. 1.3.6.4.*
- *An Artin n-stack F is an* scheme *if there exists an n-atlas $\{U_i\}$ for F such that each morphism $U_i \longrightarrow F$ is a monomorphism.*

REMARK 2.1.1.5.
(1) An algebraic space in the sense of the definition above which is automatically a 1-geometric stack, and is nothing else than an algebraic space in the usual sense. Indeed, this can be shown by induction on n: an algebraic space which is also n-geometric is by definition the quotient of a union of affine schemes X by some étale equivalence relation $R \subset X \times X$ where R is an algebraic space which is $(n-1)$-geometric. In particular, R being a subobject in $X \times X$ we see that R is a separated algebraic space, and thus is a 0-geometric stack. This implies that X/R is a 1-geometric stack. In the same way, any scheme is automatically a 1-geometric stack. Moreover, algebraic spaces (resp. schemes) which are 0-geometric stacks are precisely algebraic spaces (resp. schemes) with an affine diagonal.
(2) Thought there is a small discrepancy between the notion of Artin n-stack and the notion of n-geometric stack in $\mathrm{St}(k)$, our notion of Artin n-stack is equivalent to the notion of *slightly geometric n-stacks* of [**S3**].

2.1.2. Comparison with Artin's algebraic stacks

Artin n-stacks as defined in the last section are simplicial presheaves, whereas Artin stacks are usually presented in the literature using the theory of fibered categories (se e.g. [**La-Mo**]). In this section we briefly explain how the theory of fibered categories in groupoids can be embedded in the theory of simplicial presheaves, and how this can be used in order to compare the original definition of Artin stacks to our definition of Artin n-stacks.

In [**Hol**], it is shown that there exists a model category $Grpd/\mathcal{S}$, of cofibered categories in groupoids over a Grothendieck site $\mathcal{S}$. The fibrant objects for this model structure are precisely the stacks in groupoids in the sense of [**La-Mo**], and the equivalences in $Grpd/\mathcal{S}$ are the morphisms of cofibered categories becoming equivalences on the associated stacks (i.e. local equivalences). There exists furthermore a Quillen equivalence

$$p : P(\mathcal{S}, Grpd) \longrightarrow Grpd/\mathcal{S} \qquad P(\mathcal{S}, Grpd) \longleftarrow Grpd/\mathcal{S} : \Gamma,$$

where $P(\mathcal{S}, Grpd)$ is the local projective model category of presheaves of groupoids on $\mathcal{S}$. Finally, there exists a Quillen adjunction

$$\Pi_1 : SPr_\tau(\mathcal{S}) \longrightarrow P(\mathcal{S}, Grpd) \qquad SPr_\tau(\mathcal{S}) \longleftarrow P(\mathcal{S}, Grpd) : B,$$

where Π_1 is the natural extension of the functor sending a simplicial set to its fundamental groupoid, B is the natural extension of the nerve functor, and the $SPr_\tau(\mathcal{S})$ is the local projective model structure of simplicial presheaves on the site $\mathcal{S}$ (also denoted by $\mathcal{S}^{\sim,\tau}$ in our context, at least when $\mathcal{S}$ has limits and colimits and thus can be considered as a model category with the trivial model structure). The functor B preserves equivalences and the induced functor

$$B : \mathrm{Ho}(P(\mathcal{S}, Grpd)) \longrightarrow \mathrm{Ho}(SPr_\tau(\mathcal{S}))$$

is fully faithful and its image consists of all 1-truncated objects (in the sense of [**HAGI**, §3.7]). Put in another way, the model category $P(\mathcal{S}, Grpd)$ is Quillen equivalent to the S^2-nullification of $SPr_\tau(\mathcal{S})$ (denoted by $SPr_\tau^{\leq 1}(\mathcal{S})$ in [**HAGI**, §3.7]). In conclusion, there exists a chain of Quillen equivalences

$$Grpd/\mathcal{S} \rightleftarrows P(\mathcal{S}, Grpd) \rightleftarrows SPr_\tau^{\leq 1}(\mathcal{S}),$$

and therefore a well defined adjunction

$$t : \mathrm{Ho}(SPr_\tau(\mathcal{S})) \longrightarrow \mathrm{Ho}(Grpd/\mathcal{S}) \qquad \mathrm{Ho}(SPr_\tau(\mathcal{S})) \longleftarrow \mathrm{Ho}(Grpd/\mathcal{S}) : i,$$

such that the right adjoint

$$i : \mathrm{Ho}(Grpd/\mathcal{S}) \longrightarrow \mathrm{Ho}(SPr_\tau(\mathcal{S}))$$

is fully faithful and its image consists of all 1-truncated objects.

The category $\mathrm{Ho}(Grpd/\mathcal{S})$ can also be described as the category whose objects are stacks in groupoids in the sense of [**La-Mo**], and whose morphisms are given by 1-morphisms of stacks up to 2-isomorphisms. In other words, for two given stacks F and G in $Grpd/\mathcal{S}$, the set of morphisms from F to G in $\mathrm{Ho}(Grpd/\mathcal{S})$ is the set of isomorphism classes of the groupoid $\mathcal{H}om(F, G)$, of morphisms of stacks. This implies that the usual category of stacks in groupoids, up to 2-isomorphisms, can be identified through the functor i with the full subcategory of 1-truncated objects in $\mathrm{Ho}(SPr_\tau(\mathcal{S}))$. Furthermore, the functor i being defined as the composite of right derived functors and derived Quillen equivalences will commutes with homotopy limits. As homotopy limits in $Grpd/\mathcal{S}$ can also be identified with the 2-limit of stacks as defined in [**La-Mo**], the functor i will send 2-limits to homotopy limits. As a particular case we obtain that i sends the 2-fiber product of stacks in groupoids to the homotopy fiber product.

The 2-categorical structure of stacks in groupoids can also be recovered from the model category $SPr_\tau(S)$. Indeed, applying the simplicial localization techniques of [**D-K**1] to the Quillen adjunctions described above, we get a well defined diagram of S-categories

$$L(Grpd/\mathcal{S}) \longrightarrow LP(\mathcal{S}, Grpd) \longrightarrow LSPr_\tau(\mathcal{S}),$$

which is fully faithful in the sense of [**HAGI**, Def. 2.1.3]. In particular, the S-category $L(Grpd/\mathcal{S})$ is naturally equivalent to the full sub-S-category of $LSPr_\tau(\mathcal{S})$ consisting of 1-truncated objects. Using [**D-K**3], the S-category $L(Grpd/\mathcal{S})$ is also equivalent to the S-category whose objects are stacks in groupoids, cofibrant as objects in $Grpd/\mathcal{S}$, and whose morphisms simplicial sets are given by the simplicial Hom's sets of $Grpd/\mathcal{S}$. These simplicial Hom's sets are simply the nerves of the groupoid of functors between cofibered categories in groupoids. In other words, replacing the simplicial sets of morphisms in $L(Grpd/\mathcal{S})$ by their fundamental groupoids, we find a 2-category naturally 2-equivalent to the usual 2-category of stacks in groupoids on $\mathcal{S}$. Therefore, we see that the 2-category of stacks in groupoids can be identified, up to a natural 2-equivalence, as the 2-category obtained from the full sub-S-category of $LSPr_\tau(\mathcal{S})$ consisting of 1-truncated objects, by replacing its simplicial sets of morphisms by their fundamental groupoids.

We now come back to the case where $\mathcal{S} = (k-Aff, \acute{e}t)$, the Grothendieck site of affine k-schemes with the étale pre-topology. We have seen that there exists a fully faithful functor

$$i : \mathrm{Ho}(Grpd/k-Aff^{\sim,\acute{e}t}) \longrightarrow \mathrm{St}(k),$$

from the category of stacks in groupoids up to 2-isomorphisms, to the homotopy category of stacks. The image of this functor consists of all 1-truncated objects and it is compatible with the simplicial structure (i.e. possesses a natural lifts as a morphism of S-categories). We also have seen that i sends 2-fiber products of stacks to homotopy fiber products.

Using the functor i, every stack in groupoids can be seen as an object in our category of stacks $\mathrm{St}(k)$. For example, all examples of stacks presented in [**La-Mo**] give rise to stacks in our sense. The proposition below subsumes the main properties of the functor i, relating the usual notion of scheme, algebraic space and stack to the one of our definition Def. 2.1.1.4. Recall that a stack in groupoids X is separated (resp. quasi-separated) if its diagonal is a proper (resp. separated) morphism.

PROPOSITION 2.1.2.1. (1) *For any commutative k-algebra A, there exists a natural isomorphism*

$$i(Spec\, A) \simeq Spec\, A.$$

(2) *If X is a scheme (resp. algebraic space, resp. Deligne-Mumford stack, resp. Artin stack) with an affine diagonal in the sense of* [**La-Mo**], *then $i(X)$ is an Artin 0-stack which is 0-geometric (resp. an Artin 0-stack which is 0-geometric, resp. a Deligne-Mumford 1-stack which is 0-geometric, resp. an Artin 1-stack which is 0-geometric) in the sense of Def. 2.1.1.4.*

(3) *If X is a scheme (resp. algebraic space, resp. Deligne-Mumford stack, resp. Artin stack) in the sense of* [**La-Mo**], *then $i(X)$ is an Artin 0-stack which is 1-geometric (resp. an Artin 0-stack which is 1-geometric, resp. a Deligne-Mumford 1-stack which is 1-geometric, resp. an Artin 1-stack which is 1-geometric) in the sense of Def. 2.1.1.4.*

(4) *Let* $f : F \longrightarrow G$ *be a morphism between Artin stacks in the sense of* [**La-Mo**]. *Then the morphism* f *is flat (resp. smooth, resp. étale, resp. unramified, resp. Zariski open immersion) if and only if* $i(f) : i(F) \longrightarrow i(G)$ *is so.*

PROOF. This readily follows from the definition using the fact that i preserves affine schemes, epimorphisms of stacks, and sends 2-fiber products to homotopy pullbacks. □

REMARK 2.1.2.2. If we try to apply the general infinitesimal and obstruction theory developed in §1.4 to the present HAG-context, we immediately see that this is impossible because the suspension functor $S : \mathrm{Ho}(\mathcal{C}) = k - Mod \to k - Mod = \mathrm{Ho}(\mathcal{C})$ is trivial. On the other hand, the reader might object that there is already a well established infinitesimal theory, at least in the case of schemes, algebraic spaces and for a certain class of algebraic stacks in groupoids, and that our theory does not seem to be able to reproduce it. The answer to this question turns out to be both conceptually and technically relevant. First of all, if we look at e.g. the definition of the cotangent complex of a scheme ([**Ill**, 2.1.2]) we realize that a basic and necessary step is to enlarge the category of rings to the category of simplicial rings in order to be able to consider free (or more generally cofibrant) resolutions of maps between rings. In our setup, this can be reformulated by saying that in order to get the correct infinitesimal theory, even for ordinary schemes, it is necessary to view them as geometric objects in *derived algebraic geometry*, i.e. on homotopical algebraic geometry over the base category $\mathcal{C} = sk - Mod$ of simplicial k-modules (so that $Comm(\mathcal{C})$ is exactly the category of commutative simplicial k-algebras). In other words, the usual infinitesimal theory of schemes is already "secretly" a part of derived algebraic geometry, that will be studied in detail in the next chapter 2.2. Moreover, as it will be shown, this approach has, even for classical objects like schemes or Artin stacks in groupoids, both conceptual advantages (like e.g. the fact that the cotangent complex of a scheme can be interpreted geometrically as a *genuine* cotangent space to the scheme when viewed as a derived stack, satisfying a natural universal property, while the cotangent complex of a scheme do not have any universal property inside the theory of schemes), and technical advantages (like the fact, proved in Cor. 2.2.4.5, that *any* Artin stack in groupoids has an obstruction theory).

CONVENTION 2.1.2.3. From now on we will omit mentioning the functor i, and will simply view stacks in groupoids as objects in $\mathrm{St}(k)$. In particular, we will allow ourselves to use the standard notions and vocabulary of the general theory of schemes.

CHAPTER 2.2

Derived algebraic geometry

All along this chapter k will be a fixed commutative (associative and unital) ring.

2.2.1. The HA context

In this section we specialize our general theory of Part I to the case where $\mathcal{C} = sk-Mod$, is the category of simplicial k-modules in the universe $\mathbb{U}$. The category $sk-Mod$ is endowed with its standard model category structure, for which the fibrations and equivalences are defined on the underlying simplicial sets (see for example [**Goe-Ja**]). The tensor product of k-modules extends naturally to a levelwise tensor product on $sk-Mod$, making it into a symmetric monoidal model category. Finally, $sk-Mod$ is known to be a $\mathbb{U}$-combinatorial proper and simplicial model category.

The model category $sk-Mod$ is known to satisfy assumptions 1.1.0.1, 1.1.0.2 and 1.1.0.3 (see [**Schw-Shi**]). Finally, it follows easily from [**Q1**, II.4, II.6] that $sk-Mod$ also satisfies assumption 1.1.0.4.

The category $Comm(sk-Mod)$ will be denoted by $sk-Alg$, and its objects will be called simplicial commutative k-algebras. More generally, for $A \in sk-Alg$, the category $A-Comm(\mathcal{C})$ will be denoted by $A-Alg_s$. For any $A \in sk-Alg$ we will denoted by $A-Mod_s$ the category of A-modules in $sk-Mod$, which is nothing else than the category of simplicial modules over the simplicial ring A. The model structure on $sk-Alg$, $A-Alg_s$ and $A-Mod_s$ is the usual one, for which the equivalences and fibrations are defined on the underlying simplicial sets. For an object $A \in sk-Alg$, we will denote by $\pi_i(A)$ its homotopy group (pointed at 0). The graded abelian group $\pi_*(A)$ inherits a structure of a commutative graded algebra from A, which defines a functor $A \mapsto \pi_*(A)$ from $sk-Alg$ to the category of commutative graded k-algebras. More generally, if A is a simplicial commutative k-algebra, and M is an A-module, the graded abelian group $\pi_*(M)$ has a natural structure of a graded $\pi_*(A)$-module.

There exists a Quillen adjunction

$$\pi_0 : sk-Alg \longrightarrow k-Alg \qquad sk-Alg \longleftarrow k-Alg : i,$$

where i sends a commutative k-algebra to the corresponding constant simplicial commutative k-algebra. This Quillen adjunction induces a fully faithful functor

$$i : k-Alg \longrightarrow \mathrm{Ho}(sk-Alg).$$

From now on we will omit to mention the functor i, and always consider $k-Alg$ as embedded in $sk-Alg$, except if the contrary is specified. Note that when $A \in k-Alg$, also considered as an object in $sk-Alg$, we have two different notions of A-modules, $A-Mod$, and $A-Mod_s$. The first one is the usual category of A-modules, whereas the second one is the category of simplicial objects in $A-Mod$.

For any morphism of simplicial commutative k-algebras $A \longrightarrow B$, the B-module $\mathbb{L}_{B/A}$ constructed in 1.2.1.2 is naturally isomorphic in $\mathrm{Ho}(B-Mod)$ to D. Quillen's cotangent complex introduced in [**Q2**]. In particular, if $A \longrightarrow B$ is a morphism between (non-simplicial) commutative k-algebras, then we have $\pi_0(\mathbb{L}_{B/A}) \simeq \Omega^1_{B/A}$. More generally, we find by adjunction

$$\pi_0(\mathbb{L}_{B/A}) \simeq \Omega^1_{\pi_0(B)/\pi_0(A)}.$$

Recall also that $A \longrightarrow B$ in $k - Alg$ is étale in the sense of [**EGAIV**, 17.1.1] if and only if $\mathbb{L}_{B/A} \simeq 0$ and B is finitely presented as a commutative A-algebra. In the same way, a morphism $A \longrightarrow B$ in $k - Alg$ is smooth in the sense of [**EGAIV**, 17.1.1] if and only if $\Omega^1_{B/A}$ is a projective B-module, $\pi_i(\mathbb{L}_{B/A}) \simeq 0$ for $i > 0$ and B is finitely presented as a commutative A-algebra.. Finally, recall the existence of a natural first quadrant spectral sequence (see [**Q1**, II.6 Thm. 6(b)])

$$Tor^p_{\pi_*(A)}(\pi_*(M), \pi_*(N))_q \Rightarrow \pi_{p+q}(M \otimes^{\mathbb{L}}_A N),$$

for $A \in sk - Alg$ and any objects M and N in $A - Mod_s$.

We set $\mathcal{C}_0 := \mathcal{C} = sk - Mod$, and $\mathcal{A} := sk - Alg$. Then, clearly, assumption 1.1.0.6 is also satisfied. The triplet $(sk - Mod, sk - Mod, sk - Alg)$ is then a HA context in the sense of Def. 1.1.0.11. Note that for any $A \in sk - Alg$ we have $(A - Mod_s)_0 = A - Mod$, whereas $(A - Mod_s)_1$ consists of all A-modules M such that $\pi_0(M) = 0$, also called *connected modules*.

For an integer $n \geq 0$ we define the n-th sphere k-modules by $S^n_k := S^n \otimes k \in sk - Mod$. The free commutative monoid on S^n_k is an object $k[S^n] \in sk - Alg$, such that for any $A \in sk - Alg$ there are functorial isomorphisms

$$[k[S^n], A]_{sk-Alg} \simeq \pi_n(A).$$

In the same way we define $\Delta^n \otimes k \in sk - Mod$ and its associated free commutative monoid $k[\Delta^n]$. There are natural morphisms $k[S^n] \longrightarrow k[\Delta^{n+1}]$, coming from the natural inclusions $\partial\Delta^{n+1} = S^n \hookrightarrow \Delta^{n+1}$. The set of morphisms

$$\{k[S^n] \longrightarrow k[\Delta^{n+1}]\}_{n \geq 0}$$

form a generating set of cofibrations in $sk - Alg$. The model category $sk - Alg$ is then easily checked to be compactly generated in the sense of Def. 1.2.3.4. A finite cell object in $sk - Alg$ is then any object $A \in sk - Alg$ for which there exists a finite sequence in $sk - Alg$

$$A_0 = k \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_m = A,$$

such that for any i there exists a push-out square in $sk - Alg$

$$\begin{array}{ccc} A_i & \longrightarrow & A_{i+1} \\ \uparrow & & \uparrow \\ k[S^{n_i}] & \longrightarrow & k[\Delta^{n_i+1}] \end{array}$$

Our Prop. 1.2.3.5 implies that an object $A \in sk - Alg$ is finitely presented in the sense of Def. 1.2.3.1 if and only if it is equivalent to a retract of a finite cell object (see also [**EKMM**, III.2] or [**Kr-Ma**, Thm. III.5.7] for other proofs). More generally, for $A \in sk - Alg$, there exists a notion of finite cell object in $A - Alg_s$ using the elementary morphisms

$$A[S^n] := A \otimes_k k[S^n] \longrightarrow A[\Delta^{n+1}] := A \otimes_k k[\Delta^{n+1}].$$

In the same way, a morphism $A \longrightarrow B$ in $sk - Alg$ is finitely presented in the sense of Def. 1.2.3.1 if and only if B is equivalent to a retract of a finite cell objects in $A - Alg_s$. Prop. 1.2.3.5 also implies that any morphism $A \longrightarrow B$, considered as an object in $A-Alg_s$, is equivalent to a filtered colimit of finite cell objects, so in particular to a filtered homotopy colimit of finitely presented objects.

The Quillen adjunction between $k-Alg$ and $sk-Alg$ shows that the functor

$$\pi_0 : sk-Alg \longrightarrow k-Alg$$

does preserve finitely presented morphisms. On the contrary, the inclusion functor $i : k-Alg \longrightarrow sk-Alg$ does not preserve finitely presented objects, and the finite presentation condition in $sk-Alg$ is in general stronger than in $k-Alg$.

For $A \in sk-Alg$, we also have a notion of finite cell objects in $A-Mod_s$, based the generating set for cofibrations consisting of morphisms of the form

$$S^n_A := A \otimes_k S^n_k \longrightarrow \Delta^{n+1}_A := A \otimes_k \Delta^{n+1}_A.$$

Using Prop. 1.2.3.5 we see that the finitely presented objects in $A-Mod_s$ are the objects equivalent to a retract of a finite cell objects (see also [**EKMM**, III.2] or [**Kr-Ma**, Thm. III.5.7]). Moreover, the functor

$$\pi_0 : A-Mod_s \longrightarrow \pi_0(A)-Mod$$

is left Quillen, so preserves finitely presented objects. On the contrary, for $A \in k-Alg$, the natural inclusion functor $A-Mod \longrightarrow A-Mod_s$ from A-modules to simplicial A-modules does not preserve finitely presented objects in general.

The category $sk-Mod$ is also Quillen equivalent (actually equivalent) to the model category $C^-(k)$ of non-positively graded cochain complexes of k-modules, through the Dold-Kan correspondence ([**We**, 8.4.1]). In particular, the suspension functor

$$S : \mathrm{Ho}(sk-Mod) \longrightarrow \mathrm{Ho}(sk-Mod)$$

corresponds to the shift functor $E \mapsto E[1]$ on the level of complexes, and is a fully faithful functor. This implies that for any $A \in sk-Alg$, the suspension functor

$$S : \mathrm{Ho}(A-Mod_s) \longrightarrow \mathrm{Ho}(A-Mod_s)$$

is also fully faithful. We have furthermore $\pi_i(S(M)) \simeq \pi_{i+1}(M)$ for all $M \in A-Mod_s$. The suspension and loop functors will be denoted respectively by

$$M[1] := S(M) \qquad M[-1] := \Omega(M).$$

For any $A \in sk-Alg$, we can construct a functorial tower in $sk-Alg$, called the *Postnikov tower*,

$$A \longrightarrow \cdots \longrightarrow A_{\leq n} \longrightarrow A_{\leq n-1} \longrightarrow \cdots \longrightarrow A_{\leq 0} = \pi_0(A)$$

in such a way that $\pi_i(A_{\leq n}) = 0$ for all $i > n$, and the morphism $A \longrightarrow A_{\leq n}$ induces isomorphisms on the π_i's for all $i \leq n$. The morphism $A \longrightarrow A_{\leq n}$ is characterized by the fact that for any $B \in sk-Alg$ which is n-truncated (i.e. $\pi_i(B) = 0$ for all $i > n$), the induced morphism

$$Map_{sk-Alg}(A_{\leq n}, B) \longrightarrow Map_{sk-Alg}(A, B)$$

is an isomorphism in $\mathrm{Ho}(SSet)$. This implies in particular that the Postnikov tower is furthermore unique up to equivalence (i.e. unique as an object in the homotopy category of diagrams). There exists a natural isomorphism in $\mathrm{Ho}(sk-Alg)$

$$A \simeq Holim_n A_{\leq n}.$$

For any integer n, the homotopy fiber of the morphism

$$A_{\leq n} \longrightarrow A_{\leq n-1}$$

is isomorphic in $\mathrm{Ho}(sk-Mod)$ to $S^n \otimes_k \pi_n(A)$, and is also denoted by $\pi_n(A)[n]$. The k-module $\pi_n(A)$ has a natural structure of a $\pi_0(A)$-module, and this induces a natural structure of a simplicial $\pi_0(A)$-module on each $\pi_n(A)[i]$ for all i. Using the natural projection $A_{\leq n-1} \longrightarrow \pi_0(A)$, we thus see the object $\pi_n(A)[i]$ as an object in $A_{\leq n-1} - Mod_s$. Note that there is a natural isomorphism in $\mathrm{Ho}(A_{\leq n-1} - Mod_s)$

$$S(\pi_n(A)[i]) \simeq \pi_n(A)[i+1] \qquad \Omega(\pi_n(A)[i]) \simeq \pi_n(A)[i-1],$$

where $\pi_n(A)[i]$ is understood to be 0 for $i < 0$. We recall the following important and well known fact.

LEMMA 2.2.1.1. *With the above notations, there exists a unique derivation*

$$d_n \in \pi_0(\mathbb{D}er_k(A_{\leq n-1}, \pi_n(A)[n+1]))$$

such that the natural projection

$$A_{\leq n-1} \oplus_{d_n} \pi_n(A)[n] \longrightarrow A_{\leq n-1}$$

is isomorphic in $\mathrm{Ho}(sk - Alg/A_{\leq n-1})$ *to the natural morphism*

$$A_{\leq n} \longrightarrow A_{\leq n-1}.$$

SKETCH OF PROOF. (See also [**Ba**] for more details).

The uniqueness of d_n follows easily from our lemma 1.4.3.7, and the fact that the natural morphism

$$\mathbb{L}QZ(\pi_n(A)[n+1]) \longrightarrow \pi_n(A)[n+1]$$

induces an isomorphism on π_i for all $i \leq n+1$ (this follows from our lemma 2.2.2.7 below). To prove the existence of d_n, we consider the homotopy push-out diagram in $sk - Alg$

$$\begin{array}{ccc} A_{\leq n-1} & \longrightarrow & B \\ \uparrow & & \uparrow \\ A_{\leq n} & \longrightarrow & A_{\leq n-1}. \end{array}$$

The identity of $A_{\leq n-1}$ induces a morphism $B \longrightarrow A_{\leq n-1}$, which is a retraction of $A_{\leq n-1} \longrightarrow B$. Taking the $(n+1)$-truncation gives a commutative diagram

$$\begin{array}{ccc} A_{\leq n-1} & \longrightarrow & B_{\leq n+1} \\ \uparrow & & \uparrow{\scriptstyle s} \\ A_{\leq n} & \longrightarrow & A_{\leq n-1}. \end{array}$$

in such a way that s has a retraction. This easily implies that the morphism s is isomorphic, in a non-canonical way, to the zero derivation $A_{\leq n-1} \longrightarrow A_{\leq n-1} \oplus \pi_n(A)[n+1]$. The top horizontal morphism of the previous diagram then gives rise to a derivation

$$d_n : A_{\leq n-1} \longrightarrow A_{\leq n-1} \oplus \pi_n(A)[n+1].$$

The diagram

$$\begin{array}{ccc} A_{\leq n-1} & \stackrel{d_n}{\longrightarrow} & B_{\leq n+1} \\ \uparrow & & \uparrow{\scriptstyle s} \\ A_{\leq n} & \longrightarrow & A_{\leq n-1}, \end{array}$$

is then easily checked to be homotopy cartesian, showing that $A_{\leq n} \longrightarrow A_{\leq n-1}$ is isomorphic to $A_{\leq n-1} \oplus_{d_n} \pi_n(A)[n] \longrightarrow A_{\leq n-1}$. □

Finally, the truncation construction also exists for modules. For any $A \in sk-Alg$, and $M \in A-Mod_s$, there exists a natural tower of morphisms in $A-Mod_s$

$$M \longrightarrow \cdots \longrightarrow M_{\leq n} \longrightarrow M_{\leq n-1} \longrightarrow \cdots \longrightarrow M_{\leq 0} = \pi_0(M),$$

such a way that $\pi_i(M_{\leq n}) = 0$ for all $i > n$, and the morphism $M \longrightarrow M_{\leq n}$ induces an isomorphisms on π_i for $i \leq n$. The natural morphism

$$M \longrightarrow Holim_n M_{\leq n}$$

is an isomorphism in $\mathrm{Ho}(A-Mod_s)$. Furthermore, the A-module $M_{\leq n}$ is induced by a natural $A_{\leq n}$-module, still denoted by $M_{\leq n}$, through the natural morphism $A \longrightarrow A_{\leq n}$. The natural projection $M \longrightarrow M_{\leq n}$ is again characterized by the fact that for any A-module N which is n-truncated, the induced morphism

$$Map_{A-Mod_s}(M_{\leq n}, N) \longrightarrow Map_{A-Mod_s}(M, N)$$

is an isomorphism in $\mathrm{Ho}(SSet)$.

For an object $A \in sk-Alg$, the homotopy category $\mathrm{Ho}(Sp(A-Mod_s))$, of stable A-modules can be described in the following way. By normalization, the commutative simplicial k-algebra A can be transformed into a commutative dg-algebra over k, $N(A)$ (because N is lax symmetric monoidal). We can therefore consider its model category of unbounded $N(A)$-dg-modules, and its homotopy category $\mathrm{Ho}(N(A)-Mod)$. The two categories $\mathrm{Ho}(Sp(A-Mod_s))$ and $\mathrm{Ho}(N(A)-Mod)$ are then naturally equivalent. In particular, when A is a commutative k-algebra, then $N(A) = A$, and one finds that $\mathrm{Ho}(Sp(A-Mod_s))$ is simply the unbounded derived category of A, or equivalently the homotopy category of the model category $C(A)$ of unbounded complexes of A-modules

$$\mathrm{Ho}(Sp(A-Mod_s)) \simeq D(A) \simeq \mathrm{Ho}(C(A)).$$

Finally, using our Cor. 1.2.3.8 (see also [**EKMM**, III.7]), we see that the perfect objects in the symmetric monoidal model category $\mathrm{Ho}(Sp(A-Mod_s))$ are exactly the finitely presented objects.

We now let $k-D^-Aff$ be the opposite model category of $sk-Alg$. We use our general notations, $Spec\, A \in k-D^-Aff$ being the object corresponding to $A \in sk-Alg$. We will also sometimes use the notation

$$t_0(Spec\, A) := Spec\, \pi_0(A).$$

2.2.2. Flat, smooth, étale and Zariski open morphisms

According to our general definitions presented in §1.2 we have various notions of projective and perfect modules, flat, smooth, étale, unramified ... morphisms in $sk-Alg$. Our first task, before visiting our general notions of stacks, will be to give concrete descriptions of these notions.

Any object $A \in sk-Alg$ gives rise to a commutative graded k-algebra of homotopy $\pi_*(A)$, which is functorial in A. In particular, $\pi_i(A)$ is always endowed with a natural structure of a $\pi_0(A)$-module, functorially in A. For a morphism $A \longrightarrow B$ in $sk-Alg$ we obtain a natural morphism

$$\pi_*(A) \otimes_{\pi_0(A)} \pi_0(B) \longrightarrow \pi_*(B).$$

More generally, for $A \in sk-Alg$ and M an A-module, one has a natural morphism of $\pi_0(A)$-modules $\pi_0(M) \longrightarrow \pi_*(M)$, giving rise to a natural morphism

$$\pi_*(A) \otimes_{\pi_0(A)} \pi_0(M) \longrightarrow \pi_*(M).$$

These two morphisms are the same when the commutative A-algebra B is considered as an A-module in the usual way.

DEFINITION 2.2.2.1. *Let $A \in sk - Alg$ and M be an A-module. The A-module M is* strong *if the natural morphism*

$$\pi_*(A) \otimes_{\pi_0(A)} \pi_0(M) \longrightarrow \pi_*(M)$$

is an isomorphism.

LEMMA 2.2.2.2. *Let $A \in sk - Alg$ and M be an A-module.*

(1) *The A-module M is projective if and only if it is strong and $\pi_0(M)$ is a projective $\pi_0(A)$-module.*
(2) *The A-module M is flat if and only if it is strong and $\pi_0(M)$ is a flat $\pi_0(A)$-module.*
(3) *The A-module M is perfect if and only if it is strong and $\pi_0(M)$ is a projective $\pi_0(A)$-module of finite type.*
(4) *The A-module M is projective and finitely presented if and only if it is perfect.*

PROOF. (1) Let us suppose that M is projective.We first notice that a retract of a strong module A-module is again a strong A-module. This allows us to suppose that M is free, which clearly implies that M is strong and that $\pi_0(M)$ is a free $\pi_0(A)$-module (so in particular projective). Conversely, let M be a strong A-module with $\pi_0(M)$ projective over $\pi_0(A)$. We write $\pi_0(M)$ as a retract of a free π_0-module

$$\pi_0(M) \xrightarrow{\ i\ } \pi_0(A)^{(I)} = \oplus_I \pi_0(A) \xrightarrow{\ r\ } \pi_0(M).$$

The morphism r is given by a family of elements $r_i \in \pi_0(M)$ for $i \in I$, and therefore can be seen as a morphism $r' : A^{(I)} \longrightarrow M$, well defined in $\mathrm{Ho}(A - Mod_s)$. In the same way, the projector $p = i \circ r$ of $\pi_0(A)^{(I)}$, can be seen as a projector p' of $A^{(I)}$ in the homotopy category $\mathrm{Ho}(A - Mod_s)$. By construction, this projector gives rise to a split fibration sequence

$$K \longrightarrow A^{(I)} \longrightarrow C,$$

and the morphism r' induces a well defined morphism in $\mathrm{Ho}(A - Mod_s)$

$$r' : C \longrightarrow M.$$

By construction, this morphism induces an isomorphisms on π_0, and as C and M are strong modules, r' is an isomorphism in $\mathrm{Ho}(A - Mod_s)$.

(2) Let M be a strong A-module with $\pi_0(M)$ flat over $\pi_0(A)$, and N be any A-module. Clearly, $\pi_*(M)$ is flat as a $\pi_*(A)$-module. Therefore, the Tor spectral sequence of [**Q2**]

$$Tor^*_{\pi_*(A)}(\pi_*(M), \pi_*(N)) \Rightarrow \pi_*(M \otimes^{\mathbb{L}}_A N)$$

degenerates and gives a natural isomorphism

$$\pi_*(M \otimes^{\mathbb{L}}_A N) \simeq \pi_*(M) \otimes_{\pi_*(A)} \pi_*(N) \simeq$$

$$\simeq (\pi_0(M) \otimes_{\pi_0(A)} \pi_*(A)) \otimes_{\pi_*(A)} \pi_*(N) \simeq \pi_0(M) \otimes_{\pi_0(A)} \pi_*(N).$$

As $\pi_0(M) \otimes_{\pi_0(A)} -$ is an exact functor its transform long exact sequences into long exact sequences. This easily implies that $M \otimes^{\mathbb{L}}_A -$ preserves homotopy fiber sequences, and therefore that M is a flat A-module.

Conversely, suppose that M is a flat A-module. Any short exact sequence $0 \to N \to P$ of $\pi_0(A)$-modules can also be seen as a homotopy fiber sequence of A-modules, as any morphism $N \to P$ is always a fibration. Therefore, we obtain a homotopy fiber sequence

$$0 \longrightarrow M \otimes^{\mathbb{L}}_A N \longrightarrow M \otimes^{\mathbb{L}}_A P$$

which on π_0 gives a short exact sequence

$$0 \longrightarrow \pi_0(M) \otimes_{\pi_0(A)} N \longrightarrow \pi_0(M) \otimes_{\pi_0(A)} P.$$

This shows that $\pi_0(M) \otimes_{\pi_0(A)} -$ is an exact functor, and therefore that $\pi_0(M)$ is a flat $\pi_0(A)$-module. Furthermore, taking $N = 0$ we get that for any $\pi_0(A)$-module P one has $\pi_i(M \otimes^{\mathbb{L}}_A P) = 0$ for any $i > 0$. In other words, we have an isomorphism in $\mathrm{Ho}(A - Mod_s)$

$$M \otimes^{\mathbb{L}}_A P \simeq \pi_0(M) \otimes_{\pi_0(A)} P.$$

By shifting P we obtain that for any $i \geq 0$ and any $\pi_0(A)$-module P we have

$$M \otimes^{\mathbb{L}}_A (P[i]) \simeq (\pi_0(M) \otimes_{\pi_0(A)} P)[i].$$

Passing to Postnikov towers we see that this implies that for any A-module P we have

$$\pi_i(M \otimes^{\mathbb{L}}_A P) \simeq \pi_0(M) \otimes_{\pi_0(A)} \pi_i(P).$$

Applying this to $P = A$ we find that M is a strong A-module.

(3) Let us first suppose that M is strong with $\pi_0(M)$ projective of finite type over $\pi_0(A)$. By point (1) we know that M is a projective A-module. Moreover, the proof of (1) also shows that M is a retract in $\mathrm{Ho}(A - Mod_s)$ of some A^I with I finite. By our general result Prop. 1.2.4.2 this implies that A is perfect. Conversely, let M be a perfect A-module. By (2) and Prop. 1.2.4.2 we know that M is strong and that $\pi_0(M)$ is flat over $\pi_0(A)$. Furthermore, the unit k of $sk - Mod$ is finitely presented, so by Prop. 1.2.4.2 M is finitely presented object in $A - Mod_s$. Using the left Quillen functor π_0 from $A - Mod_s$ to $\pi_0(A)$-modules, we see that this implies that $\pi_0(M)$ is a finitely presented $\pi_0(A)$-module, and therefore is projective of finite type.

(4) Follows from (1), (3) and Prop. 1.2.4.2. □

DEFINITION 2.2.2.3. (1) *A morphism $A \longrightarrow B$ in $sk - Alg$ is* strong *if the natural morphism*

$$\pi_*(A) \otimes_{\pi_0(A)} \pi_0(B) \longrightarrow \pi_*(B)$$

is an isomorphism (i.e. B is strong as an A-module).

(2) *A morphism $A \longrightarrow B$ in $sk - Alg$ is* strongly flat *(resp.* strongly smooth, *resp.* strongly étale, *resp.* a strong Zariski open immersion*) if it is strong and if the morphism of affine schemes*

$$Spec\, \pi_0(B) \longrightarrow Spec\, \pi_0(A)$$

is flat (resp. smooth, resp. étale, resp. a Zariski open immersion).

We start by a very useful criterion in order to recognize finitely presented morphisms. We have learned this proposition from J. Lurie (see [**Lu1**]).

PROPOSITION 2.2.2.4. *Let $f : A \longrightarrow B$ be a morphism in $sk - Alg$. Then, f is finitely presented if and only if it satisfies the following two conditions.*

(1) *The morphism $\pi_0(A) \longrightarrow \pi_0(B)$ is a finitely presented morphism of commutative rings.*
(2) *The cotangent complex $\mathbb{L}_{B/A} \in \mathrm{Ho}(B - Mod_s)$ is finitely presented.*

PROOF. Let us assume first that f is finitely presented. Then (1) and (2) are easily seen to be true by fact that π_0 is left adjoint and by definition of derivations.

Let us now assume that $f : A \longrightarrow B$ is a morphism in $sk - Alg$ such that (1) and (2) are satisfied. Let $k \geq 0$ be an integer, and let $P(k)$ be the following property: for any filtered diagram C_i in $A - Alg_s$, such that $\pi_j(C_i) = 0$ for all $j > k$, the natural morphism

$$Hocolim_i Map_{A-Alg_s}(B, C_i) \longrightarrow Map_{A-Alg_s}(B, Hocolim_i C_i)$$

is an isomorphism in $\mathrm{Ho}(SSet)$.

We start to prove by induction on k that $P(k)$ holds for all k. For $k = 0$ this is hypothesis (1). Suppose $P(k-1)$ holds, and let C_i be a any filtered diagram in $A - Alg_s$, such that $\pi_j(C_i) = 0$ for all $j > k$. We consider $C = Hocolim_i C_i$, as well as the k-th Postnikov towers

$$C \longrightarrow C_{\leq k-1} \qquad (C_i) \longrightarrow (C_i)_{\leq k-1}.$$

There is a commutative square of simplicial sets

$$\begin{array}{ccc} Hocolim_i Map_{A-Alg_s}(B, C_i) & \longrightarrow & Hocolim_i Map_{A-Alg_s}(B, (C_i)_{k-1}) \\ \downarrow & & \downarrow \\ Map_{A-Alg_s}(B, C) & \longrightarrow & Map_{A-Alg_s}(B, C_{k-1}). \end{array}$$

By induction, the morphism on the right is an equivalence. Furthermore, using Prop. 1.4.2.6, Lem. 2.2.1.1 and the fact that the cotangent complex $\mathbb{L}_{B/A}$ is finitely presented, we see that the morphism induced on the homotopy fibers of the horizontal morphisms is also an equivalence. By the five lemma this implies that the morphism

$$Hocolim_i Map_{A-Alg_s}(B, C_i) \longrightarrow Map_{A-Alg_s}(B, C)$$

is an equivalence. This shows that $P(k)$ is satisfied.

As $\mathbb{L}_{B/A}$ is finitely presented, it is a retract of a finite cell B-module (see Prop. 1.2.3.5). In particular, there is an integer $k_0 > 0$, such that $[\mathbb{L}_{B/A}, M]_{B-Mod_s} = 0$ for any B-module M such that $\pi_i(M) = 0$ for all $i < k_0$ (one can chose k_0 strictly bigger than the dimension of the cells of a module of which $\mathbb{L}_{B/A}$ is a retract). Once again, Prop. 1.4.2.6 and Lem. 2.2.1.1 implies that for any commutative A-algebra C, the natural projection $C \longrightarrow C_{\leq k_0}$ induces a bijection

$$[B, C]_{A-Alg_s} \simeq [B, C_{\leq k_0}]_{A-Alg_s}.$$

Therefore, as the property $P(k_0)$ holds, we find that for any filtered system C_i in $A - Alg_s$, the natural morphism

$$Hocolim_i Map_{A-Alg_s}(B, C_i) \longrightarrow Map_{A-Alg_s}(B, Hocolim_i C_i)$$

induces an isomorphism in π_0. As this is valid for any filtered system, this shows that the morphism

$$Hocolim_i Map_{A-Alg_s}(B, C_i) \longrightarrow Map_{A-Alg_s}(B, Hocolim_i C_i)$$

induces an isomorphism on all the π_i's, and therefore is an equivalence. This shows that f is finitely presented. □

PROPOSITION 2.2.2.5. *Let $f : A \longrightarrow B$ be a morphism in $sk - Alg$.*

(1) *The morphism f is smooth if and only if it is perfect. The morphism f is formally smooth if it is formally perfect. The morphism f is formally i-smooth if and only if it is formally smooth.*
(2) *The morphism f is (formally) unramified if and only if it is (formally) étale.*

(3) *The morphism f is (formally) étale if and only if it is (formally) thh-étale.*
(4) *A morphism $A \longrightarrow B$ in $sk-Alg$ is* flat *(resp.* a Zariski open immersion*) if and only if it is* strongly flat *(resp.* a strong Zariski open immersion*).*

PROOF. (1) The first two assumptions follow from Lem. 2.2.2.2 (4). For the comparison between formally i-smooth and formally smooth morphism we notice that a B-module $P \in B-Mod$ is projective if and only if $[P, M[1]] = 0$ for any $M \in B-Mod$. This and Prop. 1.2.8.3 imply the statement (note that in our present context $\mathcal{A} = sk-Alg$ and $\mathcal{C}_0 = \mathcal{C}$).

(2) Follows from the fact that the suspension functor of $sk-Mod$ is fully faithful and from Prop. 1.2.6.5 (1).

(3) By Prop. 1.2.6.5 (2) (formally) thh-étale morphisms are (formally) étale. Conversely, we need to prove that a formally étale morphism $A \longrightarrow B$ is thh-étale. For this, we use the well known spectral sequence

$$\pi_*(Sym^*(\mathbb{L}_{B/A}[1])) \Rightarrow \pi_*(THH(B/A))$$

of [**Q2**, 8]. Using this spectral sequence we see that $B \simeq THH(B/A)$ if and only if $\mathbb{L}_{B/A} \simeq 0$. In particular a formally étale morphism is always formally thh-étale.

(4) For flat morphism this is Lem. 2.2.2.2 (2). Let $f : A \longrightarrow B$ be a Zariski open immersion. By definition, f is a flat morphism, and therefore is strongly flat by what we have seen. Moreover, for any commutative k-algebra C, considered as an object $C \in sk-Alg$ concentrated in degree 0, we have natural isomorphisms in $\mathrm{Ho}(SSet)$

$$Map_{sk-Alg}(A, C) \simeq Hom_{k-Alg}(\pi_0(A), C)$$

$$Map_{sk-Alg}(B, C) \simeq Hom_{k-Alg}(\pi_0(B), C).$$

In particular, as f is a epimorphism in $sk-Alg$ the induced morphism of affine schemes $\varphi : Spec\,\pi_0(B) \longrightarrow Spec\,\pi_0(A)$ is a monomorphism of schemes. This last morphism is therefore a flat monomorphism of affine schemes. Moreover, as f is finitely presented so is φ, which is therefore a finitely presented flat monomorphism. By [**EGAIV**, 2.4.6], a finitely presented flat morphism is open, so φ is an open flat monomorphism and thus a Zariski open immersion. This implies that f is a strong Zariski open immersion.

Conversely, let $f : A \longrightarrow B$ be a strong Zariski open immersion. By (1) it is a flat morphism. It only remains to show that it is also an epimorphism in $sk-Alg$ and that it is finitely presented. For the first of these properties, we use the Tor spectral sequence to see that the natural morphism $B \longrightarrow B \otimes^{\mathbb{L}}_A B$ induces an isomorphism on homotopy groups

$$\pi_*(B \otimes^{\mathbb{L}}_A B) \simeq \pi_*(B) \otimes_{\pi_*(A)} \pi_*(B) \simeq (\pi_0(B) \otimes_{\pi_0(A)} \pi_0(B)) \otimes_{\pi_0(A)} \pi_*(A) \simeq \pi_*(B).$$

In other words, the natural morphism $B \otimes^{\mathbb{L}}_A B \longrightarrow B$ is an isomorphism in $\mathrm{Ho}(sk-Alg)$, which implies that f is an epinomorphism. It remain to be shown that f is furthermore finitely presented, but this follows from Prop. 2.2.2.4 and Prop. 1.2.6.5. □

THEOREM 2.2.2.6. (1) *A morphism in $sk-Alg$ is étale if and only if it is strongly étale.*
(2) *A morphism in $sk-Alg$ is smooth if and only if it is strongly smooth.*

PROOF. We start by two fundamental lemmas.

Recall from §1.2.1 the Quillen adjunction

$$Q : A - Comm^{nu}(\mathcal{C}) \longrightarrow A - Mod \qquad A - Comm^{nu}(\mathcal{C}) \longleftarrow A - Mod : Z.$$

LEMMA 2.2.2.7. *Let $A \in sk - Alg$, and $M \in A - Mod_s$ be a A-module such that $\pi_i(M) = 0$ for all $i < k$, for some fixed integer k. Then, the adjunction morphism*

$$\mathbb{L}QZ(M) \longrightarrow M$$

induces isomorphisms

$$\pi_i(\mathbb{L}QZ(M)) \simeq \pi_i(M)$$

for all $i \leq k$. In particular

$$\pi_i(\mathbb{L}QZ(M)) \simeq 0 \; for \; i < k \qquad \pi_k(\mathbb{L}QZ(M)) \simeq \pi_k(M).$$

PROOF. The non-unital A-algebra $Z(M)$ being $(k-1)$-connected, we can write it, up to an equivalence, as a CW object in $A - Comm^{nu}(sk - Mod)$ with cells of dimension at least k (in the sense of [**EKMM**]). In other words $Z(M)$ is equivalent to a filtered colimit

$$\cdots \longrightarrow A_i \longrightarrow A_{i+1} \longrightarrow \cdots$$

where at each step there is a push-out diagram in $A - Comm^{nu}(sk - Mod)$

$$\begin{array}{ccc} A_i & \longrightarrow & A_{i+1} \\ \uparrow & & \uparrow \\ \coprod k[S^{n_i}]^{nu} & \longrightarrow & \coprod k[\Delta^{n_i+1}]^{nu}, \end{array}$$

where $n_{i+1} > n_i \geq k-1$, and $k[K]^{nu}$ is the free non-unital commutative simplicial k-algebra generated by a simplicial set K. Therefore, Q being left Quillen, the A-module $\mathbb{L}QZ(M)$ is the homotopy colimit of

$$\cdots \longrightarrow Q(A_i) \longrightarrow Q(A_{i+1}) \longrightarrow \cdots$$

where at each step there exists a homotopy push-out diagram in $A - Mod_s$

$$\begin{array}{ccc} Q(A_i) & \longrightarrow & Q(A_{i+1}) \\ \uparrow & & \uparrow \\ \oplus Q(k[S^{n_i}]^{nu}) & \longrightarrow & 0. \end{array}$$

Computing homotopy groups using long exact sequences, we see that the statement of the lemma can be reduced to prove that for a free non-unital commutative simplicial k-algebra $k[S^n]^{nu}$, we have

$$\pi_i(Q(k[S^n]^{nu})) \simeq 0 \; for \; i < n \qquad \pi_n(Q(k[S^n]^{nu})) \simeq k.$$

But, using that Q is left Quillen we find

$$Q(k[S^n]^{nu}) \simeq S^n \otimes k,$$

which implies the result. □

LEMMA 2.2.2.8. *Let A be any object in $sk - Alg$ and let us consider the k-th stage of its Postnikov tower*

$$A_{\leq k} \longrightarrow A_{\leq k-1}.$$

There exist natural isomorphisms

$$\pi_{k+1}(\mathbb{L}_{A_{\leq k-1}/A_{\leq k}}) \simeq \pi_k(A)$$

$$\pi_i(\mathbb{L}_{A_{\leq k-1}/A_{\leq k}}) \simeq 0 \; for \; i \leq k.$$

PROOF. This follows from lemma 2.2.1.1, lemma 1.4.3.7, and lemma 2.2.2.7. □

Let us now start the proof of theorem 2.2.2.6.

(1) Let $f : A \longrightarrow B$ be a strongly étale morphism. By definition of strongly étale the morphism f is strongly flat. In particular, the square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ \pi_0(A) & \longrightarrow & \pi_0(B) \end{array}$$

is homotopy cocartesian in $sk - Alg$. Therefore, Prop. 1.2.1.6 (2) implies

$$\mathbb{L}_{B/A} \otimes_B \pi_0(B) \simeq \mathbb{L}_{\pi_0(B)/\pi_0(A)} \simeq 0.$$

Using the Tor spectral sequence

$$Tor^*_{\pi_*(B)}(\pi_0(B), \pi_*(\mathbb{L}_{B/A})) \Rightarrow \pi_*(\mathbb{L}_{B/A} \otimes_B \pi_0(B)) = 0$$

we find by induction on k that $\pi_k(\mathbb{L}_{B/A}) = 0$. This shows that the morphism f is formally étale. The fact that it is also finitely presented follows then from Prop. 2.2.2.4.

Conversely, let $f : A \longrightarrow B$ be an étale morphism. As $\pi_0 : sk - Alg \longrightarrow k - Alg$ is left Quillen, we deduce immediately that $\pi_0(A) \longrightarrow \pi_0(B)$ is an étale morphism of commutative rings. We will prove by induction that all the truncations

$$f_k : A_{\leq k} \longrightarrow B_{\leq k}$$

are strongly étale and thus étale as well by what we have seen before. We thus assume that f_{k-1} is strongly étale (and thus also étale by what we have seen in the first part of the proof). By adjunction, the truncations

$$f_k : A_{\leq k} \longrightarrow B_{\leq k}$$

are such that

$$(\mathbb{L}_{B_{\leq k}/A_{\leq k}})_{\leq k} \simeq (\mathbb{L}_{B/A})_{\leq k} \simeq 0.$$

Furthermore, let M be any $\pi_0(B)$-module, and d be any morphism in $\mathrm{Ho}(B_{\leq k} - Mod_s)$

$$d : \mathbb{L}_{B_{\leq k}/A_{\leq k}} \longrightarrow M[k+1].$$

We consider the commutative diagram in $\mathrm{Ho}(A_{\leq k}/sk - Alg/B_{\leq k})$

$$\begin{array}{ccc} A_{\leq k} & \longrightarrow & B_{\leq k} \oplus_d M[k] \\ \downarrow & & \downarrow \\ B_{\leq k} & \longrightarrow & B_{\leq k}. \end{array}$$

The obstruction of the existence of a morphism

$$u : B_{\leq k} \longrightarrow B_{\leq k} \oplus_d M[k]$$

in $\mathrm{Ho}(A_{\leq k}/sk - Alg/B_{\leq k})$ is precisely $d \in [\mathbb{L}_{B_{\leq k}/A_{\leq k}}, M[k+1]]$. On the other hand, by adjunction, the existence of such a morphism f is equivalent to the existence of a morphism

$$u' : B \longrightarrow B_{\leq k} \oplus_d M[k]$$

in $\mathrm{Ho}(A/sk{-}Alg/B_{\leq k})$. The obstruction of the existence of u' itself is $d' \in [\mathbb{L}_{B/A}, M[k+1]]$, the image of d by the natural morphism, which vanishes as $A \to B$ is formally étale. This implies that u' and thus u exists, and therefore that for any $\pi_0(B)$-module M we have

$$[\mathbb{L}_{B_{\leq k}/A_{\leq k}}, M[k+1]] = 0.$$

As the object $\mathbb{L}_{B_{\leq k}/A_{\leq k}}$ is already known to be k-connected, we conclude that it is furthermore is $(k+1)$-connected.

Now, there exists a morphism of homotopy cofiber sequences in $sk-Mod$

$$\begin{array}{ccccc} \mathbb{L}_{A_{\leq k}} \otimes^{\mathbb{L}}_{A_{\leq k}} A_{\leq k-1} & \longrightarrow & \mathbb{L}_{A_{\leq k-1}} & \longrightarrow & \mathbb{L}_{A_{\leq k-1}/A_{\leq k}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{L}_{B_{\leq k}} \otimes^{\mathbb{L}}_{B_{\leq k}} B_{\leq k-1} & \longrightarrow & \mathbb{L}_{B_{\leq k-1}} & \longrightarrow & \mathbb{L}_{B_{\leq k-1}/B_{\leq k}}. \end{array}$$

Base changing the first row by $A_{\leq k-1} \longrightarrow B_{\leq k-1}$ gives another morphism of homotopy cofiber sequences in $sk-Mod$

$$\begin{array}{ccccc} \mathbb{L}_{A_{\leq k}} \otimes^{\mathbb{L}}_{A_{\leq k}} B_{\leq k-1} & \longrightarrow & \mathbb{L}_{A_{\leq k-1}} \otimes^{\mathbb{L}}_{A_{\leq k-1}} B_{\leq k-1} & \longrightarrow & \mathbb{L}_{A_{\leq k-1}/A_{\leq k}} \otimes^{\mathbb{L}}_{A_{\leq k-1}} B_{\leq k-1} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{L}_{B_{\leq k}} \otimes^{\mathbb{L}}_{B_{\leq k}} B_{\leq k-1} & \longrightarrow & \mathbb{L}_{B_{\leq k-1}} & \longrightarrow & \mathbb{L}_{B_{\leq k-1}/B_{\leq k}}. \end{array}$$

Passing to the long exact sequences in homotopy, and using that f_{k-1} is étale, as well as the fact that $\mathbb{L}_{B_{\leq k}/A_{\leq k}} \otimes^{\mathbb{L}}_{A_{\leq k-1}} B_{\leq k-1}$ is $(k+1)$-connected, it is easy to see that the vertical morphism on the right induces an isomorphism

$$\pi_{k+1}(\mathbb{L}_{A_{\leq k-1}/A_{\leq k}} \otimes^{\mathbb{L}}_{A_{\leq k-1}} B_{\leq k-1}) \simeq \pi_{k+1}(\mathbb{L}_{B_{\leq k-1}/B_{\leq k}}).$$

Therefore, Lemma 2.2.2.8 implies that the natural morphism

$$\pi_k(A) \otimes_{\pi_0(A)} \pi_0(B) \longrightarrow \pi_k(B)$$

is an isomorphism. By induction on k this shows that f is strongly étale.

(2) Let $f : A \longrightarrow B$ be a strongly smooth morphism. As the morphism f is strongly flat, there is a homotopy push-out square in $sk-Alg$

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ \pi_0(A) & \longrightarrow & \pi_0(B). \end{array}$$

Together with Prop. 1.2.1.6 (2), we thus have a natural isomorphism

$$\mathbb{L}_{B/A} \otimes^{\mathbb{L}}_B \pi_0(B) \simeq \mathbb{L}_{\pi_0(B)/\pi_0(A)}.$$

As the morphism $\pi_0(A) \longrightarrow \pi_0(B)$ is smooth, we have $\mathbb{L}_{\pi_0(B)/\pi_0(A)} \simeq \Omega^1_{\pi_0(B)/\pi_0(A)}[0]$. Using that $\Omega^1_{\pi_0(B)/\pi_0(A)}$ is a projective $\pi_0(B)$-module, we see that the isomorphism

$$\Omega^1_{\pi_0(B)/\pi_0(A)}[0] \simeq \mathbb{L}_{B/A} \otimes^{\mathbb{L}}_B \pi_0(B)$$

can be lifted to a morphism

$$P \longrightarrow \mathbb{L}_{B/A}$$

in $\mathrm{Ho}(B-Mod_s)$, where P is a projective B-module such that $P \otimes^{\mathbb{L}}_B \pi_0(B) \simeq \Omega^1_{\pi_0(B)/\pi_0(A)}[0]$. We let K be the homotopy cofiber of this last morphism. By construction, we have

$$K \otimes^{\mathbb{L}}_B \pi_0(B) \simeq 0,$$

which easily implies by induction on k that $\pi_k(K)=0$. Therefore, the morphism

$$P \longrightarrow \mathbb{L}_{B/A}$$

is in fact an isomorphism in $\mathrm{Ho}(B-Mod_s)$, showing that $\mathbb{L}_{B/A}$ is a projective B-module. Moreover, the homotopy cofiber sequence in $B-Mod_s$

$$\mathbb{L}_A \otimes^{\mathbb{L}}_A B \longrightarrow \mathbb{L}_B \longrightarrow \mathbb{L}_{B/A}$$

gives rise to a homotopy cofiber sequence

$$\mathbb{L}_B \longrightarrow \mathbb{L}_{B/A} \longrightarrow S(\mathbb{L}_A \otimes^{\mathbb{L}}_A B).$$

But, $\mathbb{L}_{B/A}$ being a retract of a free B-module, we see that $[\mathbb{L}_{B/A}, S(\mathbb{L}_A \otimes^{\mathbb{L}}_A B)]$ is a retract of a product of $\pi_0(S(\mathbb{L}_A \otimes^{\mathbb{L}}_A B))=0$ and thus is trivial. This implies that the morphism $\mathbb{L}_{B/A} \longrightarrow S(\mathbb{L}_A \otimes^{\mathbb{L}}_A B)$ is trivial in $\mathrm{Ho}(B-Mod_s)$, and therefore that the homotopy cofiber sequence

$$\mathbb{L}_A \otimes^{\mathbb{L}}_A B \longrightarrow \mathbb{L}_B \longrightarrow \mathbb{L}_{B/A},$$

which is also a homotopy fiber sequence, splits. In particular, the morphism $\mathbb{L}_A \otimes^{\mathbb{L}}_A B \longrightarrow \mathbb{L}_B$ has a retraction. We have thus shown that f is a formally smooth morphism. The fact that f is furthermore finitely presented follows from Prop. 2.2.2.4 and the fact that $\mathbb{L}_{B/A}$ if finitely presented because $\Omega^1_{B/A}$ is so (see Lem. 2.2.2.2).

Conversely, let $f : A \longrightarrow B$ be a smooth morphism in $sk-Alg$ and let us prove it is strongly smooth. First of all, using that $\pi_0 : sk-Alg \longrightarrow k-Alg$ is left Quillen, we see that $\pi_0(A) \longrightarrow \pi_0(B)$ has the required lifting property for being a formally smooth morphism. Furthermore, it is a finitely presented morphism, so is a smooth morphism of commutative rings. We then form the homotopy push-out square in $sk-Alg$

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ \pi_0(A) & \longrightarrow & C. \end{array}$$

By base change, $\pi_0(A) \longrightarrow C$ is a smooth morphism. We will start to prove that the natural morphism $C \longrightarrow \pi_0(C) \simeq \pi_0(B)$ is an isomorphism. Suppose it is not, and let i be the smallest integer $i>0$ such that $\pi_i(C)\neq 0$. Considering the homotopy cofiber sequence

$$\mathbb{L}_{C_{\leq i}/\pi_0(A)} \otimes^{\mathbb{L}}_{C_{\leq i}} \pi_0(C) \longrightarrow \mathbb{L}_{\pi_0(C)/\pi_0(A)} \simeq \Omega^1_{\pi_0(C)/\pi_0(A)}[0] \longrightarrow \mathbb{L}_{C_{\leq i}/\pi_0(C)},$$

and using lemma 2.2.2.8, we see that

$$\pi_i(\mathbb{L}_{C/\pi_0(A)} \otimes^{\mathbb{L}}_C \pi_0(C)) \simeq \pi_i(\mathbb{L}_{C_{\leq i}/\pi_0(A)} \otimes^{\mathbb{L}}_{C_{\leq i}} \pi_0(C)) \simeq \pi_{i+1}(\mathbb{L}_{C_{\leq i}/\pi_0(C)}) \simeq \pi_i(C) \neq 0.$$

But this contradicts the fact that $\mathbb{L}_{C/\pi_0(A)} \otimes^{\mathbb{L}}_C \pi_0(C)$ is a projective (and thus strong by lemma 2.2.2.2 (1)) $\pi_0(C)$-module, and thus the fact that $\pi_0(A) \longrightarrow C$ is a smooth morphism. We therefore have a homotopy push-out diagram in $sk-Alg$

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ \pi_0(A) & \longrightarrow & \pi_0(B). \end{array}$$

Using that the bottom horizontal morphism in flat and the Tor spectral sequence, we get by induction that the natural morphism

$$\pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \longrightarrow \pi_i(B)$$

is an isomorphism. We thus have seen that f is a strongly smooth morphism. □

An important corollary of theorem 2.2.2.6 is the following topological invariance of étale morphisms.

COROLLARY 2.2.2.9. *Let $A \in sk - Alg$ and $t_0(X) = Spec\,(\pi_0 A) \longrightarrow X = Spec\,A$ be the natural morphism. Then, the base change functor*

$$\mathrm{Ho}(k - D^- Aff/X) \longrightarrow \mathrm{Ho}(k - D^- Aff/t_0(X))$$

induces an equivalence from the full subcategory of étale morphism $Y \to X$ to the full subcategory of étale morphisms $Y' \to t_0(X)$. Furthermore, this equivalence preserves epimorphisms of stacks.

PROOF. We consider the Postnikov tower

$$A \longrightarrow \cdots \longrightarrow A_{\leq k} \longrightarrow A_{\leq k-1} \longrightarrow \cdots \longrightarrow A_{\leq 0} = \pi_0(A),$$

and the associated diagram in $k - D^- Aff$

$$X_{\leq 0} = t_0(X) \longrightarrow X_{\leq 1} \longrightarrow \cdots \longrightarrow X_{\leq k-1} \longrightarrow X_{\leq k} \longrightarrow \cdots \longrightarrow X.$$

We define a model category $k - D^- Aff/X_{\leq *}$, whose objects are families of objects $Y_k \longrightarrow X_{\leq k}$ in $k - D^- Aff/X_{\leq k}$, together with transitions morphisms $Y_{k-1} \longrightarrow Y_k \times_{X_{\leq k}} X_{\leq k-1}$ in $k - D^- Aff/X_{\leq k-1}$. The morphisms in $k - D^- Aff/X_{\leq *}$ are simply the families of morphisms $Y_k \longrightarrow Y'_k$ in $k - D^- Aff/X_{\leq k}$ that commute with the transition morphisms. The model structure on $k - D^- Aff/X_{\leq *}$ is such that a morphism $f : Y_* \longrightarrow Y'_*$ in $k - D^- Aff/X_{\leq *}$ is an equivalence (resp. a fibration) if all morphisms $Y_k \longrightarrow Y'_k$ are equivalences (resp. fibrations) in $k - D^- Aff$.

There exists a Quillen adjunction

$$G : k{-}D^- Aff/X_{\leq *} \longrightarrow k{-}D^- Aff/X \qquad k{-}D^- Aff/X_{\leq *} \longleftarrow k{-}D^- Aff/X : F,$$

defined by $G(Y_*)$ to be the colimit in $k - D^- Aff/X$ of the diagram

$$Y_0 \longrightarrow Y_1 \longrightarrow \ldots Y_k \longrightarrow \cdots \longrightarrow X.$$

The right adjoint F sends a morphism $Y \longrightarrow X$ to the various pullbacks

$$F(Y)_k := Y \times_X X_{\leq k} \longrightarrow X_k$$

together with the obvious transition isomorphisms. Using Thm. 2.2.2.6 (1) it is not hard to check that the derived adjunction

$$\mathbb{L}G : \mathrm{Ho}(k - D^- Aff/X_{\leq *}) \longrightarrow \mathrm{Ho}(k - D^- Aff/X)$$

$$\mathrm{Ho}(k - D^- Aff/X_{\leq *}) \longleftarrow \mathrm{Ho}(k - D^- Aff/X) : \mathbb{R}F,$$

induces an equivalence from the full subcategory of $\mathrm{Ho}(k - D^- Aff/X)$ consisting of étale morphisms $Y \longrightarrow X$, and the full subcategory of $\mathrm{Ho}(k{-}D^- Aff/X_{\leq *})$ consisting of objects Y_* such that each $Y_k \longrightarrow X_{\leq k}$ is étale and each transition morphism

$$Y_{k-1} \longrightarrow Y_k \times^h_{X_{\leq k}} X_{\leq k-1}$$

is an isomorphism in $\mathrm{Ho}(k - D^- Aff/X_{\leq k-1})$. Let us denote these two categories respectively by $\mathrm{Ho}(k - D^- Aff/X)^{et}$ and $\mathrm{Ho}(k - D^- Aff/X_{\leq *})^{cart,et}$. Using Cor. 1.4.3.12 and Lem. 2.2.1.1, we know that each base change functor

$$\mathrm{Ho}(k - D^- Aff/X_{\leq k}) \longrightarrow \mathrm{Ho}(k - D^- Aff/X_{\leq k-1})$$

induces an equivalence on the full sub-categories of étale morphisms. This easily implies that the natural projection functor

$$\mathrm{Ho}(k-D^{-}Aff/X_{\leq *})^{cart,et} \longrightarrow \mathrm{Ho}(k-D^{-}Aff/X_{\leq 0})^{et}$$

is an equivalence of categories. Therefore, by composition, we find that the base change functor

$$\mathrm{Ho}(k-D^{-}Aff/X)^{et} \longrightarrow \mathrm{Ho}(k-D^{-}Aff/X_{\leq *})^{cart,et} \longrightarrow \mathrm{Ho}(k-D^{-}Aff/X_{\leq 0})^{et}$$

is an equivalence of categories.

Finally, the statement concerning epimorphism of stacks is obvious, as a flat morphism $Y \longrightarrow X$ in $k-D^{-}Aff$ induces an epimorphism of stacks if and only if $t_0(Y) \longrightarrow t_0(X)$ is a surjective morphism of affine schemes. □

A direct specialization of 2.2.2.9 is the following.

COROLLARY 2.2.2.10. *Let $A \in sk-Alg$ and $t_0(X) = Spec\,(\pi_0 A) \longrightarrow X = Spec\,A$ be the natural morphism. Then, the base change functor*

$$\mathrm{Ho}(k-D^{-}Aff/X) \longrightarrow \mathrm{Ho}(k-D^{-}Aff/t_0(X))$$

induces an equivalence from full sub-categories of Zariski open immersions $Y \to X$ to the full subcategory of Zariski open immersions $Y' \to t_0(X)$. Furthermore, this equivalence preserves epimorphisms of stacks.

PROOF. Indeed, using 2.2.2.9 it is enough to see that an étale morphism $A \longrightarrow B$ is a Zariski open immersion if and only if $\pi_0(A) \longrightarrow \pi_0(B)$ is. But this true by Thm. 2.2.2.6 and Prop. 2.2.2.5. □

From the proof of Thm. 2.2.2.6 we also extract the following more precise result.

COROLLARY 2.2.2.11. *Let $f : A \longrightarrow B$ be a morphism in $sk-Alg$. The following are equivalent.*

1. *The morphism f is smooth (resp. étale).*
2. *The morphism f is flat and $\pi_0(A) \longrightarrow \pi_0(B)$ is smooth (resp. étale).*
3. *The morphism f is formally smooth (resp. formally étale) and $\pi_0(B)$ is a finitely presented $\pi_0(A)$-algebra.*

We are now ready to define the étale model topology (Def. 1.3.1.1) on $k-D^{-}Aff$.

DEFINITION 2.2.2.12. *A family of morphisms $\{Spec\,A_i \longrightarrow Spec\,A\}_{i\in I}$ in $k-D^{-}Aff$ is an* étale covering family *(or simply* ét-covering family*) if it satisfies the following two conditions.*

1. *Each morphism $A \longrightarrow A_i$ is étale.*
2. *There exists a finite sub-set $J \subset I$ such that the family $\{A \longrightarrow A_i\}_{i\in J}$ is a formal covering family in the sense of 1.2.5.1.*

Using that étale morphisms are precisely the strongly étale morphisms (see Corollary 2.2.2.11) we immediately deduce that a family of morphisms $\{Spec\,A_i \longrightarrow Spec\,A\}_{i\in I}$ in $k-D^{-}Aff$ is an ét-covering family if and only if there exists a finite sub-set $J \subset I$ satisfying the following two conditions.

- For all $i \in I$, the natural morphism

$$\pi_*(A) \otimes_{\pi_0(A)} \pi_0(A_i) \longrightarrow \pi_*(A_i)$$

is an isomorphism.

- The morphism of affine schemes

$$\coprod_{i \in J} Spec\, \pi_0(A_i) \longrightarrow Spec\, \pi_0(A)$$

is étale and surjective.

LEMMA 2.2.2.13. *The ét-covering families define a model topology (Def. 1.3.1.1) on $k - D^- Aff$, which satisfies assumption 1.3.2.2.*

PROOF. That ét-covering families defines a model topology simply follows from the general properties of étale morphisms and formal coverings described in propositions 1.2.5.2 and 1.2.6.3. It only remain to show that the étale topology satisfies assumption 1.3.2.2.

The étale topology is quasi-compact by definition, so (1) of 1.3.2.2 is satisfied. In the same way, property (2) of 1.3.2.2 is obviously satisfied according to the explicit definition of étale coverings given above. Finally, let us check property (3) of 1.3.2.2. For this, let $A \longrightarrow B_*$ be a co-simplicial object in $sk - Alg$, corresponding to a representable étale-hypercover in $k - D^- Aff$ in the sense of 1.3.2.2 (3). We consider the adjunction

$$B_* \otimes^{\mathbb{L}}_A - : \mathrm{Ho}(A - Mod_s) \longrightarrow \mathrm{Ho}(csB_* - Mod)$$

$$\mathrm{Ho}(A - Mod_s) \longleftarrow \mathrm{Ho}(csB_* - Mod) : \int$$

defined in §1.2. We restrict it to the full subcategory $\mathrm{Ho}(csB_* - Mod)^{cart}$ of $\mathrm{Ho}(csB_* - Mod)$ consisting of cartesian objects

$$B_* \otimes^{\mathbb{L}}_A - : \mathrm{Ho}(A - Mod_s) \longrightarrow \mathrm{Ho}(csB_* - Mod)^{cart}$$

$$\mathrm{Ho}(A - Mod_s) \longleftarrow \mathrm{Ho}(csB_* - Mod)^{cart} : \int$$

and we need to prove that this is an equivalence. By definition of formal coverings, the base change functor

$$B_* \otimes^{\mathbb{L}}_A - : \mathrm{Ho}(A - Mod_s) \longrightarrow \mathrm{Ho}(csB_* - Mod)^{cart}$$

is clearly conservative, so it only remains to show that the adjunction map

$$Id \longrightarrow \int \circ (B_* \otimes^{\mathbb{L}}_A -)$$

is an isomorphism.

For this, let $E_* \in \mathrm{Ho}(csB_* - Mod)^{cart}$, and let us consider the adjunction morphism

$$E_1 \longrightarrow (Holim_n E_n) \otimes^{\mathbb{L}}_A B_1.$$

We need to show that this morphism is an isomorphism in $\mathrm{Ho}(B_1 - Mod_s)$. For this, we first use that $A \longrightarrow B_1$ is a strongly flat morphism, and thus

$$\pi_*((Holim_n E_n) \otimes^{\mathbb{L}}_A B_1) \simeq \pi_*((Holim_n E_n)) \otimes_{\pi_0(A)} \pi_0(B_1).$$

We then apply the spectral sequence

$$H^p(Tot(\pi_q(E_*))) \Rightarrow \pi_{q-p}((Holim_n E_n).$$

The object $\pi_q(E_*)$ is now a co-simplicial module over the co-simplicial commutative ring $\pi_0(B_*)$, which is furthermore cartesian as all the coface morphisms $B_n \longrightarrow B_{n+1}$ are flat. The morphism $\pi_0(A) \longrightarrow \pi_0(B_*)$ being an étale, and thus faithfully flat, hypercover in the usual sense, we find by the usual flat descent for quasi-coherent sheaves that $H^p(Tot(\pi_q(E_*))) \simeq 0$ for $p \neq 0$. Therefore, the above spectral sequence degenerates and gives an isomorphism

$$\pi_p((Holim_n E_n) \simeq Ker\,(\pi_p(E_0) \rightrightarrows \pi_p(E_1)).$$

In other words, $\pi_p((Holim_n E_n))$ is the $\pi_0(A)$-module obtained by descent from $\pi_p(E_*)$ on $\pi_p(B_*)$. In particular, the natural morphism

$$\pi_p((Holim_n E_n) \otimes_{\pi_0(A)} \pi_0(B_1) \longrightarrow \pi_p(E_1)$$

is an isomorphism. Putting all of this together we find that

$$E_1 \longrightarrow (Holim_n E_n) \otimes^{\mathbb{L}}_A B_1$$

is an isomorphism in $\mathrm{Ho}(B_1 - Mod_s)$. □

We have now the model site (Def. 1.3.1.1) $(k - D^-Aff, \acute{e}t)$, with the étale model topology, and we make the following definition.

DEFINITION 2.2.2.14. (1) *A D^--stack is an object $F \in k - D^-Aff^{\sim,\acute{e}t}$ which is a stack in the sense of Def. 1.3.2.1.*
(2) *The model category of D^--stacks is $k - D^-Aff^{\sim,\acute{e}t}$, and its homotopy category will be simply denoted by $D^-\mathrm{St}(k)$.*

The following result is a corollary of Proposition 2.2.2.4. It states that the property of being finitely presented is local for the étale topology defined above.

COROLLARY 2.2.2.15. *Let $f : A \longrightarrow B$ be a morphism in $sk - Alg$.*
(1) *If there exists an étale covering $B \longrightarrow B'$ such that $A \longrightarrow B'$ is finitely presented, then f is finitely presented.*
(2) *If there exists an étale covering $A \longrightarrow A'$, such that*

$$A' \longrightarrow A' \otimes^{\mathbb{L}}_A B$$

is finitely presented, then f is finitely presented.

PROOF. This follows from proposition 2.2.2.4. Indeed, it suffices to prove that both conditions of proposition 2.2.2.4 have the required local property. The first one is well known, and the second one is a consequence of corollary 1.3.7.8, as finitely presented objects in $\mathrm{Ho}(Sp(B - Mod_s))$ are precisely the perfect objects. □

2.2.3. The HAG context: Geometric D^--stacks

We now let $\mathbf{P}$ be the class of smooth morphisms in $sk - Alg$.

LEMMA 2.2.3.1. *The class $\mathbf{P}$ of smooth morphisms and the étale model topology satisfy assumption 1.3.2.11.*

PROOF. As étale morphisms are also smooth we see that assumption 1.3.2.11 (1) is satisfied. Assumption 1.3.2.11 (2) is satisfied as smooth morphisms are stable by homotopy pullbacks, compositions and equivalences. Let us prove that smooth morphisms satisfy 1.3.2.11 (3) for the étale model topology.

Let $X \longrightarrow Y$ be a morphism in $k - D^-Aff$, and $\{U_i \longrightarrow X\}$ a finite étale covering family such that each $U_i \longrightarrow X$ and $U_i \longrightarrow Y$ is smooth. First of all, using that $\{U_i \longrightarrow X\}$ is a flat formal covering, and that each morphism $U_i \longrightarrow Y$ is flat, we see that the morphism $X \longrightarrow Y$ is flat. Using our results Prop. 2.2.2.5 and Thm. 2.2.2.6 it only remain to show that the morphism $t_0(X) \longrightarrow t_0(Y)$ is a smooth morphism between affine schemes. But, passing to t_0 we find a smooth covering family $\{t_0(U_i) \longrightarrow t_0(X)\}$, such that each morphism $t_0(U_i) \longrightarrow t_0(Y)$ is smooth, and we know (see e.g. [**EGAIV**]) that this implies that $t_0(X) \longrightarrow t_0(Y)$ is a smooth morphism of affine schemes.

Finally, property (4) of 1.3.2.11 is obvious. □

We have verified our assumptions 1.3.2.2 and 1.3.2.11 for the étale model topology and **P** the class of smooth morphisms. We thus have that

$$(sk-Mod, sk-Mod, sk-Alg, \text{ét}, \mathbf{P})$$

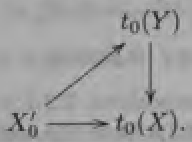
is a HAG context in the sense of Def. 1.3.2.13. We can therefore apply our general definitions to obtain a notion of n-geometric D^--stacks in $D^-\mathrm{St}(k)$, as well as the notion of n-smooth morphisms. We then check that Artin's conditions of Def. 1.4.3.1 are satisfied.

PROPOSITION 2.2.3.2. *The étale model topology and the smooth morphisms satisfy Artin's conditions relative to the HA context $(sk-Mod, sk-Mod, sk-Alg)$ in the sense of Def. 1.4.3.1.*

PROOF. We will show that the class **E** of étale morphisms satisfies conditions (1) to (5) of Def. 1.4.3.1.

(1) is clear as **P** is exactly the class of all i-smooth morphisms.
(2) and (3) are clear by the choice of **E** and $\mathcal{A}$.

To prove (4), let $p : Y \longrightarrow X$ be a smooth morphism in $k-D^-Aff$, and let us consider the smooth morphism of affine schemes $t_0(p) : t_0(Y) \longrightarrow t_0(X)$. As p is a smooth epimorphism of stacks, $t_0(p)$ is a smooth and surjective morphism of affine schemes. It is known (see e.g. [**EGAIV**]) that there exists a étale covering of affine schemes $X_0' \longrightarrow t_0(X)$ and a commutative diagram



By Cor. 2.2.2.9, we know that there exists an étale covering $X' \longrightarrow X$ in $k-D^-Aff$, inducing $X_0' \longrightarrow t_0(X)$. Taking the homotopy pullback

$$Y \times^h_X X' \longrightarrow X'$$

we can replace X by X', and therefore assume that the morphism $t_0(p) : t_0(Y) \longrightarrow t_0(X)$ has a section. We are going to show that this section can be extended to a section of p, which will be enough to prove what we want.

We use the same trick as in the proof of Cor. 2.2.2.9, and consider once again the model category $k-D^-Aff/X_{\leq *}$. Using Thm. 2.2.2.6 (2) we see that the functor

$$\mathrm{Ho}(k-D^-Aff/X) \longrightarrow \mathrm{Ho}(k-D^-Aff/X_{\leq *})$$

induces an equivalences from the full subcategory $\mathrm{Ho}(k-D^-Aff/X)^{sm}$ of smooth morphisms $Z \to X$ to the full subcategory $\mathrm{Ho}(k-D^-Aff/X_{\leq *})^{cart,sm}$ consisting of objects Z_* such that each $Z_k \longrightarrow X_{\leq k}$ is smooth, and each morphism

$$Z_{k-1} \longrightarrow Z_k \times^h_{Z_{\leq k}} Z_{\leq k-1}$$

is an isomorphism in $\mathrm{Ho}(k-D^-Aff)$. From this we deduce that the space of sections of p can be described in the following way

$$Map_{k-D^-Aff/X}(X,Y) \simeq Holim_k Map_{k-D^-Aff/X_{\leq k}}(X_{\leq k}, Y_k).$$

where $Y_k \simeq Y_{\leq k} \simeq Y \times^h_X X_{\leq k}$. But, as there always exists a surjection

$$\pi_0(Holim_k Map_{k-D^-Aff/X_{\leq k}}(X_{\leq k}, Y_k)) \longrightarrow Lim_k \pi_0(Map_{k-D^-Aff/X_{\leq k}}(X_{\leq k}, Y_k)),$$

we only need to check that $Lim_k \pi_0(Map_{k-D^-Aff/X_{\leq k}}(X_{\leq k}, Y_k)) \neq \emptyset$. For this, it is enough to prove that for any $k \geq 1$, the restriction map

$$\pi_0(Map_{k-D^-Aff/X_{\leq k}}(X_{\leq k}, Y_k)) \longrightarrow \pi_0(Map_{k-D^-Aff/X_{\leq k-1}}(X_{\leq k}, Y_{k-1}))$$

is surjective. In other words, we need to prove that a morphism in $\mathrm{Ho}(k-D^-Aff/X_{\leq k})$

$$\begin{array}{ccc} X_{\leq k-1} & \longrightarrow & Y_k \\ \downarrow & & \\ X_{\leq k} & & \end{array}$$

can be filled up to a commutative diagram in $\mathrm{Ho}(k-D^-Aff/X_{\leq k})$

$$\begin{array}{ccc} X_{\leq k-1} & \longrightarrow & Y_k \\ \downarrow & \nearrow & \\ X_{\leq k}. & & \end{array}$$

Using Prop. 1.4.2.6 and Lem. 2.2.1.1, we see that the obstruction for the existence of this lift lives in the group

$$[\mathbb{L}_{B_k/A_{\leq k}}, \pi_k(A)[k+1]]_{A_{\leq k}-Mod_s},$$

where $Y_k = Spec\, B_k$ and $X = Spec\, A$. But, as $Y_k \longrightarrow X_{\leq k}$ is a smooth morphism, the $A_{\leq k}$-module $\mathbb{L}_{B_k/A_{\leq k}}$ is projective, and therefore

$$[\mathbb{L}_{B_k/A_{\leq k}}, \pi_k(A)[k+1]]_{A_{\leq k}-Mod_s} \simeq 0,$$

by Lem. 2.2.2.2 (5).

It remains to prove that property (5) of Def. 1.4.3.1 is satisfied. Let $X \longrightarrow X_d[\Omega M]$ be as in the statement of Def. 1.4.3.1 (4), with $X = Spec\, A$ and some connected A-module $M \in A-Mod_s$, and let $U \longrightarrow X_d[\Omega M]$ be an étale morphism. We know by Lem. 1.4.3.8 (or rather its proof) that this morphism is of the form

$$X'_{d'}[\Omega M'] \longrightarrow X_d[\Omega M]$$

for some étale morphism $X' = Spec\, A' \longrightarrow X = Spec\, A$, and furthermore $X' \longrightarrow X$ is equivalent to the homotopy pullback

$$U \times_{X_d[\Omega M]} X \longrightarrow X.$$

Therefore, using Cor. 2.2.2.11, it is enough to prove that

$$t_0(X'_{d'}[\Omega M']) \longrightarrow t_0(X_d[\Omega M])$$

is a surjective morphism of affine schemes if and only if

$$t_0(X') \longrightarrow t_0(X)$$

is so. But, M being connected, $t_0(X_d[\Omega M])_{red} \simeq t_0(X)$, and in the same way $t_0(X'_{d'}[\Omega M'])_{red} \simeq t_0(X')$, this gives the result, since being a surjective morphism is topologically invariant. □

COROLLARY 2.2.3.3. (1) *Any n-geometric D^--stack has an obstruction theory. In particular, any n-geometric D^--stack has a cotangent complex.*

(2) *Any n-representable morphism of D^--stacks $f : F \longrightarrow G$ has a relative obstruction theory. In particular, any n-representable morphism of D^--stacks has a relative cotangent complex.*

PROOF. Follows from Prop. 2.2.3.2 and theorem 1.4.3.2. □

We finish by some properties of morphisms, as in Def. 1.3.6.2.

LEMMA 2.2.3.4. *Let $\mathbf{Q}$ be one the following class of morphisms in $k - D^-Aff$.*

(1) *Flat.*
(2) *Smooth.*
(3) *Etale.*
(4) *Finitely presented.*

Then, morphisms in $\mathbf{Q}$ are compatible with the étale topology and the class $\mathbf{P}$ of smooth morphisms in the sense of Def. 1.3.6.1.

PROOF. Using the explicit description of flat, smooth and étale morphisms given in Prop. 2.2.2.5 (1), (2), and (3) reduce to the analog well known facts for morphism between affine schemes. Finally, Cor. 2.2.2.15 implies that the class of finitely presented morphisms in $k - D^-Aff$ is compatible with the étale topology and the class $\mathbf{P}$. □

Lemma 2.2.3.4 and definition 1.3.6.2 allows us to define the notions of flat, smooth, étale and locally finitely presented morphisms of D^--stacks, which are all stable by equivalences, compositions and homotopy pullbacks. Recall that by definition, a flat (resp. smooth, resp. étale, resp. locally finitely presented) is always n-representable for some n. Using our general definition Def. 1.3.6.4 we also have notions of quasi-compact morphisms, finitely presented morphisms, and monomorphisms between D^--stacks. We also make the following definition.

DEFINITION 2.2.3.5. (1) *A morphism of D^--stacks is a* Zariski open immersion *if it is a locally finitely presented flat monomorphism.*

(2) *A morphism of D^--stacks $F \longrightarrow G$ is a* closed immersion *if it is representable, and if for any $A \in sk-Alg$ and any morphism $X = \mathbb{R}\underline{Spec}\, A \longrightarrow G$ the induced morphism of representable D^--stacks*

$$F \times^h_G X \simeq \mathbb{R}\underline{Spec}\, B \longrightarrow \mathbb{R}\underline{Spec}\, A$$

induces an epimorphism of rings $\pi_0(A) \longrightarrow \pi_0(B)$.

2.2.4. Truncations

We consider the natural inclusion functor

$$i : k - Aff \longrightarrow k - D^-Aff$$

right adjoint to the functor

$$\pi_0 : k - D^-Aff \longrightarrow k - Aff.$$

The pair (π_0, i) is a Quillen adjunction (for the trivial model structure on $k - Aff$), and as usual we will omit to mention the inclusion functor i, and simply consider commutative k-algebras as constant simplicial objects. Furthermore, both functors preserve equivalences and thus induce a Quillen adjunction on the model category of pre-stacks (using notations of [**HAGI**, §4.8])

$$i_! : k - Aff^\wedge = SPr(k - Aff) \longrightarrow k - D^-Aff^\wedge \qquad k - Aff^\wedge \longleftarrow k - D^-Aff^\wedge : i^*.$$

The functor i is furthermore continuous in the sense of [**HAGI**, §4.8], meaning that the right derived functor

$$\mathbb{R}i^* : \mathrm{Ho}(k - D^-Aff^\wedge) \longrightarrow \mathrm{Ho}(k - Aff^\wedge)$$

preserves the sub-categories of stacks. Indeed, by Lem. 1.3.2.3 (2) and adjunction this follows from the fact that $i : k - Aff \longrightarrow k - D^-Aff$ preserves co-products, equivalences and étale hypercovers. By the general properties of left Bousfield localizations we therefore get a Quillen adjunction on the model categories of stacks

$$i_! : k - Aff^{\sim,\acute{e}t} \longrightarrow k - D^-Aff^{\sim,\acute{e}t} \qquad k - Aff^{\sim,\acute{e}t} \longleftarrow k - D^-Aff^{\sim,\acute{e}t} : i^*.$$

From this we get a derived adjunction on the homotopy categories of stacks

$$\mathbb{L}i_! : \mathrm{St}(k) \longrightarrow D^-\mathrm{St}(k)$$
$$\mathrm{St}(k) \longleftarrow D^-\mathrm{St}(k) : \mathbb{R}i^*.$$

LEMMA 2.2.4.1. *The functor* $\mathbb{L}i_!$ *is fully faithful.*

PROOF. We need to show that for any $F \in \mathrm{St}(k)$ the adjunction morphism

$$F \longrightarrow \mathbb{R}i^* \circ \mathbb{L}i_!(F)$$

is an isomorphism. The functor $\mathbb{R}i^*$ commutes with homotopy colimits, as these are computed in the model category of simplicial presheaves and thus levelwise. Moreover, as $i_!$ is left Quillen the functor $\mathbb{L}i_!$ also commutes with homotopy colimits. Now, any stack $F \in \mathrm{St}(k)$ is a homotopy colimit of representable stacks (i.e. affine schemes), and therefore we can suppose that $F = Spec\,A$, for $A \in k - Alg$. But then $\mathbb{L}i_!(Spec\,A) \simeq \mathbb{R}\underline{Spec}\,A$. Furthermore, for any $B \in k - Alg$ there are natural isomorphisms in $\mathrm{Ho}(SSet)$

$$\mathbb{R}\underline{Spec}\,A(B) \simeq Map_{sk-Alg}(A, B) \simeq Hom_{k-Alg}(A, B) \simeq (Spec\,A)(B).$$

This shows that the adjunction morphism

$$Spec\,A \longrightarrow \mathbb{R}i^* \circ \mathbb{L}i_!(Spec\,A)$$

is an isomorphism. □

Another useful remark is the following

LEMMA 2.2.4.2. *The functor* $i^* : k - D^-Aff^{\sim,\acute{e}t} \longrightarrow k - Aff^{\sim,\acute{e}t}$ *is right and left Quillen. In particular it preserves equivalences.*

PROOF. The functor i^* has a right adjoint

$$\pi_0^* : k - Aff^{\sim,\acute{e}t} \longrightarrow k - D^-Aff^{\sim,\acute{e}t}.$$

Using lemma 1.3.2.3 (2) we see that π_0^* is a right Quillen functor. Therefore i^* is left Quillen. □

DEFINITION 2.2.4.3. (1) *The* truncation functor *is*

$$t_0 := i^* : D^-\mathrm{St}(k) \longrightarrow \mathrm{St}(k).$$

(2) *The* extension functor *is the left adjoint to* t_0

$$i := \mathbb{L}i_! : \mathrm{St}(k) \longrightarrow D^-\mathrm{St}(k).$$

(3) *A* D^-*-stack* F *is* truncated *if the adjunction morphism*

$$it_0(F) \longrightarrow F$$

is an isomorphism in $D^-\mathrm{St}(k)$.

By lemmas 2.2.4.1, 2.2.4.2 we know that the truncation functor t_0 commutes with homotopy limits and homotopy colimits. The extension functor i itself commutes with homotopy colimits and is fully faithful. An important remark is that the extension i does not commute with homotopy limits, as the inclusion functor $k-Alg \longrightarrow sk-Alg$ does not preserve homotopy push-outs.

Concretely, the truncation functor t_0 sends a functor

$$F : sk - Alg \longrightarrow SSet_{\mathbb{V}}$$

to

$$\begin{array}{cccc} t_0(F): & k - Alg & \longrightarrow & SSet_{\mathbb{V}} \\ & A & \mapsto & F(A). \end{array}$$

By adjunction we clearly have

$$t_0\mathbb{R}\underline{Spec}\, A \simeq Spec\, \pi_0(A)$$

for any $A \in sk - Alg$, showing that the notation is compatible with the one we did use before for $t_0(Spec\, A) = Spec\, \pi_0(A)$ as objects in $k - D^- Aff$. The extension functor i is characterized by

$$i(Spec\, A) \simeq \mathbb{R}\underline{Spec}\, A,$$

and the fact that it commutes with homotopy colimits.

Proposition 2.2.4.4. (1) *The functor t_0 preserves epimorphisms of stacks.*

(2) *The functor t_0 sends n-geometric D^--stacks to n-geometric stacks, and flat (resp. smooth, resp. étale) morphisms between D^--stacks to flat (resp. smooth, resp. étale) morphisms between stacks.*

(3) *The functor i preserves homotopy pullbacks of n-geometric stacks along a flat morphism, sends n-geometric stacks to n-geometric D^--stacks, and flat (resp. smooth, resp. étale) morphisms between n-geometric stacks to flat (resp. smooth, resp. étale) morphisms between $n-$-geometric D^--stacks.*

(4) *Let $F \in \mathrm{St}(k)$ be an n-geometric stack, and $F' \longrightarrow i(F)$ be a flat morphism of n-geometric D^--stacks. Then F' is truncated (and therefore is the image by i of an n-geometric stack by (2)).*

Proof. (1) By adjunction, this follows from the fact that $i : k-Alg \longrightarrow sk-Alg$ reflects étale covering families (by Prop. 2.2.2.2).

(2) The proof is by induction on n. For $n = -1$, this simply follows from the formula

$$t_0\mathbb{R}\underline{Spec}\, A \simeq Spec\, \pi_0(A),$$

and Prop. 2.2.2.5 and Thm. 2.2.2.6. Assume that the property is known for any $m < n$ and let us prove it for n. Let F be an n-geometric D^--stack, which by Prop. 1.3.4.2 can be written as $|X_*|$ for some $(n-1)$-smooth Segal groupoid X_* in $k-D^-Aff^{\sim,\acute{e}t}$. Using our property at rank $n-1$ and that t_0 commutes with homotopy limits shows that $t_0(X_*)$ is also a $(n-1)$-smooth Segal groupoid object in $k-Aff^{\sim,\acute{e}t}$. Moreover, as t_0 commutes with homotopy colimits we have $t_0(F) \simeq |t_0(X_*)|$, which by Prop. 1.3.4.2 is an n-geometric stack. This shows that t_0 sends n-geometric D^--stacks to n-geometric stacks and therefore preserves n-representable morphisms. Let $f : F \longrightarrow G$ be a flat (resp. smooth, resp. étale) morphism of D^--stacks which is n-representable, and let us prove by induction on n that $t_0(f)$ is flat (resp. smooth, resp. étale). We let $X \in \mathrm{St}(k)$ be an affine scheme, and $X \longrightarrow t_0(G)$ be a morphism of stacks. By adjunction between i and t_0 we have

$$X \times^h_{t_0(G)} t_0(F) \simeq t_0(X \times^h_G F),$$

showing that we can assume that $G = X$ is an affine scheme and thus F to be an n-geometric D^-. By definition of being flat (resp. smooth, resp. étale) there exists a smooth n-atlas $\{U_i\}$ of F such that each composite morphism $U_i \longrightarrow X$ is $(n-1)$-representable and flat (resp. smooth, resp. étale). By induction and (1) the family $\{t_0(U_i)\}$ is a smooth n-atlas for $t_0(F)$, and by induction each composition $t_0(U_i) \longrightarrow t_0(X) = X$ is flat (resp. smooth, resp. étale). This implies that $t_0(F) \longrightarrow t_0(X)$ is flat (resp. smooth, resp. étale).

(3) The proof is by induction on n. For $n = -1$ this follows from the formula $i(Spec\,A) \simeq \mathbb{R}Spec\,A$, the description of flat, smooth and étale morphisms (Prop. 2.2.2.5 and Thm. 2.2.2.6), and the fact that for any flat morphism of commutative k-algebras $A \longrightarrow B$ and any commutative A-algebra C there is a natural isomorphism in $\mathrm{Ho}(A - Alg_s)$

$$B \otimes_A C \simeq B \otimes^{\mathbb{L}}_A C.$$

Let us now assume the property is proved for $m < n$ and let us prove it for n. Let F be an n-geometric stack, and by Prop. 1.3.4.2 let us write it as $F \simeq |X_*|$ for some $(n-1)$-smooth Segal groupoid object X_* in $k - Aff^{\sim,\acute{e}t}$. By induction, $i(X_*)$ is again a $(n-1)$-smooth Segal groupoid objects in $k - D^-Aff^{\sim,\acute{e}t}$, and as i commutes with homotopy colimits we have $i(F) \simeq |i(X_*)|$. Another application of Prop. 1.3.4.2 shows that F is an n-geometric D^--stacks. We thus have seen that i sends n-geometric stacks to n-geometric D^--stacks.

Now, let $F \longrightarrow G$ be a flat morphism between n-geometric stacks, and $H \longrightarrow G$ any morphism between n-geometric stacks. We want to show that the natural morphism

$$i(F \times^h_G H) \longrightarrow i(F) \times^h_{i(G)} i(H)$$

is an isomorphism in $\mathrm{St}(k)$. For this, we write $G \simeq |X_*|$ for some $(n-1)$-smooth Segal groupoid object in $k - Aff^{\sim,\acute{e}t}$, and we consider the Segal groupoid objects

$$F_* := F \times^h_G X_* \qquad H_* := H \times^h_G X_*,$$

where $X_* \longrightarrow |X_*| = G$ is the natural augmentation in $\mathrm{St}(k)$. The Segal groupoid objects F_* and H_* are again $(n-1)$-smooth Segal groupoid objects in $k - Aff^{\sim,\acute{e}t}$ as G is an n-geometric stack. The natural morphisms of Segal groupoid objects

$$F_* \longrightarrow X_* \longleftarrow H_*$$

gives rise to another $(n-1)$-smooth Segal groupoid object $F_* \times^h_{X_*} H_*$. Clearly, we have

$$F \times^h_G H \simeq |F_* \times^h_{X_*} H_*|.$$

As i commutes with homotopy colimits, and by induction on n we have

$$i(F \times^h_G H) \simeq |i(F_*) \times^h_{i(X_*)} i(H_*)| \simeq |i(F_*)| \times^h_{|i(X_*)|} |i(H_*)| \simeq i(F) \times^h_{i(G)} i(H).$$

It remains to prove that if $f : F \longrightarrow G$ is a flat (resp. smooth, resp. étale) morphism between n-geometric stacks then $i(f) : i(F) \longrightarrow i(G)$ is a flat (resp. smooth, resp. étale) morphism between n-geometric D^--stacks. For this, let $\{U_i\}$ be a smooth n-atlas for G. We have seen before that $\{i(U_i)\}$ is a smooth n-atlas for $i(G)$. As we have seen that i commutes with homotopy pullbacks along flat morphisms, and because of the local properties of flat (resp. smooth, resp. étale) morphisms (see Prop. 1.3.6.3 and Lem. 2.2.3.4), we can suppose that G is one of the U_i's and thus is an affine scheme. Now, let $\{V_i\}$ be a smooth n-atlas for F. The family $\{i(V_i)\}$ is a smooth n-atlas for F, and furthermore each morphism $i(V_i) \longrightarrow i(G)$ is the image by i of a flat (resp. smooth, resp. étale) morphism between affine schemes and therefore is a flat (resp. smooth, resp. étale morphism) of D^--stacks. By definition this implies that $i(F) \longrightarrow i(G)$ is flat (resp. smooth, resp. étale).

(4) The proof is by induction on n. For $n = -1$ this is simply the description of flat morphisms of Prop. 2.2.2.5. Let us assume the property is proved for $m < n$ and let us prove it for n. Let $F' \longrightarrow i(F)$ be a flat morphism, with F an n-geometric stack and F' an n-geometric D^--stack. Let $\{U_i\}$ be a smooth n-atlas for F. Then, $\{i(U_i)\}$ is a smooth n-atlas for $i(F)$. We consider the commutative diagram of D^--stacks

$$\begin{array}{ccc} it_0(F') & \xrightarrow{f} & F' \\ & \searrow & \downarrow \\ & & i(F). \end{array}$$

We need to prove that f is an isomorphism in $D^-\mathrm{St}(k)$. As this is a local property on $i(F)$, we can take the homotopy pullback over the atlas $\{i(U_i)\}$, and thus suppose that F is an affine scheme. Let now $\{V_i\}$ be a smooth n-atlas for F', and we consider the homotopy nerve of the morphism

$$X_0 := \coprod_i V_i \longrightarrow F'.$$

This is a $(n-1)$-smooth Segal groupoid objects in $D^-Aff^{\sim,\acute{e}t}$, which is such that each morphism $X_i \longrightarrow i(G)$ is flat. Therefore, by induction on n, the natural morphism of Segal groupoid objects

$$it_0(X_*) \longrightarrow X_*$$

is an equivalence. Therefore, as i and t_0 commutes with homotopy colimits we find that the adjunction morphism

$$it_0(F') \simeq |it_0(X_*)| \longrightarrow |X_*| \simeq F'$$

is an isomorphism in $D^-\mathrm{St}(k)$. □

An important corollary of Prop. 2.2.2.5 is the following fact.

Corollary 2.2.4.5. *For any Artin n-stack, the D^--stack $i(F)$ has an obstruction theory.*

Proof. Follows from 2.2.2.5 (3) and Cor. 2.2.3.3. □

One also deduces from Prop. 2.2.2.5 and Lem. 2.1.1.2 the following corollary.

Corollary 2.2.4.6. *Let F be an n-geometric D^--stack. Then, for any $A \in sk - Alg$, such that $\pi_i(A) = 0$ for $i > k$, the simplicial set $\mathbb{R}F(A)$ is $(n+k+1)$-truncated.*

Proof. This is by induction on k. For $k = 0$ this is Lem. 2.1.1.2 and the fact that t_0 preserves n-geometric stacks. To pass from k to $k+1$, we consider for any $A \in sk - Alg$ with $\pi_i(A) = 0$ for $i > k+1$, the natural morphisms

$$\mathbb{R}F(A) \longrightarrow \mathbb{R}F(A_{\leq k}),$$

whose homotopy fibers can be described using Prop. 1.4.2.5 and Lem. 2.2.1.1. We find that this homotopy fiber is either empty, or equivalent to

$$Map_{A_{\leq k}-Mod}(\mathbb{L}_{F,x}, \pi_{k+1}(A)[k+1]),$$

which is $(k+1)$-truncated. By induction, $\mathbb{R}F(A_{\leq k})$ is $(k+1+n)$-truncated and the homotopy fibers of

$$\mathbb{R}F(A) \longrightarrow \mathbb{R}F(A_{\leq k}),$$

are $(k+1)$-truncated, and therefore $\mathbb{R}F(A)$ is $(k+n+2)$-truncated. $\square$

Another important property of the truncation functor is the following local description of the truncation $t_0(F)$ sitting inside the D^--stack F itself.

PROPOSITION 2.2.4.7. *Let F be an n-geometric D^--stack. The adjunction morphism $it_0(F) \longrightarrow F$ is a representable morphism. Moreover, for any $A \in sk-Alg$, and any flat morphism $\mathbb{R}\underline{Spec}\,A \longrightarrow F$, the square*

$$\begin{array}{ccc} it_0(F) & \longrightarrow & F \\ \uparrow & & \uparrow \\ \mathbb{R}\underline{Spec}\,\pi_0(A) & \longrightarrow & \mathbb{R}\underline{Spec}\,A \end{array}$$

is homotopy cartesian. In particular, the morphism $it_0(F) \longrightarrow F$ is a closed immersion in the sense of Def. 2.2.3.5.

PROOF. By the local character of representable morphisms it is enough to prove that for any flat morphism $X = \mathbb{R}\underline{Spec}\,A \longrightarrow F$, the square

$$\begin{array}{ccc} it_0(F) & \longrightarrow & F \\ \uparrow & & \uparrow \\ it_0(X) = \mathbb{R}\underline{Spec}\,\pi_0(A) & \longrightarrow & X \end{array}$$

is homotopy cartesian. The morphism

$$it_0(F) \times^h_F X \longrightarrow it_0(F)$$

is flat, and by Prop. 2.2.2.5 (4) this implies that the D^--stack $it_0(F) \times^h_F X$ is truncated. In other words the natural morphism

$$it_0(it_0(F) \times^h_F X) \simeq it_0(F) \times^h_{it_0(F)} it_0(X) \simeq it_0(X) \longrightarrow it_0(F) \times^h_F X$$

is an isomorphism. $\square$

Using our embedding

$$i : \mathrm{St}(k) \longrightarrow D^-\mathrm{St}(k)$$

we will see stacks in $\mathrm{St}(k)$, and in particular Artin n-stacks, as D^--stacks. However, as the functor i does not commute with homotopy pullbacks we will still mention it in order to avoid confusions.

2.2.5. Infinitesimal criteria for smooth and étale morphisms

Recall that $sk-Mod_1$ denotes the full subcategory of $sk-Mod$ of connected simplicial k-modules. It consists of all $M \in sk-Mod$ for which the adjunction $S(\Omega M) \longrightarrow M$ is an isomorphism in $\mathrm{Ho}(sk-Mod)$, or equivalently for which $\pi_0(M) = 0$.

PROPOSITION 2.2.5.1. *Let $f : F \longrightarrow G$ be an n-representable morphism between D^--stacks. The morphism f is smooth if and only if it satisfies the following two conditions*

(1) *The morphism $t_0(f) : t_0(F) \longrightarrow t_0(G)$ is a locally finitely presented morphism in $k-Aff^{\sim,et}$.*

(2) *For any $A \in sk-Alg$, any connected $M \in A-Mod_s$, and any derivation $d \in \pi_0(\mathbb{D}er(A,M))$, the natural projection $A \oplus_d \Omega M \longrightarrow A$ induces a surjective morphism*

$$\pi_0(\mathbb{R}F(A \oplus_d \Omega M)) \longrightarrow \pi_0 \left(\mathbb{R}G(A \oplus_d \Omega M) \times^h_{\mathbb{R}F(A)} \mathbb{R}G(A)\right).$$

PROOF. First of all we can suppose that F and G are fibrant objects in $k-D^-Aff^{\sim,\acute{e}t}$.

Suppose first that the morphism f is smooth and let us prove that is satisfies the two conditions of the proposition. We know by 2.2.4.4 (3) that $t_0(f)$ is then a smooth morphism in $k-Aff^{\sim,\acute{e}t}$, so condition (1) is satisfied. The proof that (2) is also satisfied goes by induction on n. Let us start with the case $n=-1$, and in other words when f is a smooth and representable morphism.

We fix a point x in $\pi_0\left(G(A \oplus_d \Omega M) \times^h_{F(A)} G(A)\right)$, and we need to show that the homotopy fiber taken at x of the morphism

$$F(A \oplus_d \Omega M) \longrightarrow G(A \oplus_d \Omega M) \times^h_{F(A)} G(A)$$

is non empty. The point x corresponds via Yoneda to a commutative diagram in $\mathrm{Ho}(k-D^-Aff^{\sim,\acute{e}t}/G)$

$$\begin{array}{ccc} X & \longrightarrow & F \\ \downarrow & & \downarrow \\ X_d[\Omega M] & \longrightarrow & G, \end{array}$$

where $X := \mathbb{R}\underline{Spec}\,A$, and $X_d[\Omega M] := \mathbb{R}\underline{Spec}\,(A \oplus_d \Omega M)$. Making a homotopy base change

$$\begin{array}{ccc} X & \longrightarrow & F \times^h_G X_d[\Omega M] \\ \downarrow & & \downarrow \\ X_d[\Omega M] & \longrightarrow & X_d[\Omega M], \end{array}$$

we see that we can replace G be $X_d[\Omega M]$ and f by the projection $F \times^h_G X_d[\Omega M] \longrightarrow X_d[\Omega M]$. In particular, we can assume that G is a representable D^--stack. The morphism f can then be written as

$$f : F \simeq \mathbb{R}\underline{Spec}\,C \longrightarrow \mathbb{R}\underline{Spec}\,B \simeq G,$$

and corresponds to a morphism of commutative simplicial k-algebras $B \longrightarrow C$. Then, using Prop. 1.4.2.6 and Cor. 2.2.3.3 we see that the obstruction for the point x to lifts to a point in $\pi_0(F(A \oplus_d \Omega M))$ lives in the abelian group $[\mathbb{L}_{C/B} \otimes^{\mathbb{L}}_C A, M]$. But, as $B \longrightarrow C$ is assumed to be smooth, the A-module $\mathbb{L}_{C/B} \otimes^{\mathbb{L}}_C A$ is a retract of a free A-module. This implies that $[\mathbb{L}_{C/B} \otimes^{\mathbb{L}}_C A, M]$ is a retract of a product of $\pi_0(M)$, and therefore is 0 by hypothesis on M. This implies that condition (2) of the proposition is satisfied when $n=-1$.

Let us now assume that condition (2) is satisfied for all smooth m-representable morphisms for $m < n$, and let us prove it for a smooth n-representable morphism $f : F \longrightarrow G$. Using the same trick as above, se see that we can always assume that G is a representable D^--stack, and therefore that F is an n-geometric D^--stack. Then, let us chose a point x in $\pi_0\left(G(A \oplus_d \Omega M) \times^h_{F(A)} G(A)\right)$, and we need to show that x lifts to a point in $\pi_0(F(A \oplus_d \Omega M))$. For this, we use Cor. 2.2.3.3 for F, and consider its cotangent complex $\mathbb{L}_{F,y} \in \mathrm{Ho}(Sp(A-Mod_s))$, where $y \in F(A)$ is the image of x. There exists a natural functoriality morphism

$$\mathbb{L}_{G,f(y)} \longrightarrow \mathbb{L}_{F,y}$$

whose homotopy cofiber is $\mathbb{L}_{F/G,y} \in \mathrm{Ho}(Sp(A-Mod_s))$. Then, Prop. 1.4.2.6 tell us that the obstruction for the existence of this lift lives in $[\mathbb{L}_{F/G,y}, M]$. It is therefore enough to show that $[\mathbb{L}_{F/G,y}, M] \simeq 0$ for any $M \in A-Mod_s$ such that $\pi_0(M) = 0$.

LEMMA 2.2.5.2. *Let $F \longrightarrow G$ be a smooth morphism between n-geometric D^--stacks with G a representable stack. Let $A \in sk-Alg$ and $y : Y = \mathbb{R}\underline{Spec}\,A \longrightarrow F$ be a point. Then the object*

$$\mathbb{L}_{F/G,y} \in \mathrm{Ho}(Sp(A-Mod_s))$$

is perfect, and its dual $\mathbb{L}^{\vee}_{F/G,y} \in \mathrm{Ho}(Sp(A-Mod_s))$ is 0-connective (i.e. belongs to the image of $\mathrm{Ho}(A-Mod_s) \hookrightarrow \mathrm{Ho}(Sp(A-Mod_s))$.

PROOF. Recall first that an object in $\mathrm{Ho}(Sp(A-Mod_s))$ is perfect if and only if it is finitely presented, and if and only if it is a retract of a finite cell stable A-module (see Cor. 1.2.3.8, and also [**EKMM**, III.2] or [**Kr-Ma**, Thm. III.5.7]).

The proof is then by induction on n. When F is representable, this is by definition of smooth morphisms, as then $\mathbb{L}_{F/G,y}$ is a projective A-module of finite presentation, and so is its dual. Let us suppose the lemma proved for all $m < n$, and lets prove it for n. First of all, the conditions on $\mathbb{L}_{F/G,y}$ we need to prove are local for the étale topology on A, because of Cor. 1.3.7.8. Therefore, one can assume that the point y lifts to a point of an n-atlas for F. One can thus suppose that there exists a representable D^--stack U, a smooth morphism $U \longrightarrow F$, such that $y \in \pi_0(\mathbb{R}F(A))$ is the image of a point $z \in \mathbb{R}U(A)$. There exists an fibration sequence of stable A-modules

$$\mathbb{L}_{F/G,y} \longrightarrow \mathbb{L}_{U/G,z} \longrightarrow \mathbb{L}_{U/F,z}.$$

As $U \longrightarrow G$ is smooth, $\mathbb{L}_{U/G,z}$ is a projective A-module. Furthermore, the stable A-module $\mathbb{L}_{U/F,y}$ can be identified with $\mathbb{L}_{U\times^h_F Y/Y,s}$, where s is the natural section $Y \longrightarrow U \times^h_F Y$ induced by the point $z : Y \longrightarrow U$. The morphism $U \times^h_F Y \longrightarrow Y$ being a smooth and $(n-1)$-representable morphism, induction tells us that the stable A-module $\mathbb{L}_{U/F,y}$ satisfies the conditions of the lemma. Therefore, $\mathbb{L}_{F/G,y}$ is the homotopy fiber of a morphism between stable A-module satisfying the conditions of the lemma, and is easily seen to satisfies itself these conditions. □

By the above lemma, we have

$$[\mathbb{L}_{F/G,y}, M] \simeq \pi_0(\mathbb{L}^{\vee}_{F/G,y} \otimes^{\mathbb{L}}_A M) \simeq \pi_0(\mathbb{L}^{\vee}_{F/G,y}) \otimes_{\pi_0(A)} \pi_0(M) \simeq 0$$

for any A-module M such that $\pi_0(M) = 0$. This finishes the proof of the fact that f satisfies the conditions of Prop. 2.2.5.1 when it is smooth.

Conversely, let us assume that $f : F \longrightarrow G$ is a morphism satisfying the lifting property of 2.2.5.1, and let us show that f is smooth. Clearly, one can suppose that G is a representable stack, and thus that F is an n-geometric D^--stack. We need to show that for any representable D^--stack U and any smooth morphism $U \longrightarrow F$ the composite morphism $U \longrightarrow G$ is smooth. By what we have seen in the first part of the proof, we known that $U \longrightarrow F$ also satisfies the lifting properties, and thus so does the composition $U \longrightarrow G$. We are therefore reduced to the case where f is a morphism between representable D^--stacks, and thus corresponds to a morphism of commutative simplicial k-algebras $A \longrightarrow B$. By hypothesis on f, $\pi_0(A) \longrightarrow \pi_0(B)$ is a finitely presented morphism of commutative rings. Furthermore, Prop. 1.4.2.6 and

Cor. 2.2.3.3 show that for any B-module M with $\pi_0(M) = 0$, we have $[\mathbb{L}_{B/A}, M] = 0$. Let $B^{(I)} \longrightarrow \mathbb{L}_{B/A}$ be a morphism of B-modules, with $B^{(I)}$ free over some set I, and such that the induced morphism $\pi_0(B)^{(I)} \longrightarrow \pi_0(\mathbb{L}_{B/A})$ is surjective. Let K be the homotopy fiber of the morphism $B^{(I)} \longrightarrow \mathbb{L}_{B/A}$, that, according to our choice, induces a homotopy fiber sequence of A-modules

$$B^{(I)} \longrightarrow \mathbb{L}_{B/A} \longrightarrow K[1].$$

The short exact sequence

$$[\mathbb{L}_{B/A}, B^{(I)}] \longrightarrow [\mathbb{L}_{B/A}, \mathbb{L}_{B/A}] \longrightarrow [\mathbb{L}_{B/A}, K[1]] = 0,$$

shows that $\mathbb{L}_{B/A}$ is a retract of $B^{(I)}$, and thus is a projective B-module. Furthermore, the homotopy cofiber sequence

$$\mathbb{L}_A \otimes^{\mathbb{L}}_A B \longrightarrow \mathbb{L}_B \longrightarrow \mathbb{L}_{B/A},$$

induces a short exact sequence

$$[\mathbb{L}_B, \mathbb{L}_A \otimes^{\mathbb{L}}_A B] \longrightarrow [\mathbb{L}_A \otimes^{\mathbb{L}}_A B, \mathbb{L}_A \otimes^{\mathbb{L}}_A B] \longrightarrow [\mathbb{L}_{B/A}, \mathbb{L}_A \otimes^{\mathbb{L}}_A B[1]] = 0,$$

shows that the morphism $\mathbb{L}_A \otimes^{\mathbb{L}}_A B \longrightarrow \mathbb{L}_B$ has a retraction. We conclude that f is a formally smooth morphism such that $\pi_0(A) \longrightarrow \pi_0(B)$ is finitely presented, and by Cor. 2.2.2.11 that f is a smooth morphism. □

From the proof of Prop. 2.2.5.1 we extract the following corollary.

COROLLARY 2.2.5.3. *Let $F \longrightarrow G$ be an n-representable morphism of D^--stacks such that the morphism $t_0(F) \longrightarrow t_0(G)$ is a locally finitely presented morphism of stacks. The following three conditions are equivalent.*

(1) *The morphism f is smooth.*
(2) *For any $A \in sk-Alg$ and any morphism of stacks $x : X = \mathbb{R}\underline{Spec}\, A \longrightarrow F$, the object*

$$\mathbb{L}_{F/G,x} \in \mathrm{Ho}(Sp(A - Mod_s))$$

is perfect, and its dual $\mathbb{L}^{\vee}_{F/G,y} \in \mathrm{Ho}(Sp(A - Mod_s))$ is 0-connective (i.e. belongs to the image of $\mathrm{Ho}(A - Mod_s) \hookrightarrow \mathrm{Ho}(Sp(A - Mod_s))$.
(3) *For any $A \in sk - Alg$, any morphism of stacks $x : X = \mathbb{R}\underline{Spec}\, A \longrightarrow F$, and any A-module M in $sk - Mod_1$, we have*

$$[\mathbb{L}_{F/G,x}, M] = 0.$$

PROOF. That (1) implies (2) follows from lemma 2.2.5.2. Conversely, if $\mathbb{L}_{F/G,x}$ satisfies the conditions of the corollary, then for any A-module M such that $\pi_0(M)$ one has $[\mathbb{L}_{F/G,x}, M] = 0$. Therefore, Prop. 1.4.2.6 shows that the lifting property of Prop. 2.2.5.1 holds, and thus that (2) implies (1). Furthermore, clearly (2) implies (3), and conversely (3) together with Prop. 1.4.2.6 implies the lifting property of Prop. 2.2.5.1. □

PROPOSITION 2.2.5.4. *Let $f : F \longrightarrow G$ be an n-representable morphism between D^--stacks. The morphism f is étale if and only if it satisfies the following two conditions*

(1) *The morphism $t_0(f) : t_0(F) \longrightarrow t_0(G)$ is locally finitely presented as a morphism in $k - Aff^{\sim,\tau}$.*

(2) *For any* $A \in sk-Alg$, *any* $M \in A-Mod_s$ *whose underlying k-module is in* $sk-Mod_1$, *and any derivation* $d \in \pi_0(\mathbb{D}er(A,M))$, *the natural projection* $A \oplus_d \Omega M \longrightarrow A$ *induces an isomorphism in* $\mathrm{Ho}(SSet)$

$$\mathbb{R}F(A \oplus_d \Omega M) \longrightarrow \mathbb{R}G(A \oplus_d \Omega M) \times^h_{\mathbb{R}F(A)} \mathbb{R}G(A).$$

PROOF. First of all one can suppose that F and G are fibrant objects in $k-D^-Aff^{\sim,ét}$.

Suppose first that the morphism f is étale and let us prove that is satisfies the two conditions of the proposition. We know by 2.2.4.4 (3) that $t_0(f)$ is then a étale morphism in $k-Aff^{\sim,ét}$, so condition (1) is satisfied. The proof that (2) is also satisfied goes by induction on n. Let us start with the case $n=-1$, and in other words when f is an étale and representable morphism. In this case the result follows from Cor. 1.2.8.4 (2). We now assume the result for all $m<n$ and prove it for n. Let $A \in sk-Alg$, M an A-module with $\pi_0(M)=0$, and $d \in \pi_0(\mathbb{D}er(A,M))$ be a derivation. Let x be a point in $\pi_0(\mathbb{R}G(A \oplus_d \Omega M) \times^h_{\mathbb{R}F(A)} \mathbb{R}G(A))$, with image y in $\pi_0(\mathbb{R}F(A))$. By Prop. 1.4.2.6 the homotopy fiber of the morphism

$$\mathbb{R}F(A \oplus_d \Omega M) \longrightarrow \mathbb{R}G(A \oplus_d \Omega M) \times^h_{\mathbb{R}F(A)} \mathbb{R}G(A)$$

at the point x is non empty if and only if a certain obstruction in $[\mathbb{L}_{F/G,y}, M]$ vanishes. Furthermore, if nonempty this homotopy fiber is then equivalent to

$$Map_{A-Mod}(\mathbb{L}_{F/G,y}, \Omega(M)).$$

It is then enough to prove that $\mathbb{L}_{F/G,y}=0$, and this is contained in the following lemma.

LEMMA 2.2.5.5. *Let* $F \longrightarrow G$ *be a étale morphism between n-geometric* D^-*-stacks. Let* $A \in sk-Alg$ *and* $y: Y = \mathbb{R}\underline{Spec}\,A \longrightarrow F$ *be a point. Then* $\mathbb{L}_{F/G,y} \simeq 0$.

PROOF. We easily reduce to the case when G is a representable D^--stack. The proof is then by induction on n. When F is representable, this is by definition of étale morphisms. Let us suppose the lemma proved for all $m<n$, and lets prove it for n. First of all, the vanishing of $\mathbb{L}_{F/G,y}$ is clearly a local condition for the étale topology on A. Therefore, we can assume that the point y lifts to a point of an n-atlas for F. We can thus suppose that there exists a representable D^--stack U, a smooth morphism $U \longrightarrow F$, such that $y \in \pi_0(\mathbb{R}F(A))$ is the image of a point $z \in \mathbb{R}U(A)$. There exists an fibration sequence of stable A-modules

$$\mathbb{L}_{F/G,y} \longrightarrow \mathbb{L}_{U/G,z} \longrightarrow \mathbb{L}_{U/F,z}.$$

As $U \longrightarrow G$ is étale, $\mathbb{L}_{U/G,z} \simeq 0$. Furthermore, the stable A-module $\mathbb{L}_{U/F,y}$ can be identified with $\mathbb{L}_{U\times^h_F Y/Y,s}$, where s is the natural section $Y \longrightarrow U \times^h_F Y$ induced by the point $z: Y \longrightarrow U$. The morphism $U \times^h_F Y \longrightarrow Y$ being an étale and $(n-1)$-representable morphism, induction tells us that the stable A-module $\mathbb{L}_{U/F,y}$ vanishes. We conclude that $\mathbb{L}_{F/G,y} \simeq 0$. □

The lemma finishes the proof that if f is étale then it satisfies the two conditions of the proposition.

Conversely, let us assume that f satisfies the properties (1) and (2) of the proposition. To prove that f is étale we can suppose that G is a representable D^--stack. We then need to show that for any representable D^--stack U and any smooth morphism $u: U \longrightarrow F$, the induced morphism $v: U \longrightarrow G$ is étale. But Prop. 1.4.2.6 and our assumption (2) easily implies $\mathbb{L}_{F/G,v} \simeq 0$. Furthermore, the obstruction for the homotopy cofiber sequence

$$\mathbb{L}_{U/G,v} \longrightarrow \mathbb{L}_{F/G,u} \longrightarrow \mathbb{L}_{U/F,u},$$

to splits lives in $[\mathbb{L}_{U/F,u}, S(\mathbb{L}_{U/G,v})]$, which is zero by Cor. 2.2.5.3. Therefore, $\mathbb{L}_{U/G,v}$ is a retract of $\mathbb{L}_{F/G,u}$ and thus vanishes. This implies that $U \longrightarrow G$ is a formally étale morphism of representable D^--stacks. Finally, our assumption (1) and Cor. 2.2.2.11 implies that $U \longrightarrow G$ is an étale morphism as required. □

COROLLARY 2.2.5.6. *Let $F \longrightarrow G$ be an n-representable morphism of D^--stacks such that the morphism $t_0(F) \longrightarrow t_0(G)$ is a locally finitely presented morphism of stacks. The following two conditions are equivalent.*

(1) *The morphism f is étale.*
(2) *For any $A \in sk-Alg$ and any morphism of stacks $x : X = \mathbb{R}\underline{Spec}\, A \longrightarrow F$, one has $\mathbb{L}_{F/G,x} \simeq 0$.*

PROOF. This follows from Prop. 2.2.5.4 and Prop. 1.4.2.6. □

2.2.6. Some examples of geometric D^--stacks

We present here some basic examples of geometric D^--stacks. Of course we do not claim to be exhaustive, and many other interesting examples will not be discussed here and will appear in future works (see e.g. [**Go, To-Va1**]).

2.2.6.1. Local systems. Recall from Def. 1.3.7.5 the existence of the D^--stack $\mathbf{Vect}_n$, of rank n vector bundles. Recall by 2.2.2.2 that for $A \in sk-Alg$, an A-module $M \in A-Alg_s$ is a rank n vector bundle, if and only if M is a strong A-module and $\pi_0(M)$ is a projective $\pi_0(A)$-module of rank n. Recall also from Lem. 2.2.2.2 that vector bundles are precisely the locally perfect modules. The conditions of 1.3.7.12 are all satisfied in the present context and therefore we know that $\mathbf{Vect}_n$ is a smooth 1-geometric D^--stack. As a consequence of Prop. 2.2.4.4 (4) we deduce that the D^--stack $\mathbf{Vect}_n$ is truncated in the sense of 2.2.4.3. We also have a stack of rank n vector bundles $\mathbf{Vect}_n$ in $\mathrm{St}(k)$. Using the same notations for these two different objects is justified by the following lemma.

LEMMA 2.2.6.1. *There exists a natural isomorphism in $D^-\mathrm{St}(k)$*

$$i(\mathbf{Vect}_n) \simeq \mathbf{Vect}_n.$$

PROOF. As we know that the D^--stack $\mathbf{Vect}_n$ is truncated, it is equivalent to show that there exists a natural isomorphism

$$\mathbf{Vect}_n \simeq t_0(\mathbf{Vect}_n)$$

in $\mathrm{Ho}(k-Aff^{\sim,\tau})$.

We start by defining a morphism of stacks

$$\mathbf{Vect}_n \longrightarrow t_0(\mathbf{Vect}_n).$$

For this, we construct for a commutative k-algebra A, a natural functor

$$\phi_A : A - QCoh^c_W \longrightarrow i(A) - QCoh^c_W,$$

where $i(A) \in sk-Alg$ is the constant simplicial commutative k-algebra associated to A, and where $A - QCoh^c_W$ and $i(A) - QCoh^c_W$ are defined in §1.3.7. Recall that $A - QCoh^c_W$ is the category whose objects are the data of a B-module M_B for any morphism $A \longrightarrow B$ in $k-Alg$, together with isomorphisms $M_B \otimes_B B' \simeq M_{B'}$ for any

$A \longrightarrow B \longrightarrow B'$ in $k - Alg$, satisfying the usual cocycle conditions. The morphisms $M \to M'$ in $A - QCoh^c_W$ are simply the families of isomorphisms $M_B \simeq M'_B$ of B-modules which commute with the transitions isomorphisms. In the same way, the objects in $i(A) - QCoh^c_W$ are the data a cofibrant B-module $M_B \in B - Mod_s$ for any morphism $i(A) \longrightarrow B$ in $sk - Alg$, together with equivalences $M_B \otimes_B B' \longrightarrow M_{B'}$ for any $i(A) \longrightarrow B \longrightarrow B'$ in $sk - Alg$, satisfying the usual cocycle conditions. The morphisms $M \to M'$ in $i(A) - QCoh^c_W$ are simply the equivalences $M_B \longrightarrow M'_B$ which commutes with the transition equivalences. The functor

$$\phi_A : A - QCoh^c_W \longrightarrow i(A) - QCoh^c_W,$$

sends an object M to the object $\phi_A(M)$, where $\phi_A(M)_B$ is the simplicial B-module defined by

$$(\phi_A(M)_B)_n := M_{B_n}$$

(note that B can be seen as a simplicial commutative A-algebra). The functor ϕ_A is clearly functorial in A and thus defines a morphism of simplicial presheaves

$$\mathbf{QCoh} \longrightarrow t_0(\mathbf{QCoh}).$$

We check easily that the sub-stack $\mathbf{Vect}_n$ of $\mathbf{QCoh}$ is sent to the sub-stack $t_0(\mathbf{Vect}_n)$ of $t_0(\mathbf{QCoh})$, and therefore we get a morphism of stacks

$$\mathbf{Vect}_n \longrightarrow t_0(\mathbf{Vect}_n).$$

To see that this morphism is an isomorphism of stacks we construct a morphism in the other direction by sending a simplicial $i(A)$-module M to the $\pi_0(A)$-module $\pi_0(M)$. By 2.2.2.2 we easily see that this defines an inverse of the above morphism. $\square$

For a simplicial set $K \in SSet_{\mathbb{U}}$, and an object $F \in k - D^-Aff^{\sim,\acute{e}t}$, we can use the simplicial structure of the category $k - D^-Aff^{\sim,\acute{e}t}$ in order to define the exponential $F^K \in k - D^-Aff^{\sim,\acute{e}t}$. The model category $k - D^-Aff^{\sim,\acute{e}t}$ being a simplicial model category the functor

$$(-)^K : k - D^-Aff^{\sim,\acute{e}t} \longrightarrow k - D^-Aff^{\sim,\acute{e}t}$$

is right Quillen, and therefore can be derived on the right. Its right derived functor will be denoted by

$$\begin{array}{ccc} D^-\mathrm{St}(k) & \longrightarrow & D^-\mathrm{St}(k) \\ F & \mapsto & F^{\mathbb{R}K}. \end{array}$$

Explicitly, we have

$$F^{\mathbb{R}K} \simeq (RF)^K$$

where RF is a fibrant replacement of F in $k - D^-Aff^{\sim,\acute{e}t}$.

DEFINITION 2.2.6.2. *Let K be a* $\mathbb{U}$*-small simplicial set. The* derived moduli stack of rank n local systems on K *is defined to be*

$$\mathbb{R}\mathbf{Loc}_n(K) := \mathbf{Vect}_n^{\mathbb{R}K}.$$

We start by the following easy observation.

LEMMA 2.2.6.3. *Assume that K is a finite dimensional simplicial set. Then, the D^--stack $\mathbb{R}\mathbf{Loc}_n(K)$ is a finitely presented* 1*-geometric D^--stack.*

PROOF. We consider the following homotopy co-cartesian square of simplicial sets

$$\begin{array}{ccc} Sk_iK & \longrightarrow & Sk_{i+1}K \\ \uparrow & & \uparrow \\ \coprod_{Hom(\partial\Delta^{i+1},K)} \partial\Delta^{i+1} & \longrightarrow & \coprod_{Hom(\Delta^{i+1},K)} \Delta^{i+1}, \end{array}$$

where Sk_iK is the skeleton of dimension of i of K. This gives a homotopy pullback square of D^--stacks

$$\begin{array}{ccc} \mathbb{R}\mathbf{Loc}_n(Sk_{i+1}K) & \longrightarrow & \mathbb{R}\mathbf{Loc}_n(Sk_iK) \\ \downarrow & & \downarrow \\ \prod^h_{Hom(\Delta^{i+1},K)} \mathbb{R}\mathbf{Loc}_n(\Delta^{i+1}) & \longrightarrow & \prod^h_{Hom(\partial\Delta^{i+1},K)} \mathbb{R}\mathbf{Loc}_n(\partial\Delta^{i+1}). \end{array}$$

As finitely presented 1-geometric D^--stacks are stable by homotopy pullbacks, we see by induction on the skeleton that it only remains to show that $\mathbb{R}\mathbf{Loc}_n(\partial\Delta^{i+1})$ is a finitely presented 1-geometric D^--stack. But there is an isomorphism in $\mathrm{Ho}(SSet)$,

$$\partial\Delta^{i+1} \simeq * \coprod^{\mathbb{L}}_{\partial\Delta^i} *,$$

giving rise to an isomorphism of D^--stacks

$$\mathbb{R}\mathbf{Loc}_n(\partial\Delta^{i+1}) \simeq \mathbb{R}\mathbf{Loc}_n \times^h_{\mathbb{R}\mathbf{Loc}_n(\partial\Delta^i)} \mathbb{R}\mathbf{Loc}_n.$$

By induction on i, we see that it is enough to show that $\mathbf{Vect}_n$ is a finitely presented 1-geometric D^--stack which is known from Cor. 1.3.7.12. □

Another easy observation is the description of the truncation $t_0\mathbb{R}\mathbf{Loc}_n(K)$. For this, recall that the simplicial set K has a fundamental groupoid $\Pi_1(K)$. The usual Artin stack of rank n local systems on K is the stack in groupoids defined by

$$\begin{array}{rccc} \mathbf{Loc}_n(K): & k-Alg & \longrightarrow & \{Groupoids\} \\ & A & \mapsto & \underline{Hom}(\Pi_1(K), \mathbf{Vect}_n(A)). \end{array}$$

In other words, $\mathbf{Loc}_n(K)(A)$ is the groupoid of functors from $\Pi_1(K)$ to the groupoid of rank n projective A-modules. As usual, this Artin stack is considered as an object in $\mathrm{St}(k)$.

LEMMA 2.2.6.4. *There exists a natural isomorphism in* $\mathrm{St}(k)$

$$\mathbf{Loc}_n(K) \simeq t_0\mathbb{R}\mathbf{Loc}_n(K).$$

PROOF. The truncation t_0 being the right derived functor of a right Quillen functor commutes with derived exponentials. Therefore, we have

$$t_0\mathbb{R}\mathbf{Loc}_n(K) \simeq (t_0\mathbb{R}\mathbf{Loc}_n)^{\mathbb{R}K} \simeq (\mathbf{Vect}_n)^{\mathbb{R}K} \in \mathrm{St}(k).$$

The stack $\mathbf{Vect}_n$ is 1-truncated, and therefore we also have natural isomorphisms

$$(\mathbf{Vect}_n)^{\mathbb{R}K}(A) \simeq Map_{SSet}(K, \mathbf{Vect}_n(A)) \simeq \underline{Hom}(\Pi_1(K), \Pi_1(\mathbf{Vect}_n(A))).$$

This equivalence is natural in A and provides the isomorphism of the lemma. □

Our next step is to give a more geometrical interpretation of the D^--stack $\mathbb{R}\mathbf{Loc}_n(K)$, in terms of certain local systems of objects on the topological realization of K.

Let now X be a $\mathbb{U}$-small topological space. Let $A \in sk-Mod$, and let us define a model category $A-Mod_s(X)$, of A-modules over X. The category $A-Mod_s(X)$ is simply the category of presheaves on X with values in $A-Mod_s$. The model structure on $A-Mod_s(X)$ is of the same type as the local projective model structure on simplicial presheaves. We first define an intermediate model structure on $A-Mod_s(X)$ for which equivalences (resp. fibrations) are morphism $\mathcal{E} \longrightarrow \mathcal{F}$ in $A-Mod_s(X)$ such that for any open subset $U \subset X$ the induced morphism $\mathcal{E}(U) \longrightarrow \mathcal{F}(U)$ is an equivalence (resp. a fibration). This model structure exists as $A-Mod_s$ is a $\mathbb{U}$-cofibrantly generated model category, and let us call it the *strong* model structure. The final model structure on $A-Mod_s(X)$ is the one for which cofibrations are the same cofibrations as for the strong model structure, and equivalences are the morphisms $\mathcal{E} \longrightarrow \mathcal{F}$ such that for any point $x \in X$ the induced morphism on the stalks $\mathcal{E}_x \longrightarrow \mathcal{F}_x$ is an equivalence in $A-Mod_s$. The existence of this model structure is proved the same way as for the case of simplicial presheaves (we can also use the forgetful functor $A-Mod_s(X) \longrightarrow SPr(X)$ to lift the local projective model structure on $SPr(X)$ in a standard way).

For a commutative simplicial k-algebra A, we consider $A-Mod_s(X)^c_W$, the subcategory of $A-Mod_s(X)$ consisting of cofibrant objects and equivalences between them. For a morphism of commutative simplicial k-algebras $A \longrightarrow B$, we have a base change functor

$$\begin{array}{ccc} A-Mod_s(X) & \longrightarrow & B-Mod_s(X) \\ \mathcal{E} & \mapsto & \mathcal{E}\otimes_A B \end{array}$$

which is a left Quillen functor, and therefore induces a well defined functor

$$-\otimes_A B : A-Mod_s(X)^c_W \longrightarrow B-Mod_s(X)^c_W.$$

This defines a lax functor $A \mapsto A-Mod_s(X)^c_W$, from $sk-Alg$ to Cat, which can be strictified in the usual way. We will omit to mention explicitly this strictification here and will do as if $A \mapsto A-Mod_s(X)^c_W$ does define a genuine functor $sk-Alg \longrightarrow Cat$.

We then define a sub-functor of $A \mapsto A-Mod_s(X)^c_W$ in the following way. For $A \in sk-Alg$, let $A-Loc_n(X)$ be the full subcategory of $A-Mod_s(X)^c_W$ consisting of objects $\mathcal{E}$, such that there exists an open covering $\{U_i\}$ on X, such that each restriction $\mathcal{E}_{|U_i}$ is isomorphic in $\mathrm{Ho}(A-Mod_s(U_i))$ to a constant presheaf with fibers a projective A-module of rank n (i.e. projective A-module E such that $\pi_0(E)$ is a projective $\pi_0(A)$-module of rank n). This defines a sub-functor of $A \mapsto A-Mod_s(X)^c_W$, and thus a functor from $sk-Alg$ to Cat. Applying the nerve functor we obtain a simplicial presheaf $\mathbb{R}\mathbf{Loc}_n(X) \in k-D^-Aff^{\sim,et}$, defined by

$$\mathbb{R}\mathbf{Loc}_n(X)(A) := N(A-Loc_n(X)).$$

PROPOSITION 2.2.6.5. *Let K be a simplicial set in $\mathbb{U}$ and $|K|$ be its topological realization. The simplicial presheaf $\mathbb{R}\mathbf{Loc}_n(|K|)$ is a D^--stack, and there exists an isomorphism in $D^-\mathrm{St}(k)$*

$$\mathbb{R}\mathbf{Loc}_n(|K|) \simeq \mathbb{R}\mathbf{Loc}_n(K).$$

PROOF. We first remark that if $f : X \longrightarrow X'$ is a homotopy equivalence of topological spaces, then the induced morphism

$$f^* : \mathbb{R}\mathbf{Loc}_n(X') \longrightarrow \mathbb{R}\mathbf{Loc}_n(X')$$

is an equivalence of simplicial presheaves. Indeed, a standard argument reduces to the case where f is the projection $X \times [0,1] \longrightarrow X$, and then to $[0,1] \longrightarrow *$, for which one can use the same argument as in [**To3**, Lem. 2.16].

Let us first prove that $\mathbb{R}\mathbf{Loc}_n(|K|)$ is a D^--stack, and for this we will prove that the simplicial presheaf $\mathbb{R}\mathbf{Loc}_n(|K|)$ can be written, in $SPr(k-D^-Aff)$, as a certain homotopy limit of D^--stacks. As D^--stacks in $SPr(k-D^-Aff)$ are stable by homotopy limits this will prove what we want.

Let U_* an open hypercovering of $|K|$ such that each U_i is a coproduct of contractible open subsets in $|K|$ (such a hypercover exists as $|K|$ is a locally contractible space). It is not hard to show using Cor. B.0.8 and standard cohomological descent, that for any $A \in sk - Alg$, the natural morphism

$$N(A - Mod_s(|K|)^c_W) \longrightarrow Holim_{m\in\Delta} N(A - Mod_s(U_m)^c_W)$$

is an isomorphism in $\mathrm{Ho}(SSet)$. From this, we easily deduce that the natural morphism

$$\mathbb{R}\mathbf{Loc}_n(|K|) \longrightarrow Holim_{m\in\Delta}\mathbb{R}\mathbf{Loc}_n(U_m)$$

is an isomorphism in $\mathrm{Ho}(SPr(k - D^- Aff))$. Therefore, we are reduced to show that $\mathbb{R}\mathbf{Loc}_n(U_m)$ is a D^--stack. But, U_m being a coproduct of contractible topological spaces, $\mathbb{R}\mathbf{Loc}_n(U_m)$ is a product of some $\mathbb{R}\mathbf{Loc}_n(U)$ for some contractible space U. Moreover $\mathbb{R}\mathbf{Loc}_n(U)$ is naturally isomorphic in $\mathrm{Ho}(SPr(k-D^- Aff))$ to $\mathbb{R}\mathbf{Loc}_n(*) = \mathbf{Vect}_n$. As we know that $\mathbf{Vect}_n$ is a D^--stack, this shows that the simplicial presheaf $\mathbb{R}\mathbf{Loc}_n(|K|)$ is a homotopy limit of D^--stacks and thus is itself a D^--stack.

We are left to prove that $\mathbb{R}\mathbf{Loc}_n(|K|)$ and $\mathbb{R}\mathbf{Loc}_n(K)$ are isomorphic. But, we have seen that

$$\mathbb{R}\mathbf{Loc}_n(|K|) \simeq Holim_{m\in\Delta}\mathbb{R}\mathbf{Loc}_n(U_m),$$

for an open hypercover U_* of $|K|$, such that each U_m is a coproduct of contractible open subsets. We let $K' := \pi_0(U_*)$ be the simplicial set of connected components of U_*, and thus

$$Holim_{m\in\Delta}\mathbb{R}\mathbf{Loc}_n(U_m) \simeq Holim_{m\in\Delta}\mathbb{R}\mathbf{Loc}_n(K'_m),$$

where K'_m is considered as a discrete topological space. We have thus proved that

$$\mathbb{R}\mathbf{Loc}_n(|K|) \simeq Holim_{m\in\Delta}\prod^h_{K'_m}\mathbf{Vect}_n \simeq (\mathbf{Vect}_n)^{\mathbb{R}K'}.$$

But, by [**To3**, Lem. 2.10], we know that $|K|$ is homotopically equivalent to $|K'|$, and thus that K is equivalent to K'. This implies that

$$\mathbb{R}\mathbf{Loc}_n(|K|) \simeq (\mathbf{Vect}_n)^{\mathbb{R}K'} \simeq (\mathbf{Vect}_n)^{\mathbb{R}K} \simeq \mathbb{R}\mathbf{Loc}_n(|K|).$$

□

We will now describe the cotangent complex of $\mathbb{R}\mathbf{Loc}_n(K)$. For this, we fix a global point

$$E : * \longrightarrow \mathbb{R}\mathbf{Loc}_n(K),$$

which by Lem. 2.2.6.4 corresponds to a functor

$$E : \Pi_1(K) \longrightarrow \Pi_1(\mathbf{Vect}_n(k)),$$

where $\Pi_1(\mathbf{Vect}_n(k))$ can be identified with the groupoid of rank n projective k-modules. The object E is thus a local system of rank n projective k-modules on K in the usual sense. We will compute the cotangent complex $\mathbb{L}_{\mathbb{R}\mathbf{Loc}_n(K),E} \in \mathrm{Ho}(Sp(sk - Mod))$ (recall that $\mathrm{Ho}(Sp(sk - Mod))$ can be naturally identified with the unbounded derived category of k). For this, we let $E \otimes_k E^\vee$ be the local system on K of endomorphisms of E, and $C_*(K, E \otimes_k E^\vee)$ will be the complex of homology of K with coefficients in the local system $E \otimes_k E^\vee$. We consider $C_*(K, E \otimes_k E^\vee)$ as an unbounded complex of k-modules, and therefore as an object in $\mathrm{Ho}(Sp(sk - Mod))$.

Proposition 2.2.6.6. *There exists an isomorphism in* $\mathrm{Ho}(Sp(sk - Mod)) \simeq \mathrm{Ho}(C(k))$

$$\mathbb{L}_{\mathbb{R}\mathbf{Loc}_n(K),E} \simeq C_*(K, E \otimes_k E^\vee)[-1].$$