Homotopical Algebraic Geometry II: geometric stacks and applications

PROOF. Let $M \in sk - Mod$, and let us consider the simplicial set

$$\mathbb{D}er_E(\mathbb{R}\mathbf{Loc}_n(K), M),$$

of derivations of $\mathbb{R}\mathbf{Loc}_n(K)$ at the point E and with coefficients in M. By definition of $\mathbb{R}\mathbf{Loc}_n(K)$ we have

$$\mathbb{D}er_E(\mathbb{R}\mathbf{Loc}_n(K), M) \simeq Map_{SSet/\mathbf{Vect}_n(k)}(K, \mathbf{Vect}_n(k \oplus M)),$$

where $K \longrightarrow \mathbf{Vect}_n(k)$ is given by the object E, and $\mathbf{Vect}_n(k \oplus M) \longrightarrow \mathbf{Vect}_n(k)$ is the natural projection. At this point we use Prop. A.0.6 in order to describe, functorially in M, the morphism $\mathbf{Vect}_n(k \oplus M) \longrightarrow \mathbf{Vect}_n(k)$. For this, we let $\mathcal{G}(k)$ to be the groupoid of projective k-modules of rank n. We also define an S-category $\mathcal{G}(k \oplus M)$ in the following way. Its objects are projective k-modules of rank n. The simplicial set of morphisms in $\mathcal{G}(k \oplus M)$ between two such k-modules E and E' is defined to be

$$\mathcal{G}(k \oplus M)_{(E,E')} := \underline{Hom}^{Eq}_{(k \oplus M)-Mod_s}(E \oplus (E \otimes_k M), E' \oplus (E' \otimes_k M)),$$

the simplicial set of equivalences from $E \oplus (E \otimes_k M)$ to $E' \oplus (E' \otimes_k M)$, in the model category $(k \oplus M) - Mod_s$. It is important to note that $E \oplus (E \otimes_k M)$ is isomorphic to $E \otimes_k (k \oplus M)$, and therefore is a cofibrant object in $(k \oplus M) - Mod_s$ (as the base change of a cofibrant object E in $sk - Mod$). There exists a natural morphism of S-categories

$$\mathcal{G}(k \oplus M) \longrightarrow \mathcal{G}(k)$$

being the identity on the set of objects, and the composition of natural morphisms

$$\underline{Hom}^{Eq}_{(k \oplus M)-Mod_s}(E \oplus (E \otimes_k M), E' \oplus (E' \otimes_k M)) \longrightarrow \underline{Hom}^{Eq}_{k-Mod_s}(E, E') \longrightarrow$$

$$\pi_0(\underline{Hom}^{Eq}_{k-Mod_s}(E, E')) \simeq \mathcal{G}(k)_{(E,E')}.$$

on the simplicial sets of morphisms. Clearly, Prop. A.0.6 and its functorial properties, show that the morphism

$$\mathbf{Vect}_n(k \oplus M) \longrightarrow \mathbf{Vect}_n(k)$$

is equivalent to

$$N(\mathcal{G}(k \oplus M)) \longrightarrow N(\mathcal{G}(k)),$$

and in a functorial way in M.

For any $E \in \mathcal{G}(k)$, we can consider the classifying simplicial set $K(E \otimes_k E^\vee \otimes_k M, 1)$ of the simplicial abelian group $E \otimes_k E^\vee \otimes_k M$, and for any isomorphism of projective k-modules of rank n, $E \simeq E'$, the corresponding isomorphism of simplicial set

$$K(E \otimes_k E^\vee \otimes_k M, 1) \simeq K(E' \otimes_k (E')^\vee \otimes_k M, 1).$$

This defines a local system L of simplicial sets on the groupoid $\mathcal{G}(k)$, for which the total space

$$Hocolim_{\mathcal{G}(k)} L \longrightarrow N(\mathcal{G}(k))$$

is easily seen to be isomorphic to the projection

$$N(\mathcal{G}(k \oplus M)) \longrightarrow N(\mathcal{G}(k)).$$

The conclusion is that the natural projection

$$\mathbf{Vect}_n(k \oplus M) \longrightarrow \mathbf{Vect}_n(k)$$

is equivalent, functorially in M, to the total space of the local system $E \mapsto K(E \otimes_k E^\vee \otimes_k M, 1)$ on $\mathbf{Vect}_n(k)$. Therefore, one finds a natural equivalence of simplicial sets

$$\mathbb{D}er_E(\mathbb{R}\mathbf{Loc}_n(K), M) \simeq Map_{SSet/\mathbf{Vect}_n(k)}(K, \mathbf{Vect}_n(k \oplus M)) \simeq$$

$$C^*(K, E\otimes_k E^\vee \otimes_k M[1]) \simeq Map_{C(k)}(C_*(K, E\otimes_k E^\vee)[-1], M).$$

As this equivalence is functorial in M, this shows that

$$\mathbb{L}_{\mathbb{R}\mathbf{Loc}_n(K),E} \simeq C_*(K, E\otimes_k E^\vee)[-1],$$

as required. □

REMARK 2.2.6.7. An important consequence of Prop. 2.2.6.6 is that the D^--stack depends on strictly more than the fundamental groupoid of K. Indeed, the tangent space of $\mathbb{R}\mathbf{Loc}_n(K)$ at a global point corresponding to a local system E on K is $C^*(K, E\otimes_k E^\vee)[1]$, which can be non-trivial even when K is simply connected. In general, there is a closed immersion of D^--stacks (Prop. 2.2.4.7)

$$it_0(\mathbb{R}\mathbf{Loc}_n(K)) \longrightarrow \mathbb{R}\mathbf{Loc}_n(K),$$

which on the level of tangent spaces induces the natural morphism

$$\tau_{\leq 0}(C^*(K, E\otimes_k E^\vee)[1]) \longrightarrow C^*(K, E\otimes_k E^\vee)[1]$$

giving an isomorphism on H^0 and H^1. This shows that the D^--stack $\mathbb{R}\mathbf{Loc}_n(K)$ contains strictly more information than the usual Artin stack of local systems on K, and does encode some higher homotopical invariants of K.

2.2.6.2. Algebras over an operad. Recall (e.g. from [**Re, Sp**]) the notions of operads and algebras over them, as well as their model structures. Let $\mathcal{O}$ be an operad in $k-Mod$, the category of k-modules. We assume that for any n, the k-module $\mathcal{O}(n)$ is projective. Then, for any $A \in sk-Alg$, we can consider $\mathcal{O}\otimes_k A$, which is an operad in the symmetric monoidal category $A-Mod_s$ of A-modules. We can therefore consider $\mathcal{O}\otimes_k A-Alg_s$, the category of $\mathcal{O}$-algebras in $A-Mod_s$. According to [**Hin1**] the category $\mathcal{O}\otimes_k A-Alg_s$ can be endowed with a natural structure of a $\mathbb{U}$-cofibrantly generated model category for which equivalences and fibrations are defined on the underlying objects in $sk-Mod$. For a morphism $A \longrightarrow B$ in $sk-Alg$, there is a Quillen adjunction

$$B\otimes_A - : \mathcal{O}\otimes_k A-Alg_s \longrightarrow \mathcal{O}\otimes_k B-Alg_s \qquad \mathcal{O}\otimes_k A-Alg_s \longleftarrow \mathcal{O}\otimes_k B-Alg_s : F,$$

where F is the forgetful functor. The rule $A \mapsto \mathcal{O}\otimes_k A-Alg_s$ together with base change functors $B\otimes_A -$ is almost a left Quillen presheaf in the sense of Appendix B, except that the associativity of composition of base change is only valid up to a natural isomorphism. However, the standard strictification techniques can be applied in order to replace, up to a natural equivalence, this by a genuine left Quillen presheaf. We will omit to mention this replacement and will proceed as if $A \mapsto \mathcal{O}\otimes_k A-Alg_s$ actually defines a left Quillen presheaf on $k-D^-Aff$.

For any $A \in sk-Alg$, we consider $\mathcal{O}-Alg(A)$, the category of cofibrant objects in $\mathcal{O}\otimes_k A-Alg_s$ and equivalences between them. Finally, we let $\mathcal{O}-Alg_n(A)$ be the full subcategory consisting of objects $B \in \mathcal{O}-Alg(A)$, such that the underlying A-module of B is a vector bundle of rank n (i.e. that B is a strong A-module, and $\pi_0(B)$ is a projective $\pi_0(A)$-module of rank n). The base change functors clearly preserves the sub-categories $\mathcal{O}-Alg_n(A)$, and we get this way a well defined presheaf of $\mathbb{V}$-small categories

$$\begin{array}{ccc} k-D^-Aff & \longrightarrow & Cat_{\mathbb{V}} \\ A & \mapsto & \mathcal{O}-Alg_n(A). \end{array}$$

Applying the nerve functor we obtain a simplicial presheaf

$$\begin{array}{ccc} k-D^-Aff & \longrightarrow & SSet_{\mathbb{V}} \\ A & \mapsto & N(\mathcal{O}-Alg_n(A)). \end{array}$$

This simplicial presheaf will be denoted by $\mathbf{Alg}_n^{\mathcal{O}}$, and is considered as an object in $k-D^-Aff^{\sim,\acute{e}t}$.

PROPOSITION 2.2.6.8. (1) *The object* $\mathbf{Alg}_n^{\mathcal{O}} \in k-D^-Aff^{\sim,\acute{e}t}$ *is a* D^--*stack.*

(2) *The* D^--*stack* $\mathbf{Alg}_n^{\mathcal{O}}$ *is* 1-*geometric and quasi-compact.*

PROOF. (1) The proof relies on the standard argument based on Cor. B.0.8, and is left to the reader.

(2) We consider the natural morphism of D^--stacks

$$p : \mathbf{Alg}_n^{\mathcal{O}} \longrightarrow \mathbf{Vect}_n,$$

defined by forgetting the $\mathcal{O}$-algebra structure. Precisely, it sends an object $B \in \mathcal{O}-Alg_n(A)$ to its underlying A-modules, which is a rank n-vector bundle by definition. We are going to prove that the morphism p is a representable morphism, and this will imply the result as $\mathbf{Vect}_n$ is already known to be 1-geometric and quasi-compact. For this, we consider the natural morphism $* \longrightarrow \mathbf{Vect}_n$, which is a smooth 1-atlas for $\mathbf{Vect}_n$, and consider the homotopy pullback

$$\widetilde{\mathbf{Alg}_n^{\mathcal{O}}} := \mathbf{Alg}_n^{\mathcal{O}} \times^h_{\mathbf{Vect}_n} *.$$

It is enough by Prop. 1.3.3.4 to show that $\widetilde{\mathbf{Alg}_n^{\mathcal{O}}}$ is a representable stack. For this, we use [**Re**, Thm. 1.1.5] in order to show that the D^--stack $\widetilde{\mathbf{Alg}_n^{\mathcal{O}}}$ is isomorphic in $D^-\mathrm{St}(k)$ to $Map(\mathcal{O}, \underline{End}(k^n))$, defined as follows. For any $A \in sk-Alg$, we consider $\underline{End}(A^n)$, the operad in $A-Mod_s$ of endomorphisms of the object A^n (recall that for $M \in A-Mod_s$, the operad $\underline{End}(M)$ is defined by $\underline{End}(M)(n) := \underline{Hom}_A(M^{\otimes_k n}, M)$). We let $Q\mathcal{O}$ be a cofibrant replacement of the operad $\mathcal{O}$ in the model category $sk-Mod$. Finally, $Map(\mathcal{O}, \underline{End}(k^n))$ is defined as

$$\begin{array}{ccc} sk-Alg & \longrightarrow & SSet \\ A & \mapsto & Map(\mathcal{O}, \underline{End}(k^n))(A) := \underline{Hom}_{Op}(Q\mathcal{O}, \underline{End}(A^n)), \end{array}$$

where $\underline{Hom}_{Op}$ denotes the simplicial set of morphism in the simplicial category of operads in $sk-Mod$. As we said, [**Re**, Thm. 1.1.5] implies that $\widetilde{\mathbf{Alg}_n^{\mathcal{O}}}$ is isomorphic to $Map(\mathcal{O}, \underline{End}(k^n))$. Therefore, it remain to show that $Map(\mathcal{O}, \underline{End}(k^n))$ is a representable D^--stack.

For this, we can write $\mathcal{O}$, up to an equivalence, as the homotopy colimit of free operads

$$\mathcal{O} \simeq Hocolim_i \mathcal{O}_i.$$

Then, we have

$$Map(\mathcal{O}, \underline{End}(k^n)) \simeq Holim_i Map(\mathcal{O}_i, \underline{End}(k^n))$$

and as representable D^--stacks are stable by homotopy limits, we are reduced to the case where $\mathcal{O}$ is a free operad. This means that there exists an integer $m \geq 0$, such that for any other operad $\mathcal{O}'$ in $sk-Mod$, we have a natural isomorphism of sets

$$Hom_{Op}(\mathcal{O}, \mathcal{O}') \simeq Hom(k, \mathcal{O}'(m)).$$

In particular, we find that the D^--stack $Map(\mathcal{O}, \underline{End}(k^n))$ is isomorphic to the D^--stack sending $A \in sk-Alg$ to the simplicial set $\underline{Hom}_{A-Mod_s}((A^n)^{\otimes_k m}, A)$, of morphisms from the A-module $(A^n)^{\otimes_k m}$ to A. But this last D^--stack is clearly isomorphic to $\mathbb{R}\underline{Spec}\, B$, where B is a the free commutative simplicial k-algebra on k^{n^m}, or in other words a polynomial algebra over k with n^m variables. □

We still fix an operad $\mathcal{O}$ in $k-Mod$ (again requiring $\mathcal{O}(m)$ to be a projective k-module for any m), and we let B be an $\mathcal{O}$-algebra in $k-Mod$, such that B is projective of rank n as a k-module. This defines a well defined morphism of stacks

$$B : * \longrightarrow \mathbf{Alg}_n^{\mathcal{O}}.$$

We are going to describe the cotangent complex of $\mathbf{Alg}_n^{\mathcal{O}}$ at B using the notion of (derived) derivations for $\mathcal{O}$-algebras.

For this, recall from [**Goe-Hop**] the notion of $\mathcal{O}$-derivations from B and with coefficients in a B-module. For any B-module M, one can define the square zero extension $B \oplus M$ of B by M, which is another $\mathcal{O}$-algebra together with a natural projection $B \oplus M \longrightarrow M$. The k-module of derivations from B to M is defined by

$$Der_k^{\mathcal{O}}(B, M) := Hom_{\mathcal{O}-Alg/B}(B, B \oplus M).$$

In the same way, for any $\mathcal{O}$-algebra B' with a morphism $B' \longrightarrow B$ we set

$$Der_k^{\mathcal{O}}(B', M) := Hom_{\mathcal{O}-Alg/B}(B', B \oplus M).$$

The functor $B' \mapsto Der_k^{\mathcal{O}}(B', M)$ can be derived on the left, to give a functor

$$\mathbb{R}Der_k^{\mathcal{O}}(-, M) : \mathrm{Ho}(\mathcal{O}-Alg_s/B)^{op} \longrightarrow \mathrm{Ho}(SSet_{\mathbb{V}}).$$

As shown in [**Goe-Hop**], the functor $M \mapsto \mathbb{R}Der_k^{\mathcal{O}}(B, M)$ is co-represented by an object $\mathbb{L}_B^{\mathcal{O}} \in \mathrm{Ho}(sB-Mod)$, thus there is a natural isomorphism in $\mathrm{Ho}(SSet_{\mathbb{V}})$

$$\mathbb{R}Der_k^{\mathcal{O}}(B, M) \simeq Map_{sB-Mod}(\mathbb{L}_B^{\mathcal{O}}, M).$$

The category $sB-Mod$ of simplicial B-modules is a closed model category for which equivalences and fibrations are detected in $sk-Mod$. Furthermore, the category $sB-Mod$ is tensored and co-tensored over $sk-Mod$ making it into a $sk-Mod$-model category in the sense of [**Ho1**]. Passing to model categories of spectra, we obtain a model category $Sp(sB-Mod)$ which is a $Sp(sk-Mod)$-model category. We will then set

$$\mathbb{R}\underline{Der}_k^{\mathcal{O}}(B, M) := \mathbb{R}\underline{Hom}_{Sp(sB-Mod)}(\mathbb{L}_B^{\mathcal{O}}, M) \in \mathrm{Ho}(Sp(sk-Mod)),$$

where $\mathbb{R}\underline{Hom}_{Sp(sB-Mod)}$ denotes the $\mathrm{Ho}(Sp(sk-Mod))$-enriched derived Hom of $Sp(sB-Mod)$. Note that $\mathrm{Ho}(sB-Mod)$ is equivalent to the unbounded derived category of B-modules, and as well that $\mathrm{Ho}(Sp(sk-Mod))$ is equivalent to the unbounded derived category of k-modules.

PROPOSITION 2.2.6.9. *With the notations as above, the tangent complex of the D^--stack $\mathbf{Alg}_n^{\mathcal{O}}$ at the point B is given by*

$$\mathbb{T}_{\mathbf{Alg}_n^{\mathcal{O}}, B} \simeq \mathbb{R}\underline{Der}_k^{\mathcal{O}}(B, B)[1] \in \mathrm{Ho}(Sp(sk-Mod)).$$

PROOF. We have an isomorphism in $\mathrm{Ho}(Sp(sk-Mod))$

$$\mathbb{T}_{\mathbf{Alg}_n^{\mathcal{O}}, B} \simeq \mathbb{T}_{\Omega_B \mathbf{Alg}_n^{\mathcal{O}}, B}[1].$$

Using Prop. A.0.6, we see that the D^--stack $\Omega_B \mathbf{Alg}_n^{\mathcal{O}}$ can be described as

$$\begin{array}{ccc} sk-Alg & \longrightarrow & SSet \\ A & \mapsto & Map_{\mathcal{O}-Alg_s}^{eq}(B, B \otimes_k A) \end{array}$$

where $Map_{\mathcal{O}-Alg_s}^{eq}$ denotes the mapping space of equivalences in the category of $\mathcal{O}$-algebras in $sA-Mod$. In particular, we see that for $M \in sk-Mod$, the simplicial set $\mathbb{D}er_{\Omega_B \mathbf{Alg}_n^{\mathcal{O}}}(B, M)$ is naturally equivalent to the homotopy fiber, taken at the identity, of the morphism

$$Map_{\mathcal{O}-Alg_s}(B, B \otimes_k (k \oplus M)) \longrightarrow Map_{\mathcal{O}-Alg_s}(B, B).$$

The $\mathcal{O}$-algebra $B \otimes_k (k \oplus M)$ can be identified with $B \oplus (M \otimes_k B)$, the square zero extension of B by $M \otimes_k B$, as defined in [**Goe-Hop**]. Therefore, by definition of derived derivations this homotopy fiber is naturally equivalent to $\mathbb{R}Der_k^{\mathcal{O}}(B, M)$. This shows that

$$\mathbb{D}er_{\Omega_B \mathbf{Alg}_n^{\mathcal{O}}}(B, M) \simeq \mathbb{R}Der_k^{\mathcal{O}}(B, M \otimes_k B) \simeq Map_{Sp(sB-Mod)}(\mathbb{L}_B^{\mathcal{O}}, M \otimes_k B).$$

This implies that for any $N \in sk-Mod$, we have

$$Map_{Sp(sk-Mod)}(N, \mathbb{T}_{\Omega_B \mathbf{Alg}_n^{\mathcal{O}}, B}) \simeq \mathbb{D}er_{\Omega_B \mathbf{Alg}_n^{\mathcal{O}}}(B, N^{\vee}) \simeq Map_{Sp(sB-Mod)}(\mathbb{L}_B^{\mathcal{O}}, N^{\vee} \otimes_k B) \simeq$$

$$Map_{Sp(sk-Mod)}(N, \mathbb{R}\underline{Hom}_{Sp(sB-Mod)}(\mathbb{L}_B^{\mathcal{O}}, B)) \simeq Map_{Sp(sk-Mod)}(N, \mathbb{R}\underline{Der}_k(B, B)).$$

The Yoneda lemma implies the existence of a natural isomorphism in $\mathrm{Ho}(Sp(sk-Mod))$

$$\mathbb{T}_{\Omega_B \mathbf{Alg}_n^{\mathcal{O}}, B} \simeq \mathbb{R}\underline{Der}_k(B, B).$$

We thus we have

$$\mathbb{T}_{\mathbf{Alg}_n^{\mathcal{O}}, B} \simeq \mathbb{R}\underline{Der}_k(B, B)[1].$$

□

REMARK 2.2.6.10. The proof of Prop. 2.2.6.9 actually shows that for any $M \in sk-Mod$ we have

$$\mathbb{R}\underline{Hom}_k^{Sp}(\mathbb{L}_{\mathbf{Alg}_n^{\mathcal{O}}, B}, M) \simeq \mathbb{R}Der_k(B, B \otimes_k M)[1].$$

2.2.6.3. Mapping D^--stacks. We let X be a stack in $\mathrm{St}(k)$, and F be an n-geometric D^--stack. We are going to investigate the geometricity of the D^--stack of morphisms from $i(X)$ to F.

$$\mathbf{Map}(X, F) := \mathbb{R}_{\acute{e}t}\underline{\mathcal{H}om}(i(X), F) \in D^-\mathrm{St}(k).$$

Recall that $\mathbb{R}_{\acute{e}t}\underline{\mathcal{H}om}$ denotes the internal Hom of the category $D^-\mathrm{St}(k)$.

THEOREM 2.2.6.11. *With the notations above, we assume the following three conditions are satisfied.*

(1) *The stack*

$$t_0(\mathbf{Map}(X, F)) \simeq \mathbb{R}_{\acute{e}t}\underline{\mathcal{H}om}(X, t_0(F)) \in \mathrm{St}(k)$$

is n-geometric.

(2) *The D^--stack $\mathbf{Map}(X, F)$ has a cotangent complex.*

(3) *The stack X can be written in $\mathrm{St}(k)$ has a homotopy colimit $Hocolim_i U_i$, where U_i is a affine scheme, flat over $Spec\, k$.*

Then the D^--stack $\mathbf{Map}(X, F)$ is n-geometric.

PROOF. The only if part is clear. Let us suppose that $\mathbf{Map}(X, F)$ satisfies the three conditions. To prove that it is an n-geometric D^--stack we are going to lift an n-atlas of $t_0(\mathbf{Map}(X, F))$ to an n-atlas of $\mathbf{Map}(X, F)$. For this we use the following special case of J.Lurie's representability criterion, which can be proved using the material of this paper (see Appendix C).

THEOREM 2.2.6.12. *(J. Lurie, see* [**Lu1**] *and Appendix C) Let F be a D^--stack. The following conditions are equivalent.*

(1) *F is an n-geometric D^--stack.*

(2) *F satisfies the following three conditions.*
   (a) *The truncation $t_0(F)$ is an Artin $(n+1)$-stack.*
   (b) *F has an obstruction theory.*

(c) *For any $A \in sk-Alg$, the natural morphism*

$$\mathbb{R}F(A) \longrightarrow Holim_k \mathbb{R}F(A_{\leq k})$$

is an isomorphism in $\mathrm{Ho}(SSet)$.

We need to prove that $\mathbf{Map}(X,F)$ satisfies the conditions $(a)-(c)$ of theorem 2.2.6.12. Condition (a) is clear by assumption. The existence of a cotangent complex for $\mathbf{Map}(X,F)$ is guaranteed by assumption. The fact that $\mathbf{Map}(X,F)$ is moreover inf-cartesian follows from the general fact.

LEMMA 2.2.6.13. *Let F be a D^--stack which is inf-cartesian. Then, for any D^--stack F', the D^--stack $\mathbb{R}_{\acute{e}t}\underline{\mathcal{H}om}(F',F)$ is inf-cartesian.*

PROOF. Writing F' has a homotopy colimit of representable D^--stacks

$$F' \simeq Hocolim_i U_i,$$

we find

$$\mathbb{R}_{\acute{e}t}\underline{\mathcal{H}om}(F',F) \simeq Holim_i \mathbb{R}_{\acute{e}t}\underline{\mathcal{H}om}(U_i,F).$$

As being inf-cartesian is clearly stable by homotopy limits, we reduce to the case where $F' = \mathbb{R}\underline{Spec}\,B$ is a representable D^--stack. Let $A \in sk-Alg$, M be an A-module with $\pi_0(M)=0$, and $d \in \pi_0(\mathbb{D}er(A,M))$ be a derivation. Then, the commutative square

$$\begin{array}{ccc} \mathbb{R}_{\acute{e}t}\underline{\mathcal{H}om}(F',F)(A\oplus_d \Omega M) & \longrightarrow & \mathbb{R}_{\acute{e}t}\underline{\mathcal{H}om}(F',F)(A) \\ \downarrow & & \downarrow \\ \mathbb{R}_{\acute{e}t}\underline{\mathcal{H}om}(F',F)(A) & \longrightarrow & \mathbb{R}_{\acute{e}t}\underline{\mathcal{H}om}(F',F)(A\oplus M) \end{array}$$

is equivalent to the commutative square

$$\begin{array}{ccc} \mathbb{R}F((A\oplus_d \Omega M)\otimes^{\mathbb{L}}_k B) & \longrightarrow & \mathbb{R}F(A\otimes^{\mathbb{L}}_k B) \\ \downarrow & & \downarrow \\ \mathbb{R}F(A\otimes^{\mathbb{L}}_k B) & \longrightarrow & \mathbb{R}F((A\oplus M)\otimes^{\mathbb{L}}_k B). \end{array}$$

Using the F is inf-cartesian with respect to $A\otimes^{\mathbb{L}}_k B \in sk-Alg$, and the derivation $d\otimes_k B \in \pi_0(\mathbb{D}er(A\otimes^{\mathbb{L}}_k B, M\otimes^{\mathbb{L}}_k B))$ implies that this last square is homotopy cartesian. This shows that $\mathbb{R}_{\acute{e}t}\underline{\mathcal{H}om}(F',F)$ is inf-cartesian. □

In order to finish the proof of Thm. 2.2.6.11 it only remain to show that $\mathbf{Map}(X,F)$ also satisfies the condition (c) of Thm. 2.2.6.12. For this, we write X as $Hocolim_i U_i$ with U_i affine and flat over k, and therefore $\mathbf{Map}(X,F)$ can be written as the homotopy limit $Holim_i \mathbf{Map}(U_i,F)$. In order to check condition (c) we can therefore assume that X is an affine scheme, flat over k. Let us write $X = Spec\,B$, with B a commutative flat k-algebra. Then, for any $A \in sk-Mod$, the morphism

$$\mathbb{R}\mathbf{Map}(X,F)(A) \longrightarrow Holim_k \mathbb{R}\mathbf{Map}(X,F)(A_{\leq k})$$

is equivalent to

$$\mathbb{R}F(A\otimes_k B) \longrightarrow Holim_k \mathbb{R}F((A_{\leq k})\otimes_k B).$$

But, as B is flat over k, we have $(A_{\leq k})\otimes_k B \simeq (A\otimes_k B)_{\leq k}$, and therefore the above morphism is equivalent to

$$\mathbb{R}F(A\otimes_k B) \longrightarrow Holim_k \mathbb{R}F((A\otimes_k B)_{\leq k})$$

and is therefore an equivalence by $(1)\Rightarrow(2)$ of Thm. 2.2.6.12 applied to F. □

The following corollaries are direct consequences of Thm. 2.2.6.11. The only non trivial part consists of proving the existence of a cotangent complex, which we will assume in the present version of this work.

COROLLARY 2.2.6.14. *Let X be a projective and flat scheme over* $Spec\, k$, *and* Y *a projective and smooth scheme over* $Spec\, k$. *Then, the* D^-*-stack* $\mathbf{Map}(i(X), i(Y))$ *is a 1-geometric* D^-*-scheme. Furthermore, for any morphism of schemes* $f : X \longrightarrow Y$, *the cotangent complex of* $\mathbf{Map}(i(X), i(Y))$ *at the point* f *is given by*

$$\mathbb{L}_{\mathbf{Map}(i(X),i(Y)),f} \simeq (C^*(X, f^*(T_Y)))^\vee,$$

where T_Y *is the tangent sheaf of* $Y \longrightarrow Spec\, k$.

Let us now suppose that $k = \mathbb{C}$ is the field of complex numbers, and let X be a smooth and projective variety. We will be interested in the sheaf X_{DR} of [**S1**], defined by $X_{DR}(A) := A_{red}$, for a commutative $\mathbb{C}$-algebra A. Recall that the stack $\mathcal{M}_{DR}(X)$ is defined as the stack of morphisms from X_{DR} to $\mathbf{Vect}_n$, and is identified with the stack of flat bundles on X (see [**S1**]). It is known that $\mathcal{M}_{DR}(X)$ is an Artin 1-stack.

COROLLARY 2.2.6.15. *The* D^-*-stack* $\mathbb{R}\mathcal{M}_{DR}(X) := \mathbf{Map}(i(X_{DR}), \mathbf{Vect}_n)$ *is 1-geometric. For a point* $E : * \longrightarrow \mathbb{R}\mathcal{M}_{DR}(X)$, *corresponding to a flat vector bundle* E *on* X, *the cotangent complex of* $\mathbb{R}\mathcal{M}_{DR}(X)$ *at* E *is given by*

$$\mathbb{L}_{\mathbb{R}\mathcal{M}_{DR}(X),E} \simeq (C^*_{DR}(X, E \otimes E^*))^\vee[-1],$$

where $C^*_{DR}(X, E \otimes E^*)$ *is the algebraic de Rham cohomology of* X *with coefficients in the flat bundle* $E \otimes E^*$.

Corollary 2.2.6.15 is only the beginning of the story; in fact we can also produce, in a similar way, $\mathbb{R}\mathcal{M}_{Dol}(X)$ and $\mathbb{R}\mathcal{M}_{Hod}(X)$, which are derived versions of the moduli stacks of Higgs bundles and λ-connections of [**S1**], and this would lead us to a *derived version of non-abelian Hodge theory.* We think this is very interesting research direction because these derived moduli also encode higher homotopical data in their tangent complexes. We hope to come back to this topic in a future work.

The following corollary is a direct consequence of Thm. 2.2.6.11. The only non trivial part consists of proving the existence of a cotangent complex, which we will assume in the present version of this work.

Corollary 2.2.6.14. *Let X be a projective and flat scheme over $Spec\,k$, and Y a projective and smooth scheme over $Spec\,k$. Then, the D^--stack $\mathbb{R}\mathrm{Map}(X, Y)$ is a 1-geometric D^--stack. Furthermore, for any morphism $f : X \longrightarrow Y$, the cotangent complex of $\mathbb{R}\mathrm{Map}(X, Y)$ at the point f is given by*

$$\mathbb{L}_{\mathbb{R}\mathrm{Map}(X,Y),f} \simeq (C^*(X, f^*(T_Y)))^\vee,$$

where T_Y is the tangent sheaf of $Y \longrightarrow Spec\,k$.

Let us now suppose that $k = \mathbb{C}$ is the field of complex numbers, and let X be a smooth and projective variety. We will be interested in the simplicial stack X_{DR} of [S1], defined by $X_{DR}(A) := A_{red}$, for a commutative $\mathbb{C}$-algebra A. Recall that the stack $\mathcal{M}_{DR}(X)$ is defined as the stack of morphisms from X_{DR} to Vect_n, and is identified with the stack of flat bundles on X (see [S1]). It is known that $\mathcal{M}_{DR}(X)$ is an Artin 1-stack.

Corollary 2.2.6.15. *The D^--stack $\mathbb{R}\mathcal{M}_{DR}(X) := \mathbb{R}\mathrm{Map}(X_{DR}, \mathrm{Vect}_n)$ is 1-geometric. For a point $E : * \longrightarrow \mathbb{R}\mathcal{M}_{DR}(X)$, corresponding to a flat vector bundle E on X, the cotangent complex of $\mathbb{R}\mathcal{M}_{DR}(X)$ at E is given by*

$$\mathbb{L}_{\mathbb{R}\mathcal{M}_{DR}(X),E} \simeq (C^*_{DR}(X, E \otimes E^\vee))^\vee[-1],$$

where $C^*_{DR}(X, E \otimes E^\vee)$ is the algebraic de Rham cohomology of X with coefficients in the flat bundle $E \otimes E^\vee$.

Corollary 2.2.6.15 is only the beginning of the story; in fact we can also produce, in a similar way, $\mathbb{R}\mathcal{M}_{Dol}(X)$ and $\mathbb{R}\mathcal{M}_{Hod}(X)$, which are derived versions of the moduli stacks of Higgs bundles and λ-connections of [S1], and this would lead to a *derived version of non-abelian Hodge theory*. We think this is very interesting research direction because these derived moduli also contain higher homotopical data in their tangent complexes. We hope to come back to this topic in a future work.

CHAPTER 2.3

Complicial algebraic geometry

In this chapter we present a second context of application of the general formalism of Part I, in which the base model category is $C(k)$, the category of unbounded complexes over some ring k of characteristic zero. Contrary to the previously considered applications, the general notions presented in §1.2 does not produce here notions which are very close to the usual ones for commutative rings. As a consequence the geometric intuition is here only a very loose guide.

We will present two different HAG contexts over $C(k)$. The first one is very weak in the sense that it is very easy for a stack to be geometric in this context (these geometric stacks will be called *weakly geometric*). The price to pay for this abundance of geometric stacks is that this context does not satisfy Artin's conditions and thus there is no good infinitesimal theory.

The second HAG context we consider is a bit closer to the geometric intuition, and satisfies the Artin's conditions so it behaves well infinitesimally.

Both of these "unbounded" contexts seem interesting as we are able to produce examples of geometric stacks which cannot be represented by geometric D^--stacks, i.e. by the kind of geometric stacks studied in the previous chapter.

In this chapter k will be a commutative $\mathbb{Q}$-algebra.

2.3.1. Two HA contexts

We let $C := C(k)$, the model category of unbounded complexes of k-modules in $\mathbb{U}$. The model structure on $C(k)$ is the projective one, for which fibrations are epimorphisms and equivalences are quasi-isomorphisms. The model category $C(k)$ is a symmetric monoidal model category for the tensor product of complexes. Furthermore, it is well known that our assumptions 1.1.0.1, 1.1.0.3, 1.1.0.2 and 1.1.0.4 are satisfied.

The category $Comm(C(k))$ is the usual model category of unbounded commutative differential graded algebras over k, for which fibrations are epimorphisms and equivalences are quasi-isomorphisms. The category $Comm(C(k))$ will be denoted by $k-cdga$, and its objects will simply be called cdga's. For $A \in k-cdga$, the category $A-Mod$ is the category of unbounded A-dg-modules, again with its natural model structure. In order to avoid confusion of notations we will denote the category $A-Mod$ by $A-Mod_{dg}$. Note that for a usual commutative k-algebra k' we have $k'-Mod_{dg} = C(k')$, whereas $k'-Mod$ will denote the usual category of k'-modules. Objects in $A-Mod_{dg}$ will be called *A-dg-modules*.

As for the case of simplicial algebras, we will set for any $E \in C(k)$

$$\pi_i(E) := H^{-i}(E).$$

When A is a k-cdga, the $\mathbb{Z}$-graded k-module $\pi_*(A)$ has a natural structure of a commutative graded k-algebra. In the same way for M an A-dg-module, $\pi_*(M)$ becomes a graded $\pi_*(A)$-module. Objects $A \in k-cdga$ such that $\pi_i(A) = 0$ for any

$i < 0$ will be called (-1)-connected. Any $A \in k - cdga$ possesses a (-1)-connected cover $A' \longrightarrow A$, which is such that $\pi_i(A') \simeq \pi_i(A)$ for any $i \geq 0$.

As in the context of derived algebraic geometry we will denote by $M \mapsto M[1]$ the suspension functor. In the same way, $M \mapsto M[n]$ is the n-times iterated suspension functor (when n is negative this means the n-times iterated loop functor).

The first new feature of complicial algebraic geometry is the existence of two interesting choices for the subcategory $\mathcal{C}_0$, both of them of particular interest depending on the context. We let $C(k)_{\leq 0}$ be the full subcategory of $C(k)$ consisting of complexes E such that $\pi_i(E) = 0$ for any $i < 0$ (or equivalently, such that $H^i(E) = 0$ for all $i > 0$, which explains better the notation). We also set $k - cdga_0$ to be the full subcategory of $k - cdga$ consisting of $A \in k - cdga$ such that $\pi_i(A) = 0$ for any $i \neq 0$. In the same way we denote by $k - cdga_{\leq 0}$ for the full subcategory of $k - cdga$ consisting of A such that $\pi_i(A) \equiv H^{-i}(A) = 0$ for any $i < 0$.

LEMMA 2.3.1.1. (1) *The triplet $(C(k), C(k), k - cdga)$ is a HA context.*
(2) *The triplet $(C(k), C(k)_{\leq 0}, k - cdga_0)$ is a HA context.*

PROOF. The only non-trivial point is to show that any $A \in k - cdga_0$ is $C(k)_{\leq 0}$-good in the sense of Def. 1.1.0.10. By definition A is equivalent to some usual commutative k-algebra, so we can replace A by k itself. We are then left to prove that the natural functor

$$\mathrm{Ho}(C(k))^{op} \longrightarrow \mathrm{Ho}((C(k)^{op}_{\leq 0})^\wedge)$$

is fully faithful.

To prove this, we consider the restriction functor

$$i^* : \mathrm{Ho}((C(k)^{op})^\wedge) \longrightarrow \mathrm{Ho}((C(k)^{op}_{\leq 0})^\wedge)$$

induced by the inclusion $i : C(k)_{\leq 0} \subset C(k)$. We restrict the functor i^* to the full sub-categories of corepresentable objects

$$i^* : \mathrm{Ho}((C(k)^{op})^\wedge)^{corep} \longrightarrow \mathrm{Ho}((C(k)^{op}_{\leq 0})^\wedge)^{corep}.$$

A precision here: we say that a functor $F : C(k)_{\leq 0} \longrightarrow SSet$ is *corepresentable* if it is of the form

$$D \mapsto Map_{C(k)}(E, i(D))$$

for some object $E \in C(k)$ (i.e. it belongs the essential image of $\mathrm{Ho}(C(k))^{op} \longrightarrow \mathrm{Ho}((C(k)^{op}_{\leq 0})^\wedge)$. We claim that this restricted functor i^* is an equivalence of categories. Indeed, an inverse f can be constructed by sending a functor $F : C(k)_{\leq 0} \longrightarrow SSet$ to the functor

$$\begin{array}{rccc} f(F): & C(k) & \longrightarrow & SSet \\ & D & \mapsto & Holim_{n \geq 0} \Omega^n F(D(\leq n)[n]), \end{array}$$

where $D(\leq n)$ is the naive truncation of D defined by $D(\leq n)_m = D_m$ if $m \leq n$ and $D(\leq n)_m = 0$ if $m > n$, and $\Omega^n F(D(\leq n)[n])$ is the n-fold loop space of the simplicial set $F(D(\leq n)[n])$, based at the natural point $* \simeq F(0) \to F(D(\leq n)[n])$. As any $D \in C(k)$ is functorially equivalent to the homotopy limit $Holim_n D(\leq n)$, it is easy to check that the functor f and i^* are inverse to each other.

To finish the proof, it is enough to notice that there exists a commutative diagram (up to a natural isomorphism)

$$\begin{array}{ccc} \mathrm{Ho}(C(k))^{op} & \xrightarrow{\mathbb{R}\underline{h}} & \mathrm{Ho}((C(k))^{\wedge})^{corep} \\ & \searrow & \downarrow i^* \\ & & \mathrm{Ho}((C(k)^{op}_{\leq 0})^{\wedge})^{corep}. \end{array}$$

The Yoneda embedding $\mathbb{R}\underline{h}$ being fully faithful, this implies that the functor

$$\mathrm{Ho}(C(k))^{op} \longrightarrow \mathrm{Ho}((C(k)^{op}_{\leq 0})^{\wedge})$$

is also fully faithful. □

REMARK 2.3.1.2. There are objects $A \in k - cdga$ which are *not* $C(k)_{\leq 0}$-good. For example, we can take A with $\pi_*(A) \simeq k[T, T^{-1}]$ where T is in degree 2. Then, there is no nontrivial A-dg-module M such that $M \in C(k)_{\leq 0}$: this clearly implies that A can not be $C(k)_{\leq 0}$-good.

Another example is given by $A = k[T]$ where T is in degree -2, and M be the A-module $A[T^{-1}]$. As there are no morphisms from M to A-modules in $C(k)_{\leq 0}$, then M is sent to zero by the functor

$$\mathrm{Ho}(A - Mod)^{op} \longrightarrow \mathrm{Ho}((A - Mod^{op}_{\leq 0})^{\wedge}).$$

DEFINITION 2.3.1.3. (1) *Let $A \in k - cdga$, and M be an A-dg-module. The A-dg-module M is* strong *if the natural morphism*

$$\pi_*(A) \otimes_{\pi_0(A)} \pi_0(M) \longrightarrow \pi_*(M)$$

is an isomorphism.

(2) *A morphism $A \longrightarrow B$ in $k - cdga$ is* strongly flat *(resp.* strongly smooth, *resp.* strongly étale, *resp.* a strong Zariski open immersion*) if B is strong as an A-dg-module, and if the morphism of affine schemes*

$$Spec\, \pi_0(B) \longrightarrow Spec\, \pi_0(A)$$

is flat (resp. smooth, resp. étale, resp. a Zariski open immersion).

One of the main difference between derived algebraic geometry and complicial algebraic geometry lies in the fact that the strong notions of flat, smooth, étale and Zariski open immersion are not as easily related to the general notions presented in §1.2. We have the following partial comparison result.

PROPOSITION 2.3.1.4. *Let $f : A \longrightarrow B$ be a morphism in $k - cdga$.*

(1) *If A and B are (-1)-connected, the morphism f is smooth (resp. i-smooth, resp. resp. étale, resp. a Zariski open immersion) in the sense of Def. 1.2.6.1, 1.2.7.1, if and only if f is strongly smooth (resp. strongly étale, resp. a strong Zariski open immersion).*
(2) *If the morphism f is strongly flat (resp. strongly smooth, resp. strongly étale, resp. a strong Zariski open immersion), then it is flat (resp. smooth, resp. étale, resp. a Zariski open immersion) in the sense of Def. 1.2.6.1, 1.2.7.1.*

PROOF. (1) The proof is precisely the same as for Thm. 2.2.2.6, as the homotopy theory of (-1)-connected $k - cdga$ is equivalent to the one of commutative simplicial k-algebras, and as this equivalence preserves cotangent complexes.

(2) For flat morphisms there is nothing to prove, as all morphisms are flat in the sense of Def. 1.2.6.1 since the model category $C(k)$ is stable. Let us suppose that $f : A \longrightarrow B$ is strongly smooth (resp. strongly étale, resp. a strong Zariski open immersion). we can consider the morphism induced on the (-1)-connected covers

$$f' : A' \longrightarrow B',$$

where we recall that the (-1)-connected cover $A' \longrightarrow A$ induces isomorphims on π_i for all $i \geq 0$, and is such that $\pi_i(A') = 0$ for all $i < 0$. As the morphism f is strongly flat that the square

$$\begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ \downarrow & & \downarrow \\ A & \xrightarrow[f]{} & B \end{array}$$

is homotopy co-cartesian. Therefore, as our notions of smooth, étale and Zariski open immersion are stable by homotopy push-out, it is enough to show that f' is smooth (resp. étale, resp. a Zariski open immersion). But this follows from (1). □

EXAMPLE 2.3.1.5. Before going further into complicial algebraic geometry we would like to present an example illustrating the difference between the strong notions of flat, smooth, étale and Zariski open immersion and the general notions presented in §1.2, showing in particular that proposition 2.3.1.4 (2) does not have a converse.

Let A be any commutative k-algebra, $X = Spec\, A$ the corresponding affine scheme, and $U \subset X$ be a quasi-compact open subscheme. It is easy to see that there exists a perfect complex of A-modules K, such that U is the open subscheme of X on which K is acyclic. By Prop. 1.2.10.1 there exists then a morphism $A \longrightarrow A_K$ in $k - cdga$ which is a Zariski open immersion. Moreover, the universal property of $A \longrightarrow A_K$ shows that if A_K is cohomologically concentrated in degree 0 then U is affine and we have $U \simeq Spec\, \pi_0(A_K)$. Therefore, as soon as U is not affine, A_K cannot be concentrated in degree 0. The morphism $A \longrightarrow A_K$ is thus a Zariski open immersion, and therefore étale, but is not a strong morphism.

This example also shows that if the scheme U is considered as a scheme over $C(k)$ (see §2.3.5.1), then U is equivalent to $\mathbb{R}\underline{Spec}\, A_K$, and thus is affine as a stack over $C(k)$, even though U is not necessarily an affine subscheme of X over k.

The opposite model category $k - cdga^{op}$ will be denoted by $k - DAff$. We will endow it with the strong étale model topology.

DEFINITION 2.3.1.6. *A family of morphisms* $\{Spec\, A_i \longrightarrow Spec\, A\}_{i\in I}$ *in* $k - DAff$ *is a* strong étale covering family *(or simply* s-ét covering family*) if it satisfies the following two conditions.*

1. *Each morphism* $A \longrightarrow A_i$ *is strongly étale.*
2. *There exists a finite subset* $J \subset I$ *such that the family* $\{A \longrightarrow A_i\}_{i\in J}$ *is a formal covering family in the sense of 1.2.5.1.*

Using the definition of strong étale morphisms, we immediately check that a family of morphisms $\{Spec\, A_i \longrightarrow Spec\, A\}_{i\in I}$ in $k - DAff$ is a s-ét covering family if and only if there exists a finite sub-set $J \subset I$ satisfying the following two conditions.

- For all $i \in I$, the natural morphism

$$\pi_*(A) \otimes_{\pi_0(A)} \pi_0(A_i) \longrightarrow \pi_*(A_i)$$

is an isomorphism.

- The morphism of affine schemes
$$\coprod_{i\in J} Spec\,\pi_0(A_i) \longrightarrow Spec\,\pi_0(A)$$
is étale and surjective.

LEMMA 2.3.1.7. *The s-ét covering families define a model topology on $k-DAff$, which satisfies assumption 1.3.2.2.*

PROOF. The same as for Lem. 2.2.2.13. □

The model topology s-ét gives rise to a model category of stacks $k - DAff^{\sim,s\text{-ét}}$.

DEFINITION 2.3.1.8. (1) *A D-stack is an object $F \in k - DAff^{\sim,s\text{-ét}}$ which is a stack in the sense of Def. 1.3.2.1.*
(2) *The model category of D-stacks is $k - DAff^{\sim,s\text{-ét}}$, and its homotopy category is simply denoted by $\mathrm{DSt}(k)$.*

From Prop. 2.3.1.4 we get the following generalization of Cor. 2.2.2.9.

COROLLARY 2.3.1.9. *Let $A \in k-cdga$ and $A' \longrightarrow A$ be its (-1)-connected cover. Let us consider the natural morphisms*
$$t_0(X) := Spec\,(\pi_0 A) \longrightarrow X' = Spec\,A' \qquad X = Spec\,A \longrightarrow X' := Spec A'.$$
Then, the homotopy base change functors
$$\mathrm{Ho}(k - DAff/X') \longrightarrow \mathrm{Ho}(k - DAff/t_0(X))$$
$$\mathrm{Ho}(k - DAff/X') \longrightarrow \mathrm{Ho}(k - DAff/X)$$
induces equivalences between the full sub-categories of strongly étale morphisms. Furthermore, these equivalences preserve epimorphisms of stacks.

PROOF. Using Cor. 2.2.2.9 we see that it is enough to show that the base change along the connective cover $A' \longrightarrow A$
$$\mathrm{Ho}(k - DAff/X') \longrightarrow \mathrm{Ho}(k - DAff/X)$$
induces an equivalences from the full sub-categories of strongly étale morphism $Y' \to X'$ to the full subcategory of étale morphisms $Y \to X$. But an inverse to this functor is given by sending a strongly étale morphism $A \longrightarrow B$ to its connective cover $A' \longrightarrow B'$. □

2.3.2. Weakly geometric D-stacks

We now let $\mathbf{P}_w$ be the class of formally perfect morphisms in $k - DAff$, also called in the present context *weakly smooth morphisms.*

LEMMA 2.3.2.1. *The class $\mathbf{P}_w$ of fp morphisms and the s-ét model topology satisfy assumptions 1.3.2.11.*

PROOF. The only non-trivial thing to show is that the notion of fp morphism is local for the s-ét topology on the source. As strongly étale morphisms are also formally étale, this easily reduces to showing that being a perfect module is local for the s-ét model topology. But this follows from corollary 1.3.7.4. □

We can now state that $(C(k), C(k), k - cdga, \text{s-ét}, \mathbf{P}_w)$ is a HAG context in the sense of Def. 1.3.2.13. From our general definitions we obtain a first notion of geometric D-stacks.

DEFINITION 2.3.2.2. (1) *A weakly n-geometric D-stack is a a D-stack $F \in \mathrm{DSt}(k)$ which is n-geometric for the HAG context $(C(k), C(k), k-cdga, s\text{-ét}, \mathbf{P}_w)$.*

(2) *A weakly n-representable morphism of D-stacks is an n-representable morphism for the HAG context* $(C(k), C(k), k-cdga, s\text{-}ét, \mathbf{P}_w)$.

In the context $(C(k), C(k), k-cdga, \text{s-ét}, \mathbf{P}_w)$ the i-smooth morphisms are the formally etale morphisms, and therefore Artin's conditions of Def. 1.4.3.1 can not be satisfied. There are actually many interesting weakly geometric D-stacks which do not have cotangent complexes, as we will see.

2.3.3. Examples of weakly geometric D-stacks

2.3.3.1. Perfect modules. We consider the D-stack **Perf**, as defined in Def. 1.3.7.5. As we have seen during the proof of Lem. 2.3.2.1 the notion of being perfect is local for the s-ét topology. In particular, for any $A \in k-cdga$, the simplicial set $\mathbf{Perf}(A)$ is naturally equivalent to the nerve of the category of equivalences between perfect A-dg-modules.

PROPOSITION 2.3.3.1. (1) *The D-stack* **Perf** *is categorically locally of finite presentation in the sense of Def. 1.3.6.4.*

(2) *The D-stack* **Perf** *is weakly 1-geometric. Furthermore its diagonal is (-1)-representable.*

PROOF. (1) Let $A = Colim_{i\in I} A_i$ be a filtered colimit of objects in $k-cdga$. We need to prove that the morphism

$$Colim_i \mathbf{Perf}(A_i) \longrightarrow \mathbf{Perf}(A)$$

is an equivalence. Let $j \in I$, and two perfect A_j-dg-modules P and Q. Then, we have

$$Map_{A-Mod}(A \otimes^{\mathbb{L}}_{A_j} P, A \otimes^{\mathbb{L}}_{A_j} Q) \simeq Map_{A_j-Mod}(P, A \otimes^{\mathbb{L}}_{A_j} Q) \simeq$$

$$\simeq Map_{A_j-Mod}(P, Colim_{i\in j/J} A_i \otimes^{\mathbb{L}}_{A_j} Q).$$

As P is perfect it is finitely presented in the sense of Def. 1.2.3.1, and thus we get

$$Map_{A-Mod}(A \otimes^{\mathbb{L}}_{A_j} P, A \otimes^{\mathbb{L}}_{A_j} Q) \simeq Colim_{i\in j/I} Map_{A_i-Mod}(A_i \otimes^{\mathbb{L}}_{A_j} P, A_i \otimes^{\mathbb{L}}_{A_j} Q).$$

Furthermore, as during the proof of Prop. 1.3.7.14 we can show that the same formula holds when Map is replaced by the sub-simplicial set Map^{eq} of equivalences. Invoking the relations between mapping spaces and loop spaces of nerves of model categories (see Appendix B), this clearly implies that the morphism

$$Colim_i \mathbf{Perf}(A_i) \longrightarrow \mathbf{Perf}(A)$$

induces isomorphisms on π_i for $i > 0$ and an injective morphism on π_0.

It only remains to show that for any perfect A-dg-module P, there exists $j \in I$ and a perfect A_j-dg-module Q such that $P \simeq A \otimes^{\mathbb{L}}_{A_j} Q$. For this, we use that perfect modules are precisely the retract of finite cell modules. By construction, it is clear that a finite cell A-dg-module is defined over some A_j. We can therefore write P as a direct factor of $A \otimes^{\mathbb{L}}_{A_j} Q$ for some $j \in I$ and Q a perfect A_j-dg-module. The direct factor P is then determined by a projector

$$p \in [A \otimes^{\mathbb{L}}_{A_j} Q, A \otimes^{\mathbb{L}}_{A_j} Q] \simeq Colim_{i\in j/I} [A_i \otimes^{\mathbb{L}}_{A_j} Q, A_i \otimes^{\mathbb{L}}_{A_j} Q].$$

Thus, p defines a projector p_i in $[A_i \otimes^{\mathbb{L}}_{A_j} Q, A_i \otimes^{\mathbb{L}}_{A_j} Q]$ for some i, corresponding to a direct factor P_i of $A_i \otimes^{\mathbb{L}}_{A_j} Q$. Clearly, we have

$$P \simeq A \otimes^{\mathbb{L}}_{A_i} P_i.$$

(2) We start by showing that the diagonal of **Perf** is (-1)-representable. In other words, we need to prove that for any $A \in k-cdga$, and any two perfect A-dg-modules M and N, the D-stack

$$\begin{array}{cccc} Eq(M,N): & A-cdga & \longrightarrow & SSet_{\mathbb{V}} \\ & (A \to B) & \mapsto & Map^{eq}_{B-Mod}(M \otimes^{\mathbb{L}}_A B, N \otimes^{\mathbb{L}}_A B), \end{array}$$

(where Map^{eq} denotes the sub-simplicial set of Map consisting of equivalences) is a representable D-stack. This is true, and the proof is exactly the same as the proof of Prop. 1.3.7.14.

To construct a 1-atlas, let us chose a $\mathbb{U}$-small set $\mathcal{F}$ of representative for the isomorphism classes of finitely presented objects in $\mathrm{Ho}(k-cdga)$. Then, for any $A \in \mathcal{F}$, let $\mathcal{M}_A$ be a $\mathbb{U}$-small set of representative for the isomorphism classes of perfect objects in $\mathrm{Ho}(A-Mod_{dg})$. The fact that these $\mathbb{U}$-small sets $\mathcal{F}$ and $\mathcal{M}$ exists follows from the fact finitely presented objects are retracts of finite cell objects (see Cor. 1.2.3.8).

We consider the natural morphism

$$p: U := \coprod_{A \in \mathcal{F}} \coprod_{P \in \mathcal{M}_A} U_{A,P} := \mathbb{R}\underline{Spec}\, A \longrightarrow \mathbf{Perf}.$$

By construction the morphism p is an epimorphism of stacks. Furthermore, as the diagonal of **Perf** is (-1)-representable each morphism $U_{A,P} = \mathbb{R}\underline{Spec}\, A \longrightarrow \mathbf{Perf}$ is (-1)-representable. Finally, as $U_{A,P} = \mathbb{R}\underline{Spec}\, A$ and **Perf** are both categorically locally of finite presentation, we see that for any $B \in k-cdga$ and any $Y := \mathbb{R}\underline{Spec}\, B \longrightarrow \mathbf{Perf}$, the morphism

$$U_{A,P} \times^h_{\mathbf{Perf}} Y \longrightarrow Y$$

is a finitely presented morphism between representable D-stacks. In particular, it is a perfect morphism. We therefore conclude that

$$p: U := \coprod_{A \in \mathcal{F}} \coprod_{P \in \mathcal{M}_A} U_{A,P} := \mathbb{R}\underline{Spec}\, A \longrightarrow \mathbf{Perf}$$

is a 1-atlas for **Perf**. This finishes the proof that **Perf** is weakly 1-geometric. □

2.3.3.2. The D-stacks of dg-algebras and dg-categories. We consider for any $A \in k-cdga$ the model category $A-dga$, of associative and unital A-algebras (in $\mathbb{U}$). The model structure on $A-dga$ is the usual one for which equivalences are quasi-isomorphisms and fibrations are epimorphisms. We consider $Ass(A)$ to be the subcategory of $A-dga$ consisting of equivalences between objects $B \in A-dga$ satisfying the following two conditions.

- The object B is cofibrant in $A-dga$.
- The underlying A-dg-module of B is perfect.

For a morphism $A \longrightarrow A'$ in $k-cdga$, we have a base change functor

$$A' \otimes_A - : Ass(A) \longrightarrow Ass(A'),$$

making $A \mapsto Ass(A)$ into a pseudo-functor on $k-cdga$. Using the usual strictification procedure, and passing to the nerve we obtain a simplicial presheaf

$$\begin{array}{cccc} \mathbf{Ass}: & k-DAff & \longrightarrow & SSet_{\mathbb{V}} \\ & A & \mapsto & N(Ass(A)). \end{array}$$

PROPOSITION 2.3.3.2. (1) *The simplicial presheaf* **Ass** *is a D-stack.*
(2) *The natural projection* **Ass** $\longrightarrow$ **Perf**, *which forget the algebra structure, is (-1)-representable.*

PROOF. (1) This is exactly the same proof as Thm. 1.3.7.2.

(2) Let $A \in k-cdga$ and $\mathbb{R}\widetilde{Spec\,A} \longrightarrow \mathbf{Perf}$ be a point corresponding to a perfect A-dg-module E. We denote by $\widetilde{\mathbf{Ass}_E}$ the homotopy fiber taken at E of the morphism $\mathbf{Ass} \longrightarrow \mathbf{Perf}$.

LEMMA 2.3.3.3. *The D-stack* $\widetilde{\mathbf{Ass}_E}$ *is representable.*

PROOF. This is the same argument as for Prop. 2.2.6.8. We see using [**Re**, Thm. 1.1.5] that $\widetilde{\mathbf{Ass}_E}$ is the D-stack (over $Spec\,A$) $Map(\mathcal{A}ss, \mathbb{R}\underline{End}(E))$, of morphisms from the associative operad to the the (derived) endomorphism operad of E, defined the same way as in the proof of 2.2.6.8. Again the same argument as for 2.2.6.8, consisting of writing $\mathcal{A}ss$ as a homotopy colimit of free operads, reduces the statement of the lemma to prove that for a perfect A-dg-module K of, the D-stack

$$\begin{array}{ccc} A-cdga & \longrightarrow & SSet \\ A' & \mapsto & Map_{(C(k))}(K, A') \end{array}$$

is representable. But this is true as it is equivalent to $\mathbb{R}\underline{Spec}\,B$, where B is the (derived) free A-cdga over K. □

The previous lemma shows that $\mathbf{Ass} \longrightarrow \mathbf{Perf}$ is a (-1)-representable morphism, and finishes the proof of Prop. 2.3.3.2. □

COROLLARY 2.3.3.4. *The D-stack* **Ass** *is weakly* 1*-geometric.*

PROOF. Follows from Prop. 2.3.3.1 and Prop. 2.3.3.2. □

We now consider a slight modification of **Ass**, by considering dg-algebras as dg-categories with only one objects. For this, let $A \in k-cdga$. Recall that a A-dg-category is by definition a category enriched over the symmetric monoidal category $A-Mod_{dg}$, of A-dg-modules. More precisely, a A-dg-category $\mathcal{D}$ consists of the following data

- A set of objects $Ob(\mathcal{D})$.
- For any pair of objects (x, y) in $Ob(\mathcal{D})$ an A-dg-module $\mathcal{D}(x, y)$.
- For any triple of objects (x, y, z) in $Ob(\mathcal{D})$ a composition morphism

$$\mathcal{D}(x, y) \otimes_A \mathcal{D}(y, z) \longrightarrow \mathcal{D}(x, z),$$

  which satisfies the usual unital and associativity conditions.

The A-dg-categories (in the universe $\mathbb{U}$) form a category $A-dgCat$, with the obvious notion of morphisms. For an $A-dg$-category $\mathcal{D}$, we can form a category $\pi_0(\mathcal{D})$, sometimes called the *homotopy category of* $\mathcal{D}$, whose objects are the same as $\mathcal{D}$ and for which morphisms from x to y is the set $\pi_0(\mathcal{D}(x, y))$ (with the obvious induced compositions). The construction $\mathcal{D} \mapsto \pi_0(\mathcal{D})$ defines a functor from $A-dgCat$ to the category of $\mathbb{U}$-small categories. Recall that a morphism $f : \mathcal{D} \longrightarrow \mathcal{E}$ is then called a *quasi-equivalence* (or simply an *equivalence*) if it satisfies the following two conditions.

- For any pair of objects (x, y) in $\mathcal{D}$ the induced morphism

$$f_{x,y} : \mathcal{D}(x, y) \longrightarrow \mathcal{E}(f(x), f(y))$$

  is an equivalence in $A-Mod_{dg}$.
- The induced functor

$$\pi_0(f) : \pi_0(\mathcal{D}) \longrightarrow \pi_0(\mathcal{E})$$

  is an equivalence of categories.

We also define a notion of fibration, as the morphisms $f : \mathcal{D} \longrightarrow \mathcal{E}$ satisfying the following two conditions.

- For any pair of objects (x, y) in $\mathcal{D}$ the induced morphism
$$f_{x,y} : \mathcal{D}(x, y) \longrightarrow \mathcal{E}(f(x), f(y))$$
is a fibration in $A - Mod_{dg}$.
- For any object x in $\mathcal{D}$, and any isomorphism $v : f(x) \to z$ in $\pi_0(\mathcal{E})$, there exists an isomorphism $u : x \to y$ in $\pi_0(\mathcal{D})$ such that $\pi_0(f)(u) = v$.

With these notions of fibrations and equivalences, the category $A - dgCat$ is a model category. This is proved in [**Tab**] when A is a commutative ring. The general case of categories enriched in a well behaved monoidal model category has been worked out recently by J. Tapia (private communication).

For $A \in k - cdga$, we denote by $Cat_*(A)$ the category of equivalences between objects $\mathcal{D} \in A - dgCat$ satisfying the following two conditions.

- For any two objects x and y in $\mathcal{D}$ the A-dg-module $\mathcal{D}(x, y)$ is perfect and cofibrant in $A - Mod_{dg}$.
- The category $\pi_0(\mathcal{D})$ possesses a unique object up to isomorphism.

For a morphism $A \longrightarrow A'$ in $k - cdga$, we have a base change functor
$$- \otimes_A A' : Cat_*(A) \longrightarrow Cat_*(A')$$
obtained by the formula
$$Ob(\mathcal{D} \otimes_A A') := Ob(\mathcal{D}) \qquad (\mathcal{D} \otimes_A A')(x, y) := \mathcal{D}(x, y) \otimes_A A'.$$
This makes $A \mapsto Cat_*(A)$ into a pseudo-functor on $k - cdga$. Strictifying and applying the nerve construction we get a well defined simplicial presheaf
$$\begin{array}{rccc} \mathbf{Cat}_* : & k - cdga & \longrightarrow & SSet_{\mathbb{V}} \\ & A & \mapsto & N(Cat_*(A)). \end{array}$$

It is worth mentioning that $\mathbf{Cat}_*$ is not a stack, since there are non trivial twisted forms of objects in $Cat_*(k)$ with respect to the étale topology on k. These twisted forms can be interpreted as certain *stacks in dg-categories* having *locally* a unique object up to equivalences, but they might have either no global objects or several nonequivalent global objects. We will not explicitly describe the stack associated to $\mathbf{Cat}_*$, as this will be irrelevant for the sequel, and will simply consider $\mathbf{Cat}_*$ as an object in $D\mathrm{St}(k)$.

There exists a morphism of simplicial presheaves
$$B : \mathbf{Ass} \longrightarrow \mathbf{Cat}_*,$$
which sends an associative A-algebra C to the A-dg-category BC, having a unique object $*$ and C as the endomorphism A-algebra of $*$. The morphism B will be considered as a morphism in $D\mathrm{St}(k)$.

Proposition 2.3.3.5. *The morphism* $B : \mathbf{Ass} \longrightarrow \mathbf{Cat}_*$ *is weakly* 1*-representable, fp and an epimorphism of D-stacks.*

Proof. We will prove a more precise result, giving explicit description of the homotopy fibers of B. For this, we start by some model category considerations relating associative dga to dg-categories. Recall that for any $A \in k - cdga$, we have two model categories, $A - dga$ and $A - dgCat$, of associative A-algebras and A-dg-categories. We consider $\mathbf{1}/A - dgCat$, the model category of dg-categories together with a distinguised object. More precisely, $\mathbf{1}$ is the A-dg-category with a unique object and A as its endomorphism dg-algebra, and $\mathbf{1}/A - dgCat$ is the comma model category. The functor $B : A - dga \longrightarrow \mathbf{1}/A - dgCat$ is a left Quillen functor. Indeed, its right adjoint Ω_*, sends a pointed A-dg-category $\mathcal{D}$ to the A-algebra $\mathcal{D}(*,*)$ of endomorphisms of the distinguished point $*$. Clearly, the adjunction morphism $\Omega_* B \Rightarrow Id$ is an equivalence, and thus the functor

$$B : \mathrm{Ho}(A - dga) \longrightarrow \mathrm{Ho}(\mathbf{1}/A - dgCat)$$

is fully faithful. As a consequence, we get that for any $C, C' \in A - dga$ there exists a natural homotopy fiber sequence of simplicial sets

$$Map_{A-dgCat}(\mathbf{1}, BC') \longrightarrow Map_{A-dga}(B, B') \longrightarrow Map_{A-dgCat}(BC, BC').$$

Now, let $A \in k - cgda$ and B be an associative A-algebra, corresponding to a morphism

$$x : X := \mathbb{R}\underline{Spec}\, A \longrightarrow \mathbf{Ass}.$$

We consider the D-stack

$$F := \mathbf{Ass} \times^h_{\mathbf{Cat}_*} X.$$

Using the relations between mapping spaces in $A - dga$ and $A - dgCat$ above we easily see that the D-stack F is connected (i.e. $\pi_0(F) \simeq *$). In particular, in order to show that F is weakly 1-geometric, it is enough to prove that $\Omega_B F$ is a representable D-stack. Using again the homotopy fiber sequence of mapping spaces above we see that the D-stack $\Omega_B F$ can be described as

$$\begin{array}{rccc} \Omega_B F : & A - cdga & \longrightarrow & SSet_{\mathbb{V}} \\ & (A \to A') & \mapsto & Map_{\mathbf{1}/A'-dgCat}(S^1 \otimes^{\mathbb{L}} \mathbf{1}, B(B \otimes^{\mathbb{L}}_A A')) \end{array}$$

where $S^1 \otimes^{\mathbb{L}} \mathbf{1}$ is computed in the model category $A - dgCat$. One can easily check that one has an isomorphism in $\mathbf{1}/A - dgCat$

$$S^1 \otimes^{\mathbb{L}} \mathbf{1} \simeq B(A[T, T^{-1}]),$$

where $A[T, T^{-1}] := A \otimes_k k[T, T^{-1}]$. Therefore, there is a natural equivalence

$$Map_{\mathbf{1}/A'-dgCat}(S^1 \otimes^{\mathbb{L}} \mathbf{1}, B(B \otimes^{\mathbb{L}}_A A')) \simeq Map_{A-dga}(A[T, T^{-1}], B \otimes^{\mathbb{L}}_A A'),$$

and thus the D-stack $\Omega_B F$ can also be described by

$$\begin{array}{rccc} \Omega_B F : & A - cdga & \longrightarrow & SSet_{\mathbb{V}} \\ & (A \to A') & \mapsto & Map_{A-dga}(A[T, T^{-1}], B \otimes^{\mathbb{L}}_A A'). \end{array}$$

The morphism of k-algebras $k[T] \longrightarrow k[T, T^{-1}]$ induces natural morphisms (here $B^\vee$ is the dual of the perfect A-dg-module B)

$$Map_{A-dga}(A[T, T^{-1}], B \otimes^{\mathbb{L}}_A A') \longrightarrow Map_{A-dga}(A[T], B \otimes^{\mathbb{L}}_A A') \simeq Map_{A-Mod}(B^\vee, A').$$

It is not difficult to check that this gives a morphism of D-stacks

$$\Omega_B F \longrightarrow \mathbb{R}\underline{Spec}\, \mathbb{L}F(B^\vee),$$

where $\mathbb{L}F(B^\vee)$ is the derived free A-cdga over the A-dg-module $B^\vee$. Furthermore, applying Prop. 1.2.10.1 this morphism is easily seen to be representable by an open Zariski immersion, showing that $\Omega_B F$ is thus a representable D-stack. □

COROLLARY 2.3.3.6. *The D-stack $\mathbf{Cat}_*$ is weakly 2-geometric.*

PROOF. This follows immediately from Cor. 2.3.3.4, Prop. 2.3.3.5 and the general criterion of Cor. 1.3.4.5. □

Let $A \in k-cdga$ and B be an associative A-algebra corresponding to a morphism of D-stacks

$$B : X := \mathbb{R}\underline{Spec}\, A \longrightarrow \mathbf{Ass}.$$

We define a D-stack B^* on $A-cdga$ in the following way

$$\begin{array}{cccc} B^* : & A-cdga & \longrightarrow & SSet_{\mathbb{V}} \\ & (A \to A') & \mapsto & Map_{A-dga}(A[T,T^{-1}], B \otimes^{\mathbb{L}}_A A'). \end{array}$$

The D-stack B^* is called the *D-stack of invertible elements in B*. The D-stack B^* possesses in fact a natural loop stack structure (i.e. the above functor factors naturally, up to equivalence, through a functor from $A-cdga$ to the category of loop spaces), induced by the Hopf algebra structure on $A[T,T^{-1}]$. This loop stack structure is also the one induced by the natural equivalences

$$B^*(A') \simeq \Omega_* Map_{A-dgCat}(*, B \otimes^{\mathbb{L}}_A A').$$

Delooping gives another D-stack

$$\begin{array}{cccc} K(B^*,1) : & A-cdga & \longrightarrow & SSet_{\mathbb{V}} \\ & (A \to A') & \mapsto & K(B^*(A'),1). \end{array}$$

The following corollary is a reformulation of Prop. 2.3.3.5 and of its proof.

COROLLARY 2.3.3.7. *Let $A \in k-cdga$ and $B \in \mathbf{Ass}(A)$ corresponding to an associative A-algebra B. Then, there is a natural homotopy cartesian square of D-stacks*

$$\begin{array}{ccc} \mathbf{Ass} & \longrightarrow & \mathbf{Cat}_* \\ \uparrow & & \uparrow B \\ K(B^*,1) & \longrightarrow & \mathbb{R}\underline{Spec}\, A. \end{array}$$

2.3.4. Geometric D-stacks

We now switch to the HA context $(C(k), C(k)_{\leq 0}, k-cdga_0)$. Recall that $C(k)_{\leq 0}$ is the subcategory of $C(k)$ consisting of (-1)-connected object, and $k-cdga_0$ is the subcategory of $k-cdga$ of objects cohomologically concentrated in degree 0. Within this HA context we let $\mathbf{P}$ to be the class of formally perfect and formally i-smooth morphisms in $k-DAff$. Morphisms in $\mathbf{P}$ will simply be called *fip-smooth morphisms*. There are no easy description of them, but Prop. 1.2.8.3 implies that a morphism $f : A \longrightarrow B$ be a morphism is fip-smooth if it satisfies the following two conditions.

- The cotangent complex $\mathbb{L}_{B/A} \in \mathrm{Ho}(A-Mod_{dg})$ is perfect.
- For any $R \in k-cdga_0$, any connected module $M \in R-Mod_{dg}$ and any morphism $B \longrightarrow R$, one has

$$\pi_0(\mathbb{L}^{\vee}_{B/A} \otimes^{\mathbb{L}}_B M) = 0.$$

The converse is easily shown to be true if A and B are both (-1)-connected.

LEMMA 2.3.4.1. *The class $\mathbf{P}$ of fip-smooth morphisms and the s-ét model topology satisfy assumptions 1.3.2.11.*

PROOF. We see that the only non trivial part is to show that the notion of fip-smooth morphism is local for the s-ét topology. For fp-morphisms this is Cor. 1.3.7.4. For fi-smooth morphisms this is an easy consequence of the definitions. □

From Lem. 2.3.4.1 we get a HAG context $(C(k), C(k)_{\leq 0}, k-cdga_0, s-et, \mathbf{P})$.

DEFINITION 2.3.4.2. (1) *An n-geometric D-stack is a D-stack $F \in \mathrm{DSt}(k)$ which is n-geometric for the HAG context $(C(k), C(k)_{\leq 0}, k-cdga_0, s-et, \mathbf{P})$.*

(2) *A strongly n-representable morphism of D-stacks is an n-representable morphism of D-stacks for the HAG context* $(C(k), C(k)_{\leq 0}, k-cdga_0, s-et, \mathbf{P})$.

Note that $\mathbf{P} \subset \mathbf{P}_w$ and therefore that any n-geometric D-stack is weakly n-geometric.

LEMMA 2.3.4.3. *The s-ét topology and the class* $\mathbf{P}$ *of fip-smooth morphisms satisfy Artin's conditions of Def. 1.4.3.1.*

PROOF. This is essentially the same proof as Prop. 2.2.3.2, and is even more simple as one uses here the strongly étale topology. □

COROLLARY 2.3.4.4. *Any n-geometric D-stack possesses an obstruction theory (relative to the HA context* $(C(k), C(k)_{\leq 0}, k-cdga_0)$*).*

PROOF. This follows from Lem. 2.3.4.3 and Thm. 1.4.3.2. □

2.3.5. Examples of geometric D-stacks

2.3.5.1. D^--stacks and D-stacks. We consider the normalization functor $N : sk-Alg \longrightarrow k-cdga$, sending a simplicial commutative k-algebra A to its normalization $N(A)$, with its induced structure of commutative differential graded algebra. The pullback functor gives a Quillen adjunction

$$N_! : k\text{-}D^-Aff^{\sim,et} \longrightarrow k\text{-}DAff^{\sim,s-et} \qquad k\text{-}D^-Aff^{\sim,et} \longleftarrow k\text{-}DAff^{\sim,s-et} : N^*.$$

As the functor N is known to be homotopically fully faithful, the left derived functor

$$j := \mathbb{L}N_! : D^-\mathrm{St}(k) \longrightarrow D\mathrm{St}(k)$$

is fully faithful. We can actually characterize the essential image of the functor j as consisting of all D-stacks F for which for any $A \in k-cdga$ with (-1)-connected cover $A' \longrightarrow A$, the morphism $\mathbb{R}F(A') \longrightarrow \mathbb{R}F(A)$ is an equivalence. In other words, for any $F \in k-D^-Aff^{\sim,et}$, and any $A \in k-cdga$, we have

$$\mathbb{R}j(F)(A) \simeq \mathbb{R}F(D(A')),$$

where $D(A') \in sk-Alg$ is a denormalization of A', the (-1)-connected cover of A.

PROPOSITION 2.3.5.1. (1) *For any $A \in sk-Alg$, we have*

$$j(\mathbb{R}\underline{Spec}\,A) \simeq \mathbb{R}\underline{Spec}\,N(A).$$

(2) *The functor j commutes with homotopy limits and homotopy colimits.*
(3) *The functor j sends n-geometric D^--stacks to n-geometric D-stacks.*

PROOF. This is clear. □

The previous proposition shows that any n-geometric D^--stack gives rise to an n-geometric D-stack, and thus provides us with a lot of examples of those.

2.3.5.2. CW-perfect modules. Let $A \in k-cdga$. We define by induction on $n = b-a$ the notion of a perfect CW-A-dg-module of amplitude contained in $[a, b]$.

DEFINITION 2.3.5.2. (1) *A perfect CW-A-dg-module of amplitude contained in $[a, a]$ is an A-dg-module M isomorphic in* $\mathrm{Ho}(A-Mod_{dg})$ *to $P[-a]$, with P a projective and finitely presented A-dg-module (as usual in the sense of definitions 1.2.3.1 and 1.2.4.1).*

(2) *Assume that the notion of perfect CW-A-dg-module of amplitude contained in $[a,b]$ has been defined for any $a \leq b$ such that $b-a=n-1$.* A perfect CW-A-dg-module of amplitude contained in $[a,b]$, *with $b-a=n$, is an A-dg-module M isomorphic in $\mathrm{Ho}(A-Mod_{dg})$ to the homotopy cofiber of a morphism*

$$P[-a-1] \longrightarrow N,$$

where P is projective and finitely presented, and N is a perfect CW-A-dg-module of amplitude contained in $[a+1,b]$.

The perfect CW-A-dg-modules satisfy the following stability conditions.

LEMMA 2.3.5.3. (1) *If M is a perfect CW-A-dg-module of amplitude contained in $[a,b]$, and $A \longrightarrow A'$ is a morphism in $k-cdga$, then $A' \otimes^{\mathbb{L}}_A M$ is a perfect CW-A-dg-module of amplitude contained in $[a,b]$.*

(2) *Let $A \in k-cdga_{\leq 0}$ be a (-1)-connected $k-cdga$. Then, any perfect A-dg-module is a perfect CW-A-dg-module of amplitude $[a,b]$ for some integer $a \leq b$.*

(3) *Let $A \in k-cdga_{\leq 0}$ be a (-1)-connected $k-cdga$, and $M \in A-Mod_{dg}$. If there exists a s-et covering $A \longrightarrow A'$ such that $A' \otimes^{\mathbb{L}}_A M$ is a perfect CW-A-dg-module of amplitude contained in $[a,b]$, then so is M.*

PROOF. Only (2) and (3) requires a proof. Furthermore, (3) clearly follows from (2) and the local nature of perfect modules (see Cor. 1.3.7.4), and thus it only remains to prove (2). But this is proved in [**EKMM**, III.7]. □

We define a sub-D-stack $\mathbf{Perf}^{CW}_{[a,b]} \subset \mathbf{Perf}$, consisting of all perfect modules locally equivalent to some CW-dg-modules of amplitude contained in $[a,b]$. Precisely, for $A \in k-cdga$, $\mathbf{Perf}^{CW}_{[a,b]}(A)$ is the sub-simplicial set of $\mathbf{Perf}(A)$ which is the union of all connected components corresponding to A-dg-modules M such that there is an s-ét covering $A \longrightarrow A'$ with $A' \otimes^{\mathbb{L}}_A M$ a perfect CW-A'-dg-module of amplitude contained in $[a,b]$.

PROPOSITION 2.3.5.4. *The D-stack $\mathbf{Perf}^{CW}_{[a,b]}$ is 1-geometric.*

PROOF. We proceed by induction on $n=b-a$. For $n=0$, the D-stack $\mathbf{Perf}^{CW}_{[a,a]}$ is simply the stack of vector bundles $\mathbf{Vect}$, which is 1-geometric by our general result Cor. 1.3.7.12. Assume that $\mathbf{Perf}^{CW}_{[a,b]}$ is known to be 1-geometric for $b-a<n$. Shifting if necessary, we can clearly assume that $a=0$, and thus that $b=n-1$.

That the diagonal of $\mathbf{Perf}^{CW}_{[a,b]}$ is (-1)-representable comes from the fact that $\mathbf{Perf}^{CW}_{[a,b]} \longrightarrow \mathbf{Perf}$ is a monomorphisms and from Prop. 2.3.3.1. It remains to construct a 1-atlas for $\mathbf{Perf}^{CW}_{[a,b]}$.

Let $A \in k-cdga$. We consider the model category $Mor(A-Mod_{dg})$, whose objects are morphisms in $A-Mod_{dg}$ (and with the usual model category structure, see e.g. [**Ho1**]). We consider the subcategory $Mor(A-Mod_{dg})'$ consisting cofibrant objects $u : M \to N$ in $Mor(A-Mod_{dg})$ (i.e. u is a cofibration between cofibrant A-dg-modules) such that M is projective of finite presentation and N is locally a perfect CW-A-dg-module of amplidtude contained in $[0,n-1]$. Morphisms in $Mor(A-Mod_{dg})'$ are taken to be equivalences in $Mor(A-Mod_{dg})$. The correspondence $A \mapsto Mor(A-Mod_{dg})'$ defines a pseudo-functor on $k-cdga$, and after strictification and passing to the nerve we get this way a simplicial presheaf

$$\begin{array}{rccc} F: & k-cdga & \longrightarrow & SSet_{\mathbb{V}} \\ & A & \mapsto & N(Mor(A-Mod_{dg})'). \end{array}$$

There exists a morphism of D-stacks

$$F \longrightarrow \mathbf{Vect} \times^h \mathbf{Perf}^{CW}_{[0,n-1]}$$

that sends an object $M \to N$ to (M, N). This morphism is easily seen to be (-1)-representable, as its fiber at an A-point (M, N) is the D-stack of morphisms from M to N, or equivalently is $\mathbb{R}\underline{Spec}\, B$ where B is the derived free A-cdga over $M^\vee \otimes^{\mathbb{L}}_A N$. By induction hypothesis we deduce that the D-stack F itself if 1-geometric.

Finally, there exists a natural morphism of D-stacks

$$p : F \longrightarrow \mathbf{Perf}^{CW}_{[-1,n-1]}$$

sending a morphism $u : M \to N$ in $Mor(A - Mod_{dg})'$ to its homotopy cofiber. The morphism p is an epimorphism of D-stacks by definition of being locally a perfect CW-dg-module, thus it only remains to show that p is fip-smooth. Indeed, this would imply the existence of a 1-atlas for $\mathbf{Perf}^{CW}_{[-1,n-1]}$, and thus that it is 1-geometric. Translating we get that $\mathbf{Perf}^{CW}_{[a,b]}$ is 1-geometric for $b - a = n$.

In order to prove that p is fip-smooth, let $A \in k - cdga$, and $K : \mathbb{R}\underline{Spec}\, A \longrightarrow \mathbf{Perf}^{CW}_{[-1,n-1]}$ be an A-point, corresponding to a perfect CW-A-dg-module of amplitude contained in $[-1, n-1]$. We consider the homotopy cartesian square

$$\begin{array}{ccc} F & \xrightarrow{p} & \mathbf{Perf}^{CW}_{[-1,n-1]} \\ \uparrow & & \uparrow \\ F' & \xrightarrow[p']{} & \mathbb{R}\underline{Spec}\, A. \end{array}$$

We need to prove that p is a fip-smooth morphism. The D-stack F' has natural projection $F' \longrightarrow \mathbf{Vect}$ given by $F' \longrightarrow F \longrightarrow \mathbf{Vect}$, where the second morphism sends a morphism $M \to N$ to the vector bundle M. We therefore get a morphism of D-stacks

$$F' \longrightarrow \mathbf{Vect} \times^h \mathbb{R}\underline{Spec}\, A \longrightarrow \mathbb{R}\underline{Spec}\, A.$$

As the morphism $\mathbf{Vect} \longrightarrow *$ is smooth, it is enough to show that the morphism

$$F' \longrightarrow \mathbf{Vect} \times^h \mathbb{R}\underline{Spec}\, A$$

is fip-smooth. For this we consider the homotopy cartesian square

$$\begin{array}{ccc} F' & \longrightarrow & \mathbf{Vect} \times^h \mathbb{R}\underline{Spec}\, A \\ \uparrow & & \uparrow \\ F'_0 & \longrightarrow & \mathbb{R}\underline{Spec}\, A, \end{array}$$

where the section $\mathbb{R}\underline{Spec}\, A \longrightarrow \mathbf{Vect}$ correspond to a trivial rank r vector bundle A^r. It only remains to show that the morphism $F'_0 \longrightarrow \mathbb{R}\underline{Spec}\, A$ is fip-smooth. The D-stack F'_0 over A can then be easily described (using for example our general Cor. B.0.8) as

$$\begin{array}{rccc} F'_0 : & A - cdga & \longrightarrow & SSet_{\mathbb{V}} \\ & (A \to A') & \mapsto & Map_{A-Mod}(K[-1], A^r). \end{array}$$

In other words, we can write $F'_0 \simeq \mathbb{R}\underline{Spec}\, B$, where B is the derived free $A - cdga$ over $(K[-1])^r$. But, as K is a perfect CW-A-dg-module of amplitude contained in $[-1, n-1]$, $(K[-1])^r$ is a perfect CW A-dg-module of amplitude contained in $[0, n]$. The proposition will then follow from the general lemma.

Let $A \in k - cdga$, and recall that the forgetful functor $A - cdga \longrightarrow A - Mod_{dg}$ is right Quillen, and that its derived left adjoint

$$\mathrm{Ho}(A - Mod_{dg}) \longrightarrow \mathrm{Ho}(A - cdga)$$

sends, by definition, an A-module E to the *derived free A-cdga over E.*

LEMMA 2.3.5.5. *Let $A \in k - cdga$, and K be a perfect CW-A-dg-module of amplitude contained in $[0, n]$. Let B be the derived free $A - cdga$ over K, and*

$$p : Y := \mathbb{R}\underline{Spec}\, B \longrightarrow X := \mathbb{R}\underline{Spec}\, A$$

be the natural projection. Then p is fip-smooth.

PROOF. Let k' be any commutative k-algebra, and $B \longrightarrow k$ be any morphism in $\mathrm{Ho}(k - cdga)$, corresponding to a point $x : Spec\, k \longrightarrow X$. Then, we have an isomorphism in $D(k')$

$$\mathbb{L}_{Y/X,x} \simeq K \otimes^{\mathbb{L}}_A k'.$$

We thus see that $\mathbb{L}_{Y/X,x}$ is a perfect complex of k'-modules of Tor amplitude concentrated in degrees $[0, n]$. In particular, it is clear that for any complex M of k'-modules such that $\pi_i(M) = 0$ for any $i > 0$, then we have

$$[\mathbb{L}_{Y/X,x}, M]_{D(k')} \simeq \pi_0(\mathbb{L}^{\vee}_{Y/X,x} \otimes^{\mathbb{L}}_{k'} M) \simeq 0.$$

This shows that the A-algebra B is fip-smooth, and thus shows the lemma. □

The previous lemma finishes the proof of the proposition. □

A direct consequence of Prop. 2.3.5.4 and Thm. 1.4.3.2 is the following.

COROLLARY 2.3.5.6. *The D-stack $\mathbf{Perf}^{CW}_{[a,b]}$ has an obstruction theory (relative to the HA context $(C(k), C(k)_{\leq 0}, k - cdga_0)$. For any $A \in k - cdga_0$ and any point $E : X := \mathbb{R}\underline{Spec}\, A \longrightarrow \mathbf{Perf}^{CW}_{[a,b]}$ corresponding to a perfect CW-A-dg-module E, there are natural isomorphisms in* $\mathrm{Ho}(A - Mod_{dg})$

$$\mathbb{L}_{\mathbf{Perf}^{CW}_{[a,b]}, E} \simeq E^{\vee} \otimes^{\mathbb{L}}_A E[-1]$$

$$\mathbb{T}_{\mathbf{Perf}^{CW}_{[a,b]}, E} \simeq E^{\vee} \otimes^{\mathbb{L}}_A E[1].$$

PROOF. The first part of the corollary follows from our Thm. 1.4.3.2 and Prop. 2.3.5.4. Let A and $E : X := \mathbb{R}\underline{Spec}\, A \longrightarrow \mathbf{Perf}^{CW}_{[a,b]}$ as in the statement. We have

$$\mathbb{L}_{\mathbf{Perf}^{CW}_{[a,b]}, E} \simeq \mathbb{L}_{\Omega_E \mathbf{Perf}^{CW}_{[a,b]}, E}[-1].$$

Moreover, $\Omega_E \mathbf{Perf}^{CW}_{[a,b]} \simeq \mathbb{R}\underline{Aut}(E)$, where $\mathbb{R}\underline{Aut}(E)$ is the D-stack of self-equivalences of the perfect module E as defined in §1.3.7. By Prop. 1.3.7.14 we know that $\mathbb{R}\underline{Aut}(E)$ is representable, and furthermore that the natural inclusion morphism

$$\mathbb{R}\underline{Aut}(E) \longrightarrow \mathbb{R}\underline{End}(E)$$

is a formally étale morphism of representable D-stacks. Therefore, we have

$$\mathbb{L}_{\Omega_E \mathbf{Perf}^{CW}_{[a,b]}, E} \simeq \mathbb{L}_{\mathbb{R}\underline{End}(E), Id}.$$

Finally, we have

$$\mathbb{R}\underline{End}(E) \simeq \mathbb{R}\underline{Spec}\, B,$$

where B is the derived free A-cdga over $E^{\vee} \otimes^{\mathbb{L}}_A E$. This implies that

$$\mathbb{L}_{\mathbb{R}\underline{End}(E), Id} \simeq E^{\vee} \otimes^{\mathbb{L}}_A E,$$

and by what we have seen that

$$\mathbb{L}_{\mathbf{Perf}^{CW}_{[a,b]}, E} \simeq E^{\vee} \otimes^{\mathbb{L}}_A E[-1].$$

□

REMARK 2.3.5.7. (1) It is important to note that, if

$$N^* : DSt(k) \longrightarrow D^-St(k)$$

denotes the restriction functor, we have

$$N^*(\mathbf{Perf}^{CW}_{[a,b]}) \simeq \mathbf{Perf}_{[a,b]},$$

where $\mathbf{Perf}_{[a,b]}$ is the sub-stack of $\mathbf{Perf} \in D^-St(k)$ consisting of perfect modules of Tor-amplitude contained in $[a,b]$. However, the two D-stacks $\mathbf{Perf}^{CW}_{[a,b]}$ and $j(\mathbf{Perf}_{[a,b]})$ are not the same. Indeed, for any $A \in k - cdga$ with A' as (-1)-connective cover, we have

$$j(\mathbf{Perf}_{[a,b]})(A) \simeq \mathbf{Perf}_{[a,b]}(A').$$

In general the natural morphism

$$- \otimes^{\mathbb{L}}_A A' : \mathbf{Perf}_{[a,b]}(A') \longrightarrow \mathbf{Perf}^{CW}_{[a,b]}(A)$$

is not an equivalence. For example, let us suppose that there exist a non zero element $x \in \pi_{-1}(A) \simeq [A, A[1]]$, then the matrix

$$\begin{pmatrix} Id_A & x \\ 0 & Id_{A[1]} \end{pmatrix}$$

defines an equivalence $A \bigoplus A[1] \simeq A \bigoplus A[1]$ of A-dg-modules which is not induced by an equivalence $A' \bigoplus A'[1] \simeq A' \bigoplus A'[1]$ of A'-dg-modules.

(2) We can also show that the D^--stack $\mathbf{Perf}_{[a,b]} \in D^-St(k)$ is $(n+1)$-geometric for $n = b - a$. The proof is essentially the same as for Prop. 2.3.5.4 and will not be reproduced here. Of course, the formula for the cotangent complex remains the same. See [**To-Va1**] for more details.

2.3.5.3. CW-dg-algebras. Recall the existence of the following diagram of D-stacks

$$\begin{array}{ccc} & & \mathbf{Ass} \\ & & \downarrow \\ \mathbf{Perf}^{CW}_{[a,b]} & \longrightarrow & \mathbf{Perf}. \end{array}$$

DEFINITION 2.3.5.8. *The D-stack of CW-dg-algebras of amplitude contained in $[a,b]$ is defined by the following homotopy cartesian square*

$$\begin{array}{ccc} \mathbf{Ass}^{CW}_{[a,b]} & \longrightarrow & \mathbf{Ass} \\ \downarrow & & \downarrow \\ \mathbf{Perf}^{CW}_{[a,b]} & \longrightarrow & \mathbf{Perf}. \end{array}$$

Let B be an associative k-dga, and let M be a B-bi-dg-module (i.e. a $B \otimes^{\mathbb{L}}_k B^{op}$-dg-module). We can form the square zero extension $B \oplus M$, which is another associative k-dga together with a natural projection $B \oplus M \longrightarrow B$. The simplicial set of (derived) derivations from B to M is then defined as

$$\mathbb{R}Der_k(B, M) := Map_{k-dga/B}(B, B \oplus M).$$

The same kind of proof as for Prop. 1.2.1.2 shows that there exists an object $\mathbb{L}^{Ass}_B \in \mathrm{Ho}(B \otimes^{\mathbb{L}}_k B^{op} - Mod_{dg})$ an natural isomorphisms in $\mathrm{Ho}(SSet_{\mathbb{V}})$

$$\mathbb{R}Der_k(B, M) \simeq Map_{B \otimes^{\mathbb{L}}_k B^{op} - Mod}(\mathbb{L}^{Ass}_B, M).$$

We then set

$$\mathbb{R}\underline{Der}_k(B,M) := \mathbb{R}\underline{Hom}_{B\otimes^{\mathbb{L}}_k B^{op}-Mod}(\mathbb{L}^{Ass}_B, M) \in \mathrm{Ho}(C(k)),$$

where $\mathbb{R}\underline{Hom}_{B\otimes^{\mathbb{L}}_k B^{op}-Mod}$ denotes the $\mathrm{Ho}(C(k))$-enriched derived Hom's of the $C(k)$-model category $B\otimes^{\mathbb{L}}_k B^{op} - Mod_{dg}$.

COROLLARY 2.3.5.9. *The D-stack $\mathbf{Ass}^{CW}_{[a,b]}$ is 1-geometric. For any global point $B : Spec\, k \longrightarrow \mathbf{Ass}^{CW}_{[a,b]}$, corresponding to an associative k-dga B, one has a natural isomorphism in* $\mathrm{Ho}(C(k))$

$$\mathbb{T}_{\mathbf{Ass}^{CW}_{[a,b]},B} \simeq \mathbb{R}\underline{Der}_k(B,B)[1].$$

PROOF. Using Prop. 2.3.3.2 we see that the natural projection

$$\mathbf{Ass}^{CW}_{[a,b]} \longrightarrow \mathbf{Perf}^{CW}_{[a,b]}$$

is (-1)-representable. Therefore, by Prop. 2.3.5.4 we know that $\mathbf{Ass}^{CW}_{[a,b]}$ is 1-geometric. In patricular, Thm. 1.4.3.2 implies that it has an obstruction theory relative to the HA context $(C(k), C(k)_{\leq 0}, k-cdga_0)$. We thus have

$$\mathbb{T}_{\mathbf{Ass}^{CW}_{[a,b]},B} \simeq \mathbb{T}_{\Omega_B \mathbf{Ass}^{CW}_{[a,b]},B}[1].$$

The identification

$$\mathbb{T}_{\Omega_B \mathbf{Ass}^{CW}_{[a,b]},B} \simeq \mathbb{R}\underline{Der}_k(B,B)$$

follows from the exact same argument as Prop. 2.2.6.9, using $\mathbb{L}^{Ass}_B$ instead of $\mathbb{L}^{\mathcal{O}}_B$. □

2.3.5.4. The D-stack of negative CW-dg-categories. Recall from §2.3.3.2 the existence of the morphism of D-stacks

$$B : \mathbf{Ass} \longrightarrow \mathbf{Cat}_*,$$

sending an associative k-algebra C to the dg-category BC having a unique object and C as endomorphisms of this object.

DEFINITION 2.3.5.10. *Let $n \leq 0$ be an integer. The D-stack of* CW-dg-categories *of amplitude contained in $[n,0]$ is defined as the full sub-D-stack $\mathbf{Cat}^{CW}_{*,[n,0]}$ of $\mathbf{Cat}_*$ consisting of the essential image of the morphism*

$$\mathbf{Ass}^{CW}_{[n,0]} \longrightarrow \mathbf{Ass} \longrightarrow \mathbf{Cat}_*.$$

More precisely, for $A \in k-cdga$, one sets $\mathbf{Cat}^{CW}_{*,[n,0]}(A)$ to be the sub-simplicial set of $\mathbf{Cat}_*(A)$ consisting of A-dg-categories $\mathcal{D}$ such that for any $* \in Ob(\mathcal{D})$ the A-dg-module $\mathcal{D}(*,*)$ is (locally) a perfect CW-A-dg-module of amplitude contained in $[n,0]$.

Recall that for any associative k-dga B, one has a model category $B\otimes^{\mathbb{L}}_k B^{op} - Mod_{dg}$ of B-bi-dg-modules. This model category is naturally tensored and co-tensored over the symmetric monoidal model category $C(k)$, making it into a $C(k)$-model category in the sense of [**Ho1**]. The derived $\mathrm{Ho}(C(k))$-enriched Hom's of $B\otimes^{\mathbb{L}}_k B^{op} - Mod_{dg}$ will then be denoted by $\mathbb{R}\underline{Hom}_{B\otimes^{\mathbb{L}}_k B^{op}}$.

Finally, for any associative k-dga B, one sets

$$\mathbb{HH}_k(B,B) := \mathbb{R}\underline{Hom}_{B\otimes^{\mathbb{L}}_k B^{op}}(B,B) \in \mathrm{Ho}(C(k)),$$

where B is considered as B-bi-dg-module in the obvious way.

THEOREM 2.3.5.11. (1) *The morphism*

$$B : \mathbf{Ass}^{CW}_{[n,0]} \longrightarrow \mathbf{Cat}^{CW}_{*,[n,0]}$$

is a 1*-representable fip-smooth covering of D-stacks.*

(2) *The associated D-stack to* $\mathbf{Cat}^{CW}_{*,[n,0]}$ *is a 2-geometric D-stack.*
(3) *If* C *is an associative k-dg-algebra, corresponding to a point* $C : Spec\,k \longrightarrow \mathbf{Ass}^{CW}_{[n,0]}$, *then one has a natural isomorphism in* $\mathrm{Ho}(C(k))$

$$\mathbb{T}_{\mathbf{Cat}^{CW}_{*,[n,0]},BC} \simeq \mathbb{HH}_k(C,C)[2].$$

Proof. (1) Using Cor. 2.3.3.7, it is enough to show that for any $A \in k-cdga$, and any associative A-algebra C, which is a perfect CW-A-dg-module of amplitude contained in $[n,0]$, the morphism of D-stack $K(C^*,1) \longrightarrow \mathbb{R}\underline{Spec}\,A$ is 1-representable and fip-smooth. For this it is clearly enough to show that the D-stack C^* is representable and that the morphism $C^* \longrightarrow \mathbb{R}\underline{Spec}\,A$ is fip-smooth. We already have seen during the proof of Prop. 2.3.3.5 that C^* is representable and a Zariski open sub-D-stack of $\mathbb{R}\underline{Spec}\,F$, where F is the free A-cdga over $C^\vee$. It is therefore enough to see that $A \longrightarrow F$ is fip-smooth, which follows from Lem. 2.3.5.5 as by assumption $C^\vee$ is a perfect CW-A-dg-module of amplitude contained in $[0,n]$.

(2) Follows from Cor. 2.3.5.9 and Cor. 1.3.4.5.

(3) We consider a point $C : Spec\,k \longrightarrow \mathbf{Ass}^{CW}[n,0]$, and the homotopy cartesian square

$$\begin{array}{ccc}
\mathbf{Ass}^{CW}_{[n,0]} & \longrightarrow & \mathbf{Cat}^{CW}_{*,[n,0]} \\
\uparrow & & \uparrow{\scriptstyle C} \\
K(C^*,1) & \longrightarrow & Spec\,k.
\end{array}$$

By (2), Cor. 2.3.5.9 and Thm. 1.4.3.2 we know that all the stacks in the previous square have an obstruction theory, and thus a cotangent complex (relative to the HA context $(C(k), C(k)_{\leq 0}, k-cdga_0)$). Therefore, one finds a homotopy fibration sequence of complexes of k-modules

$$\mathbb{T}_{K(C^*,1),*} \longrightarrow \mathbb{T}_{\mathbf{Ass}^{CW}_{[n,0]},C} \longrightarrow \mathbb{T}_{\mathbf{Cat}^{CW}_{*,[n,0]},BC}$$

that can also be rewritten as

$$C[1] \longrightarrow \mathbb{R}\underline{Der}_k(C,C)[1] \longrightarrow \mathbb{T}_{\mathbf{Cat}^{CW}_{*,[n,0]},BC}.$$

The morphism $C \longrightarrow \mathbb{R}\underline{Der}_k(C,C)$ can be described in the following way. The C-bi-dg-module $\mathbb{L}^{Ass}_C$ can be easily identified with the homotopy fiber (in the model category of $C \otimes^{\mathbb{L}}_k C^{op}$-dg-modules) of the multiplication morphism

$$C \otimes^{\mathbb{L}}_k C^{op} \longrightarrow C.$$

The natural morphism $\mathbb{L}^{Ass}_C \longrightarrow C \otimes^{\mathbb{L}}_k C^{op}$ then induces our morphism on the level of derivations

$$C \simeq \mathbb{R}\underline{Hom}_{C\otimes^{\mathbb{L}}_k C^{op}-Mod}(C\otimes^{\mathbb{L}}_k C^{op}, C) \longrightarrow \mathbb{R}\underline{Hom}_{C\otimes^{\mathbb{L}}_k C^{op}-Mod}(\mathbb{L}^{Ass}_C, B) \simeq \mathbb{R}\underline{Der}_k(C,C).$$

In particular, we see that there exists a natural homotopy fiber sequence in $C(k)$

$$\mathbb{HH}_k(C,C) = \mathbb{R}\underline{Hom}_{C\otimes^{\mathbb{L}}_k C^{op}-Mod}(C,C) \longrightarrow C \longrightarrow \mathbb{R}\underline{Der}_k(C,C).$$

We deduce that there exists a natural isomorphism in $\mathrm{Ho}(C(k))$

$$\mathbb{T}_{\mathbf{Cat}^{CW}_{*,[n,0]},BC}[-2] \simeq \mathbb{HH}_k(C,C).$$

□

An important corollary of Thm. 2.3.5.11 is given by the following fact. It appears in many places in the literature but we know of no references including a proof of it. For this, we recall that for any commutative k-algebra k' we denote by $Cat_*(k')$ the category of equivalences between k'-dg-categories satisfying the following two conditions.

- For any two objects x and y in $\mathcal{D}$ the complex of k'-module $\mathcal{D}(x,y)$ is perfect (and cofibrant in $C(k')$).
- The category $\pi_0(\mathcal{D})$ possesses a unique object up to isomorphism.

We finally let $Cat_*^{[n,0]}(k')$ be the full subcategory of $Cat_*(k')$ consisting of objects $\mathcal{D}$ such that for any two objects x and y the perfect complex $\mathcal{D}(x,y)$ has Tor amplitude contained in $[n,0]$ for some $n \leq 0$.

COROLLARY 2.3.5.12. *Let $\mathcal{D} \in Cat_*^{[n,0]}(k')$. Then, the homotopy fiber $\mathbb{D}ef_{\mathcal{D}}$, taken at the point $\mathcal{D}$, of the morphism of simplicial sets*

$$N(Cat_*^{[n,0]}(k[\epsilon])) \longrightarrow N(Cat_*^{[n,0]}(k))$$

is given by

$$\mathbb{D}ef_{\mathcal{D}} \simeq Map_{C(k)}(k, \mathbb{HH}_k(\mathcal{D},\mathcal{D})[2]).$$

In particular, we have

$$\pi_i(\mathbb{D}ef_{\mathcal{D}}) \simeq \mathbb{HH}_k^{2-i}(\mathcal{D},\mathcal{D}).$$

In the above corollary we have used $\mathbb{HH}_k(\mathcal{D},\mathcal{D})$, the Hochschild complex of a dg-category $\mathcal{D}$. It is defined the same way as for associative dg-algebras, and when $\mathcal{D}$ is equivalent to BC for an associative dg-alegbra C we have

$$\mathbb{HH}_k(\mathcal{D},\mathcal{D}) \simeq \mathbb{HH}_k(C,C).$$

Finally, we would like to mention that restricting to negatively graded dg-categories seems difficult to avoid if we want to keep the existence of a cotangent complex.

COROLLARY 2.3.5.13. *Assume that k is a field. Let $C \in \mathbf{Ass}_{[a,b]}^{CW}(k)$ be a k-point corresponding to an associative dg-algebra C. If we have $H^i(C) \neq 0$ for some $i > 0$ then the D-stack $\mathbf{Cat}_*$ does not have a cotangent complex at the point BC.*

PROOF. Suppose that $\mathbf{Cat}_*$ does have a cotangent complex at the point BC. Then, as so does the D-stack $\mathbf{Ass}_{[a,b]}^{CW}$ (by Cor. 2.3.5.9), we see that the homotopy fiber, taken at the point BC, of the morphism $B : \mathbf{Ass} \longrightarrow \mathbf{Cat}_*$ has a cotangent complex at C. This homotopy fiber is $K(C^*,1)$ (see Cor. 2.3.3.7), and thus we would have

$$\mathbb{L}_{K(C^*,1),C} \simeq C^{\vee}[-1].$$

Let $k[\epsilon_{i-1}]$ be the square zero extension of k by $k[i-1]$ (i.e. ϵ_{i-1} is in degree $-i+1$), for some $i > 0$ as in the statement. We now consider the homotopy fiber sequence

$$Map_{C(k)}(C^{\vee}[-1], k[i-1]) \longrightarrow K(C^*,1)(k[\epsilon_{i-1}]) \longrightarrow K(C^*,1)(k).$$

Considering the long exact sequence in homotopy we find

$$\pi_1(K(C^*,1)(k[\epsilon_{i-1}])) = H^0(C)^* \oplus H^{i-1}(C) \longrightarrow \pi_1(K(C^*,1)(k)) = H^0(C)^* \longrightarrow$$

$$\pi_0(Map_{C(k)}(C^{\vee}[-1], k[i-1])) = H^i(C) \longrightarrow \pi_0(K(C^*,1)(k[\epsilon_{i-1}])) = *.$$

This shows that the last morphism must be injective, which can not be the case as soon as $H^i(C) \neq 0$. ∎

REMARK 2.3.5.14. (1) Some of the results in Thm. 2.3.5.11 were announced in [**To-Ve2**, Thm. 5.6]. We need to warn the reader that [**To-Ve2**, Thm. 5.6] is not correct for the description of $\widetilde{\mathbb{R}Cat_{\mathcal{O}}}$ briefly given before that theorem (the same mistake appears in [**To2**, Thm. 4.4]). Indeed, $\widetilde{\mathbb{R}Cat_{\mathcal{O}}}$ would correspond to *isotrivial* deformations of dg-categories, for which the underlying complexes of morphisms stays locally constant. Therefore, the tangent complex of $\widetilde{\mathbb{R}Cat_{\mathcal{O}}}$ can not be the full Hochschild complex as stated in [**To-Ve2**, Thm. 5.6]. Our theorem 2.3.5.11 corrects this mistake.

(2) We like to consider our Thm. 2.3.5.11 (3) and Cor. 2.3.5.12 as a possible explanation of the following sentence in [**Ko-So**, p. 266]:
"In some sense, the full Hochschild complex controls deformations of the A_∞-category with one object, such that its endomorphism space is equal to A."

Furthermore, our homotopy fibration sequence

$$K(C^*,1) \longrightarrow \mathbf{Ass}^{CW}_{[n,0]} \longrightarrow \mathbf{Cat}^{CW}_{*,[n,0]}$$

is the geometric global counter-part of the well known exact triangles of complexes (see e.g. [**Ko**, p. 59])

$$C[1] \longrightarrow \mathbb{R}\underline{Der}_k(C,C)[2] \longrightarrow \mathbb{HH}_k(C,C)[2] \xrightarrow{+1}$$

as we pass from the former to the latter by taking tangent complexes at the point C.

(3) We saw in Cor. 2.3.5.13 that the full D-stack $\mathbf{Cat}_*$ can not have a reasonable infinitesimal theory. We think it is important to mention that even Cor. 2.3.5.12 cannot reasonably be true if we remove the assumption that $\mathcal{D}(x,y)$ is of Tor-amplitude contained in $[n,0]$ for some $n \leq 0$. Indeed, for any commutative k-algebra k', the morphism

$$B : \mathbf{Ass}(k') \longrightarrow \mathbf{Cat}_*(k')$$

is easily seen to induce an isomorphism on π_0 and a surjection on π_1. From this and the fact that $\mathbb{T}_{\mathbf{Ass},C} = \mathbb{R}\underline{Der}_k(C,C)[1]$, we easily deduce that the natural morphism

$$\pi_0(\mathbb{R}\underline{Der}_k(C,C)[1]) \longrightarrow \pi_0(\mathbb{D}ef_{BC})$$

is surjective. Therefore, if Cor. 2.3.5.12 were true for $\mathcal{D} = BC$, we would have that the morphism

$$H^1(\mathbb{R}\underline{Der}_k(C,C)) \longrightarrow \mathbb{HH}^2(C,C)$$

is surjective, or equivalently that the natural morphism

$$H^2(C) \longrightarrow H^2(\mathbb{R}\underline{Der}_k(C,C))$$

is injective. But this is not the case in general, as the morphism $C \longrightarrow \mathbb{R}\underline{Der}_k(C,C)$ can be zero (for example when C is commutative). It is therefore not strictly correct to state that the Hochschild cohomology of an associative dg-algebra controls its deformation as a dg-category, contrary to what appears in several references (including some of the authors !).

(4) For $n=0$, we can easily show that the restriction $N^*(\mathbf{Cat}^{CW}_{*,[0,0]}) \in D^-\mathrm{St}(k)$ is a 2-geometric D^--stack in the sense of §2.2. Furthermore, its truncation $t_0N^*(\mathbf{Cat}^{CW}_{*,[0,0]}) \in \mathrm{Ho}(k-Aff^{\sim,et})$ is naturally equivalent to the Artin 2-stack of k-linear categories with one object. The tangent complex of

$N^*(\mathbf{Cat}^{CW}_{*,[0,0]})$ at a point C corresponding to a k-alegbra projective of finite type over k is then the usual Hochschild complex $HH(C,C)$ computing the Hochschild cohomology of C.

Of course $\mathbf{Cat}^{CW}_{*,[n,0]}$ is only a rough approximation to what should be the stack of dg-categories, and in particular we think that $\mathbf{Cat}^{CW}_{*,[n,0]}$ is not suited for dealing with dg-categories coming from algebraic geometry. Let for example X be a smooth and projective variety over a field k; it is known that the the derived category $D_{qcoh}(X)$ is of the form $\mathrm{Ho}(B - Mod_{dg})$ for some associative dg-algebra B which is perfect as a complex of k-modules. It is however very unlikely that B can be chosen to be concentrated in degrees $[n,0]$ for some $n \leq 0$ (by construction $H^i(B) = Ext^i(E,E)$ for some compact generator $E \in D_{qcoh}(X)$). So the dg-algebra B when considered as a dg-category will *not* define a point in $\mathbf{Cat}^{CW}_{*,[n,0]}$. Another, more serious problem comes from the fact that the dg-algebra B is not uniquely determined, but is only unique up to Morita equivalence. As a consequence, the variety X can deform and B might not follow this deformation (though another, Morita equivalent, dg-algebra will follow the deformation), and thus $X \mapsto B$ will not be a morphism of stacks (even locally around X). Therefore, it seems very important to consider the stack $\mathbf{Cat}_*$ modulo Morita equivalences. We also think that passing to Morita equivalences will solve the problem of the non geometricity of $\mathbf{Cat}_*$ mentioned above. This direction is currently being investigated by M. Anel.

$\mathcal{N}(\mathbf{Cat}^{CW}_{*,[n,0]})$ at a point C corresponding to a k-algebra projective of finite type over k is given by the usual Hochschild complex $HH(C,C)$ computing the Hochschild cohomology of C.

Of course $\mathbf{Cat}^{CW}_{*,[n,0]}$ is only a rough approximation to what should be the stack of dg-categories, and in particular we think that $\mathbf{Cat}^{CW}_{*,[n,0]}$ is not suited for dealing with dg-categories coming from algebraic geometry. Let for example X be a smooth and projective variety over a field k; it is known that the derived category $D_{parf}(X)$ is of the form $Ho(B^{op}-Mod_{dg})$ for some associative dg-algebra B which is perfect as a complex of k-modules. It is however very unlikely that B can be chosen to be concentrated in degrees $[n,0]$ for some $n \leq 0$ (by construction $H^i(B) = Ext^i(E,E)$ for some compact generator $E \in D_{parf}(X)$). So, the dg-algebra B when considered as a dg-category will not define a point in $\mathbf{Cat}^{CW}_{*,[n,0]}$. Another, more serious problem comes from the fact that the dg-algebra B is not uniquely determined, but is only unique up to Morita equivalence. As a consequence, the variety X can deform and B might not follow this deformation (though another, Morita equivalent, dg-algebra will follow the deformation), and thus $X \mapsto B$ will not be a morphism of stacks (even locally around X). Therefore, it seems very important to consider the stack $\mathbf{Cat}_*$ modulo Morita equivalences. We also think that passing to Morita equivalences will solve the problem of the non geometricity of $\mathbf{Cat}_*$ mentioned above. This direction is currently being investigated by M. Anel.

CHAPTER 2.4

Brave new algebraic geometry

In this final chapter we briefly present brave new algebraic geometry[1], i.e. algebraic geometry over ring spectra. We will emphasize the main differences with derived algebraic geometry, and the subject will be studied in more details in future works.

As in the case of complicial algebraic geometry, we will present two distinct HAG contexts (see Cor. 2.4.1.11) which essentially differs in the choice of the class **P**. The first one, where **P** is chosen to be the class of strongly étale morphisms (see Def. 2.4.1.3), is suited for defining *brave new* Deligne-Mumford stacks, and, as all the contexts based on "strong" morphisms, it is geometrically very close to usual algebraic geometry. The second HAG context, where **P** is chosen to be the class of fip-smooth morphisms (i.e. formally perfect and formally i-smooth morphisms), is weaker and is similar to the corresponding weak context already presented in complicial algebraic geometry: it allows to define *brave new* geometric stacks which are not Deligne-Mumford. Since the notion of fip-smooth morphism for brave new rings behaves differently from commutative rings (see Prop. 2.4.1.5), the geometric intuition in this context is once again a bit far from standard algebraic geometry (e.g. smooth morphisms are not necessary flat). Nonetheless we think that this context is very interesting, as it is not only geometrically reasonable (e.g. it satisfies Artin's conditions), but it is also able to "see" some of the interesting new phenomena arising in the theory of structured ring spectra.

2.4.1. Two HAG contexts

We let $C := Sp^{\Sigma}$, the category of symmetric spectra in $\mathbb{U}$. The model structure we are going to use on $Sp\Sigma$ is the so called positive stable model structure described in [**Shi**]. This model structure is Quillen equivalent to the usual model structure, but is much better behaved with respect to homotopy theory of monoids and modules objects. The model category Sp^{Σ} is a symmetric monoidal model category for the smash product of symmetric spectra. Furthermore, all our assumptions 1.1.0.1, 1.1.0.3, 1.1.0.2 and 1.1.0.4 are satisfied thanks to [**MMSS**, Theorem 14.5], [**Shi**, Thm. 3.1, Thm. 3.2], and [**Shi**, Cor. 4.3] in conjunction with Lemma [**HSS**, 5.4.4].

The category $Comm(Sp^{\Sigma})$ is the category of commutative symmetric ring spectra, together with the positive stable model structure. The category $Comm(Sp^{\Sigma})$ will be denoted by $S - Alg$, and its objects will simply be called *commutative S-algebras* or also *bn rings* (where *bn* stands for *brave new*). For any $E \in Sp^{\Sigma}$, we will set

$$\pi_i(E) := \pi_i^{stab}(RE),$$

where RE is a fibrant replacement of E in Sp^{Σ}, and $\pi_i^{stab}(RE)$ are the naive stable homotopy groups of the Ω-spectrum RE. Note that if $E \in Sp^{\Sigma}$ is fibrant, then there is a natural isomorphism $\pi_*^{stab}E \simeq \pi_* E$, and that a map $f : E' \to E''$ is a weak equivalence in Sp^{Σ} if and only if $\pi_* f$ is an isomorphism.

[1] The term "brave new rings" was invented by F. Waldhausen to describe structured ring spectra; we have only adapted it to our situation.

When A is a commutative S-algebra, the $\mathbb{Z}$-graded abelian group $\pi_*(A)$ has a natural structure of a commutative graded algebra. In the same way, when M an A-module, $\pi_*(M)$ becomes a graded $\pi_*(A)$-module. An object E will be called connective, or (-1)-connected, if $\pi_i(E) = 0$ for all $i < 0$. We let $\mathcal{C}_0$ be Sp_c^{Σ}, be full subcategory of connective objects in Sp^{Σ}. We let $\mathcal{A}$ be $S - Alg_0$, the full subcategory of $Comm(Sp^{\Sigma})$ consisting of commutative S-algebras A with $\pi_i(A) = 0$ for any $i \neq 0$. If we denote by $H : CommRings \longrightarrow Comm(\mathcal{C})$ the Eilenberg-MacLane functor, then $S - Alg_0$ is the subcategory of $S - Alg$ formed by all the commutative S-algebras equivalent to some Hk for some commutative ring k.

LEMMA 2.4.1.1. *The triplet* $(Sp^{\Sigma}, Sp_c^{\Sigma}, S - Alg_0)$ *is a HA context.*

PROOF. The only thing to check is that any object $A \in S - Alg_0$ is Sp_c^{Σ}-good in the sense of Def. 1.1.0.10. Thanks to the equivalence between the homotopy theory of Hk-modules and of complexes of k-modules (see [**EKMM**, Thm. IV.2.4]), this has already been proved during the proof of Lem. 2.3.1.1 (2) . □

The following example lists some classes of formally étale maps in brave new algebraic geometry, according to our general definitions in Chapter 1.1.

EXAMPLE 2.4.1.2.

(1) If A and B are connective S-algebras, a morphism $A \to B$ is formally *thh*-étale if and only if it is formally étale ([**Min**, Cor. 2.8]).
(2) A morphism of (discrete) commutative rings $R \to R'$ is formally étale if and only if the associated morphism $HR \to HR'$ of bn rings is formally étale if and only if the associated morphism $HR \to HR'$ of bn rings is formally *thh*-étale ([**HAGI**, §5.2]).
(3) the complexification map $KO \to KU$ is *thh*-formally étale (by [**Ro**, p. 3]) hence formally étale. More generally, the same argument shows that any *Galois extension* of bn rings, according to J. Rognes [**Ro**], is formally *thh*-étale, hence formally étale.
(4) There exist examples of formally étale morphisms of bn-rings which are not *thh*-étale (see [**Min**] or [**HAGI**, §5.2]).

As in the case of complicial algebraic geometry (Ch. 2.3) we find it useful to introduce also *strong* versions for properties of morphisms between commutative S-algebras.

DEFINITION 2.4.1.3. (1) *Let* $A \in S - Alg$, *and* M *be an* A-*module. The* A-*module* M *is* strong *if the natural morphism*
$$\pi_*(A) \otimes_{\pi_0(A)} \pi_0(M) \longrightarrow \pi_*(M)$$
is an isomorphism.
(2) *A morphism* $A \longrightarrow B$ *in* $S - Alg$ *is* strongly flat *(resp.* strongly (formally) smooth, *resp.* strongly (formally) étale, *resp.* a strong Zariski open immersion*) if* B *is strong as an* A-*module, and if the morphism of affine schemes*
$$Spec\, \pi_0(B) \longrightarrow Spec\, \pi_0(A)$$
is flat (resp. (formally) smooth, resp. (formally) étale, resp. a Zariski open immersion).

One of the main difference between derived algebraic geometry and unbounded derived algebraic geometry was that the strong notions of flat, smooth, étale and Zariski open immersion are not as easily related to the corresponding general notions presented in §1.2. In the present situation, the comparison is even more loose as typical phenomena arising from the existence of Steenrod operations in characteristic p, make the notion of smooth morphisms of S-algebras rather subtle, and definitely different from the above notion of strongly smooth morphisms. We do not think this is a problem of the theory, but rather we think of this as an interesting new feature of brave new algebraic geometry, as compared to derived algebraic geometry, and well worthy of investigation.

PROPOSITION 2.4.1.4. *Let $f : A \longrightarrow B$ be a morphism in $S-Alg$.*

(1) *If A and B are connective, the morphism f is étale (resp. a Zariski open immersion) in the sense of Def. 1.2.6.1, if and only if f is strongly étale (resp. a strong Zariski open immersion).*
(2) *If the morphism f is strongly flat (resp. strongly étale, resp. a strong Zariski open immersion), then it is flat (resp. étale, resp. a Zariski open immersion) in the sense of Def. 1.2.6.1.*

PROOF. (1) The proof is the same as for Thm. 2.2.2.6.

(2) The proof is the same as for Prop. 2.3.1.4. □

The reader will notice that the proof of Thm. 2.2.2.6 (2) does not apply to the present context as a smooth morphism of commutative rings is in general not smooth when considered as a morphism of commutative S-algebras. The typical example of this phenomenon is the following.

PROPOSITION 2.4.1.5.
- *The canonical map $H\mathbb{Q} \to H(\mathbb{Q}[T])$ is smooth, i-smooth and perfect.*
- *The canonical map $H\mathbb{F}_p \to H(\mathbb{F}_p[T])$ is strongly smooth but* not *formally smooth, nor formally i-smooth (Def. 1.2.7.1).*

PROOF. For any discrete commutative ring k, we have a canonical map $a_k : Hk[T] := F_{Hk}(Hk) \to H(k[T])$ of commutative Hk-algebras, corresponding to the map $H(k \to k[T])$ pointing the element T. Now

$$\pi_*(Hk[T]) \simeq \bigoplus_{r\geq 0} \mathrm{H}_*(\Sigma_r, k)$$

where in the group homology $\mathrm{H}_*(\Sigma_r, k)$, k is a trivial Σ_r-module. Since $\mathrm{H}_n(\Sigma_r, \mathbb{Q}) = 0$, for $n \neq 0$, and $\mathrm{H}_0(\Sigma_r, \mathbb{Q}) \simeq \mathbb{Q}$, for any $r \geq 0$ we see that $a_\mathbb{Q}$ is a stable homotopy equivalence, and therefore a weak equivalence ([**HSS**, Thm. 3.1.11]). In other words $H(\mathbb{Q}[T])$ "is" the free commutative $H\mathbb{Q}$-algebra on one generator; therefore it is finitely presented over $H\mathbb{Q}$ and, since for any (discrete) commutative ring k we have

$$\mathbb{L}_{Hk[T]/Hk} \simeq Hk[T],$$

the cotangent complex $\mathbb{L}_{H(\mathbb{Q}[T])/H\mathbb{Q}}$ is free of rank one over $H(\mathbb{Q}[T])$, hence projective and perfect. So $H\mathbb{Q} \to H(\mathbb{Q}[T])$ is smooth and perfect.

Let's move to the char $p > 0$ case. It is clear that $H\mathbb{F}_p \to H(\mathbb{F}_p[T])$ is strongly smooth; let's suppose that it is formally smooth. In particular $\mathbb{L}_{H(\mathbb{F}_p[T])/H\mathbb{F}_p}$ is a projective $H(\mathbb{F}_p[T])$-module. Therefore $\pi_*\mathbb{L}_{H(\mathbb{F}_p[T])/H\mathbb{F}_p}$ injects into

$$\pi_* \coprod_E H(\mathbb{F}_p[T]) \simeq \prod_E \mathbb{F}_p[T]$$

(concentrated in degree 0). But, by [**Ba-McC**, Thm. 4.2] and [**Ri-Rob**, Thm. 4.1],

$$\pi_*\mathbb{L}_{H(\mathbb{F}_p[T])/H\mathbb{F}_p} \simeq (H(\mathbb{F}_p[T]))_*(H\mathbb{Z})$$

and the last ring has $(H\mathbb{F}_p)_*(H\mathbb{Z})$ as a direct summand (using the augmentation $H(\mathbb{F}_p[T]) \to H\mathbb{F}_p$). Now, it is known that $(H\mathbb{F}_p)_*(H\mathbb{Z})$ is not concentrated in degree 0 (it is a polynomial $\mathbb{F}_p$-algebra in positive degrees generators, for $p = 2$, and the tensor product of such an algebra with an exterior $\mathbb{F}_p$-algebra for odd p). Therefore $H\mathbb{F}_p \to H(\mathbb{F}_p[T])$ cannot be formally smooth. □

We remark again that, as now made clear by the proof above, the conceptual reason for the non-smoothness of $H\mathbb{F}_p \to H(\mathbb{F}_p[T])$ is essentially the existence of (non-trivial) Steenrod operations in characteristic $p > 0$. Since formal smoothness is stable under base-change, we also conclude that $H\mathbb{Z} \to H(\mathbb{Z}[T])$ is not formally smooth. The same argument also shows that this morphism is not formally i-smooth.

The following example shows that the converse of Prop. 2.4.1.4 (2) is false in general.

EXAMPLE 2.4.1.6. The complexification map $m : KO \to KU$ is formally étale but not strongly formally étale. In fact, we have

$$\pi_* m : \pi_*(KO) = \mathbb{Z}[\eta, \beta, \lambda^{\pm 1}]/(\eta^3, 2\eta, \eta\beta, \beta^2 - 4\lambda) \longrightarrow \pi_*(KO) = \mathbb{Z}[\nu^{\pm 1}],$$

with $deg(\eta) = 1, deg(\beta) = 4, deg(\lambda) = 8, deg(\nu) = 2$, $\pi_* m(\eta) = 0$, $\pi_* m(\beta) = 2\nu^2$ and $\pi_* m(\lambda) = \nu^4$. In particular $\pi_0 m$ is an isomorphism (hence étale) but m is not strong. We address the reader to [**HAGI**, Rmk. 5.2.9] for an example, due to M. Mandell, of a non connective formally étale extension of $H\mathbb{F}_p$, which is therefore not strongly formally étale. There also exist examples of Zariski open immersion $HR \longrightarrow A$, here R is a smooth commutative k-algebra, such that A possesses non trivial negative homotopy groups (see [**HAGI**, §5.2] for more details).

The opposite model category $S - Alg$ will be denoted by $SAff$. We will endow it with the following *strong étale model topology*.

DEFINITION 2.4.1.7. *A family of morphisms* $\{Spec\, A_i \longrightarrow Spec\, A\}_{i\in I}$ *in* $SAff$ *is a* strong étale covering family *(or simply* s-ét covering family*) if it satisfies the following two conditions.*

(1) *Each morphism* $A \longrightarrow A_i$ *is strongly étale.*
(2) *There exists a finite sub-set* $J \subset I$ *such that the family* $\{A \longrightarrow A_i\}_{i\in J}$ *is a formal covering family in the sense of 1.2.5.1.*

Using the definition of strong étale morphisms, we immediately check that a family of morphisms $\{Spec\, A_i \longrightarrow Spec\, A\}_{i\in I}$ in $SAff$ is a s-ét covering family if and only if there exists a finite sub-set $J \subset I$ satisfying the following two conditions.

- For all $i \in I$, the natural morphism
$$\pi_*(A) \otimes_{\pi_0(A)} \pi_0(A_i) \longrightarrow \pi_*(A_i)$$
is an isomorphism.
- The morphism of affine schemes
$$\coprod_{i\in J} Spec\, \pi_0(A_i) \longrightarrow Spec\, \pi_0(A)$$
is étale and surjective.

LEMMA 2.4.1.8. *The s-ét covering families define a model topology on* $SAff$, *that satisfies assumption 1.3.2.2.*

PROOF. The same as for Lem. 2.2.2.13. □

The model topology s-ét gives rise to a model category of stacks $SAff^{\sim, \text{s-ét}}$.

DEFINITION 2.4.1.9. (1) *An* S-stack *is an object* $F \in SAff^{\sim,s\text{-ét}}$ *which is a stack in the sense of Def. 1.3.2.1.*

(2) *The* model category of S-stacks *is* $SAff^{\sim,s\text{-ét}}$, *and its homotopy category will be simply denoted by* $\mathrm{St}(S)$.

We now set $\mathbf{P}$ to be the class of fip-smooth morphisms (i.e. formally perfect and formally i-smooth morphisms) in $SAlg$, and $\mathbf{P}_{s\text{-ét}}$ be the class of strongly étale morphisms.

LEMMA 2.4.1.10. (1) *The class* $\mathbf{P}_{s\text{-ét}}$ *of strongly étale morphisms and the s-ét model topology satisfy assumptions 1.3.2.11.*

(2) *The class* $\mathbf{P}$ *of fip-smooth morphisms and the s-ét model topology satisfy assumptions 1.3.2.11.*

PROOF. It is essentially the same as for Lem. 2.3.2.1. □

COROLLARY 2.4.1.11. (1) *The 5-tuple* $(Sp^{\Sigma}, Sp^{\Sigma}, S - Alg, s\text{-ét}, \mathbf{P}_{s\text{-ét}})$ *is a HAG context.*

(2) *The 5-tuple* $(Sp^{\Sigma}, Sp^{\Sigma}_c, S - Alg_0, s\text{-ét}, \mathbf{P})$ *is a HAG context.*

According to our general theory, the notions of morphisms in $\mathbf{P}$ and $\mathbf{P}_{s\text{-ét}}$ gives two notions of geometric stacks in $SAff^{\sim,s\text{-ét}}$.

DEFINITION 2.4.1.12. (1) A n-geometric Deligne-Mumford S-stack *is an n-geometric S-stack with respect to the class* $\mathbf{P}_{s\text{-ét}}$ *of strongly étale morphisms.*

(2) An n-geometric S-stack *is an n-geometric S-stack with respect to the class* $\mathbf{P}$ *of fip-smooth morphisms.*

Of course, as $\mathbf{P}_{s\text{-ét}}$ is included in $\mathbf{P}$ any strong n-geometric Deligne-Mumford S-stack is an n-geometric S-stack.

Finally, the reader can easily check the following proposition.

PROPOSITION 2.4.1.13. (1) *The topology s-ét and the class* $\mathbf{P}_{s\text{-ét}}$ *satisfy Artin condition (for the HA context* $(Sp^{\Sigma}, Sp^{\Sigma}, S - Alg)$.

(2) *The topology s-ét and the class* $\mathbf{P}$ *satisfy Artin's condition (for the HA context* $(Sp^{\Sigma}, Sp^{\Sigma}_c, S - Alg_0)$.

In particular, we obtain as a corollary of Thm. 1.4.3.2 that any n-geometric S-stack has an obstruction theory relative to the HA context $(Sp^{\Sigma}, Sp^{\Sigma}_c, S - Alg_0)$. In the same way, any strong n-geometric Deligne-Mumford S-stack has an obstruction theory relative to the context $(Sp^{\Sigma}, Sp^{\Sigma}, S - Alg)$.

Without going into details, we mention that all the examples of geometric D-stacks given in the previous chapter can be generalized to examples of geometric S-stacks. One fundamental example is $\mathbf{Perf}^{CW}_{[a,b]}$ of perfect CW-modules of amplitude contained in $[a, b]$, which by a similar argument as for Prop. 2.3.5.4 is a 1-geometric S-stack.

2.4.2. Elliptic cohomology as a Deligne-Mumford S-stack

In this final section we present the construction of a 1-geometric Deligne-Mumford S-stack using the sheaf of spectra of topological modular forms.

The Eilenberg-MacLane spectrum construction (see [**HSS**, Ex. 1.2.5]) gives rise to a fully faithful functor

$$\mathbb{L}H_! : \mathrm{St}(\mathbb{Z}) \longrightarrow \mathrm{St}(S),$$

which starts from the homotopy category of stacks on the usual étale site of affine schemes, i.e.

$$\mathrm{St}(\mathbb{Z}) := \mathrm{Ho}(\mathbb{Z} - Aff^{\sim,\text{ét}}).$$

This functor has a right adjoint, called the *truncation functor*

$$h^0 := H^* : \mathrm{St}(S) \longrightarrow \mathrm{St}(\mathbb{Z}),$$

simply given by composing a simplicial presheaf $F : SAff^{op} \longrightarrow SSet$ with the functor $H : Aff \longrightarrow S - Aff$.

Let us denote by $\overline{\mathcal{E}}$ the moduli stack of generalized elliptic curves with integral geometric fibers, which is the standard compactification of the moduli stack of elliptic curves by adding the nodal curves at infinity (see e.g. [**Del-Rap**, IV], where it is denoted by $\mathcal{M}_{(1)}$); recall that $\overline{\mathcal{E}}$ is a Deligne-Mumford stack, proper and smooth over $Spec\,\mathbb{Z}$ ([**Del-Rap**, Prop. 2.2]).

As shown by recent works of M. Hopkins, H. Miller, P. Goerss, N. Strickland, C. Rezk and M. Ando, there exists a natural presheaf of commutative S-algebras on the small étale site $\overline{\mathcal{E}}_{\text{ét}}$ of $\overline{\mathcal{E}}$. We will denote this presheaf by *tmf*. Recall that by construction, if $U = Spec\,A \longrightarrow \overline{\mathcal{E}}$ is an étale morphism, corresponding to an elliptic curve E over the ring A, then $tmf(U)$ is the (connective) elliptic cohomology theory associated to the formal group of E (in particular, one has $\pi_0(tmf(U)) = A$). Recall also that the (derived) global sections $\mathbb{R}\Gamma(\overline{\mathcal{E}}, tmf)$, form a commutative S-algebra, well defined in $\mathrm{Ho}(S - Alg)$, called the *spectrum of topological modular forms*, and denoted by tmf [2].

Let $U \longrightarrow \overline{\mathcal{E}}$ be a surjective étale morphism with U an affine scheme, and let us consider its nerve

$$\begin{array}{rccc} U_* : & \Delta^{op} & \longrightarrow & Aff \\ & [n] & \mapsto & U_n := \underbrace{U \times_{\overline{\mathcal{E}}} U \times_{\overline{\mathcal{E}}} \cdots \times_{\overline{\mathcal{E}}} U}_{n\ times}. \end{array}$$

This is a simplicial object in $\overline{\mathcal{E}}_{\text{ét}}$, and by applying *tmf* we obtain a co-simplicial object in $S - Alg$

$$\begin{array}{rccc} tmf(U_*) : & \Delta & \longrightarrow & S - Alg \\ & [n] & \mapsto & tmf(U_n). \end{array}$$

Taking $\mathbb{R}\underline{Spec}$ of this diagram we obtain a simplicial object in the model category $\overline{S - Aff^{\sim,s\text{-ét}}}$

$$\begin{array}{rccc} \mathbb{R}\underline{Spec}\,(tmf(U_*)) : & \Delta^{op} & \longrightarrow & S - Aff^{\sim,s\text{-ét}} \\ & [n] & \mapsto & \mathbb{R}\underline{Spec}\,(tmf(U_n)). \end{array}$$

The homotopy colimit of this diagram will be denoted by

$$\overline{\mathcal{E}}_{\mathbf{S}} := \mathrm{hocolim}_{n\in\Delta^{op}} \mathbb{R}\underline{Spec}\,(tmf(U_*)) \in \mathrm{St}(S).$$

The following result is technically just a remark as there is essentially nothing to prove; however, we prefer to state it as a theorem to emphasize its importance.

THEOREM 2.4.2.1. *The stack $\overline{\mathcal{E}}_{\mathbf{S}}$ defined above is a strong Deligne-Mumford 1-geometric S-stack. Furthermore $\overline{\mathcal{E}}_{\mathbf{S}}$ is a "brave new derivation" of the moduli stack $\overline{\mathcal{E}}$ of elliptic curves, i.e. there exists a natural isomorphism in* $\mathrm{St}(\mathbb{Z})$

$$h^0(\overline{\mathcal{E}}_{\mathbf{S}}) \simeq \overline{\mathcal{E}}.$$

[2] The notation here is a bit nonstandard: what is usually called the spectrum of topological modular forms is actually the connective cover of the spectrum we have denoted by tmf

PROOF. To prove that $\overline{\mathcal{E}}_{\mathbf{S}}$ is geometric, it is enough to check that the simplicial object $\mathbb{R}\underline{Spec}\,(tmf(U_*))$ is a strongly étale Segal groupoid. For this, recall that for any morphism $U = Spec\,B \to V = Spec\,A$ in $\overline{\mathcal{E}}_{\acute{e}t}$, the natural morphism

$$\pi_*(tmf(V)) \otimes_{\pi_0(tmf(V))} \pi_0(tmf(U)) \simeq \pi_*(tmf(V)) \otimes_A B \longrightarrow \pi_*(tmf(U))$$

is an isomorphism. This shows that the functor

$$\mathbb{R}\underline{Spec}\,(tmf(-)) : \overline{\mathcal{E}}_{\acute{e}t} \longrightarrow S - Aff^{\sim, s\text{-}\acute{e}t}$$

preserves homotopy fiber products and therefore sends Segal groupoid objects to Segal groupoid objects. In particular, $\mathbb{R}\underline{Spec}\,(tmf(U_*))$ is a Segal groupoid object. The same fact also shows that for any morphism $U = Spec\,B \to V = Spec\,A$ in $\overline{\mathcal{E}}_{\acute{e}t}$, the induced map $tmf(V) \longrightarrow tmf(U)$ is a strong étale morphism. This implies that $\mathbb{R}\underline{Spec}\,(tmf(U_*))$ is a strongly étale Segal groupoid object in representable S-stacks, and thus shows that $\overline{\mathcal{E}}_{\mathbf{S}}$ is indeed a strong Deligne-Mumford 1-geometric S-stack.

The truncation functor h^0 clearly commutes with homotopy colimits, and therefore

$$h^0(\overline{\mathcal{E}}_{\mathbf{S}}) \simeq \mathrm{hocolim}_{n \in \Delta^{op}} h^0(\mathbb{R}\underline{Spec}\,(tmf(U_n))) \in \mathrm{St}(\mathbb{Z}).$$

Furthermore, for any connective representable S-stack, $\mathbb{R}\underline{Spec}\,A$, one has a natural isomorphism $h^0(\mathbb{R}\underline{Spec}\,A) \simeq Spec\,\pi_0(A)$. Therefore, one sees immediately that there is a natural isomorphism of simplicial objects in $\mathbb{Z} - Aff^{\sim, \acute{e}t}$

$$h^0(\mathbb{R}\underline{Spec}\,(tmf(U_*))) \simeq U_*.$$

Therefore, we get

$$h^0(\overline{\mathcal{E}}_{\mathbf{S}}) \simeq \mathrm{hocolim}_{n \in \Delta^{op}} h^0(\mathbb{R}\underline{Spec}\,(tmf(U_n))) \simeq \mathrm{hocolim}_{n \in \Delta^{op}} U_n \simeq \overline{\mathcal{E}},$$

as U_* is the nerve of an étale covering of $\overline{\mathcal{E}}$. □

Theorem 2.4.2.1 tells us that the presheaf of topological modular forms tmf provides a natural geometric S-stack $\overline{\mathcal{E}}_{\mathbf{S}}$ whose truncation is the usual stack of elliptic curves $\overline{\mathcal{E}}$. Furthermore, as one can show that the small strong étale topoi of $\overline{\mathcal{E}}_{\mathbf{S}}$ and $\overline{\mathcal{E}}$ coincide (this is a general fact about strong étale model topologies), we see that

$$\mathsf{tmf} := \mathbb{R}\Gamma(\overline{\mathcal{E}}, tmf) \simeq \mathbb{R}\Gamma(\overline{\mathcal{E}}_{\mathbf{S}}, \mathcal{O}),$$

and therefore that topological modular forms can be simply interpreted as *functions on the geometric S-stack $\overline{\mathcal{E}}_{\mathbf{S}}$*. Of course, our construction of $\overline{\mathcal{E}}_{\mathbf{S}}$ has essentially been rigged to make this true, so this is not a surprise. However, we have gained a bit from the conceptual point of view: since after all $\overline{\mathcal{E}}$ is a moduli stack, now that we know the existence of the geometric S-stack $\overline{\mathcal{E}}_{\mathbf{S}}$ we can ask for a *modular interpretation* of it, or in other words for a direct geometric description of the corresponding simplicial presheaf on $S - Aff$. An answer to this question not only would provide a direct construction of tmf, but would also give a conceptual interpretation of it in a geometric language closer the usual notion of modular forms.

QUESTION 2.4.2.2. *Find a modular interpretation of the S-stack $\overline{\mathcal{E}}_{\mathbf{S}}$.*

Essentially, we are asking for the brave new "objects" that the S-stack $\overline{\mathcal{E}}_{\mathbf{S}}$ classifies. We could also consider the non-connective version of the S-stack $\overline{\mathcal{E}}_{\mathbf{S}}$ (defined through the non-connective version of tmf) for which a modular interpretation seems much more accessible.

Very recent work by J. Lurie (see [**Lu2**] for a detailed announcement of his results) answers in fact to Question 2.4.2.2; he shows that such a variant of $\overline{\mathcal{E}}_{\mathbf{S}}$ classifies brave new versions of [**AHS**]'s elliptic spectra plus additional data (called orientations). This moduli-theoretic point of view makes use of some very interesting notions of *brave new abelian varieties*, *brave new formal groups* and their geometry. The complete picture (possibly extended to higher chromatic levels) does not only give an alternative construction of the spectrum tmf (and a better functoriality) but it could be the starting point of a rather new[3] and deep interaction between stable homotopy theory and homotopical algebraic geometry, involving many new questions and objects, and probably also new insights on classical objects of algebraic topology.

[3]J. Lurie's approach has as a byproduct also a natural construction of *G-equivariant* versions of elliptic cohomology, for any compact G.

APPENDIX A

Classifying spaces of model categories

The classifying space of a model category M is defined to be $N(M_W)$, the nerve of its subcategory of equivalences. More generally, if $C \subset M_W$ is a full subcategory of the category of equivalences in M, which is closed by equivalences in M, the classifying space of C is $N(C)$, the nerve of C.

We fix a $\mathbb{V}$-small model category M and a full subcategory $C \subset M_W$ closed by equivalences. We consider the model category $(C,C)^\wedge$, defined in [**HAGI**, §2.3.2]. Recall that the underlying category of $(C,C)^\wedge$ is the category $SSet_{\mathbb{V}}^{C^{op}}$, of $\mathbb{V}$-small simplicial presheaves on C. The model structure of $(C,C)^\wedge$ is defined as the left Bousfield localization of the levelwise projective model structure on $SSet_{\mathbb{V}}^{C^{op}}$, by inverting all the morphisms in C. The important fact is that local objects in $(C,C)^\wedge$ are functors $F : C^{op} \longrightarrow SSet_{\mathbb{V}}$ sending all morphisms in C to equivalences.

We define an adjunction

$$N : (C,C)^\wedge \longrightarrow SSet/N(C) \qquad (C,C)^\wedge \longleftarrow SSet/N(C) : S$$

in the following way. A functor $F : C^{op} \longrightarrow SSet$ is sent to the simplicial set $N(F)$, for which the set of n-simplices is the set of parirs

$$N(F)_n := \{(c_0 \to c_1 \to \cdots \to c_n, \alpha)\}$$

where $(c_0 \to c_1 \to \cdots \to c_n)$ is an n-simplex in $N(C)$ and $\alpha \in F(c_n)_n$ is an n-simplex in $F(c_n)$. Put in an other way, $N(F)$ is the diagonal of the bi-simplicial set

$$(n,m) \mapsto N(C/F_n)_m$$

where C/F_n is the category of objects of the presheaf of n-simplices in F. The functor N has a right adjoint

$$S : SSet/N(C) \longrightarrow (C,C)^\wedge$$

sending $X \longrightarrow N(C)$ to the simplicial presheaf

$$\begin{array}{rccc} S(X): & C^{op} & \longrightarrow & SSet \\ & x & \mapsto & \underline{Hom}_{SSet/N(C)}(N(h_x), X), \end{array}$$

where h_x is the presheaf of sets represented by $x \in C$ (note that $N(h_x) \to N(C)$ is isomorphic to $N(C/x) \to N(C)$).

PROPOSITION A.0.3. *The adjunction (N,S) is a Quillen equivalence.*

PROOF. First of all we need to check that (N,S) is a Quillen adjunction. For this we use the standard properties of left Bousfield localizations, and we see that it is enough to check that

$$N : SPr(C) \longrightarrow SSet/N(C) \qquad SPr(C) \longleftarrow SSet/N(C) : S$$

is a Quillen adjunction (where $SPr(C)$ is the projective model structure of simplicial presheaves on C), and that S preserves fibrant objects. These two facts are clear by definition of S and the description of fibrant objects in $N(C,C)^\wedge$.

For any $x \in C$, the morphism $N(h_x) = N(C/x) \longrightarrow N(C)$ is isomorphic in $\mathrm{Ho}(SSet/N(C))$ to $x \longrightarrow N(C)$. Therefore, for $X \in \mathrm{Ho}(SSet/N(C))$ and $x \in C$, the simplicial set

$$\mathbb{R}S(X)(x) \simeq \mathbb{R}\underline{Hom}_{SSet/N(C)}(x, X)$$

is naturally isomorphic in $\mathrm{Ho}(SSet)$ to the homotopy fiber of $X \longrightarrow N(C)$ taken at x. This clearly implies that the right derived functor

$$\mathbb{R}S : \mathrm{Ho}(SSet/N(C)) \longrightarrow \mathrm{Ho}((C,C)^{\wedge})$$

is conservative. In particular, it only remains to show that the adjunction morphism $Id \longrightarrow \mathbb{R}S\mathbb{L}N$ is an isomorphism. But this last assumption follows from the definition the functor N and from a standard lemma (se for example [**Q3**]), which shows that the homotopy fiber at $x \in C$ of $N(F) \longrightarrow N(C)$ is naturally equivalent to $F(x)$ when F is fibrant in $(C,C)^{\wedge}$. □

Recall from [**HAGI**, Lem. 4.2.2] that for any object $x \in C$, one can construct a local model for h_x as $\underline{h}_{R(x)}$, sending $y \in C$ to $Hom_C(\Gamma^*(y), R(x))$, where Γ^* is a co-simplicial replacement functor in M. The natural morphism $h_x \longrightarrow \underline{h}_{R(x)}$ being an equivalence in $(C,C)^{\wedge}$, one finds using Prop. A.0.3 natural equivalences of simplicial sets

$$\underline{Hom}(h_y, \underline{h}_{R(x)}) \simeq Map^{eq}_M(y,x) \simeq \mathbb{R}\underline{Hom}_{SSet/N(C)}(N(h_y), N(h_x)) \simeq y \times^h_{N(C)} x,$$

where $Map^{eq}_M(y,x)$ is the sub-simplicial set of the mapping space $Map_M(y,x)$ consisting of equivalences. As a corollary of this we find the important result due to Dwyer and Kan. For this, we recall that the simplicial monoid of self equivalences of an object $x \in M$ can be defined as

$$Aut(x) := \underline{Hom}_{(C,C)^{\wedge}}(\underline{h}_{R(x)}, \underline{h}_{R(x)}).$$

Corollary A.0.4. *Let $C \subset M_W$ be a full subcategory of equivalences in a model category M, which is stable by equivalences. Then, one has a natural isomorphism in* $\mathrm{Ho}(SSet)$

$$N(C) \simeq \coprod_{x \in \pi_0(N(C)))} BAut(x)$$

where $Aut(x)$ is the simplicial monoid of self equivalences of x in M.

Another important consequence of Prop. A.0.3 is the following interpretation of mapping spaces in term of homotopy fibers between classifying spaces of certain model categories.

Corollary A.0.5. *Let M be a model category and $x, y \in M$ be two fibrant and cofibrant objects in M. Then, there exists a natural homotopy fiber sequence of simplicial sets*

$$Map_M(x,y) \longrightarrow N((x/M)_W) \longrightarrow N(M_W),$$

where the homotopy fiber is taken at $y \in M$.

Proof. This follows easily from Cor. A.0.4. □

We will need a slightly more functorial interpretation of Cor. A.0.4 in the particular case where the model category M is simplicial. We assume now that M is a $\mathbb{V}$-small simplicial $\mathbb{U}$-model category (i.e. the model category M is a $SSet_{\mathbb{U}}$-model category in the sense of [**Ho1**]). We still let $C \subset M_W$ be a full subcategory of the category of equivalences in M, and still assume that $C \subset M_W$ is stable by equivalences.

We define an S-category $\mathcal{G}(C)$ is the following way (recall that an S-category is a simplicially enriched category, see for example [**HAGI**, §2.1] for more details and notations). The objects of $\mathcal{G}(C)$ are the objects of C which are furthermore fibrant and cofibrant in M. For two objects x and y, the simplicial set of morphisms is defined to be

$$\mathcal{G}(C)_{(x,y)} := \underline{Hom}_M^{eq}(x,y),$$

where by definition $\underline{Hom}_M^{eq}(x,y)_n$ is the set of equivalences in M from $\Delta^n \otimes x$ to y (i.e. $\underline{Hom}_M^{eq}(x,y)$ is the sub-simplicial set of $\underline{Hom}_M(x,y)$ consisting of equivalences). Clearly, the S-category $\mathcal{G}(C)$ is *groupoid like*, in the sense that its category of connected components $\pi_0(\mathcal{G}(C))$ (also denoted by $\mathrm{Ho}(\mathcal{G}(C))$ is a groupoid (or equivalently every morphism in $\mathcal{G}(C)$ has an inverse up to homotopy). Let $C^{c,f}$ be the full subcategory of C consisting of fibrant and cofibrant objects in C. There exist two natural morphisms of S-categories

$$C \longleftarrow C^{c,f} \longrightarrow \mathcal{G}(C),$$

where a category is considered as an S-category with discrete simplicial sets of morphisms. Passing to the nerves, one gets a diagram of simplicial sets

$$N(C) \longleftarrow N(C^{c,f}) \longrightarrow N(\mathcal{G}(C)),$$

where the nerve functor is extended diagonally to S-categories (see e.g. [**D-K**1]). Another interpretation of corollary A.0.4 is the following result.

PROPOSITION A.0.6. *With the above notations, the two morphisms*

$$N(C) \longleftarrow N(C^{c,f}) \longrightarrow N(\mathcal{G}(C)),$$

are equivalences of simplicial sets.

PROOF. It is well known that the left arrow is an equivalence as a fibrant-cofibrant replacement functor gives an inverse up to homotopy. For the right arrow, we let $N(\mathcal{G}(C))_n$ be the category of n-simplicies in $N(\mathcal{G}(C))$, defined by having the same objects and with

$$(N(\mathcal{G}(C))_n)_{x,y} := (N(\mathcal{G}(C))_{x,y})_n.$$

By definition of the nerve, one has a natural equivalence

$$N(\mathcal{G}(C)) \simeq Hocolim_{n\Delta^{op}}(N(\mathcal{G}(C))_n).$$

Furthermore, it is clear that each functor

$$C^{c,f} = \mathcal{G}(C)_0 \longrightarrow \mathcal{G}(C)_n$$

induces an equivalence of the nerves, as the 0-simplex $[0] \to [n]$ clearly induces a functor

$$N(\mathcal{G}(C)_n) \longrightarrow N(C^{c,f})$$

which is a homotopy inverse. □

Proposition A.0.6 is another interpretation of Prop. A.0.3, as the S-category $\mathcal{G}(C)$ is groupoid-like, the delooping theorem of G. Segal implies that there exists a natural equivalence of simplicial sets

$$x \times^h_{N(C)} y \longleftarrow x \times^h_{N(C^{c,f})} y \longrightarrow x \times^h_{N(\mathcal{G}(C))} y \longleftarrow \mathcal{G}(C)_{(x,y)} = \underline{Hom}_M^{eq}(x,y).$$

The advantage of Prop. A.0.6 over the more general proposition A.0.3 is that it is more easy to state a functorial property of the equivalences in the following particular context (the equivalence in Prop. A.0.3 can also be made functorial, but it requires some additional work, using for example simplicial localization techniques).

We assume that $G : M \longrightarrow N$ is a simplicial, left Quillen functor between $\mathbb{V}$-small simplicial $\mathbb{U}$-model categories. We let $C \subset M_W$ and $D \subset N_W$ be two full sub-categories stable by equivalences, and we suppose that all objects in M and N are fibrant. Finally, we assume that the functor G restricted to cofibrant objects sends $C^c := M^c \cap C$ to $D^c := N^c \cap D$. In this situation, we define an S-functor

$$G : \mathcal{G}(C) \longrightarrow \mathcal{G}(D)$$

simply by using the simplicial enrichment of G. Then, one has a commutative diagram of S-categories

$$\begin{array}{ccccc} C & \longleftarrow & C^c & \longrightarrow & \mathcal{G}(C) \\ & & \downarrow G & & \downarrow G \\ D & \longleftarrow & D^c & \longrightarrow & \mathcal{G}(D), \end{array}$$

and thus a commutative diagram of simplicial sets

$$\begin{array}{ccccc} N(C) & \longleftarrow & N(C^c) & \longrightarrow & N(\mathcal{G}(C)) \\ & & \downarrow G & & \downarrow G \\ N(D) & \longleftarrow & N(D^c) & \longrightarrow & N(\mathcal{G}(D)). \end{array}$$

The important fact here is that this construction is associative with respect to composition of the simplicial left Quillen functor G. In other words, if one has a diagram of simplicial model categories M_i and simplicial left Quillen functors, together with sub-categories $C_i \subset (M_i)_W$ satisfying the required properties, then one obtain a commutative diagram of diagrams of simplicial sets

$$\begin{array}{ccccc} N(C_i) & \longleftarrow & N(C_i^c) & \longrightarrow & N(\mathcal{G}(C_i)) \\ & & \downarrow G_i & & \downarrow G_i \\ N(D_i) & \longleftarrow & N(D_i^c) & \longrightarrow & N(\mathcal{G}(D_i)). \end{array}$$

APPENDIX B

Strictification

Let I be a $\mathbb{U}$-small category. For any $i \in I$ we let M_i be a $\mathbb{U}$-cofibrantly generated model category, and for any $u : i \to j$ morphism in I we let

$$u^* : M_j \longrightarrow M_i \qquad M_j \longleftarrow M_i : u_*$$

be a Quillen adjunction. We suppose furthermore that for any composition

$$i \xrightarrow{u} j \xrightarrow{v} k$$

one has an equality of functors

$$u^* \circ v^* = (v \circ u)^*$$

Such a data $(\{M_i\}_i, \{u^*\}_u)$ will be called, according to **[H-S]**, a *($\mathbb{U}$-)cofibrantly generated left Quillen presheaf over I*, and will be denoted simply by the letter M.

For any cofibrantly generated left Quillen presheaf M on I, we consider the category M^I, of I-diagrams in M in the following way. Objects in M^I are given by the data of objects $x_i \in M_i$ for any $i \in I$, together with morphisms $\phi_u : u^*(x_j) \longrightarrow x_i$ for any $u : i \to j$ in I, making the following diagram commutative

$$\begin{array}{ccc} u^*v^*(x_k) & \xrightarrow{u^*(\phi_v)} & u^*(x_j) \\ {\scriptstyle Id}\downarrow & & \downarrow{\scriptstyle \phi_u} \\ (v \circ u)^*(x_k) & \xrightarrow[\phi_{v \circ u}]{} & x_i \end{array}$$

for any $i \xrightarrow{u} j \xrightarrow{v} k$ in I. Morphisms in M^I are simply given by families morphisms $f_i : x_i \longrightarrow y_i$, such that $f_i \circ \phi_u = \phi_u \circ f_j$ for any $i \to j$ in I.

The category M^I is endowed with a model structure for which the fibrations or equivalences are the morphisms f such that for any $i \in I$ the induced morphism f_i is a fibration or an equivalence in M_i. As all model categories M_i are $\mathbb{U}$-cofibrantly generated, it is not hard to adapt the general argument of **[Hi**, 11.6] in order to prove that M^I is also a $\mathbb{U}$-cofibrantly generated model category.

We define an object $x \in M^I$ to be *homotopy cartesian* if for any $u : i \to j$ in I the induced morphism

$$\mathbb{L}u^*(x_i) \longrightarrow x_j$$

is an isomorphism in $\mathrm{Ho}(M_j)$. The full subcategory of cartesian objects in M^I will be denoted by M^I_{cart}.

There exists a presheaf of categories $(-/I)^{op}$ over I, having the opposite comma category $(i/I)^{op}$ as value over the object i, and the natural functor $(i/I)^{op} \longrightarrow (j/I)^{op}$ for any morphism $j \longrightarrow i$ in I. For any $\mathbb{U}$-cofibrantly left Quillen presheaf M over I, we define a morphism of presheaves of categories over I

$$M^I \times (-/I)^{op} \longrightarrow M,$$

where M^I is seen as a constant presheaf of categories, in the following way. For an object $i \in I$, the functor

$$M^I \times (i/I)^{op} \longrightarrow M_i$$

sends an object $(x, u : i \to j)$ to $u^*(x_j) \in M_i$, and a morphism $(x, u : i \to j) \longrightarrow (y, v : i \to k)$, given by a morphism $x \to y$ in M^I and a commutative diagram in I

$$\begin{array}{ccc} i & & \\ {\scriptstyle v}\downarrow & \searrow^{u} & \\ k & \xrightarrow{w} & j, \end{array}$$

is sent to the morphism in M_i

$$u^*(x_j) \simeq v^*(w^*(x_j)) \longrightarrow v^*(x_k).$$

For a diagram $u : i \to j$ in I, the following diagram

$$\begin{array}{ccc} M^I \times (j/I)^{op} & \longrightarrow & M_j \\ \downarrow & & \downarrow {\scriptstyle u^*} \\ M^I \times (i/I)^{op} & \longrightarrow & M_i \end{array}$$

clearly commutes, showing that the above definition actually defines a morphism

$$M^I \times (-/I)^{op} \longrightarrow M.$$

We let $(M^I_{cart})^{cof}_W$ be the subcategory of M^I consisting of homotopy cartesian and cofibrant objects in M^I and equivalences between them. In the same way we consider the sub-presheaf M^c_W whose value at $i \in I$ is the subcategory of M_i consisting of cofibrant objects in M_i and equivalences between them. Note that the functors $u^* : M_j \longrightarrow M_i$ being left Quillen for any $u : i \longrightarrow j$, preserves the sub-categories of equivalences between cofibrant objects.

We have thus defined a morphism of presheaves of categories

$$(M^I_{cart})^c_W \times (-/I)^{op} \longrightarrow M^c_W,$$

and we now consider the corresponding morphism of simplicial presheaves obtained by applying the nerve functor

$$N((M^I_{cart})^c_W) \times N((-/I)^{op}) \longrightarrow N(M^c_W),$$

that is considered as a morphism in the homotopy category $\mathrm{Ho}(SPr(I))$, of simplicial presheaves over I. As for any i the category $(i/I)^{op}$ has a final object, its nerve $N((i/I)^{op})$ is contractible, and therefore the natural projection

$$N((M^I_{cart})^c_W) \times N((-/I)^{op}) \longrightarrow N((M^I_{cart})^c_W)$$

is an isomorphism in $\mathrm{Ho}(SPr(I))$. We therefore have constructed a well defined morphism i $\mathrm{Ho}(SPr(I))$, from the constant simplicial presheaf $N((M^I_{cart})^c_W)$ to the simplicial presheaf $N(M^c_W)$. By adjunction this gives a well defined morphism in $\mathrm{Ho}(SSet)$

$$N((M^I_{cart})^c_W) \longrightarrow Holim_{i\in I} N(M^c_W).$$

The strictification theorem asserts that this last morphism is an isomorphism in $\mathrm{Ho}(SSet)$. As this seems to be a folklore result we will not include a proof.

THEOREM B.0.7. *For any* $\mathbb{U}$*-small category* I *and any* $\mathbb{U}$*-cofibrantly generated left Quillen presheaf* M *on* I*, the natural morphism*

$$N((M^I_{cart})^c_W) \longrightarrow Holim_{i\in I^{op}} N(M^c_W)$$

is an isomorphism in $\mathrm{Ho}(SSet)$.

PROOF. When M is the constant Quillen presheaf of simplicial sets this is proved in [**D-K**3]. The general case can treated in a similar way. See also [**H-S**, Thm. 18.6] for a stronger result. □

Let I and M be as in the statement of Thm. B.0.7, and let M_0 be a $\mathbb{U}$-cofibrantly model category. We consider M_0 as a constant left Quillen presheaf on I, for which all values are equal to M_0, and all transition functors are identities. We assume that there exists a left Quillen natural transformation $\phi : M_0 \longrightarrow M$. By this, we mean the data of left Quillen functors $\phi_i : M_0 \longrightarrow M_i$ for any $i \in I$, such that for any $u : i \to j$ one has $u^* \circ \phi_j = \phi_i$. In this case, we define a functor

$$\phi : M_0 \longrightarrow M^I,$$

by the obvious formula $\phi(x)_i := \phi_i(x)$ for $x \in M_0$, and the transition morphisms of $\phi(x)$ all being identities. The functor ϕ is not a left Quillen functor, but preserves equivalences between cofibrant objects, and thus possesses a left derived functor

$$\mathbb{L}\phi : \mathrm{Ho}(M_0) \longrightarrow \mathrm{Ho}(M^I).$$

One can even show that this functor possesses a right adjoint, sending an object $x \in M^I$ to the homotopy limit of the diagram in M_0, $i \mapsto \mathbb{R}\psi_i(x_i)$, where ψ_i is the right adjoint to ϕ_i.

One also has a natural transformation of presheaves of categories

$$(M_0)^c_W \longrightarrow M^c_W$$

inducing a natural morphism of simplicial presheaves on I

$$N((M_0)^c_W) \longrightarrow N(M^c_W),$$

and thus a natural morphism in $\mathrm{Ho}(SSet)$

$$N((M_0)^c_W) \longrightarrow Holim_{i\in I^{op}} N(M^c_W).$$

COROLLARY B.0.8. *Let* I*,* M *and* M_0 *be as above, and assume that the functor*

$$\mathbb{L}\phi : \mathrm{Ho}(M_0) \longrightarrow \mathrm{Ho}(M^I)$$

is fully faithful and that its image consists of all homotopy cartesian objects in $\mathrm{Ho}(M^I)$. *Then the induced morphism*

$$N((M_0)^c_W) \longrightarrow Holim_{i\in I^{op}} N(M^c_W)$$

is an isomorphism in $\mathrm{Ho}(SSet)$.

PROOF. Indeed, we consider the functor

$$\begin{array}{cccc} G : & (M_0)^c_W & \longrightarrow & (M^I_{cart})^c_W \\ & x & \mapsto & \phi(Qx), \end{array}$$

where Qx is a functorial cofibrant replacement of x. By hypothesis, the induced morphism on the nerves

$$N((M_0)^c_W) \longrightarrow N(M^I_{cart})^c_W$$

is an isomorphism in Ho($SSet$). We now consider the commutative diagram in Ho($SSet$)

$$\begin{array}{ccc} N((M_0)^c_W) & \longrightarrow & N(M^I_{cart})^c_W \\ & \searrow & \downarrow \\ & & Holim_{i\in I^{op}} N(M^c_W). \end{array}$$

The right vertical arrow being an isomorphism by Thm. B.0.7 we deduce the corollary. □

APPENDIX C

Representability criterion (after J. Lurie)

The purpose of this appendix is to give a sketch of a proof of the following special case of J. Lurie's representability theorem. Lurie's theorem is much deeper and out of the range of this work. We will not need it in its full generality and will content ourselves with this special case, largely enough for our applications.

THEOREM C.0.9. *(J. Lurie, see* [**Lu1**]*) Let F be a D^--stack. The following conditions are equivalent.*

(1) *F is an n-geometric D^--stack.*
(2) *F satisfies the following three conditions.*
  (a) *The truncation $t_0(F)$ is an Artin $(n+1)$-stack.*
  (b) *F has an obstruction theory relative to $sk-Mod_1$.*
  (c) *For any $A \in sk-Alg$, the natural morphism*
$$\mathbb{R}F(A) \longrightarrow Holim_k \mathbb{R}F(A_{\leq k})$$
  is an isomorphism in $\mathrm{Ho}(SSet)$.

SKETCH OF PROOF. The only if part is the easy part. (a) is true by Prop. 2.2.4.4 and (b) by Cor. 2.2.3.3. For (c) one proves the following more general lemma.

LEMMA C.0.10. *Let $f : F \longrightarrow G$ be an n-representable morphism. Then for any $A \in sk-Alg$, the natural square*

$$\begin{array}{ccc}
\mathbb{R}F(A) & \longrightarrow & Holim_k \mathbb{R}F(A_{\leq k}) \\
\downarrow & & \downarrow \\
\mathbb{R}G(A) & \longrightarrow & Holim_k \mathbb{R}G(A_{\leq k})
\end{array}$$

is homotopy cartesian.

PROOF. We prove this by induction on n. For $n = -1$, one reduces easily to the case of a morphism between representable D^--stacks, for which the result simply follows from the fact that $A \simeq Holim_k A_{\leq k}$. Let us now assume that $n \geq 0$ and the result prove for $m < n$. Let $A \in sk - Alg$ and

$$x \in \pi_0(Holim_k \mathbb{R}G(A) \times^h_{\mathbb{R}G(A_{\leq k})} \mathbb{R}F(A_{\leq k}))$$

with projections

$$x_k \in \pi_0(\mathbb{R}G(A) \times^h_{\mathbb{R}G(A_{\leq k})} \mathbb{R}F(A_{\leq k})).$$

We need to prove that the homotopy fiber H of

$$\mathbb{R}F(A) \longrightarrow Holim_k \mathbb{R}G(A) \times^h_{\mathbb{R}G(A_{\leq k})} \mathbb{R}F(A_{\leq k})$$

at x is contractible. Replacing F by $F \times^h_G X$ where $X := \mathbb{R}\underline{Spec}\, A$, and G by X, one can assume that G is a representable stack and F is an n-geometric stack. As G is representable, the morphism

$$\mathbb{R}G(A) \longrightarrow Holim_k \mathbb{R}G(A_{\leq k})$$

is an equivalence. Therefore, we are reduced to the case where $G = *$. The point x is then a point in $\pi_0(Holim_k \mathbb{R}F(A_{\leq k}))$, and we need to prove that the homotopy fiber H, taken at x, of the morphism

$$\mathbb{R}F(A) \longrightarrow Holim_k \mathbb{R}F(A_{\leq k})$$

is contractible. Using Cor. 2.2.2.9 one sees easily that this last statement is local on the small étale site of A. By a localization argument we can therefore assume that each projection $x_k \in \pi_0(\mathbb{R}F(A_{\leq k}))$ of x is the image of a point $y_k \in \pi_0(\mathbb{R}U(A_{\leq k}))$, for some representable D^--stack U and a smooth morphism $U \longrightarrow F$.

Using Prop. 1.4.2.6 and Lem. 2.2.1.1 we see that the homotopy fiber of the morphism

$$\mathbb{R}U(A_{\leq k+1}) \longrightarrow \mathbb{R}U(A_{\leq k}) \times^h_{\mathbb{R}F(A_{\leq k})} \mathbb{R}F(A_{\leq k+1})$$

taken at y_k is equivalent to $Map_{A_{\leq k}-Mod}(\mathbb{L}_{U/F,y_k}, \pi_{k+1}(A)[k+1])$. Cor. 2.2.5.3 then implies that when k is big enough, the homotopy fibers of the morphisms

$$\mathbb{R}U(A_{\leq k+1}) \longrightarrow \mathbb{R}U(A_{\leq k}) \times^h_{\mathbb{R}F(A_{\leq k})} \mathbb{R}F(A_{\leq k+1})$$

are simply connected, and thus this morphism is surjective on connected components. This easily implies that the points y_k can be thought as a point $y \in \pi_0(Holim_k \mathbb{R}U(A_{\leq k}))$ whose image in $\pi_0(Holim_k \mathbb{R}D(A_{\leq k}))$ is equal to x.

We then consider the diagram

$$\begin{array}{ccc} \mathbb{R}U(A) & \longrightarrow & Holim_k \mathbb{R}U(A_{\leq k}) \\ \downarrow & & \downarrow \\ \mathbb{R}F(A) & \longrightarrow & Holim_k \mathbb{R}F(A_{\leq k}). \end{array}$$

By induction on n we see that this diagram is homotopy cartesian, and that the top horizontal morphism is an equivalence. There, the morphism induced on the homotopy fibers of the horizontal morphisms is an equivalence, showing that H is contractible as required. □

Conversely, let F be a D^--stack satisfying conditions $(a)-(c)$ of C.0.9. The proof goes by induction on n. Let us first $n = -1$. We start by a lifting lemma.

LEMMA C.0.11. *Let F be a D^--stack satisfying the conditions $(a)-(c)$ of Thm. C.0.9. Then, for any affine scheme U_0, and any étale morphism $U_0 \longrightarrow t_0(F)$, there exists a representable D^--stack U, a morphism $u : U \longrightarrow F$, with $\mathbb{L}_{F,u} \simeq 0$, and a homotopy cartesian square in $k - D^-Aff^{\sim,et}$*

$$\begin{array}{ccc} U_0 & \longrightarrow & t_0(F) \\ \downarrow & & \downarrow \\ U & \longrightarrow & F. \end{array}$$

PROOF. We are going to construct by induction a sequence of representable D^--stacks

$$U_0 \longrightarrow U_1 \dots \longrightarrow U_k \longrightarrow U_{k+1} \dots \longrightarrow F$$

in $k - D^-Aff^{\sim,et}/F$ satisfying the following properties.

- One has $U_k = \mathbb{R}Spec\, A_k$ with $\pi_i(A_k) = 0$ for all $i > k$.
- The corresponding morphism $A_{k+1} \longrightarrow A_k$ induces isomorphisms on π_i for all $i \leq k$.

- The morphism $u_k : U_k \longrightarrow F$ are such that

$$\pi_i(\mathbb{L}_{U_k/F,u_k}) = 0 \; \forall \, i \leq k+1.$$

Assume for the moment that this sequence is constructed, and let $A := Holim_{A_k}$, and $U := \mathbb{R}\underline{Spec}\, A$. The points u_k, defines a well defined point in $\pi_0(Holim_k \mathbb{R}F(A_k))$, which by condition (d) induces a well defined morphism of stacks $u : U \longrightarrow F$. Clearly, one has a homotopy cartesian square

$$\begin{array}{ccc} U_0 & \longrightarrow & t_0(F) \\ \downarrow & & \downarrow \\ U & \longrightarrow & F. \end{array}$$

Let M be any A-module. Again, using condition (d), one sees that

$$\mathbb{D}er_F(U, M) \simeq Holim_k \mathbb{D}er_F(U_k, M_{\leq k}) \simeq Holim_k Map_{A_k-Mod_s}(\mathbb{L}_{U_k/F,u_k}, M_k) \simeq 0.$$

This implies that $\mathbb{L}_{U/F,u} = 0$.

It remains to explain how to construct the sequence of U_k. This is done by induction. For $k = 0$, the only thing to check is that

$$\pi_i(\mathbb{L}_{U_0/F,u_0}) = 0 \; \forall \, i \leq 1.$$

This follows easily from Prop. 1.4.2.6 and the fact that $u_0 : U_0 \longrightarrow t_0(F)$ is étale.

Assume now that all the U_i for $i \leq k$ have been constructed. We consider $u_k : U_k \longrightarrow F$, and the natural morphism

$$\mathbb{L}_{U_k} \longrightarrow \mathbb{L}_{U_k/F,u_k} \longrightarrow (\mathbb{L}_{U_k/F,u_k})_{\leq k+2} = N_{k+1}[k+2],$$

where $N_{k+1} := \pi_{k+2}(\mathbb{L}_{U_k/F,u_k})$. The morphism

$$\mathbb{L}_{U_k} \longrightarrow N_{k+1}[k+2]$$

defines a square zero extension of A_k by $N_{k+1}[k+1]$, $A_{k+1} \longrightarrow A_k$. By construction, and using Prop. 1.4.2.5 there exists a well defined point in

$$\mathbb{R}\underline{Hom}_{U_k/k-D^-Aff^{\sim,\text{ét}}/F}(U_{k+1}, F),$$

corresponding to a morphism in $\mathrm{Ho}(k - D^-Aff^{\sim,\text{ét}}/F)$

$$\begin{array}{ccc} U_k & \longrightarrow & U_{k+1} \\ & \searrow & \downarrow \\ & & F. \end{array}$$

It is not hard to check that the corresponding morphism $\mathbb{R}\underline{Spec}\, A_{k+1} = U_{k+1} \longrightarrow F$ has the required properties. □

Let us come back to the case $n = -1$. Lemma C.0.11 implies that there exists a representable D^--stack U, a morphism $U \longrightarrow F$ such that $\mathbb{L}_{F,u} \simeq 0$, and inducing an isomorphism $t_0(U) \simeq t_0(F)$. Using Prop. 1.4.2.6 and Lem. 2.2.1.1, it is not hard to show by induction on k, that for any $A \in sk - Alg$ with $\pi_i(A) = 0$ for $i > k$, the induced morphism

$$\mathbb{R}U(A) \longrightarrow \mathbb{R}F(A)$$

is an isomorphism. Using condition (c) for F and U one sees that this is also true for any $A \in sk - Alg$. Therefore, $U \simeq F$, and F is a representable D^--stack.

We now finish the proof of Thm. C.0.9 by induction on n. Let us suppose we know Thm. C.0.9 for $m < n$ and let F be a D^--stack satisfying conditions $(a) - (c)$ for rank n. By induction we see that the diagonal of F is $(n-1)$-representable and the hard point is to prove that F has an n-atlas. For this we lift an n-atlas of $it_0(F)$ in the following way. Starting from a smooth morphism $U_0 \longrightarrow it_0(F)$, we to construct by induction a sequence of representable D^--stacks

$$U_0 \longrightarrow U_1 \ldots \longrightarrow U_k \longrightarrow U_{k+1} \ldots \longrightarrow F$$

in $k - D^-Aff^{\sim,\text{ét}}/F$ satisfying the following properties.

- One has $U_k = \mathbb{R}\underline{Spec}\, A_k$ with $\pi_i(A_k) = 0$ for all $i > k$.
- The corresponding morphism $A_{k+1} \longrightarrow A_k$ induces isomorphisms on π_i for all $i \leq k$.
- The morphism $u_k : U_k \longrightarrow F$ is such that for any $M \in A_k - Mod_s$ with $\pi_i(M) = 0$ for all $i > k+1$ and $\pi_0(M) = 0$, one has
$$[\mathbb{L}_{U_k/F,u_k}, M]_{Sp(A_k - Mod_s)} = 0.$$

Such a sequence can be constructed by induction on k using obstruction theory as in Lem. C.0.11. We then let $A = Holim_k A_k$, and $U = \mathbb{R}\underline{Spec}\, A$. Then, condition (c) implies the existence of a well defined morphism $U \longrightarrow F$, which is seen to be smooth using Cor. 2.2.5.3. Using this lifting of smooth morphisms from it_0 to F, one produces an n-atlas for F by lifting an n-atlas of $it_0(F)$. □

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