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Interior and boundary continuity of the solution of the singular equation $(\beta(u))_t = \mathscr{L}u$

Ugo Gianazza[a,*], Bianca Stroffolini[b], Vincenzo Vespri[c]

[a] *Dipartimento di Matematica "F. Casorati", Università degli Studi di Pavia, via Ferrata 1, 27100 Pavia, Italy*

[b] *Dipartimento di Matematica e Applicazioni, Università degli Studi di Napoli Federico II, Complesso Monte S. Angelo Edificio "T", Via Cintia, 80126 Napoli, Italy*

[c] *Dipartimento di Matematica "Ulisse Dini", Università degli Studi di Firenze, Viale Morgagni 67/a, 50134 Firenze, Italy*

Abstract

We extend some results of DiBenedetto and Vespri (Arch. Rational Mech. Anal 132(3) (1995) 247) proving the interior and boundary continuity of bounded solutions of the singular equation $(\beta(u))_t = \mathscr{L}u$ where $\mathscr{L}$ is a second order elliptic operator with bounded measurable coefficients that depend both on space and time in a proper way.

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1. Introduction

Let $\beta(s)$ be a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ satisfying

$$\beta(s_1) - \beta(s_2) \geqslant \gamma_o(s_1 - s_2) \quad \forall s_i \in \mathbb{R} \;\; \gamma_o > 0. \tag{1.1}$$

We also assume that

$$\text{for every } M > 0, \qquad \sup_{-M \leqslant s \leqslant M} |\beta(s)| = \gamma_1 < \infty. \tag{1.2}$$

* Corresponding author. Tel.: +39-0382-505653; fax: +39-0382-505602.

E-mail addresses: gianazza@imati.cnr.it (U. Gianazza), bstroffo@unina.it (B. Stroffolini), vespri@dma.unifi.it (V. Vespri).

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Let Ω be a Lipschitz domain in $\mathbb{R}^N$, let Ω_T denote the cross product $\Omega \times [0,T]$ and consider the singular parabolic equation

$$(\beta(u))_t = \mathscr{L}u, \tag{1.3}$$

where $\mathscr{L}$ is an elliptic operator with principal part in divergence form.

Needless to say, if $\beta(u)=u$ we are dealing with the classical heat equation, for which we refer, for example, to [11]. Here we want to consider more general expressions for β and the reason lies in the natural connections that equations like (1.3) have with the modelling of phase transitions or the flow of fluids in porous media. For more details on these models, see, for example, [8] and [18].

Questions of regularity for weak solutions of (1.3) have been considered since the early 1980s and a certain number of results has been proved, even if things are far from being completely settled.

Without pretending to mention all the contributions, we can say that when $\beta(s)$ exhibits a single jump, say at $s=0$, continuity in the interior and at the boundary were proved in [1–3,13,19] and an explicit modulus of continuity was given.

A lot less is known for more general β and actually things lied still for a long while, from the late 1980s until the mid 1990s, when interior continuity was proved in [6] for β with superlinear growth. It is worth mentioning that for $N=2$ a general second order uniform elliptic operator is considered, whereas for $N \geqslant 3$ the method used heavily relies on the radial symmetry of $\mathscr{L}$ and therefore is limited to the case of $\mathscr{L}=\Delta$.

Adapting the techniques of [6], more general operators have been considered in [14–16], where once more, local interior continuity is proved and a quantitive estimate of the modulus of continuity is provided in the case of the p-laplacian, anisotropic p-laplacian and non-standard growth operator, respectively. In dimension $N=2$ one can even consider a maximal monotone graph $\beta=\beta_{AC}+\beta_s$ of bounded variation, with β_{AC} strictly increasing and $\beta_s \geqslant 0$ (see [7]). More recently, in [9] it was studied the case of a β which admits an arbitrary but finite number of jumps.

As it should be clear by now, a lot still needs to be done, especially under the point of view of boundary behavior. This paper is somehow a development of [6], in that we prove interior and boundary continuity of weak solutions of (1.3) under the same hypotheses on β but working with a more general elliptic operator. As a matter of fact, a lot of what we are considering here has been already proved in [17]. However, since the boundary continuity is new and some of the methods are also new, for the sake of completeness we chose to prove all the results *ex novo*. On the other hand, we decided to focus on what is really new with respect to [6], without entering too much into details when things are simply adapted.

We can finally state the explicit assumptions and our precise results. Here we assume

$$\mathscr{L}u = \sum_{ij} D_i(a_{ij}(x,t)D_j u + a_i(x,t)u) + b_i(x,t)D_i u + e(x,t)u,$$

where $a_{ij}(x,t)$, $a_i(x,t)$, $b_i(x,t)$, $e(x,t)$ are continuous functions with respect to the time variable and are measurable functions with respect to the spatial variables satisfying

$$\frac{1}{\mu_1}|\xi|^2 \leqslant \sum_{ij} a_{ij}(x,t)\xi_i\xi_j \leqslant \mu_1|\xi|^2 \tag{1.4}$$

$$\left\|\sum a_i^2, \sum b_i^2, e\right\|_{q,r,\Omega_T} \leqslant \mu_2, \tag{1.5}$$

with q and r such that

$$\frac{1}{r} + \frac{N}{2q} = 1 - \kappa_1$$

and $q \in [N/2(1-\kappa_1), \infty]$ $r \in [1/(1-\kappa_1), \infty]$ $0 < \kappa_1 < 1$, $N \geqslant 2$.

Moreover we suppose that

$$\forall 0 < t < s < T \qquad \int_t^s \int_\Omega \left(ev - \sum_i a_i D_i v \right) \mathrm{d}x\,\mathrm{d}t \leqslant 0,$$

$$\forall v \geqslant 0,\ v \in C_0^1(\Omega \times (t,s)). \tag{1.6}$$

As it will be clear in Section 4, we assume (1.6) in order to have the maximum principle in any parabolic cylinder $Q(\rho, \theta\rho^2) \subset \Omega_T$.

The equation in (1.3) is meant weakly and in the sense of inclusion of graphs, namely

Definition 1. The function $u \in L^2(0,T;W^{1,2}(\Omega))$ is a weak solution of (1.3) if there exists a selection $\xi \subset \beta(u)$, the inclusion being intended in the sense of the graphs, such that

$$t \rightarrow \xi(\cdot,t) \text{ is weakly continuous in } L^2(\Omega)$$

and

$$\int_\Omega \xi\phi\Big|_{t_1}^{t_2} + \int_\Omega \int_{t_1}^{t_2} \left\{ -\xi\phi_t + \sum_{ij}(a_{ij}D_j u + a_i u)D_i\phi + b_i D_i u\phi + eu\phi \right\} \mathrm{d}x\,\mathrm{d}t = 0$$

for all $\phi \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;W_0^{1,2}(\Omega))$ and for all intervals $(t_1,t_2) \subset (0,T)$.

Remark 1. Local summability can be considered both for u and ϕ if we are interested only in the continuity in the interior of Ω (see for example [6]).

To simplify the presentation, we assume that u is bounded in the whole Ω_T, so that

$$\|u\|_{\infty,\Omega_T} \leqslant M \text{ for some given constant } M > 0. \tag{1.7}$$

As in [6] the continuity at a point $P \in \Omega_T$ follows showing that the oscillation of u in a sequence of shrinking boxes with vertex at P tends to zero as the size of such neighborhoods tends to zero.

The following theorem is the main result of this paper:

Theorem 1. *Let* $N \geqslant 3$ *and let the structural assumptions* (1.1)–(1.2) *and* (1.4)–(1.6) *hold. Let* u *be a weak solution of* (1.3) *in the sense of Definition* 1 *and assume that* u *satisfies* (1.7). *Then* u *is continuous in* Ω_T. *Moreover, for every compact subset* $\mathscr{K} \subset \Omega_T$, *there exists a continuous, non-negative, increasing function*

$$s \mapsto \omega_{data,\mathscr{K}}(s), \qquad \omega_{data,\mathscr{K}}(0) = 0$$

that can be determined a priori only in terms of the data and the distance from $\mathscr{K}$ *to the parabolic boundary of* Ω_T *s.t.*

$$|u(x_1,t_1) - u(x_2,t_2)| \leqslant \omega_{data,\,\mathscr{K}}(|x_1 - x_2| + |t_1 - t_2|^{(1/2)})$$

for every pair of points $(x_i,t_i) \in \mathscr{K}$, $i = 1,2$.

The plan of the paper is the following: we first examine the case of time-independent coefficients (Sections 2–5) and we prove Theorem 1 under this restrictive assumption; then we achieve the general case using a proper approximation argument for coefficients continuous in t (Section 6); finally we apply the previous results to prove the boundary continuity of u for homogeneous Dirichlet conditions (For the precise statements of the assumptions and the theorem in this case and also for the difficulties presented by the treatment of boundary conditions, we directly refer to Section 7).

We conclude this section with some general remarks. First of all, even if we decided to concentrate on a linear second order elliptic operator $\mathscr{L}$ with coefficients satisfying suitable summability hypotheses as described above (see Remark 2 and Section 4 for some comments and further discussions), the same techniques allow us to prove the continuity result in a more general context. As we try to highlight in the proof, the only properties of $\mathscr{L}(x,t,u,Du)$ on which we will rely are the following:

(1) $\mathscr{L}$ satisfies the maximum principle;
(2) the coefficients of $\mathscr{L}$ are continuous in t;
(3) in the case of time-independent coefficients the elliptic operator $\mathscr{L}$ satisfies a uniform Harnack inequality t by t.

The most important of these three assumptions is the last one and this shows once more how the Harnack inequality is crucial when proving regularity results for solutions of partial differential equations.

2. Local energy estimates

We assume that u can be constructed as the limit in the weak topology of $L^2(0,T;W^{1,2}(\Omega))$ of a sequence of local smooth solutions for smooth $\beta(\cdot)$. This is the same approach used in [6], to which we refer for further details. Hence we will be working with smooth solutions of

$$\beta'(u)u_t = \sum_{ij} D_i(a_{ij}(x,t)D_ju + a_i(x,t)u) + b_i(x,t)D_iu + e(x,t)u, \tag{2.1}$$

where $\beta(\cdot)$ is regular and satisfies

$$\beta'(s) \geqslant \gamma_0, \quad \forall s \in (-M, M).$$

However all our estimates depend only upon the data. Finally, we use the same notation of [6], that we recall for sake of completeness.

For $\rho > 0$ we denote by K_ρ the cube of wedge 2ρ centered at the origin, i.e.

$$K_\rho = \left\{ x \in \mathbb{R}^N \,\middle|\, \max_{1 \leqslant i \leqslant N} |x_i| < \rho \right\}$$

and by $[y + K_\rho]$ the cube centered at y and congruent to K_ρ. For $\theta > 0$ by $Q(\rho, \theta\rho^2)$ we denote the cylinder of cross section K_ρ, height $\theta\rho^2$, and vertex at the origin, i.e.

$$Q(\rho, \theta\rho^2) = K_\rho \times (-\theta\rho^2, 0)$$

and for a point $(y, s) \in \mathbb{R}^{N+1}$ we let $[(y, s) + Q(\rho; \theta\rho^2)]$ be the cylinder of vertex at (y, s) and congruent to $Q(\rho, \theta\rho^2)$.

For $k \in \mathbb{R}$ the truncations $(u - k)_+$ and $(u - k)_-$ are defined by

$$(u - k)_+ = \max\{u - k; 0\} \quad (u - k)_- = \max\{k - u; 0\}.$$

Next we define

$$\mathscr{A}^{\pm}_{k,\rho}(t) = \{x \in K_\rho | (u(x, t) - k)_\pm > 0\},$$

introduce the numbers

$$\mathscr{H}^{\pm}_k = \|(u - k)_\pm\|_{\infty, [(y,s)+Q(\rho;\theta\rho^2)]}, \quad \hat{q} = \frac{2q(1 + \kappa)}{q - 1}, \quad \hat{r} = \frac{2r(1 + \kappa)}{r - 1},$$

$$\kappa = \frac{2}{N}\kappa_1$$

and the function

$$\begin{aligned}\Psi(\mathscr{H}^{\pm}_k, (u - k)_\pm, c) &= \ln^+ \left\{ \frac{\mathscr{H}^{\pm}_k}{\mathscr{H}^{\pm}_k - (u - k)_\pm + c} \right\} \\ &= \max\left\{ \ln\left\{ \frac{\mathscr{H}^{\pm}_k}{\mathscr{H}^{\pm}_k - (u - k)_\pm + c} \right\}, 0 \right\} \quad 0 < c < \mathscr{H}^{\pm}_k.\end{aligned}$$

Proposition 1. *There exist constants $\gamma = \gamma(data)$ and $\delta_0 = \delta_0(data)$ such that for all cylinders $[(y, s) + Q(\sigma\rho, \theta\sigma\rho^2)] \subset [(y, s) + Q(\rho; \theta\rho^2)]$, $\sigma \in (0, 1)$ and for all levels k satisfying* $\operatorname{ess\,sup}_{Q(\rho;\theta\rho^2)} |(u - k)_\pm| = \delta \leqslant \delta_0$ *we get*

$$\sup_{s-\theta\rho^2 \leqslant t \leqslant s} \int_{y+K_{\sigma\rho}} (u - k)^2_\pm(x, t)\,\mathrm{d}x + \iint_{(y,s)+Q(\sigma\rho,\sigma\theta\rho^2)} |D(u - k)_\pm|^2\,\mathrm{d}x\,\mathrm{d}t$$

$$\leqslant \frac{\gamma}{(1 - \sigma)^2\rho^2} \iint_{(y,s)+Q(\rho,\theta\rho^2)} (u - k)^2_\pm\,\mathrm{d}x\,\mathrm{d}t$$

$$+\frac{\gamma}{(1-\sigma)\theta\rho^2}\iint_{(y,s)+Q(\rho,\theta\rho^2)}(u-k)_\pm\,\mathrm{d}x\,\mathrm{d}t$$

$$+\left\|\sum a_i^2+\sum b_i^2+|e|\right\|_{q,r}\left(\int_{-\theta\rho^2}^{0}|\mathscr{A}_{k,\rho}^{\pm}(\tau)|^{\hat r/\hat q}\,\mathrm{d}\tau\right)^{2(1+\kappa)/\hat r}, \tag{2.2}$$

$$\sup_{s-\theta\rho^2\leqslant t\leqslant s}\int_{y+K_{\sigma\rho}}\Psi^2(\mathscr{H}_k^{\pm},(u-k)_\pm,c)(x,t)\,\mathrm{d}x$$

$$\leqslant\frac{\gamma(data)}{(1-\sigma^2)\rho^2}\iint_{(y,s)+Q(\rho,\theta\rho^2)}\Psi(\mathscr{H}_k^{\pm},(u-k)_\pm,c)\,\mathrm{d}x\,\mathrm{d}\tau$$

$$+\frac{\gamma(data)}{c}\int_{y+K_\rho}\Psi(\mathscr{H}_k^{\pm},(u-k)_\pm,c)(x,s-\theta\rho^2)$$

$$+\frac{\gamma}{c^2}\left(1+\ln\frac{\mathscr{H}_k^{\pm}}{c}\right)\mu_1\left\{\int_{s-\theta\rho^2}^{s}|\mathscr{A}_{k,\rho}^{\pm}(\tau)|^{\hat r/\hat q}\,\mathrm{d}\tau\right\}^{2(1+\kappa)/\hat r}. \tag{2.3}$$

Proof. See [6], Proposition 2.1. The only difference is given by the lower order terms, but these are handled in a standard way. □

Remark 2. The estimate (2.2) holds true even under more general hypotheses, namely

(a) $(a(x,t,u,Du)Du \geqslant C_0|Du|^2-\phi_0(x,t)$,
(b) $|a(x,t,u,Du)| \leqslant C_1|Du|+\phi_1(x,t)$,
(c) $|b(x,t,u,Du)| \leqslant C_2|Du|+\phi_2$;

where $\phi_0,\phi_1^2,\phi_2\in L^{q,r}(\Omega_T)$.

Remark 3. In the proofs we will find quantities of the type $A_i\rho^{N\kappa}\omega^{-2}$, where A_i are constants that can be determined a priori only in terms of the data and are independent of ω and ρ. Without loss of generality we may assume that they satisfy $A_i\rho^{N\kappa}\omega^{-2}\leqslant 1$. Indeed, if not, we would have $\omega\leqslant C\rho^{\varepsilon_0}$ for $C=\max A_i$ and $\varepsilon_0=Nk/2$ and things would be trivial.

Fix $\theta>0$ and consider $[(y,s)+Q(2\rho,2\theta\rho^2)]\subset\Omega_T$. We put

$$\mu^+=\sup_{[(y,s)+Q(2\rho,2\theta\rho^2)]}u,\qquad \mu^-=\inf_{[(y,s)+Q(2\rho,2\theta\rho^2)]}u$$

and

$$\omega=\operatorname*{osc}_{[(y,s)+Q(2\rho,2\theta\rho^2)]}u=\mu^+-\mu^-.$$

Let $\xi^{\pm} \in (0,1)$ be properly chosen in order to satisfy the assumption $\mathscr{H}_k^{\pm} \leqslant \delta_0$ and define the following level sets

$$\mathscr{A}^{+}_{\xi^{+},\rho} = \{(x,t) \in [(y,s) + Q(\rho, \theta\rho^2] : u(x,t) > \mu^{+} - \xi^{+}\omega\},$$

$$\mathscr{A}^{-}_{\xi^{-},\rho} = \{(x,t) \in [(y,s) + Q(\rho, \theta\rho^2] : u(x,t) < \mu^{-} + \xi^{-}\omega\}.$$

If we fix $\lambda \in (0,1)$ we have the following estimates

Proposition 2. *There exists a number v^{+} depending on the structure of β and λ, ξ^{+}, θ, ω such that*

$$\operatorname{meas} \mathscr{A}^{+}_{\xi^{+},\rho} < v^{+}|Q(\rho, \theta\rho^2)| \Rightarrow u(x,t) < \mu^{+} - \lambda\xi^{+}\omega$$

$$\forall (x,t) \in \left[(y,s) + Q\left(\frac{\rho}{2}, \frac{\theta\rho^2}{2}\right)\right].$$

Proposition 3. *There exists a number v^{-} depending on the structure of β and λ, ξ^{-}, θ, ω such that*

$$\operatorname{meas} \mathscr{A}^{-}_{\xi^{-},\rho} < v^{-}|Q(\rho, \theta\rho^2)| \Rightarrow u(x,t) > \mu^{-} + \lambda\xi^{-}\omega$$

$$\forall (x,t) \in \left[(y,s) + Q\left(\frac{\rho}{2}, \frac{\theta\rho^2}{2}\right)\right].$$

Proof. Let us remark that the numbers $v^{\pm}$ are given by the formula:

$$v^{\pm} = \frac{c}{\theta}\left(\frac{\theta\xi^{\pm}\omega}{1+\theta\xi^{\pm}\omega}\right)^{\frac{1+\kappa}{\sigma}}, \quad \sigma = \min\left\{\frac{2}{N}\kappa_1; \frac{2}{N+2}\right\}, \quad c = c(\mathrm{dat}, \lambda).$$

The proof is like the one given in [6] for Propositions 3.1^{+} and 3.1^{-}. The only difference is in the use of Lemma 4.2 of [5] instead of Lemma 4.1 of the same work. Once more the reason lies in the presence in our equation of first order terms. □

Fix $\theta > 0$ and consider the cylinder $[(y,s) + Q(\rho; \theta\rho^2)]$; for $\xi^{+} \in (0,1)$ we set

$$\mathscr{A}_{\xi^{+},\rho}(t) = \{x \in K_\rho : u(x,t) > \mu^{+} - \xi^{+}\omega\}.$$

We assume that the function $u(\cdot, s-\theta\rho^2)$ does not exceed the value $\mu^{+} - \xi_0^{+}\omega$ at the bottom of the cylinder for some $\xi_0^{+} \in (0,1)$ properly chosen in order to satisfy the assumption $\mathscr{H}_k^{\pm} \leqslant \delta_0$, i.e.

$$u(x, s-\theta\rho^2) \leqslant \mu^{+} - \xi_0^{+}\omega \quad \forall x \in [y + K_\rho].$$

We have

Proposition 4. *For every $v^{+} \in (0,1)$ there exists a number $\xi^{+} \in (0, \frac{1}{4}\xi_0^{+})$ depending only upon the data and the numbers ξ_0^{+} and θ such that*

$$|\mathscr{A}_{\xi^{+},\frac{1}{2}\rho}(t)| \leqslant v^{+}|K_{\frac{1}{2}\rho}| \quad \forall t \in (s-\theta\rho^2, s).$$

The number ξ^{+} is chosen to satisfy $v^{+} = \gamma\theta/\ln(\xi_0^{+}/2\xi^{+})$.

Proof. See Proposition 4.1^+ of [6]. □

We conclude this section by stating a proposition that can be found in a more general way in [6]

Proposition 5. *Let* $v \in W^{1,2}(K_\rho)$, *satisfying*

$$\int_{K_\rho} |Dv|^2 \, dx \leqslant \gamma$$

for a given constant γ *and* $\text{meas}\{x \in K_\rho : v(x) < 1\} \geqslant \alpha |K_\rho|$ *for a given* $\alpha \in (0,1)$. *Then, for every* $\eta \in (0,1)$, *and* $\lambda > 1$, *there exists* $x^* \in K_\rho$ *and a number* $\delta \in (0,1)$ *such that, within the cube* $K_{\delta\rho}(x^*)$ *centered in* x^* *with wedge* $2\delta\rho$, *there holds*:

$$\text{meas}\{x \in K_{\delta\rho}(x^*) : v(x) < \lambda\} > (1-\eta)|K_{\delta\rho}|.$$

3. On the sets where u is near μ^+ or near μ^-

If we define $\mathscr{A}^+_{\xi^+,\rho}(t) = \{x \in K_\rho : u(x,t) > \mu^+ - \xi^+\omega\}$ and $\mathscr{A}^-_{\xi^-,\rho}(t) = \{x \in K_\rho : u(x,t) < \mu^- - \xi^-\omega\}$, we have

$$\text{meas}\, \mathscr{A}^\pm_{\xi^\pm,\rho} = \int_{-\rho^2}^0 |\mathscr{A}^\pm_{\xi^\pm,\rho}(t)| \, dt.$$

In the following the quantities $v^\pm$ are the ones introduced in Propositions 2 and 3.

Proposition 6. *If*

$$\text{meas}\, \mathscr{A}^+_{\xi^+,\rho} = \int_{-\rho^2}^0 |\mathscr{A}^+_{\xi^+,\rho}(t)| \, dt > v^+ |Q(\rho,\rho^2)| \tag{3.1}$$

holds, for all $\lambda > 1$ *and for all* $\eta \in (0,1)$ *there exist a point* $(y^*_+, s^*_+) \in [(y,s) + Q(\rho;\rho^2)]$, *a number* $\delta_+ \in (0,1)$ *and a cylinder*

$$[(y^*_+, s^*_+) + Q(\delta_+\rho; \delta_+^2\rho^2)] \subset [(y,s) + Q(\rho;\rho^2)]$$

such that

$$\text{meas}\{(x,t) \in [(y^*_+, s^*_+) + Q(\delta_+\rho; \delta_+^2\rho^2)] : u(x,t) > \mu^+ - \lambda\xi^+\omega\}$$
$$> (1-\eta)|[(y^*_+, s^*_+) + Q(\delta_+\rho; \delta_+^2\rho^2)]|. \tag{3.2}$$

The number δ_+ *depends upon the data and the numbers* λ, η, ξ^+ *and* ω.

Proposition 7. *If*

$$\text{meas}\, \mathscr{A}^-_{\xi^-,\rho} = \int_{-\rho^2}^0 |\mathscr{A}^-_{\xi^-,\rho}(t)| \, dt > v^- |Q(\rho,\rho^2)| \tag{3.3}$$

holds, for all $\lambda > 1$ *and for all* $\eta \in (0,1)$ *there exist a point* $(y^*_-, s^*_-) \in [(y,s) + Q(\rho;\rho^2)]$, *a number* $\delta_- \in (0,1)$ *and a cylinder*

$$[(y^*_-, s^*_-) + Q(\delta_-\rho; \delta_-^2\rho^2)] \subset [(y,s) + Q(\rho;\rho^2)]$$

such that

$$\text{meas}\{(x,t)\in[(y_-^*,s_-^*)+Q(\delta_-\rho;\delta_-^2\rho^2)]:u(x,t)<\mu^-+\lambda\xi^-\omega\}$$
$$>(1-\eta)|[(y_-^*,s_-^*)+Q(\delta_-\rho;\delta_-^2\rho^2)]|. \tag{3.4}$$

The number δ_- depends upon the data and the numbers λ, η, ξ^- and ω.

Proof. We write (2.2) over $Q(\rho,\rho^2)$ and $Q(2\rho,2\rho^2)$ for which $\sigma=\frac{1}{2}$ and for the functions

$$(u-k^+)_+ \quad \text{with } k^+=\mu^+-\xi^+\omega$$

and

$$(u-k^-)_- \quad \text{with } k^-=\mu^-+\xi^-\omega.$$

We take into account that the term $\{\int_{-\theta\tilde{\rho}_n}^0|\mathscr{A}^+_{\xi_n,\rho_n}(\tau)|^{\hat{q}/\hat{r}}\,\mathrm{d}\tau\}^{2/\hat{r}(1+\kappa)}$ is controlled by $\gamma\omega\rho^N$ and we obtain

$$\iint_{Q(\rho,\rho^2)}|D(\mu^+-u)|^2\,\mathrm{d}x\,\mathrm{d}t\leqslant\gamma\omega\rho^N, \tag{3.5}$$

$$\iint_{Q(\rho,\rho^2)}|D(u-\mu^-)|^2\,\mathrm{d}x\,\mathrm{d}t\leqslant\gamma\omega\rho^N. \tag{3.6}$$

We rewrite (3.1) and (3.5) in terms of $v^+=(\mu^+-u)/\omega\xi^+$, (3.3) and (3.6) in terms of $v^-=(u-\mu^-)/\omega\xi^-$ to get

$$\text{meas}\{(x,t)\in Q(\rho;\rho^2):v^\pm<1\}>v^\pm|Q(\rho;\rho^2)|, \tag{3.7}$$

$$\iint_{Q(\rho,\rho^2)}|Dv^\pm|^2\,\mathrm{d}x\,\mathrm{d}t\leqslant\frac{\gamma}{\omega\xi^{\pm2}}\rho^N. \tag{3.8}$$

Now relying in a fundamental way on Proposition 5 stated in the previous Section, we conclude exactly as in [6], Propositions 5.1$^\pm$. □

Let the cylinder $[(y,s)+Q(\rho;\rho^2)]$ be fixed and consider coaxial boxes of the type

$$[(y,\tau)+Q(r;r^2)],\quad 0<r\leqslant\rho. \tag{3.9}$$

The time-location of the vertices ranges over

$$\tau\in[s-(\rho^2-r^2),s] \tag{3.10}$$

and r is a positive parameter ranging over

$$r\in[\delta\rho,\rho],\quad \text{where } \delta\in(0,1) \text{ is to be chosen.} \tag{3.11}$$

We assume that conditions (3.1) and (3.3) both hold for all cylinders defined in (3.9)–(3.11). In such a case, within $[(y,s)+Q(\rho;\rho^2)]$, we will identify two disjoint subcylinders such that in one of these u is all near μ^+ and in the other one u is all near μ^-.

Proposition 8. *Let* (3.1) *and* (3.3) *both hold for all coaxial cylinders of the type* (3.9)–(3.11). *There exist two points* $(y_1^*, s^*), (y_2^*, s^*)$, *at the same time level* s^*, *a number* $\delta \in (0,1)$ *and two cylinders*

$$[(y_1^*, s^*) + Q(r; r^2)], \quad [(y_2^*, s^*) + Q(r; r^2)] \;\; r = \delta\rho,$$

contained in $[(y,s) + Q(\rho; \rho^2)]$, *such that*

$$u(x,t) > \mu^- + \tfrac{2}{3}(1 - \lambda\xi^+)\omega \quad \forall (x,t) \in [(y_1^*, s^*) + Q(r; r^2)]$$

and

$$u(x,t) < \mu^+ - \tfrac{2}{3}(1 - \lambda\xi^-)\omega \quad \forall (x,t) \in [(y_2^*, s^*) + Q(r; r^2)].$$

The proof of this Proposition is the same as Proposition 8.1 in [6].

Using Proposition 8 it is possible to derive a local estimate for the gradient Du exactly as in [6].

Proposition 9. *Let* (3.1) *and* (3.3) *both hold for all coaxial cylinders defined in* (3.9)–(3.11) *and choose* $\xi^+ = \xi^- = \frac{1}{12}$ *and* $\lambda = \frac{3}{2}$. *There exists a constant* γ *depending only upon the data and* ω *such that*

$$\rho^N \omega^2 \gamma \leqslant \int_{s-\rho^2}^{s} \int_{\delta\rho < \|x-y\| < \rho} |Du|^2 \, dx \, dt.$$

4. Comparison function

In [6] the key point in the proof of the continuity theorem are some estimates of a proper function v, which is then compared with the solution u of the original singular parabolic equation. In these estimates the radial symmetry of the problem is heavily used, but a careful examination of the whole procedure shows that this assumption can be done away with, provided the maximum principle and a Harnack inequality for the corresponding elliptic operator hold for all time levels. This is precisely the way we will follow in the sequel. Without entering too much into details, let us just remark that the basic reason to use the comparison function is to mimic a parabolic Harnack inequality, whose validity is not known in this context.

4.1. Statement of the problem

For $d > 1$ and $\varepsilon_0 \in (0,1)$, let $\mathscr{A}_{\varepsilon_0,d}$ denote the annulus $\mathscr{A}_{\varepsilon_0,d} \equiv \{\varepsilon_0 < |x| < d\}$ and for $k > 0$ consider the cylindrical domain with annular cross section

$$\mathscr{C}(\varepsilon_0, 4d; k) \equiv \mathscr{A}_{\varepsilon_0,4d} \times (0,k).$$

We consider the elliptic operator $\mathscr{L}$ introduced in (1.3) and state some further assumptions:

(1) (H1) The coefficients a_{ij}, a_i, b_i, e considered in (1.3), (1.4), (1.5) do not depend on time [Notice that this hypothesis will be removed in the last sections].

(2) (H2) As a consequence of (H1), condition (1.6) becomes

$$\int_\Omega \left(ev - \sum_i a_i D_i v \right) \mathrm{d}x \leqslant 0, \ \forall v \geqslant 0, \ v \in C_0^1(\Omega).$$

(3) (H3) The local weak solution w of the elliptic equation

$$\mathscr{L}w = 0 \quad \text{in } \Omega$$

satisfies a Harnack inequality, namely, if $w \geqslant 0$ and $K_R(y) \subset \Omega$, for any $\sigma > 0$ we have

$$\sup_{K_{\sigma R}} w \leqslant C \left(\inf_{K_{\sigma R}} w + \chi R^\alpha \right),$$

where χ, α and C depend only on the data. Strictly speaking, the ellipticity condition and the summability hypotheses on the coefficients ensure that w belongs to an elliptic De Giorgi class, which in turns guarantees that w satisfies a Harnack inequality (under this point of view, see for example [12, Chapter 3] [5]); therefore there would be no reason to assume this explicitly. On the other hand, as we mentioned in the Introduction, we want to emphasize the structural assumptions needed in the proof, so that Theorem 1 still holds for any other operator that satisfies them.

Remark 4. As already mentioned in Section 1, the meaning of (H2) is to provide us with the maximum principle. By no means it is the most general hypothesis under this point of view: in fact more general conditions could be equally-well considered, but we will not take them into account here.

Let now v be the unique solution of the boundary problem

$$\begin{cases} (\tilde{\beta}(v))_t = \mathscr{L}v & \text{on } \mathscr{A}_{\varepsilon_0,4d} \times (0,k) \\ v(x,t) = 0 & \text{in } |x| = 4d \\ v(x,t) = 1 & \text{in } |x| = \varepsilon_0 \\ v(x,0) = 0, & \end{cases} \tag{4.1}$$

where $\tilde{\beta}$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ satisfying:

$$\frac{\tilde{\beta}(s_1) - \tilde{\beta}(s_2)}{s_1 - s_2} \geqslant \tilde{\gamma}_0, \qquad \sup_{|s| \leqslant M} |\tilde{\beta}(s)| \leqslant \tilde{\gamma}_1$$

for some given constants $\tilde{\gamma}_i$. As discussed at the beginning of this section, the function v will be used as a local comparison function for the solution u of (1.3). Therefore the quantities $\varepsilon_0, d, \tilde{\gamma}_i$ will be chosen in dependance of the local oscillation of u. As in the previous sections, in the following we assume $\tilde{\beta}$ to be smooth, so that

$$\tilde{\beta}'(s) \geqslant \tilde{\gamma}_0 \quad \forall s \in (-M, M).$$

Thanks to the previous assumptions, we have

Proposition 10. *Let v be the solution of* (4.1). *Then*

$$0 \leqslant v(x,t) \leqslant 1 \quad \textit{for a.e. } (x,t) \in \mathscr{C}(\varepsilon_0, 4d; k).$$

Proof. Take $(v-h)_\pm$ with $0 \leqslant h \leqslant 1$ as test function in (4.1) and then argue as in the proof of Theorem 7.2, Section III of [11]. Due to the smoothness assumptions on $\tilde{\beta}$, we have

$$(\tilde{\beta}(v))_t = \tilde{\beta}'(v)\, v_t$$

and

$$\pm\tilde{\beta}'(v)\, v_t\, (v-h)_\pm = \frac{\partial}{\partial t} \int_0^{(v-h)_\pm} \tilde{\beta}'(h \pm s) s \, \mathrm{d}s.$$

Hence the only difference with the non-degenerate parabolic setting is the presence of the term

$$\int_0^k \int_{A_{\varepsilon_0,4d}} \frac{\partial}{\partial t} \int_0^{(v-h)_\pm} \tilde{\beta}(h \pm s) s \, \mathrm{d}s$$

instead of the usual straight time-derivative. However relying on

$$\int_0^{(v-h)_\pm} \tilde{\beta}'(h \pm s) s \, \mathrm{d}s \geqslant \frac{\tilde{\gamma}_0}{2} (v-h)_\pm^2, \tag{4.2}$$

$$\int_0^{(v-h)_\pm} \tilde{\beta}'(h \pm s) s \, \mathrm{d}s \leqslant \sup_{|s| \leqslant M} |\tilde{\beta}(s)| (v-h)_\pm \tag{4.3}$$

we can conclude exactly as in [11].

The rest of this section will be devoted to the proof of the following

Proposition 11. *There exist numbers $\sigma_0 > 0$ and $k > 0$ such that for every y in the annulus $\{1 < |y| < 2d\}$ there exists a time level $t \in (0,k)$ such that*

$$v(y,t) > \sigma_0. \tag{4.4}$$

4.2. An auxiliary function

We consider the auxiliary function z given by the difference of the solution v of the parabolic problem in the circular cylindric section considered in (4.1) and the solution

ζ of the following elliptic problem in the circular annulus $\mathscr{A}_{\varepsilon_0,4d}$

$$\begin{cases} \mathscr{L}\zeta = 0 & \text{on } \mathscr{A}_{\varepsilon_0,4d} \\ \zeta(x) = 0 & \text{in } |x| = 4d \\ \zeta(x) = 1 & \text{in } |x| = \varepsilon_0. \end{cases} \tag{4.5}$$

Notice that by well-known classical result ζ is Hölder continuous. Moreover we have the following

Proposition 12. *Under assumptions* (H1) *and* (H2) (4.5) *admits a unique weak solution in* $W^{1,2}(\Omega)$ *and*

$$0 \leqslant \zeta(x) \leqslant 1 \quad \textit{for a.e. } x \in \Omega. \tag{4.6}$$

Proof. For the existence and uniqueness, see [10] Theorem 8.3. For the maximum principle stated in (4.6), see [10], Theorem 8.1.

If we now put $z = \zeta - v$ and we set $\Gamma(x,\cdot) = -\tilde{\beta}(\zeta(x) - \cdot)$, it is easy to see that z satisfies

$$\begin{cases} (\Gamma(z))_t = \mathscr{L}z & \text{on } \mathscr{A}_{\varepsilon_0,4d} \times (0,k) \\ z(x,t) = 0 & \text{in } |x| = 4d \\ z(x,t) = 0 & \text{in } |x| = \varepsilon_0 \\ z(x,0) = \zeta & \text{in } \mathscr{A}_{\varepsilon_0,4d} \end{cases} \tag{4.7}$$

in the sense specified in Section 2. As with v above, z satisfies a proper maximum principle, namely

Proposition 13. *Let* z *be the solution of* (4.7). *Then*

$$z(x,t) \geqslant 0 \quad \textit{for a.e. } (x,t) \in \mathscr{C}(\varepsilon_0, 4d; k).$$

Proof. Things are the same as considered above in the proof of Proposition 10, to which we refer.

4.3. Basic energy estimates

Before coming to the proof of Proposition 11, we need some introductory estimates. For a cylinder $[(y,s) + Q(2\rho; 2\theta\rho^2)] \subset \mathscr{C}(\varepsilon_0, 4d; k)$ we set

$$z^+ = \sup_{[(y,s)+Q(2\rho;2\theta\rho^2)]} z; \quad z^- = \inf_{[(y,s)+Q(2\rho;2\theta\rho^2)]} z$$

and denote by w a number satisfying:

$$w \geqslant z^+ - z^- = \operatorname*{osc}_{[(y,s)+Q(2\rho;2\theta\rho^2)]} z.$$

Relying as before on the properties of $\tilde\beta$, we get

$$\pm\Gamma'(z)z_t\,\Psi\Psi' = \frac{\partial}{\partial t}\int_0^{(z-k)_\pm} \tilde\beta'(\zeta-(k\pm s))\Psi(s)\Psi'(s)\,\mathrm{d}s.$$

Moreover

$$\int_0^{(z-k)_\pm} \tilde\beta'(\zeta-(k\pm s))s\,\mathrm{d}s \geqslant \frac{\tilde\gamma_0}{2}(z-k)_\pm^2, \tag{4.8}$$

$$\int_0^{(z-k)_\pm} \tilde\beta'(\zeta-(k\pm s))s\,\mathrm{d}s \leqslant \sup_{|s|\leqslant M}|\tilde\beta(s)|(z-k)_\pm, \tag{4.9}$$

$$\int_0^{(z-k)_\pm} \tilde\beta'(\zeta-(k\pm s))\Psi(s)\Psi'(s)\,\mathrm{d}s \geqslant \frac{\tilde\gamma_0}{2}\Psi^2. \tag{4.10}$$

Hence, by (4.8), (4.9) and (4.10), it is easy to see that the function z satisfies the energy estimates of Section 2. For the logarithmic estimate, Proposition 4 continues to hold for the function z. In this context, having fixed $v^\pm\in(0,1)$, and $\xi_0^\pm\in(0,1)$, the numbers $\xi^\pm$ for which the analogues of Propositions 2 and 3 hold are chosen from the formulae

$$v^\pm = \frac{\gamma(\mathrm{data}, w)}{\ln(\xi_0^\pm/\xi^\pm)}.$$

We extend $z(\cdot,t)$ for $|x|<\varepsilon_0$ and $t\in(0,k)$ by 0 and continue to denote by z such an extension. Denoting by B_d the ball of radius d about the origin, we have:

$$z(\cdot,t)\in W_0^{1,2}(B_{4d}) \quad \text{for a.e. } t\in(0,k).$$

We can then put z as test function in the equation to obtain

$$\int_0^k\int_{\mathscr{A}_{\varepsilon_0,4d}}\left(\frac{\partial}{\partial t}\int_0^z \tilde\beta'(\zeta+s)s\,\mathrm{d}s\right)\mathrm{d}x\,\mathrm{d}t + \frac{v}{2}\int_0^k\int_{\mathscr{A}_{\varepsilon_0,4d}}|Dz|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant \frac{1}{v}\int_0^k\int_{\mathscr{A}_{\varepsilon_0,4d}}\left(\sum a_i^2+\sum b_i^2+|e|\right)z^2\,\mathrm{d}x\,\mathrm{d}t.$$

If we now set $\mathscr{D}=(1/v)\sum a_i^2+(1/v)\sum b_i^2+|e|$, arguing as in [11] Chapter III, page 139–140 and taking into account the analog of (4.8) and (4.9) we obtain:

$$\min\{\gamma_0,v\}\left[\sup_{0<t\leqslant k}\int_{\mathscr{A}_{\varepsilon_0,4d}} z^2(x,t)\,\mathrm{d}x + \int_0^k\int_{\mathscr{A}_{\varepsilon_0,4d}}|Dz|^2\,\mathrm{d}x\,\mathrm{d}t\right]$$
$$\leqslant C\int_{\mathscr{A}_{\varepsilon_0,4d}} z(x,0)\,\mathrm{d}x + \gamma\|\mathscr{D}\|_{q,r}\left[\sup_{0<t\leqslant k}\int_{\mathscr{A}_{\varepsilon_0,4d}} z^2(x,t)\,\mathrm{d}x + \int_0^k\int_{\mathscr{A}_{\varepsilon_0,4d}}|Dz|^2\,\mathrm{d}x\,\mathrm{d}t\right].$$

If $\gamma\|\mathscr{D}\|_{q,r}$ is less than $\min\{\gamma_0, v\}$ it is possible to estimate

$$\sup_{0\leqslant t\leqslant k}\int_{\mathscr{A}_{\varepsilon_0,4d}} z^2(x,t)\,\mathrm{d}x + \int_0^k\int_{\mathscr{A}_{\varepsilon_0,4d}} |Dz|^2\,\mathrm{d}x\,\mathrm{d}t \quad \text{in terms of} \quad \int_{\mathscr{A}_{\varepsilon_0,4d}} z(x,0)\,\mathrm{d}x,$$

but in general this is not the case. Once more we argue as in [11]: we consider a partition of $(0,k)$ in a finite number of intervals in such a way that $\gamma\|\mathscr{D}\|_{q,r}\leqslant v/2$ and we get

$$\sup_{0\leqslant t\leqslant k}\int_{\mathscr{A}_{\varepsilon_0,4d}} z^2(x,t)\,\mathrm{d}x + \int_0^k\int_{\mathscr{A}_{\varepsilon_0,4d}} |Dz|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C\int_{\mathscr{A}_{\varepsilon_0,4d}} z(x,0)\,\mathrm{d}x. \tag{4.11}$$

4.4. Proof of Proposition 11

We now have all the estimates we need to conclude. We can rewrite (4.11) in the following way

$$\int_0^k\int_{\mathscr{A}_{\varepsilon_0,4d}} |Dz|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C. \tag{4.12}$$

There must exist a time level t^* such that

$$\int_{\mathscr{A}_{\varepsilon_0,4d}} |Dz|^2\,\mathrm{d}x = \int_{B_{4d}} |Dz(x,t^*)|^2\,\mathrm{d}x \leqslant \tau_0 \tag{4.13}$$

with τ_0 a proper small quantity. In fact, if it were not so, integrating on $(0,k)$ we obtain

$$\int_0^k\int_{B_{4d}} |Dz|^2\,\mathrm{d}x \geqslant k\tau_0$$

and it's enough to choose k large enough to get a contradiction. Now we claim that a consequence of (4.13) is that

$$\forall x\in B_{2d} \quad (\zeta - v)(x,t^*) \leqslant \eta_0 \tag{4.14}$$

with η_0 a positive quantity very close to zero. In fact, if it were not true, reproducing the same argument of [6], Sections 6–**??**, and using Proposition 5, we conclude that there exist a $y^*\in B_{4d}$ and a small cube $K_\rho(y^*)\subset B_{4d}$ such that

$$\forall x\in K_\rho(y^*) \quad (\zeta - v)(x,t^*) > \frac{\eta_0}{2}. \tag{4.15}$$

Due to the previous zero-extension of z, $\forall x\in B_{\varepsilon_0}$ we have that $(\zeta-v)(x,t^*)=0$. Hence, reducing the diameter if necessary, we can single out a cube $K_\rho(0)$ with the same radius as the one in (4.15) s.t. $\forall x\in K_\rho(0)$ we have $(\zeta - v)(x,t^*)=0$. Then we can work as in Section 9 of [6] and get a lower bound for $\int_{\mathscr{A}_{\varepsilon_0,4d}} |Dz(x,t^*)|^2\,\mathrm{d}x$. Provided τ_0 is chosen sufficiently small (or k sufficiently large), we end up with a contradiction, so that eventually (4.14) must hold.

Hence we have

$$\forall x \in B_{2d} \quad 0 < (\zeta - v)(x, t^*) \leqslant \eta_0 \Rightarrow \forall x \in B_{2d} \quad \zeta(x) - \eta_0 < v(x, t^*). \tag{4.16}$$

On the other hand, if we consider $\mathscr{A}_{2\varepsilon_0,2d}$, there must exist $y_0 \in \mathscr{A}_{2\varepsilon_0,2d}$ s.t.

$$0 < 2\eta_0 < \alpha \leqslant \zeta(y_0) \leqslant 1.$$

In fact, ζ is Hölder continuous up to the boundary and $0 \leqslant \zeta \leqslant 1$ by the maximum principle, as we showed before.

Then, by the Harnack inequality, we conclude that there exists $\tilde{\alpha} > 2\eta_0$ s.t.

$$\forall x \in \mathscr{A}_{2\varepsilon_0,2d} \quad \zeta(x) \geqslant \tilde{\alpha}. \tag{4.17}$$

Hence, by (4.16) and (4.17) we obtain

$$v(x, t^*) \geqslant \tilde{\alpha} - \eta_0 > \eta_0 \quad \forall x \in \mathscr{A}_{2\varepsilon_0,2d}$$

which implies that

$$v(x, t^*) > \eta_0 \quad \forall x \in K_\rho(y_0),\ \forall K_\rho(y_0) \subset \mathscr{A}_{2\varepsilon_0,2d}.$$

We can now apply the logarithmic estimate, to deduce that $\forall v \in (0, 1)$ there exists a proper ξ, dependent only on η_0 and θ s.t.

$$|\{x \in K_\rho(y_0) : v(x, t) < \xi\}| < v|K_\rho| \quad \forall t \in [t^*, t^* + \theta].$$

But then, using standard arguments (see for example [4], Chapter III, paragraph 6), we obtain that

$$v(x, t) > \tilde{\xi} \quad \forall (x, t) \in [(y_0, t^* + \theta) + Q(\rho/2, \theta)]$$

and from this follows

$$\forall t \in (t^*, t^* + \theta),\ \forall y \in 1 < |y| < 2d \quad v(y, t) > \sigma_0$$

once we set $\sigma_0 = \tilde{\xi}$.

5. Proof of the theorem for time-independent coefficients

To prove the continuity of u at a point $(y, s) \in \Omega_T$, let us first assume that such a point coincides with the origin and work within a cylinder $Q(\rho; \theta\rho^2)$, with θ a positive number to be chosen. Without loss of generality the number θ can be chosen as an integer, so that the starting cylinder will be partitioned, up to a set of measure zero, into disjoint layers of the type

$$[(0, t_i) + Q(\rho; \rho^2)], \quad t_i = -i\rho^2,\ i = 0, 1, \ldots, \theta - 1. \tag{5.1}$$

The numbers $\mu^\pm$ and ω are defined in Section 2. Within such a layer we will show that we can locate a small set where u is quantitatively bounded away, either from μ^+ or from μ^-.

We let $\xi^{\pm}$ and λ be defined as in Proposition 9 and δ be determined by Proposition 8. Notice that the number δ depends upon ω and is independent of ρ.

Fix any box of the type (5.1) and after a translation assume that its vertex coincides with the origin, so that we can rewrite it as $Q(\rho;\rho^2)$. We partition the cylinder in two steps. First we partition the cube K_ρ, up to a set of measure zero, into m^N pairwise disjoint subcubes of wedge $(2/m)\rho$, with m a positive integer to be chosen. If we denote their centres by x_ℓ, $\ell = 1,2,\dots,m^N$, we have

$$\bar{K}_\rho = \bigcup_{\ell=1}^{m^N} [x_\ell + \bar{K}_{\rho/m}].$$

Then we partition the cylinder into $m^N m^2$ pairwise disjoint subcylinders. If we denote by (x_ℓ, t_h) their vertices, each of them takes the form

$$\left[(x_\ell,t_h) + Q\left(\frac{1}{m}\rho;\frac{1}{m^2}\rho^2\right)\right],$$

where for each ℓ in the range $\ell = 1,\dots,m^N$ we have $t_h = (1-h)\rho^2/m^2$, $h = 1,2,\dots,m^2$. Therefore

$$\bar{Q}(\rho;\rho^2) = \bigcup_{h=1}^{m^2}\bigcup_{\ell=1}^{m^N}\left[(x_\ell,t_h) + \bar{Q}\left(\frac{1}{m}\rho;\frac{1}{m^2}\rho^2\right)\right].$$

Within each $[(x_\ell,t_h) + Q((1/m)\rho;(1/m^2)\rho^2)]$ consider coaxial cylinders of the type $[(x_\ell,\tau)+Q(r;r^2)]$. The time location of their vertices ranges over $\tau \in [t_h - ((1/m^2)\rho^2 - r^2), t_h]$ and r is a positive parameter ranging over the interval $[\delta(1/m)\rho,(1/m)\rho]$ where δ is the number determined in Proposition 8 for the proper choices of $\xi^{\pm}$ and λ. These are cylinders of the type (3.9), (3.10), (3.11) where ρ has been replaced by $(1/m)\rho$. For all these cylinders Propositions 2 and 3 hold true for $\theta = 1$. We assume that with the choice $\xi^+ = \xi^- = \frac{1}{12}$ the bound on $\mathscr{H}_k^{\pm}$ asked for by Proposition 1 is satisfied (otherwise we simply have to adjust the value). Since ξ^+ and ξ^- are the same, we denote by v the common value of $v^{\pm}$. Now we have

Proposition 14. *There exists a positive integer m than can be determined a priori only in terms of ω and the data, such that for some cylinder* $[(x_\ell,t_h)+Q((1/m)\rho;(1/m^2)\rho^2)]$ *and for some cylinder* $[(x_\ell,\tau)+Q(r;r^2)] \subset [(x_\ell,t_h)+Q((1/m)\rho;(1/m^2)\rho^2)]$ *either*

$$\text{meas}\left\{(x,t)\in[(x_\ell,\tau)+Q(r;r^2)] : u(x,t) > \mu^+ - \tfrac{1}{12}\omega\right\} < v|Q(r;r^2)| \tag{5.2}$$

or

$$\text{meas}\left\{(x,t)\in[(x_\ell,\tau)+Q(r;r^2)] : u(x,t) < \mu^- + \tfrac{1}{12}\omega\right\} < v|Q(r;r^2)|. \tag{5.3}$$

Proof. If both (5.2) and (5.3) are violated for every cylinder of the type $[(x_\ell,\tau) + Q(r;r^2)]$ and for every $[(x_\ell,t_h) + Q((1/m)\rho;(1/m^2)\rho^2)]$, making up the partition of $Q(\rho;\rho^2)$, by virtue of Proposition 9 there exists a constant that can be determined in terms of the data and ω and independent of ρ and m such that:

$$\left(\frac{1}{m}\rho\right)^N \omega^2 \leqslant \gamma \int_{t_h-(\rho/m)^2}^{t_h}\int_{[x_\ell+K_{\rho/m}]} |Du|^2\,\mathrm{d}x\,\mathrm{d}t \quad \forall \ell = 1,\dots,m^N,\ \forall h = 1,\dots,m^2.$$

Adding over such indices, we obtain:

$$m^2\rho^N\omega^2 \leqslant \gamma \int_{Q(\rho;\rho^2)} |Du|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

We combine this with Propositions 6 and 7 and rewrite the resulting inequality as

$$1 < \frac{\gamma(\mathrm{data},\omega)}{\omega m^2}.$$

The proposition follows by choosing m so large that the right hand side does not exceed 1. It follows also that $\omega \to m(\omega)$ is a decreasing function of ω and $\lim_{\omega\to 0} m(\omega) = \infty$. $\square$

Now let $[(x_\ell,\tau)+Q(r;r^2)]$ be a cylinder for which the alternative holds. Then by Proposition 3 with $\lambda = \frac{2}{3}$, we have:

$$u(x,t) > \mu^- + \frac{1}{18}\omega \quad \forall (x,t) \in \left[(x_\ell,\tau) + Q\left(\frac{\delta}{2m}\rho; \frac{\delta^2}{2m^2}\rho^2\right)\right].$$

On the other hand, if the alternative + holds true within $[(x_\ell,\tau)+Q(r;r^2)]$, then by Proposition 2 we get

$$u(x,t) < \mu^+ - \frac{1}{18}\,\omega \quad \forall (x,t) \in \left[(x_\ell,\tau) + Q\left(\frac{\delta}{2m}\rho; \frac{\delta^2}{2m}\rho^2\right)\right].$$

We may assume that δ^{-1} is an integer. Then we further partition the starting cube K_ρ up to a set of measure zero into

$$q(\omega) = \left(\frac{4m(\omega)}{\delta(\omega)}\right)^N$$

disjoint cubes of wedge

$$\frac{\delta(\omega)}{2m(\omega)}\rho = 2\delta_0\rho.$$

We let x_ℓ, $\ell = 1,2,\ldots,q(\omega)$ denote their centres so that

$$\bar{K}_\rho = \bigcup_{\ell=1}^{q} [x_\ell + \bar{K}_{\frac{\delta}{2m}\rho}].$$

Analogously, we subdivide the cube $Q(\rho;\rho^2)$ into

$$p(\omega) = \left(\frac{4m(\omega)}{\delta(\omega)}\right)^N \left(\frac{4m(\omega)}{\delta(\omega)}\right)^2$$

pairwise disjoint cylinders. If we denote by (x_ℓ,t_h) their vertices, they take the form:

$$\left[(x_\ell,t_h) + Q\left(\frac{\delta}{2m}\rho; \left(\frac{\delta}{2m}\right)^2\rho^2\right)\right], \tag{5.4}$$

where for each $\ell = 1,2,\ldots,q(\omega)$

$$t_h = (1-h)\left(\frac{\delta}{2m}\right)^2\rho^2 \qquad h = 1,2,\ldots,p(\omega).$$

We return to the original partition of $Q(\rho, \theta\rho^2)$ with boxes of the type (5.1) and conclude

Proposition 15. *For each box of the type* (5.1) *there exists a subcylinder of the type* (5.4) *for which either*

$$u(x,t) < \mu^+ - \tfrac{1}{18}\,\omega \quad \forall (x,t) \in [(x_\ell, t_h) + Q(\delta_0\rho; \delta_0^2\rho^2)] \tag{5.5}$$

or

$$u(x,t) > \mu^- + \tfrac{1}{18}\,\omega \quad \forall (x,t) \in [(x_\ell, t_h) + Q(\delta_0\rho; \delta_0^2\rho^2)]. \tag{5.6}$$

Let us now concentrate on the lower half of the cylinder $Q(\rho;\theta\rho^2)$ i.e. $[(0, -\frac{1}{2}\theta\rho^2) + Q(\rho; \frac{1}{2}\theta\rho^2)]$. We assume that the number θ has been chosen and let

$$[(x_\ell, \tau) + Q(\delta_0\rho; \delta_0^2\rho^2)] \subset \left[\left(0, -\tfrac{1}{2}\theta\rho^2\right) + Q\left(\rho; \tfrac{1}{2}\theta\rho^2\right)\right]$$

be a cylinder for which say (5.6) holds. We start from such a box and construct a long, thin cylinder with vertex at the top of $Q(\rho;\theta\rho^2)$ i.e.

$$[x_\ell + K_{4r}] \times [\tau, 0] \quad 4r \equiv \delta_0\rho.$$

We rewrite this as

$$[(x_\ell, 0) + Q(4r; 4\bar{\theta}r^2)],$$

where

$$2\delta_0^{-2}(\theta - 1) \leqslant \bar{\theta} \leqslant 4\delta_0^{-2}\theta.$$

Thanks to (5.6), we have

$$u(x, -4\bar{\theta}r^2) > \mu^- + \tfrac{1}{18}\omega \quad \forall x \in [x_\ell + K_{4r}]. \tag{5.7}$$

Proposition 16. *There exists a number $\xi \in (0, \frac{1}{18})$ that can be determined a priori only in terms of the data and ω such that*

$$u(x,t) > \mu^- + \xi\omega \quad \forall (x,t) \in [(x_\ell, 0) + Q(r; \bar{\theta}r^2)]. \tag{5.8}$$

Proof. See [6], Proposition 24.1. □

Thanks to Proposition 16 we have thus isolated a long, thin cylinder where u is bounded below as in (5.8). The abscissa of the vertex of such cylinder is not known; if $x_\ell \equiv 0$ then it would imply a decrease of the oscillation of a factor $(1 - \xi)$ and we would be finished. However, since the location of $x_\ell \in K_\rho$ is not known, there is the necessity to show that a version of (5.8) holds within a small thin cylinder with vertex at the origin. This is achieved into two stages. The first stage is some sort of spreading of positivity, which we can describe in this way. Assume that of the two alternatives

(5.5) and (5.6) the second one holds; then there exists a time level $t_0 \in (-\theta\rho^2, -\frac{1}{2}\theta\rho^2)$ such that u is quantitatively bounded below in the full cube $[x_\ell + K_{\delta_0\rho}]$. Such positivity spreads sidewise to a full smaller cube about the origin, after a sufficiently long time: this is precisely the content of the next Proposition

Proposition 17. *There exist numbers* $\xi_0, \delta_* \in (0,1)$ *and* $\theta > 1$, *that can be determined a priori in terms of the data and* ω, *and a time level*

$$-\theta\rho^2 \leqslant t_0 \leqslant -\tfrac{1}{2}\theta\rho^2$$

such that either

$$u(x,t_0) < \mu^+ - \xi_0\omega \quad \forall x \in [y + K_{\delta_*\rho}] \tag{5.9}$$

or

$$u(x,t_0) > \mu^- + \xi_0\omega \quad \forall x \in [y + K_{\delta_*\rho}]. \tag{5.10}$$

Proof. The essential tool is a sequential selection of blocks of positivity (see [6]). The number θ will be a product of a finite, increasing sequence of positive integers $\theta = \prod k_j$ that determine a partition of $Q(\rho;\theta\rho^2)$ into disjoint stacks.

There are two alternatives: either among the stacks there exists one where the bound $-$ (actually (5.6)) is verified for the same abscissa x_ℓ for at least one cube within a smaller stack or the same with $+$ (actually (5.5)). In fact the case of neither of the two alternatives being verified cannot occur, because otherwise we would have a contradiction with Proposition 15.

Coming to the details, everything runs exactly as in the proof of Proposition 24.2 of [6], except for the part relative to the comparison function. Hence we will concentrate on this part, referring to [6] for the rest: even if this makes this paper somehow not so self-contained, on the other hand, we avoid the perfect reproduction of Sections 26–29 of [6].

If we apply the change of variables

$$x \to \frac{4x - x_\ell}{|x_\ell|} \qquad t \to \frac{(t_{j+1} - t) + 16k_{j+1}\rho^2}{16\rho^2}$$

and introduce the function

$$\tilde{u} \equiv \frac{u - \mu^-}{\omega\xi_j(\omega)}$$

(see [6] for the definition of $\xi_j(\omega)$), we have that $\tilde{u}(x,t) \geqslant 1$ in $B_{\varepsilon_0} \times (0, k_{j+1})$; moreover $\tilde{u}$ solves the differential equation

$$(\tilde{\beta}(\tilde{u}))_t = \tilde{\mathscr{L}}(\tilde{u}),$$

where $\tilde{\beta}$ and $\tilde{\mathscr{L}}$ satisfy the same structural conditions (1.1)–(1.2) and (1.4)–(1.6), where the corresponding constants $\tilde{\gamma}_0$, $\tilde{\gamma}_1$, $\tilde{\mu}_1$ and $\tilde{\mu}_2$ all depend on ω, which however is to be considered fixed when dealing with $\tilde{u}$.

Finally we are interested in the behaviour of $\tilde{u}$ in an annulus contained in $\{1 < |x| < 2d\} \times (0, k_{j+1})$. For this reason we introduce a proper comparison function. Namely, let

$$v \in C^0(0, k_{j+1}; L^2(B_{4d})) \cap L^2(0, k_{j+1}; W_0^{1,2}(B_{4d}))$$

solve the following boundary value problem

$$\begin{cases} (\tilde{\beta}(v))_t = \tilde{\mathscr{L}} v & \text{on } \mathscr{A}_{\varepsilon_0,4d} \times (0, k_{j+1}) \\ v(x,t) = 0 & \text{in } |x| = 4d \\ v(x,t) = 1 & \text{in } |x| = \varepsilon_0 \\ v(x,0) = 0. & \end{cases}$$

This is the comparison function studied in Section 4 with k replaced by k_{j+1}. Thanks to Proposition 10, we have that $0 \leqslant v \leqslant 1$. Moreover, by Proposition 11 there exist numbers $\sigma_{0,j}$ and k_{j+1} so that

$$v(y,t) > \sigma_{0,j} \quad \forall 1 < |y| < 2d \text{ and for some } t \in (0, k_{j+1}). \tag{5.11}$$

By the maximum principle $\tilde{u} \geqslant v$. Hence the same lower bound as for v holds for $\tilde{u}$ too and returning to the original coordinates, we conclude that there exists a time level t_0 such that

$$u(x, t_0) > \mu_- + \xi_{0,j}\omega \quad \forall x \in K_{\delta_{*,j}\rho}$$

with $\xi_{0,j} \equiv \sigma_{0,j}\xi_j(\omega)$ and a proper $\delta_{*,j}$. The rest follows as in [6]. □

The second stage is the reduction of the oscillation of u near the top of the starting box $Q(\rho; \theta\rho^2)$. We have

Proposition 18. *There exists numbers $\xi_*, \delta_* \in (0,1)$ and a number $\theta > 1$ that can be determined a priori in terms of the data and ω, such that either*

$$u(x,t) < \mu^+ - \xi_*\omega \quad \forall (x,t) \in Q(\delta_*\rho; \theta\delta_*^2\rho^2) \tag{5.12}$$

or

$$u(x,t) > \mu^- + \xi_*\omega \quad \forall (x,t) \in Q(\delta_*\rho; \theta\delta_*^2\rho^2). \tag{5.13}$$

Proof. Exactly as in [6], Proposition 24.3, the proof starts from (5.10) and uses the logarithmic estimate (Proposition 4) and Propositions 3 and 2 to conclude. □

We can finally conclude with the proof of Theorem 1. The argument consists in showing the existence of a family of nested shrinking cylinders with the same vertex s.t. for each of them the oscillation is controlled by a sequence ω_n that tends to zero.

By the previous procedure, we have determined the functions $\omega \to \xi_*(\omega), \delta_*(\omega), \theta(\omega)$. Consider now a cylinder with vertex at the origin, contained in Ω_T of the form $Q(2\rho; 2\theta(\omega)\rho^2)$ where ω is any number satisfying

$$\operatorname*{osc}_{Q(2\rho, 2\theta(\omega)\rho^2)} u \leqslant \omega,$$

applying the previous Proposition, we get

$$\operatorname*{osc}_{Q[\delta_*\rho;\theta\delta_*^2\rho^2]} u \leqslant (1-\xi_*(\omega))\omega.$$

Consider the sequence

$$\omega_0 = 2M, \qquad \omega_{n+1} = (1-\xi_*(\omega_n))\omega_n.$$

As in [6] by induction one constructs a sequence of cylinders Q_n whose radii ρ_n and vertical heights $\theta(\omega_n)\rho_n^2$ decrease to zero. Therefore $\{Q_n\}_{n\in\mathbb{N}}$ is a family of nested shrinking cylinders with the same vertex at the origin. For each of them we have

$$\operatorname*{osc}_{Q_n} u \leqslant \omega_n.$$

and we are finished.

6. Coefficients that depend on *t*

Now we want to remove hypothesis (H1) of Section 4 and assume that the coefficients are continuous in time. The difficulty lies in this fact: whereas the estimates in Sections 2 and 3 hold both for time-independent and time-dependent coefficients, the properties of the comparison function, studied in Section 4 and applied in Section 5, heavily rely on the structure of an elliptic operator, whose coefficients obviously do not depend on time.

The idea to deal with the new situation is simply stated: provided we work in a sufficiently small cylinder, the time-dependent coefficients can be regarded as a small and controllable perturbation to time-independent coefficients, so that everything can be brought back to the case previously considered.

This method is widely used in the context of variable coefficients, so that we will sketch the general approach, without entering too much into details. It is worth remarking that the main point is in checking that the 'controllable perturbation' mentioned above does not destroy assumptions (H2) and (H3) of Section 4, which are crucial in proving the estimates on the comparison function v needed to conclude about the continuity of u.

Let us now fix $\rho < 1$, choose θ of the order of the unity and fix $(x_0, t_0) \in \Omega_T$ such that the parabolic cylinder $[(x_0,t_0) + Q(\rho,\theta\rho^2)] \subset \Omega \times (0,T)$. We can rewrite the operator $\mathscr{L}$ in the following way

$$\begin{aligned}\mathscr{L}u &= \sum_{ij} D_i(a_{ij}(x,t)D_ju + a_i(x,t)u) + b_i(x,t)D_iu + e(x,t)u\\ &= \sum_{ij} D_i(a_{ij}(x,t_0)D_ju + (a_{ij}(x,t) - a_{ij}(x,t_0))D_ju\\ &\quad + a_i(x,t_0)u + (a_i(x,t) - a_i(x,t_0))u) + b_i(x,t_0)D_iu\\ &\quad + (b_i(x,t) - b_i(x,t_0))D_iu + e(x,t_0)u + (e(x,t) - e(x,t_0))u.\end{aligned}$$

If we now set

$$\mathbf{A}_i(x,t,u,Du) = a_{ij}(x,t_0)D_ju + (a_{ij}(x,t) - a_{ij}(x,t_0))D_ju$$
$$+ a_i(x,t_0)u + (a_i(x,t) - a_i(x,t_0))u,$$

$$\mathbf{B}(x,t,u,Du) = b_i(x,t_0)D_iu + (b_i(x,t) - b_i(x,t_0))D_iu$$
$$+ e(x,t_0)u + (e(x,t) - e(x,t_0))u$$

we can write

$$\mathscr{L}u = D_i\mathbf{A}_i(x,t,u,Du) + \mathbf{B}(x,t,u,Du).$$

Notice that we use the kind of notation which is fairly common for quasilinear elliptic equations in divergence form for the sake of compactness, even if the operator we are dealing with is actually linear.

Thanks to the time continuity of the coefficients, with no loss of generality we can assume that

$$|a_{ij}(x,t) - a_{ij}(x,t_0)| \leqslant \omega_1(x,|t-t_0|),$$

$$|a_i(x,t) - a_i(x,t_0)| \leqslant \omega_2(x,|t-t_0|),$$

$$|b_i(x,t) - b_i(x,t_0)| \leqslant \omega_3(x,|t-t_0|),$$

$$|e(x,t) - e(x,t_0)| \leqslant \omega_4(x,|t-t_0|),$$

where $\omega_i = \omega_i(x,s) : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}_+$ are measurable in x, continuous and increasing in s such that $\omega_i(x,0) = 0$. Moreover ω_1 is bounded as the coefficients a_{ij} in (1.4), whereas ω_2, ω_3 and ω_4 satisfy the same summability conditions as stated in (1.5).

Let us now estimate $\mathbf{A}_i$ and $\mathbf{B}$. We have

$$\begin{aligned}
|\mathbf{A}_i(x,t,u,Du)| &= |a_{ij}(x,t_0)D_ju + (a_{ij}(x,t) - a_{ij}(x,t_0))D_ju \\
&\quad + a_i(x,t_0)u + (a_i(x,t) - a_i(x,t_0))u| \\
&\leqslant |a_{ij}(x,t_0)||Du| + |a_{ij}(x,t) - a_{ij}(x,t_0)||Du| \\
&\quad + \left(\sum_{i=1}^N a_i^2(x,t_0)\right)^{1/2} |u| + |a_i(x,t) - a_i(x,t_0)||u| \\
&\leqslant (\mu + \omega_1(x,\theta\rho^2))|Du| + \left(\left(\sum_{i=1}^N a_i^2(x,t_0)\right)^{1/2} + \omega_2(x,\theta\rho^2)\right)|u| \\
&= \alpha_1(x)|Du| + \alpha_2(x)|u|. \qquad (6.1)
\end{aligned}$$

Analogously

$$\begin{aligned}
|\mathbf{B}(x,t,u,Du)| &= |b_i(x,t_0)D_iu + (b_i(x,t)-b_i(x,t_0))D_iu \\
&\quad + e(x,t_0)u + (e(x,t)-e(x,t_0))u| \\
&\leqslant \left(\sum_{i=1}^N b_i^2(x,t_0)\right)^{1/2} |Du| + |b_i(x,t)-b_i(x,t_0)||Du| \\
&\quad + |e(x,t_0)||u| + |e(x,t)-e(x,t_0)||u| \\
&\leqslant \left(\left(\sum_{i=1}^N b_i^2(x,t_0)\right)^{1/2} + \omega_3(x,\theta\rho^2)\right)|Du| \\
&\quad + (|e(x,t_0)| + \omega_4(x,\theta\rho^2))|u| \\
&= \beta_1(x)|Du| + \beta_2(x)|u|. \qquad (6.2)
\end{aligned}$$

Finally

$$\begin{aligned}
\mathbf{A}_i \cdot D_iu &= a_{ij}(x,t_0)D_juD_iu + (a_{ij}(x,t)-a_{ij}(x,t_0))D_juD_iu \\
&\quad + a_i(x,t_0)uD_iu + (a_i(x,t)-a_i(x,t_0))uD_iu \\
&\geqslant \frac{1}{\mu_1}|Du|^2 - \omega_1(x,\theta\rho^2)|Du|^2 - \left(\sum_{i=1}^N a_i^2(x,t_0)\right)^{1/2} |u||Du| \\
&\quad - \omega_2(x,\theta\rho^2)|u||Du| \\
&\geqslant \left(\frac{1}{\mu_1} - \omega_1(x,\theta\rho^2) - \varepsilon_1\left(\sum_{i=1}^N a_i^2(x,t_0)\right)^{1/2} - \varepsilon_2\omega_2(x,\theta\rho^2)\right)|Du|^2 \\
&\quad + \left(C(\varepsilon_1)\left(\sum_{i=1}^N a_i^2(x,t_0)\right)^{1/2} + C(\varepsilon_2)\omega_2(x,\theta\rho^2)\right)|u|^2 \\
&\geqslant \gamma_1(x)|Du|^2 - \gamma_2(x)|u|^2. \qquad (6.3)
\end{aligned}$$

With a proper choice of ε_1 and ε_2, provided that ρ is taken sufficiently small we have that

$$\gamma_1 = \frac{1}{\mu_1} - \omega_1(x,\theta\rho^2) - \varepsilon_1\left(\sum_{i=1}^N a_i^2(x,t_0)\right)^{1/2} - \varepsilon_2\omega_2(x,\theta\rho^2) > 0$$

and thanks to (1.4) and (1.5) the coefficients α_i, β_i and γ_i satisfy the same summability hypotheses as a_{ij}, a_i, b_i, e but do not depend on t.

As we have already said at the beginning of this Section, to conclude the proof of the continuity of u in the case of time-independent coefficients, we had to compare

u with the solution v of an elliptic equation, to which we were led after a suitable change of variables, namely

$$x \to \frac{4x - x_\ell}{|x_\ell|} \qquad t \to \frac{(t_{j+1} - t) + 16k_{j+1}\rho^2}{16\rho^2}.$$

Let us repeat the same procedure in the new context of time-dependent coefficients we are considering now.

Due to the smallness of the perturbation (remember that ρ is fixed), it is easy to see that the structural conditions (6.1)–(6.3) considered above still hold.

As largely discussed in [12], these are the natural assumptions under which we have a Harnack inequality like the one discussed in (H3) in Section 4. The same can be said for the comparison principle, which is implied by (H2).

Hence we are basically dealing with a time-independent operator, so that the whole procedure of Sections 4 and 5 can be repeated here without any restriction. In this way we are brought back to the case studied in the previous Section and we can conclude.

7. Boundary regularity under homogeneous Dirichlet conditions

Boundary regularity for degenerate and singular parabolic equations is not a simple task. When dealing with equations like

$$u_t - \operatorname{div}(|Du|^{p-2}\,Du) = 0 \quad \text{in } \Omega_T,\ \ p > 1$$

the proof of the interior continuity is based only on the energy and logarithmic estimates, like the ones proved here in Section 2 (see [4] for all details under this point of view).

With this in mind, coming to boundary regularity for variational data for the same kind of problems, one realizes that the only thing to do is to show the validity of the corresponding energy and logarithmic boundary estimates.

In the case of Dirichlet data, things are slightly more difficult and a proper care has to be used with the levels k, but once more basically the idea is to mimic what has been done in the interior.

For singular equations like the ones we are dealing with here, things are much more involved. In the case of β with a single jump, interior regularity is again a matter of energy and logarithmic inequalities and this allows a complete solution of boundary regularity for variational data, as in [2].

Even if we didn't write it down explicitly anywhere, we feel that the same could be repeated for β with an arbitrary but finite number of jumps, as considered in [9].

Looking at things more properly, the main difficulty in the proof of continuity for weak solutions of

$$(\beta(u))_t = \mathscr{L}u$$

lies in the term

$$\iint_{Q(\rho,\theta\rho^2)} (u-k)_\pm \,\mathrm{d}x\,\mathrm{d}\tau$$

in the right-hand side of the energy inequality like (2.2) here, as largely discussed in [3]. Roughly speaking, this sets some constraints on the possible values of the levels k.

On the other hand the study of the boundary regularity for Dirichlet data poses further limits on the levels k.

As a consequence, when switching from homogeneous to inhomogeneous Dirichlet data, a new method is required, even if the interior regularity is purely based on the energy and logarithmic estimates and this is precisely the case dealt with in [2] and [3].

In our case things are more difficult since the proof does not rely uniquely on suitable estimates satisfied by u, but uses a proper comparison function v as discussed at a larger extent in the previous Section.

Therefore if it remains an open problem to devise a proof technique based only on energy and logarithmic estimates, more specifically, under our point of view, variational boundary conditions cannot be treated simply referring to interior regularity and the same holds for general Dirichlet data.

The only thing we could do was to consider homogeneous Dirichlet boundary conditions under mild assumptions on $\partial\Omega$ and following a strategy already outlined in [7], without taking into account initial conditions, we could prove the following

Theorem 2. *Let Ω be a bounded Lipschitz domain and let u be a weak solution of*

$$(\beta(u))_t = \mathscr{L}u \quad \text{in } \Omega \times (0,T),$$

$$u(\cdot,t)|_{\partial\Omega} = 0 \quad \text{a.e. } t \in (\varepsilon, T),$$

where with respect to Definition 1, we say that $u(\cdot,t)=0$ on $\partial\Omega$ in the sense of the traces of functions in $W^{1,2}(\Omega)$. Moreover let us assume that

$$u \in L^\infty(\bar{\Omega} \times [\varepsilon, T]), \qquad \varepsilon \in (0,T).$$

Then $u \in C(\bar{\Omega} \times [\varepsilon, T])$ and there exists a continuous non-negative, increasing function $s \to \omega_D(s) : \mathbb{R}_+ \to \mathbb{R}_+$ with $\omega_D(0)=0$ that can be determined a priori only in terms of the data such that

$$|u(x_1,t_1) - u(x_2,t_2)| \leqslant \omega_D(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}})$$

for every pair of points (x_1,t_1), $(x_2,t_2) \in \bar{\Omega} \times [\varepsilon, T]$.

Proof. Even if the theorem is stated in a global way, the proof has a typical local flavour. We limit ourselves to a simple sketch.

Due to the compactness of Ω we can cover $\partial\Omega$ with a finite number of neighborhood centered at points of $\partial\Omega$. The Lipschitz continuity of the boundary allows us to find a map from every neighborhood into a half ball of $\mathbb{R}^N$. The transformed equation via this map has coefficients which are still measurable with respect to x, properly summable with respect to the couple (x,t), continuous with respect to t and satisfy structural conditions (1.4) and (1.6). Now we reflect the operator through the entire ball and notice that this reflection does not affect the $(\beta(u))_t$ term, since it's done only with respect the x variable. We have therefore reduced ourselves to study a problem in the interior and we can apply the previous results to conclude.

References

[1] L. Caffarelli, L.C. Evans, Continuity of the temperature in the two phase Stefan problem, Arch. Rational Mech. Anal. 81 (1983) 199–220.
[2] E. DiBenedetto, Continuity of weak solutions to certain singular parabolic equations, Ann. Mat. Pura Appl. CXXI (4) (1982) 131–176.
[3] E. DiBenedetto, A boundary modulus of continuity for a class of singular parabolic equations, J. Differential Equations 63 (1986) 418–447.
[4] E. DiBenedetto, Degenerate Parabolic Equations, Springer, Series Universitext, New York, 1993.
[5] E. DiBenedetto, N.S. Trudinger, Harnack inequalities for quasi-minima of variational integrals, Ann. Inst. Herni Poincaré 1 (4) (1984) 295–308.
[6] E. DiBenedetto, V. Vespri, On the singular equation $\beta(u)_t = \Delta u$, Arch. Rational Mech. Anal. 132 (3) (1995) 247–309.
[7] L. Esposito, G. Mingione, B. Stroffolini, On the continuity of the singular equation $\beta(u)_t=\Delta u$, Nonlinear Anal. 36 (1999) 1037–1048.
[8] A. Friedman, Variational Principles and Free Boundary Problems, Wiley-Interscience, New York, 1982.
[9] U. Gianazza, V. Vespri, Continuity of weak solutions of a singular parabolic Equation, Preprint of the Istituto di Analisi Numerica del C.N.R. di Pavia # 1184, 1-30, Adv. Diff. Eq., to appear.
[10] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, Berlin-Heidelberg-New York-Tokyo, 1983.
[11] O.A. Ladyzhenskaja, V.A. Solonnikov, N.N. Ural'tzeva, Linear and quasilinear equations of parabolic type, AMS Transl. Math. Mono, Vol. 23, Providence RI, USA, 1968.
[12] J. Malý, W.P. Ziemer, Fine regularity of solutions of elliptic partial differential equations, AMS Math. Surveys Mono., Vol. 51, Providence RI, USA, 1997.
[13] P. Sachs, The initial and boundary value problem for a class of degenerate parabolic equations, Comm. Part. Diff. Eq. 8 (1983) 693–734.
[14] G. Sargenti, Continuity results for solutions of certain degenerate parabolic equations, Rend. Sem. Mat. Univ. Padova 99 (1998) 105–131.
[15] G. Sargenti, Local boundedness and regularity results for a class of singular anisotropic parabolic equations, Atti Sem. Mat. Fis. Univ. Modena XLVII (2000) 45–78.
[16] G. Sargenti, V. Vespri, Regularity results for multiphase Stefan-like equations, in: K. Malanowski, Z. Nahorski, M. Peszynska (Eds.), Modelling and Optimization of Distributed Parameter Systems, Chapman & Hall, London, 1997, pp. 111–117.
[17] B. Stroffolini, V. Vespri, On the continuity of the solution of the singular equation $(\beta(u))_t = \mathscr{L}u$, Le Matematiche 55 (Suppl. 2) (2001) 165–195.
[18] A. Visintin, A. Fasano, E. Magenes, C. Verdi, About mathematical models of phase transitions, Boll. UMI 1-B (8) (1998) 1–47, 49–69, 71–81, 83–108.
[19] W.P. Ziemer, Interior and boundary continuity of weak solutions of degenerate parabolic equations, Trans. Amer. Math. Soc. 271 (2) (1982) 733–748.