Brave New Algebraic Geometry
and global derived moduli spaces
of ring spectra

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Abstract

We develop homotopical algebraic geometry ([To-Ve 1, To-Ve 2]) in the special context where the base symmetric monoidal model category is that of spectra **S**, i.e. what might be called, after Waldhausen, *brave new algebraic geometry*. We discuss various model topologies on the model category of commutative algebras in **S**, and their associated theories of geometric **S**-stacks (a geometric **S**-stack being an analog of Artin notion of algebraic stack in Algebraic Geometry). Two examples of geometric **S**-stacks are given: a global moduli space of associative ring spectrum structures, and the stack of elliptic curves endowed with the sheaf of topological modular forms.

Contents

1 Introduction

Homotopical Algebraic Geometry is a kind of algebraic geometry where the affine objects are given by commutative ring-like objects in some homotopy theory (technically speaking, in a symmetric monoidal model category); these affine objects are then glued together according to an appropriate homotopical modification of a Grothendieck topology (a *model topology*, see [To-Ve 1, 4.3]). More generally, we allow ourselves to consider more flexible objects like stacks, in order to deal with appropriate moduli problems. This theory is developed in full generality in [To-Ve 1, To-Ve 2] (see also [To-Ve 3]). Our motivations for such a theory came from a variety of sources: first of all, on the algebro-geometric side, we wanted to produce a sufficiently functorial language in which the so called Derived Moduli Spaces foreseen by Deligne, Drinfel'd and Kontsevich could really be constructed; secondly, on the topological side, we thought that maybe the many recent results in Brave New Algebra, i.e. in (commutative) algebra over structured ring spectra (in any one of their brave new symmetric monoidal model categories, see e.g. [Ho-Sh-Sm, EKMM]), could be pushed to a kind of Brave New Algebraic Geometry in which one could take advantage of the possibility of gluing these brave new rings together into an actual geometric object, much in the same way as commutative algebra is helped (and generalized) by the existence of algebraic geometry. Thirdly, on the motivic side, following a suggestion of Y. Manin, we wished to have a sufficiently general theory in order to study algebraic geometry *over* the recent model categories of motives for smooth schemes over a field ([Hu, Ja, Sp]).

The purpose of this paper is to present the first steps in the second type of applications mentioned above, i.e. a specialization of the general framework of homotopical algebraic geometry to the context of stable homotopy theory. Our category $\mathbf{S} - \mathbf{Aff}$ of brave new affine objects will therefore be defined as the the opposite model category of the category of commutative rings in the category $\mathbf{S}$ of symmetric spectra ([Ho-Sh-Sm]).

We first define and study various model topologies defined on $\mathbf{S} - \mathbf{Aff}$. They are all extensions, to different extents, of the usual Grothendieck topologies defined on the category of (affine) schemes, like the Zariski and étale ones.

With any of these model topologies τ at our disposal, we define and give the basic properties of the corresponding model category of $\mathbf{S}$-stacks, understood in the broadest sense as not necessarily truncated presheaves of simplicial sets on $\mathbf{S} - \mathbf{Aff}$ satisfying a homotopical descent (i.e. sheaf-like) condition with respect to τ-(hyper)covers. A model topology on $\mathbf{S} - \mathbf{Aff}$ is said to be *subcanonical* if the representable simplicial presheaves, i.e. those of the form $\mathrm{Map}(A, -)$, for some commutative ring A in $\mathbf{S}$, Map being the mapping space in $\mathbf{S} - \mathbf{Aff}^{op}$, are $\mathbf{S}$-stacks.

As in algebraic geometry one finds it useful to study those stacks defined by smooth groupoids (these are called Artin algebraic stacks), we also define a brave new analog of these and call them geometric $\mathbf{S}$-stacks, to emphasize that such $\mathbf{S}$-stacks host a rich geometry very close to the geometric intuition learned in algebraic geometry. In particular, given a geometric $\mathbf{S}$-stack F, it makes sense to speak about quasi-coherent and perfect modules over F, about the K-theory of F, etc.; various properties of morphisms (e.g. smooth, étale, proper, etc.) between geometric $\mathbf{S}$-stacks can likewise be defined.

Stacks were introduced in algebraic geometry mainly to study moduli problems of various sorts; they provide actual geometric objects (rather than sets of isomorphisms classes or coarse moduli schemes) which store all the fine details of the classification problem and on which a geometry very similar to that of algebraic varieties or schemes can be developed, the two aspects having a fruitful interplay. In a similar vein, in our brave new context, we give one example of a moduli problem arising in algebraic topology (the classification of A_∞-ring spectrum structures on a given spectrum M) that can be studied geometrically through the geometric **S**-stack $\mathbb{R}\underline{Ass}_M$ it represents. We wish to emphasize that instead of a discrete homotopy type (like the ones studied, for different moduli problems, in [Re, B-D-G, G-H]), we get a full geometric object on which a lot of interesting geometry can be performed. The geometricity of the S-stack $\mathbb{R}\underline{Ass}_M$, with respect to any fixed subcanonical model topology, is actually the main theorem of this paper (see Theorem 4.2.1).

We also wish to remark that the approach presented in this paper can be extended to other, more interested and involved, moduli problems algebraic topologists are interested in, and perhaps this richer geometry could be of some help in answering, or at least in formulating in a clearer way, some of the deep questions raised by the recent progress in stable homotopy theory (see [G]). In this direction, we will explain in §4.3 how topological modular forms give rise to a natural geometric **S**-stack which is an extension in the brave new direction of the moduli stack of elliptic curves (see Theorem 4.3.1). This fact seems to us a very important remark (probably much more interesting than our Theorem 4.2.1), and we think it could be the starting point of a very interesting research program.

We also present a brave new analog of the stack of vector bundles on a scheme, called the **S**-(pre)stack Perf of perfect modules (Section 3.2), and we expect it to be a key tool in brave new algebraic geometry. The prestack Perf is a stack if and only if the model topology we are working with is subcanonical (Thm. 3.2.1 whose proof is postponed to [To-Ve 2]). This is another instance of the relevance of the *descent problem*, i.e. the question whether a given model topology is subcanonical or not (see Section 3.1). Though we prove that some of the model topologies we introduce (namely the standard and the semi-standard ones, Section 2.3) are subcanonical, at present we are not able to settle (nor in the positive nor in the negative) the descent problem for the three most promising model topologies we define, namely the Zariski, étale and thh-étale ones. Though this is at the moment quite unsatisfactory, we believe that the descent problem for these topologies is a very interesting question in itself even leaving outside its crucial role in brave new algebraic geometry.

For the Zariski model topology, we have a partial positive result in this direction. By definition, for a model topology τ the property of being subcanonical depends on the notion of stacks we consider; if instead of defining a stack as a prestack (i.e. a simplicial presheaf) satisfying homotopical descent with respect to *all* homotopy τ-hypercovers, we simply require descent with respect to all Čech τ-hypercovers (i.e. those arising as nerves of τ-covers), we obtain a notion of *Čech stacks*, recently considered by J. Lurie ([Lu]) and Dugger-Hollander-Isaksen ([DHI]). We prove (Corollary 3.1.4) that the Zariski model topology is in fact subcanonical with respect to the notion of Čech stacks. Moreover, by replacing in Theorem 4.2.1 the word "stack" with the weaker "Čech stack", the statement remains true for any model topology which is subcanonical with respect to the notion of Čech stacks.

It is therefore natural to ask why we did not choose to formulate everything only in terms of Čech stacks. We believe that at this early stage of development of homotopical algebraic geometry and, in particular, of brave new algebraic geometry, it is not advisable to make choices that could prevent some applications or obscure some of the properties of the objects involved, while it is more useful to keep in mind various options, some of which can be more useful in one context than in others. For example, it is clear that knowing that a given, geometrically meaningful, simplicial presheaf is a stack and not only a Čech stack adds a lot more informations, in fact exactly the descent property with respect to unbounded hypercovers ([DHI, Thm. A.6]). Moreover, Čech stacks fail in general to satisfy an analog of Whitehead theorem: a pointed Čech stack may have vanishing π_i sheaves for any $i \geq 0$ without being necessarily contractible. This last fact is a very inconvenient property of

Čech stacks, that makes Postnikov decompositions and spectral sequences arguments uncertain. On the other hand, the Čech descent condition is usually much easier to establish than the full descent condition, and as we have already remarked, some natural model topologies are easily seen to be subcanonical with respect to the notion of Čech stacks while it might be tricky to show that they are actually subcanonical. Finally, we would like to mention that in our experience we have never met serious troubles by using one or the other of the two notions, and in many interesting contexts it does not really matter which notion one uses, as the rather subtle differences actually tend not to appear in practice.

Notations. To fix ideas, we will work in the category $\mathbf{S} := \mathbf{Sp}^{\Sigma}$ of symmetric spectra (see [Ho-Sh-Sm]), but all the constructions of this paper will also work, possibly with minor variations (see [Sch]), for other equivalent theories (e.g for the category of S-modules of [EKMM]). We will consider $\mathbf{S}$ as a symmetric monoidal simplicial model category (for the smash product $- \wedge -$) with the Shipley-Smith positive S-model structure (see [Shi, Prop. 3.1]).

We define $\mathbf{S} - \mathbf{Alg}$ as the category of (associative and unital) commutative monoids objects in $\mathbf{S}$, endowed with the S-model structure of [Shi, Thm. 3.2]; we will simply call them *commutative S-algebras* instead of the more correct but longer, commutative symmetric ring spectra. For any commutative S-algebra A, we will denote by $A - \mathbf{Alg}$ the under-category $A/\mathbf{S} - \mathbf{Alg}$, whose objects will be called *commutative A-algebras*. Finally, if A is a commutative S-algebra, $A - \mathbf{Mod}$ will be the category of A-modules with the A-model structure ([Shi, Prop. 3.1]). This model category is also a symmetric monoidal model category for the smash product $- \wedge_A -$ over A.

For a morphism of commutative S-algebras, $f : A \longrightarrow B$ one has a Quillen adjunction

$$f^* : A - \mathbf{Mod} \longrightarrow B - \mathbf{Mod} \qquad A - \mathbf{Mod} \longleftarrow B - \mathbf{Mod} : f_*,$$

where $f^*(-) := - \wedge_A B$ is the base change functor. We will denote by

$$\mathbb{L}f^* : \mathbf{Ho}(A - \mathbf{Mod}) \longrightarrow \mathbf{Ho}(B - \mathbf{Mod}) \qquad \mathbf{Ho}(A - \mathbf{Mod}) \longleftarrow \mathbf{Ho}(B - \mathbf{Mod}) : \mathbb{R}f_*$$

the induced derived adjunction on the homotopy categories.

Our references for model category theory are [Hi, Ho]. For a model category M with equivalences W, the set of morphisms in the homotopy category $\mathbf{Ho}(M) := W^{-1}M$ will be denoted by $[-, -]_M$, or simply by $[-, -]$ if the context is clear. The (homotopy) mapping spaces in M will be denotedby $\mathrm{Map}_M(-, -)$. When M is a simplicial model category, the simplicial Hom's (resp. derived simplicial Hom's) will be denoted by $\underline{Hom}_M$ (resp. $\mathbb{R}\underline{Hom}_M$), or simply by $\underline{Hom}$ (resp. $\mathbb{R}\underline{Hom}$) if the context is clear. Recall that in this case one can compute $\mathrm{Map}_M(-, -)$ as $\mathbb{R}\underline{Hom}_M(-, -)$.

Finally, for a model category M and an object $x \in M$ we will often use the coma model categories x/M and M/x. When the model category M is not left proper (resp. is not right proper) we will always assume that x has been replaced by a cofibrant (resp. fibrant) model before considering x/M (resp. M/x). More generally, we will not always mention fibrant and cofibrant replacements and suppose implicitly that all our objects are fibrant and/or cofibrant when required.

Since we wish to concentrate on applications to stable homotopy theory, some general constructions and details about homotopical algebraic geometry will be omitted by referring to [To-Ve 1]. For

a few of the results presented we will only give here sketchy proofs; full proofs will appear in [To-Ve 2].

2 Brave new sites

In this section we present two model topologies on the (opposite) category of commutative S-algebras. They are *brave new analogs* of the Zariski and étale topologies defined on the category of usual commutative rings and will allow us to define the *brave new Zariski* and *étale sites*.

We denote by $\mathbf{S} - \mathbf{Aff}$ the opposite model category of $\mathbf{S} - \mathbf{Alg}$.

If M is a model category we say that an object x in M is *finitely presented* if, for any filtered direct system of objects $\{z_i\}_{i \in J}$ in M, the natural map

$$\mathrm{colim}_i \, \mathrm{Map}_M(x, z_i) \longrightarrow \mathrm{Map}_M(x, \mathrm{colim}_i \, z_i)$$

is an isomorphism in the homotopy category of simplicial sets.

Definition 2.0.1 *A morphism $A \to B$ of commutative S-algebras is* finitely presented *if it is a finitely presented object in the model under-category $A/(\mathbf{S} - \mathbf{Alg}) = A - \mathbf{Alg}$; in this case, we also say that B is a finitely presented commutative A-algebra. An A-module E is* finitely presented *or* perfect *if it is a finitely presented object in the model category $A - \mathbf{Mod}$.*

Perfect A-modules can also be characterized as retracts of finite cell A-modules (see [EKMM, Thm. III-7.9]); in particular, there are plenty of them. If A is a commutative S-algebra, then the free commutative A-algebra on a finite number of generators (or, more generally, on any perfect A-module) is a finitely presented A-algebra. The reader will find other examples of finitely presented morphisms of commutative S-algebras in Lemma 2.1.6.

2.1 The brave new Zariski topology

Definition 2.1.1 • *A morphism $f : A \longrightarrow B$ in $\mathbf{S} - \mathbf{Alg}$ is called a* formal Zariski open immersion *if the induced functor $\mathbb{R}f_* : \mathbf{Ho}(B - \mathbf{Mod}) \longrightarrow \mathbf{Ho}(A - \mathbf{Mod})$ is fully faithful.*

- *A morphism $f : A \longrightarrow B$ is a* Zariski open immersion *if $\mathbf{S} - \mathbf{Alg}$ is it is a formal Zariski open immersion and of finite presentation (as a morphism of commutative S-algebras).*

- *A family $\{f_i : A \longrightarrow A_i\}_{i \in I}$ of morphisms in $\mathbf{S} - \mathbf{Alg}$ is called a* (formal) Zariski open covering *if it satisfies the following two conditions.*

  - *Each morphism $A \longrightarrow A_i$ is a (formal) Zariski open immersion.*
  - *There exist a finite subset $J \subset I$ such that the family of inverse image functors*

    $$\{\mathbb{L}f_j^* : \mathbf{Ho}(A - \mathbf{Mod}) \longrightarrow \mathbf{Ho}(A_j - \mathbf{Mod})\}_{j \in J}$$

    is conservative (i.e. a morphism in $\mathbf{Ho}(A - \mathbf{Mod})$ is an isomorphism if and only if its images by all the $\mathbb{L}f_j^*$'s are isomorphisms).

Example 2.1.2 If $A \in \mathbf{S} - \mathbf{Alg}$ and E is an A-module such that the associated Bousfield localization L_E is smashing (i.e. the natural transformation $L_E(-) \to L_E A \wedge^{\mathbb{L}}_A (-)$ is an isomorphism), then $A \to L_E A$ (which is a morphism of commutative S-algebras by e.g. [EKMM, §VIII.2]) is a formal Zariski open immersion. This follows immediately from the fact that $\mathbf{Ho}(L_E A - Mod)$ is equivalent to the subcategory of $\mathbf{Ho}(A - Mod)$ consisting of L_E-local objects, by [Wo].

It is easy to check that (formal) Zariski open covering families define a *model topology* in the sense of [To-Ve 1, §4.3] on the model category $\mathbf{S} - \mathbf{Aff}$. For the reader's convenience we recall what this means in the following lemma.

Lemma 2.1.3
  • *If $A \longrightarrow B$ is an equivalence of commutative S-algebras then the one element family $\{A \longrightarrow B\}$ is a (formal) Zariski open covering.*

  • *Let $\{A \longrightarrow A_i\}_{i \in I}$ be a (formal) Zariski open covering of S-algebras and $A \longrightarrow B$ a morphism. Then, the family of homotopy push-outs $\{B \longrightarrow B \wedge^{\mathbb{L}}_A A_i\}_{i \in I}$ is also a (formal) Zariski open covering.*

  • *Let $\{A \longrightarrow A_i\}_{i \in I}$ be a (formal) Zariski open covering of S-algebras, and for any $i \in I$ let $\{A_i \longrightarrow A_{ij}\}_{j \in J_i}$ be a (formal) Zariski open covering of S-algebras. Then, the total family $\{A \longrightarrow A_{ij}\}_{i \in I, j \in J_i}$ is again a (formal) Zariski open covering.*

Proof: Left as an exercise. □

By definition, Lemma 2.1.3 shows that (formal) Zariski open coverings define a model topology on the model category $\mathbf{S} - \mathbf{Aff}$ and so, as proved in [To-Ve 1, Prop. 4.3.5], induce a Grothendieck topology on the homotopy category $\mathbf{Ho}(\mathbf{S} - \mathbf{Alg})$. This model topology is called the *brave new (formal) Zariski topology*, and endows $\mathbf{S} - \mathbf{Aff}$ with the structure of a model site in the sense of [To-Ve 1, §4]. This model site, denoted by $(\mathbf{S} - \mathbf{Aff}, \mathrm{Zar})$ for the brave new Zariski topology, and $(\mathbf{S} - \mathbf{Aff}, \mathrm{fZar})$ for the brave new formal Zariski topology. They will be called the *brave new Zariski site* and the *brave new formal Zariski site*.

Let $\mathbf{Alg}$ be the category of (associative and unital) commutative rings. Let us recall the existence of the Eilenberg-Mac Lane functor
$$H : \mathbf{Alg} \longrightarrow \mathbf{S} - \mathbf{Alg},$$
sending a commutative ring R to the commutative S-algebra HR such that $\pi_0(HR) = R$ and $\pi_i(HR) = 0$ for any $i \neq 0$. This functor is homotopically fully faithful and the following lemma shows that our brave new Zariski topology does generalize the usual Zariski topology.

Lemma 2.1.4 (1) *Let $R \longrightarrow R'$ be a morphism of commutative rings. The induced morphism $HR \longrightarrow HR'$ is a Zariski open immersion of commutative S-algebras (in the sense of Definition 2.1.1) if and only if the morphism $Spec\, R' \longrightarrow Spec\, R$ is an open immersion of schemes.*

  (2) *A family of morphisms of commutative rings, $\{R \longrightarrow R'_i\}_{i \in I}$, induces a Zariski covering family of commutative S-algebras $\{HR \longrightarrow HR'_i\}_{i \in I}$ (in the sense of Definition 2.1.1) if and only if the family $\{Spec\, R_i \longrightarrow Spec\, R\}_{i \in I}$ is a Zariski open covering of schemes.*

Proof: Let us start with the general situation of a morphism $f : A \longrightarrow B$ of commutative S-algebras such that the induced functor $\mathbb{R}f_* : \mathbf{Ho}(B - \mathbf{Mod}) \longrightarrow \mathbf{Ho}(A - \mathbf{Mod})$ is fully faithful. Let $L = \mathbb{R}f_* \circ \mathbb{L}f^*$, which comes with a natural transformation $Id \longrightarrow L$. Then, the essential image of $\mathbb{R}f_*$ consist of objects M in $\mathbf{Ho}(A - \mathbf{Mod})$ such that the localization morphism $M \longrightarrow LM$ is an isomorphism.

The Quillen adjunction (f^*, f_*) extends to a Quillen adjunction on the category of commutative algebras

$$f^* : A - \mathbf{Alg} \longrightarrow B - \mathbf{Alg} \qquad A - \mathbf{Alg} \longleftarrow B - \mathbf{Alg} : f_*,$$

also with the property that $\mathbb{R}f_* : \mathbf{Ho}(B - \mathbf{Alg}) \longrightarrow \mathbf{Ho}(A - \mathbf{Alg})$ is fully faithful. Furthermore, the essential image of this last functor consist of all objects $C \in \mathbf{Ho}(A - \mathbf{Alg})$ such that the underlying A-module of C satisfies $C \simeq LC$ (i.e. the underlying A-module of C lives in the image of $\mathbf{Ho}(B - \mathbf{Mod})$).

¿From these observations, we deduce that for any commutative A-algebra C, the mapping space $\mathbb{R}\underline{Hom}_{A-\mathbf{Alg}}(B, C)$ is either empty or contractible; it is non-empty if and only if the underlying A-module of C belongs to the essential image of $\mathbb{R}f_*$.

To prove (1), let us first suppose that $f : Spec\, R' \longrightarrow Spec\, R$ is an open immersion of schemes. The induced functor on the derived categories $f_* : D(R') \longrightarrow D(R)$ is then fully faithful. As there are natural equivalences ([EKMM, IV Thm. 2.4])

$$\mathbf{Ho}(HR - \mathbf{Mod}) \simeq D(R) \qquad \mathbf{Ho}(HR' - \mathbf{Mod}) \simeq D(R')$$

this implies that the functor $\mathbb{R}f_* : \mathbf{Ho}(HR' - \mathbf{Mod}) \longrightarrow \mathbf{Ho}(HR - \mathbf{Mod})$ is also fully faithful. It only remains to show that $HR \longrightarrow HR'$ is finitely presented in the sense of Definition 2.0.1.

We will first assume that $R' = R_f$ for some element $f \in R$. The essential image of $\mathbb{R}f_* : \mathbf{Ho}(HR' - \mathbf{Mod}) \longrightarrow \mathbf{Ho}(HR - \mathbf{Mod})$ then consists of all objects $E \in \mathbf{Ho}(HR - \mathbf{Mod}) \simeq D(R)$ such that f acts by isomorphisms on the cohomology R-module $H^*(E)$. By what we have seen at the beginning of the proof, this implies that for any commutative HR-algebra C the mapping space $\mathbb{R}\underline{Hom}_{HR-\mathbf{Alg}}(HR', C)$ is contractible if f becomes invertible in $\pi_0(C)$, and empty otherwise. ¿From this one easily deduces that $\mathbb{R}\underline{Hom}_{HR-\mathbf{Alg}}(HR', -)$ commutes with filtered colimits, or in other words that HR' is a finitely presented HR-algebra in the sense of Definition 2.0.1.

In the general case, one can write $Spec\, R'$ as a finite union of schemes of the form $Spec\, R_f$ for some elements $f \in R$. A bit of descent theory (see §3.1) then allows us to reduce to the case where $R' = R_f$ and conclude.

Let us now assume that $HR \longrightarrow HR'$ is a Zariski open immersion of commutative S-algebras. By adjunction (between H and π_0 restricted on connective S-algebras) one sees easily that $R \longrightarrow R'$ is a finitely presented morphism of commutative rings.

The induced functor on (unbounded) derived categories

$$f_* : D(R') \simeq \mathbf{Ho}(HR' - \mathbf{Mod}) \longrightarrow D(R) \simeq \mathbf{Ho}(HR - \mathbf{Mod})$$

is fully faithful. Through the Dold-Kan correspondence, this implies that the Quillen adjunction on the model category of simplicial modules (see [G-J])

$$f^* : sR - \mathbf{Mod} \longrightarrow sR' - \mathbf{Mod} \qquad sR - \mathbf{Mod} \longleftarrow sR' - \mathbf{Mod} : f_*$$

is such that $\mathbb{L}f^* \circ f_* \simeq Id$. Let $sR - \mathbf{Alg}$ and $sR' - \mathbf{Alg}$ be the categories of simplicial commutative R-algebras and simplicial commutative R'-algebras, endowed with their natural model structures (equivalences are and fibration are detected in the category of simplicial modules). Then, the Quillen adjunction

$$f_* : sR' - \mathbf{Alg} \longrightarrow sR - \mathbf{Alg} \qquad sR' - \mathbf{Alg} \longleftarrow sR - \mathbf{Alg} : f^*$$

also satisfies $\mathbb{L}f^* \circ f_* \simeq Id$, as this is true on the level on simplicial modules. In particular, for any simplicial R'-module M, the space of derived derivations

$$\mathbb{L}Der_R(R', M) := \mathbb{R}\underline{Hom}_{sR-\mathbf{Alg}/R'}(R', R' \oplus M) \simeq \mathbb{R}\underline{Hom}_{sR'-\mathbf{Alg}/R'}(R', R' \oplus M) \simeq *$$

is acyclic (here $R' \oplus M$ is the simplicial R'-algebra which is the trivial extension of R' by M). As a consequence one sees that Quillen's cotangent complex $\mathbb{L}_{R'/R}$ is acyclic, which implies that the morphism $R \longrightarrow R'$ is an étale morphism of rings.

Finally, using the fact that the functor on the category of modules $R' - \textbf{Mod} \longrightarrow R - \textbf{Mod}$ is fully faithful, one sees that $Spec\, R' \longrightarrow Spec\, R$ is a monomorphism of schemes. Therefore, the morphism of schemes $Spec\, R' \longrightarrow Spec\, R$ is an étale monomorphism, and so is an open immersion by [EGA-IV, Thm. 17.9.1].

Finally, point (2) is clear if one knows (1) and that $\textbf{Ho}(HR - \textbf{Mod}) \simeq D(R)$. $\qquad\qquad\qquad$ □

Remark 2.1.5 The argument at the beginning of the proof of Lemma 2.1.4 shows that if $f : A \to B$ is a Zariski open immersion, the functor $L(f) := \mathbb{R}f_*\mathbb{L}f^*$ is a *localization functor* on the homotopy category of A-modules in the sense of [HPS, Def. 3.1.1]. And it is also clear by definition that $L(f)$ is also *smashing* ([HPS, Def. 3.3.2]). Let us call a localization functor L on $\textbf{Ho}(A - Mod)$ a *formal Zariski localization functor over A* if $L \simeq L(f)$ for some formal Zariski open immersion f. Let us also say that a localization functor L on $\textbf{Ho}(A - Mod)$ is a *smashing algebra Bousfield localization over A* if $L \simeq L_B$ for some A-algebra B such that L_B is smashing (over A). Then it is easy to verify that in the set of equivalence classes of localization functors on $\textbf{Ho}(A - Mod)$, the subset consisting of formal Zariski localization functors over A coincides with the subset consisting of smashing algebra Bousfield localizations over A. In fact, if $f : A \to B$ is a Zariski open immersion, L_B denotes the Bousfield localization with respect to the A-module B, and $\ell_{B/A} : A \to L_BA$ the corresponding morphism of commutative A-algebras, we have $L(f) \simeq L_B \simeq L(\ell_{B/A})$ because all three localizations have the same category of acyclics. Conversely, if L_C is a smashing algebra Bousfield localization over A, and $\ell_{C/A} : A \to L_CA$ is the corresponding morphism of commutative A-algebras, one has $L_B \simeq L(\ell_{C/A})$.

Let $\textbf{Aff}$ be the opposite category of commutative rings, and $(\textbf{Aff}, \mathrm{Zar})$ the big Zariski site. The site $(\textbf{Aff}, \mathrm{Zar})$ can also be considered as a model site (for the trivial model structure on $\textbf{Aff}$). Lemma 2.1.4 implies in particular that the functor $H : \textbf{Aff} \longrightarrow \textbf{S} - \textbf{Aff}$ induces a continuous morphism of model sites ([To-Ve 1, Def. 4.8.4]). In this way, the site $(\textbf{Aff}, \mathrm{Zar})$ becomes a *sub-model site* of $(\textbf{S} - \textbf{Aff}, \mathrm{Zar})$.

To finish with the Zariski topology we will now describe a general procedure in order to construct interesting open Zariski immersions of commutative S-algebras using the techniques of Bousfield localization for model categories.

Let A be a commutative S-algebra and M be a A-module. We will assume that M is a perfect A-module (in the sense of Definition 2.0.1), or equivalently that it is a strongly dualizable object in the monoidal category $\textbf{Ho}(A - \textbf{Mod})$. As already noticed, perfect A-modules are exactly the retracts of finite cell A-modules, see [EKMM, Thm. III-7.9]). Let $M[n] = S^n \otimes^{\mathbb{L}} M$ be the n-th suspension A-module of M, for $n \in \mathbb{Z}$.

We denote by $D(M[n])$ the derived dual of $M[n]$, defined as the derived internal Hom's of A-modules

$$D(M[n]) := \mathbb{R}\mathcal{HOM}_{A - \textbf{Mod}}(M[n], A).$$

Consider now the (derived) free commutative A-algebra over $D(M[n])$, $\mathbb{L}F_A(D(M[n]))$, characterized by the usual adjunction

$$[\mathbb{L}F_A(D(M[n])), -]_{A - \textbf{Alg}} \simeq [D(M[n]), -]_{A - \textbf{Mod}}.$$

The model category $A - \textbf{Alg}$ is a combinatorial and cellular model category, and therefore one can apply the localization techniques (see e.g. [Hi, Sm]) in order to invert the natural augmentations

$F_A(D(M[n])) \longrightarrow A$ for all $n \in \mathbb{Z}$. One checks easily that, since M is strongly dualizable, the local objects for this localization are the commutative A-algebras B such that $M \wedge_A^{\mathbb{L}} B \simeq 0$ in $\mathbf{Ho}(B - \mathbf{Mod})$. The local model of A for this localization will be denoted by A_M. By definition, it is characterized by the following universal property: for any commutative A-algebra B, the mapping space $\mathbb{R}\underline{Hom}_{A-\mathbf{Alg}}(A_M, B)$ is contractible if $B \wedge_A^{\mathbb{L}} M \simeq 0$ and empty otherwise. In other words, for any commutative S-algebra B the natural morphism

$$\mathbb{R}\underline{Hom}_{\mathbf{S}-\mathbf{Alg}}(A_M, B) \longrightarrow \mathbb{R}\underline{Hom}_{\mathbf{S}-\mathbf{Alg}}(A, B)$$

is equivalent to an inclusion of connected components and its image consists of morphisms $A \longrightarrow B$ in $\mathbf{Ho}(\mathbf{S} - \mathbf{Alg})$ such that $B \wedge_A^{\mathbb{L}} M \simeq 0$.

Lemma 2.1.6 *With the above notations, the morphism $A \longrightarrow A_M$ is a Zariski open immersion.*

Proof: Let us start by showing that A_M is a finitely presented commutative A-algebra.

Let $\{B_i\}_{i \in I}$ be a filtered system of commutative A-algebras and $B = \mathrm{colim}_i B_i$. We assume that $B \wedge_A^{\mathbb{L}} M \simeq 0$, and we need to prove that there exists an $i \in I$ such that $B_i \wedge_A^{\mathbb{L}} M \simeq 0$.

By assumption, the two points Id and 0 are the same in $\pi_0(\mathbb{R}\underline{End}_{B-\mathbf{Mod}}(M \wedge_A^{\mathbb{L}} B))$. But, as M is a perfect A-module one has

$$\pi_0(\mathbb{R}\underline{End}_{B-\mathbf{Mod}}(M \wedge_A^{\mathbb{L}} B)) \simeq \mathrm{colim}_{i \in I} \pi_0(\mathbb{R}\underline{End}_{B_i-\mathbf{Mod}}(M \wedge_A^{\mathbb{L}} B_i)).$$

This implies that there is some index $i \in I$ such that Id and 0 are homotopic in $\mathbb{R}\underline{End}_{B_i-\mathbf{Mod}}(M \wedge_A^{\mathbb{L}} B_i)$, and therefore that $M \wedge_A^{\mathbb{L}} B_i$ is contractible.

It remains to prove that the induced functor $\mathbf{Ho}(A_M - \mathbf{Mod}) \longrightarrow \mathbf{Ho}(A - \mathbf{Mod})$ is fully faithful. For this, one uses that for any commutative S-algebra B, the morphism

$$\mathbb{R}\underline{Hom}_{\mathbf{S}-\mathbf{Alg}}(A_M, B) \longrightarrow \mathbb{R}\underline{Hom}_{\mathbf{S}-\mathbf{Alg}}(A, B)$$

is an inclusion of connected components. Therefore, the natural morphism

$$\mathbb{R}\underline{Hom}_{\mathbf{S}-\mathbf{Alg}}(A_M, B) \longrightarrow \mathbb{R}\underline{Hom}_{\mathbf{S}-\mathbf{Alg}}(A_M, B) \times^h_{\mathbb{R}\underline{Hom}_{\mathbf{S}-\mathbf{Alg}}(A, B)} \mathbb{R}\underline{Hom}_{\mathbf{S}-\mathbf{Alg}}(A_M, B)$$

is an isomorphism in $Ho(SSet)$. This implies that the natural morphism

$$A_M \longrightarrow A_M \wedge_A^{\mathbb{L}} A_M$$

is an equivalence of commutative S-algebras. In particular, one has for any A_M-module M

$$M \simeq M \wedge_{A_M}^{\mathbb{L}} A_M \simeq M \wedge_{A_M}^{\mathbb{L}} (A_M \wedge_A^{\mathbb{L}} A_M) \simeq M \wedge_A^{\mathbb{L}} A_M,$$

showing that the base change functor

$$\mathbf{Ho}(A_M - \mathbf{Mod}) \longrightarrow \mathbf{Ho}(A - \mathbf{Mod})$$

is fully faithful. $\qquad \square$

An important property of the localization $A \longrightarrow A_M$ is the following fact.

Lemma 2.1.7 *Let A be a commutative S-algebra, and M be a perfect A-module. Then the essential image of the fully faithful functor*

$$\mathbf{Ho}(A_M - \mathbf{Mod}) \longrightarrow \mathbf{Ho}(A - \mathbf{Mod})$$

consists of all A-modules N such that $M \wedge_A^{\mathbb{L}} N \simeq D(M) \wedge_A^{\mathbb{L}} N \simeq 0$.

Note that since M is perfect, then for any A-module N, $M \wedge_A^{\mathbb{L}} N \simeq 0$ iff $D(M) \wedge_A^{\mathbb{L}} N \simeq 0$, so the two conditions in the lemma are actually one. Moreover, $A_M \simeq A_{D(M)}$ in $\mathbf{Ho}(A - \mathbf{Alg})$.

Proof: As every A_M-module N can be constructed by homotopy colimits of free A_M-modules and $- \wedge_A^{\mathbb{L}} M$ commutes with homotopy colimits, it is clear that $A_M \wedge_A^{\mathbb{L}} M \simeq 0$ implies $N \wedge_A^{\mathbb{L}} M \simeq 0$. Since $A_M \wedge_A^{\mathbb{L}} D(M) \simeq D(A_M \wedge_A^{\mathbb{L}} M) \simeq 0$ (here the second derived dual is in the category of A_M-modules), the same argument shows that $N \wedge_A^{\mathbb{L}} D(M) \simeq 0$.

Conversely, let N be an A-module such that $N \wedge_A^{\mathbb{L}} M \simeq N \wedge_A^{\mathbb{L}} D(M) \simeq 0$. By definition, the commutative A-algebra $A \longrightarrow A_M$ is obtained as a local model of $A \to A$ when one inverts the set of morphisms $\mathbb{L}F_A(D(M[n])) \longrightarrow A$, for any $n \in \mathbb{Z}$. It is well known (see e.g. [Hi, §4]) that such a local model can be obtained by a transfinite composition of homotopy push-outs of the form

$$
\begin{array}{ccc}
A_\alpha & \longrightarrow & A_{\alpha+1} \\
\uparrow & & \uparrow \\
\partial\Delta^p \otimes^{\mathbb{L}} \mathbb{L}F_A(D(M[n])) & \longrightarrow & \Delta^p \otimes^{\mathbb{L}} \mathbb{L}F_A(D(M[n]))
\end{array}
$$

in the category of A-algebras. ¿From this description, and the fact that $- \wedge_A^{\mathbb{L}} M$ commutes with homotopy colimits, one sees that the adjunction morphism $N \longrightarrow N \wedge_A^{\mathbb{L}} A_M$ is an equivalence because by assumption on N, the natural morphism $N \simeq N \wedge_A^{\mathbb{L}} A \longrightarrow N \wedge_A^{\mathbb{L}} \mathbb{L}F_A(D(M[n]))$ is an equivalence. $\square$

Lemma 2.1.7 allows us to interpret geometrically A_M as the *open complement* of the *support* of the A-module M. Lemma 2.1.7 also has a converse whose proof is left as an exercise.

Lemma 2.1.8 *Let $f : A \longrightarrow B$ be a morphism of commutative S-algebras and M be a perfect A-module. We suppose that the functor $\mathbb{R}f_* : \mathbf{Ho}(B - \mathbf{Mod}) \longrightarrow \mathbf{Ho}(A - \mathbf{Mod})$ is fully faithful and that its essential image consists of all A-modules N such that $N \wedge_A^{\mathbb{L}} M \simeq 0$. Then, the two commutative A-algebras B and A_M are equivalent (i.e. isomorphic in $\mathbf{Ho}(A - \mathbf{Alg})$).*

Remark 2.1.9 One should note carefully that even though if the Eilenberg-Mac Lane functor H embeds $(\mathbf{Aff}, \mathrm{Zar})$ in $(\mathbf{S} - \mathbf{Aff}, \mathrm{Zar})$ as model sites, there exist commutative rings R and Zariski open coverings $HR \longrightarrow B$ in $\mathbf{S} - \mathbf{Aff}$ such that B is not of the form HR' for some commutative R-algebra R'. One example is given by taking R to be $\mathbb{C}[X,Y]$, and considering the localized commutative HR-algebra $(HR)_M$ (in the sense above), where M is the perfect R-module $R/(X,Y) \simeq \mathbb{C}$. If $(HR)_M$ were of the form HR' for a Zariski open immersion $Spec\,R' \longrightarrow Spec\,R$, then for any other commutative ring R'', the set of scheme morphisms $Hom(Spec\,R'', Spec\,R')$ would be the subset of $Hom(Spec\,R'', \mathbb{A}^2)$ consisting of morphisms factoring through $\mathbb{A}^2 - \{0\}$. This would mean that $Spec\,R' \simeq \mathbb{A}^2 - \{0\}$, which is not possible as $\mathbb{A}^2 - \{0\}$ is a not an affine scheme. This example is of course the same as the example given in [To, §2.2] of a 0-truncated affine stack which is not an affine scheme. These kind of example shows that there are many more affine objects in homotopical algebraic geometry than in usual algebraic geometry.

Remark 2.1.10

1. Note that Lemma 2.1.7 shows that the localization process $(A, M) \rightsquigarrow A_M$ is in some sense "orthogonal" to the usual Bousfield localization process $(A, M) \rightsquigarrow L_M A$ in that the local objects for the former are exactly the acyclic objects for the latter. To state everything in terms of Bousfield localizations, this says that L_{A_M}-local objects are exactly L_M-acyclic objects (compare with Remark 2.1.5). Note that however, while the Bousfield localization is always defined for any A-module M, the commutative A-algebra A_M probably does not exist unless M is perfect.

2. Let S_p be the p-local sphere. If $f : S_p \to B$ is any formal Zariski open immersion then $L :=$ $\mathbb{R}f_*\mathbb{L}f^*$ is clearly a smashing localization functor in the sense of [HPS, §3]. Its category C of perfect[1] acyclics (i.e. perfect objects X in $\mathbf{Ho}(S_p - \mathbf{Mod})$ such that LX is null) is then a localizing thick subcategory of the homotopy category $\mathbf{Ho}(S_p - \mathbf{Mod}^{\mathrm{perf}})$ of the category of perfect S_p-modules, and therefore by [H-S] it is equivalent to the category C_n of perfect $E(n)$-acyclics, for some $0 \leq n < \infty$, where $E(n)$ is the n-th Johnson-Wilson S_p-module (see e.g. [Rav]); in other words L and $L_n := L_{E(n)}$ are both smashing localization functors on $\mathbf{Ho}(S_p - \mathbf{Mod})$ having the same subcategory of finite acyclics. Therefore, if we assume (one of the form of) the Telescope conjecture (see [Mil]), we get that L_n and L have equivalent categories of acyclics and so have equivalent categories of local objects. But the category of local objects for L is equivalent to the category $\mathbf{Ho}(B - \mathbf{Mod})$ (since $\mathbb{R}f_*$ is fully faithful by hypothesis) and the category of local objects for L_n is equivalent to the category $\mathbf{Ho}((L_nS_p) - \mathbf{Mod})$, by [Wo] since L_n is smashing. This easily implies that the two commutative S_p-algebras B and L_nS_p are equivalent (i.e. isomorphic in $\mathbf{Ho}(S_p - \mathbf{Alg})$).
In conclusion, one sees that if the Telescope conjecture is true, then, up to equivalence of S_p-algebras, the only (non-trivial) formal Zariski open immersions for S_p are given by the family

$$\mathcal{U} := \{S_p \to L_nS_p\}_{0 \leq n < \infty}.$$

This example shows that the formal Zariski topology might be better suited in certain contexts than the Zariski topology itself (e.g. it is not clear that there exists any non-trivial Zariski open immersion of S_p, i.e. that the morphisms of commutative S-algebras $S_p \longrightarrow L_nS_p$ are of finite presentation). Note however that the family $\mathcal{U}$ is not a formal Zariski *covering* according to Definition 2.1.1 because the family of base-change functors

$$\left\{(-) \wedge^{\mathbb{L}}_{S_p} L_nS_p : \mathbf{Ho}(S_p - Mod) \longrightarrow \mathbf{Ho}(L_nS_p - Mod)\right\}_{0 \leq n < \infty}$$

is not conservative; in fact, as Neil Strickland pointed out to us, the Brown-Comenetz dual I of S_p is a non-perfect non-trivial S_p-module which is nonetheless L_n-acyclic for any n. However, it is true that the family of base-changes above is conservative when restricted to the (homotopy) categories of perfect modules. Therefore, one could modify the second covering condition in Definition 2.1.1, by only requiring the property of being conservative on the subcategories of perfect modules and relaxing the finiteness of J; let us call this modified covering condition *formal Zariski covering-on-finites* condition. Then, $\mathcal{U}$ is a formal Zariski covering-on-finites family and indeed the unique one, up to equivalences of S_p-algebras, if the Telescope conjecture holds.

3. The previous example also shows that the commutative S-algebras L_nS_p are *local for the formal Zariski topology* (again assuming the Telescope conjecture). Indeed, for any formal Zariski open covering $\{L_nS_p \longrightarrow B_i\}_{i \in I}$ there is an i such that $L_nS_p \longrightarrow B_i$ is an equivalence of commutative S-algebras.

2.2 The brave new étale topology

Notions of étale morphisms of commutative S-algebras has been studied by several authors ([Ro1, MC-Min]). In this paragraph we present the definition that appeared in [To-Ve 1] and was used there in order to define the étale K-theory of commutative S-algebras.

---

[1]The word *finite* instead of perfect would be more customary in this setting.

We refer to [Ba] for the notions of topological cotangent spectrum and of topological André-Quillen cohomology relative to a morphism $A \to B$ of commutative S-algebras, except for slightly different notations. We denote by $\mathbb{L}\Omega_{B/A} \in \mathbf{Ho}(B - \mathbf{Mod})$ the topological cotangent spectrum (denoted as $\Omega_{B/A}$ in [Ba]) and, for any B-module M, by

$$\mathbb{L}Der_A(B, M) := \mathbb{R}\underline{Hom}_{A-\mathbf{Alg}/B}(B, B \vee M)$$

the derived space of topological derivations from B to M ($B \vee M$ being the trivial extension of B by M). Note that there is an isomorphism $\mathbb{L}Der_A(B, M) \simeq \mathbb{R}\underline{Hom}_{B-\mathbf{Mod}}(\mathbb{L}\Omega_{B/A}, M)$, natural in M.

Definition 2.2.1   • *Let $f : A \longrightarrow B$ be a morphism of commutative S-algebras.*

- *The morphism f is called* formally étale *if $\mathbb{L}\Omega_{B/A} \simeq 0$.*
- *The morphism f is called* étale *if it is formally étale and of finite presentation (as a morphism of commutative S-algebras).*

• *A family of morphisms $\{f_i : A \longrightarrow A_i\}_{i \in I}$ in $\mathbf{S} - \mathbf{Alg}$ is called a* (formal) étale covering *if it satisfies the following two conditions.*

- *Each morphism $A \longrightarrow A_i$ is (formally) étale.*
- *There exists a finite subset $J \subset I$ such that the family of inverse image functors*

$$\{\mathbb{L}f_j^* : \mathbf{Ho}(A - \mathbf{Mod}) \longrightarrow \mathbf{Ho}(\mathbf{A_j} - \mathbf{Mod})\}_{j \in J}$$

is conservative (i.e. a morphism in $\mathbf{Ho}(A - \mathbf{Mod})$ is an isomorphism if and only if its images by all the $\mathbb{L}f_j^*$'s are isomorphisms).

As shown in [To-Ve 1, §5.2], (formal) étale covering families are stable by equivalences, compositions and homotopy push-outs, and therefore define a model topology on the model category $\mathbf{S} - \mathbf{Aff}$. Therefore one gets two model topologies called the *brave new étale topology* and the *brave new formal étale topology*. The corresponding model sites will be denoted by $(\mathbf{S} - \mathbf{Aff}, \text{ét})$ and $(\mathbf{S} - \mathbf{Aff}, \text{fét})$, and will be called the *brave new étale site* and the *brave new formal étale site*.

As for the brave new Zariski topology one proves that the brave new étale topology is a generalization of the usual étale topology.

Lemma 2.2.2   *1. Let $R \longrightarrow R'$ be a morphism of commutative rings. The induced morphism $HR \longrightarrow HR'$ is an étale morphism of commutative S-algebras (in the sense of Definition 2.2.1) if and only if the morphism $Spec\,R' \longrightarrow Spec\,R$ is an étale morphism of schemes.*

2. A family of morphisms of commutative rings, $\{R \longrightarrow R_i'\}_{i \in I}$, induces an étale covering family of commutative S-algebras $\{HR \longrightarrow HR_i'\}_{i \in I}$ (in the sense of Definition 2.2.1) if and only if the family $\{Spec\,R_i \longrightarrow Spec\,R\}_{i \in I}$ is an étale covering of schemes.

Proof: This is proved in [To-Ve 1, §5.2].   □

Let $\mathbf{Aff}$ be the opposite category of commutative rings, and $(\mathbf{Aff}, \text{ét})$ the big étale site. The site $(\mathbf{Aff}, \text{ét})$ can also be considered as a model site (for the trivial model structure on $\mathbf{Aff}$). Lemma 2.2.2 shows in particular that the Eilenberg-Mac Lane functor $H : \mathbf{Aff} \longrightarrow \mathbf{S} - \mathbf{Aff}$ induces a continuous morphism of model sites ([To-Ve 1, §4.8]). In this way, the site $(\mathbf{Aff}, \text{ét})$ becomes a *sub-model site* of $(\mathbf{S} - \mathbf{Aff}, \text{ét})$.

Another important fact is that the brave new étale topology is finer than the brave new Zariski topology.

Lemma 2.2.3   *1. Any formal Zariski open immersion of commutative S-algebras is a formally étale morphism.*

   2. Any Zariski open immersion of commutative S-algebras is an étale morphism.

   3. Any (formal) Zariski open covering of a commutative S-algebra is a (formal) étale covering.

   Proof: Only (1) requires a proof, and the proof will be similar to the one of Lemma 2.1.4 (2). Let $f : A \longrightarrow B$ be a formal Zariski open immersion of commutative S-algebras. As the functor $\mathbb{R}f_* : \mathbf{Ho}(B - \mathbf{Mod}) \longrightarrow \mathbf{Ho}(A - \mathbf{Mod})$ is a full embedding so is the induced functor $\mathbb{R}f_* : \mathbf{Ho}(B - \mathbf{Alg}) \longrightarrow \mathbf{Ho}(A - \mathbf{Alg})$. By definition of topological derivations one has for any B-module M, $\mathbb{L}Der_A(B, M) = \mathbb{R}\underline{Hom}_{A-\mathbf{Alg}/B}(B, B \vee M)$. This and the fact that $\mathbb{R}f_*$ is fully faithful imply that

$$\mathbb{L}Der_A(B, M) = \mathbb{R}\underline{Hom}_{A-\mathbf{Alg}/B}(B, B \vee M) \simeq \mathbb{R}\underline{Hom}_{B-\mathbf{Alg}/B}(B, B \vee M) \simeq *,$$

and therefore that $\mathbb{L}\Omega_{B/A} \simeq 0$. $\square$

Lemma 2.2.3 implies that the identity functor of $\mathbf{S} - \mathbf{Aff}$ defines a continuous morphism between model sites

$$(\mathbf{S} - \mathbf{Aff}, \mathrm{Zar}) \longrightarrow (\mathbf{S} - \mathbf{Aff}, \text{ét}),$$

which is a base change functor from the brave new Zariski site to the brave new étale site. The same is true for the formal versions of these sites.

To finish this part, we would like to mention a stronger version of the brave new étale topology, called the *thh-étale topology*, which is sometimes more convenient to deal with.

Definition 2.2.4   • *Let $f : A \longrightarrow B$ be a morphism of commutative S-algebras.*

   – *The morphism f is called* formally *thh-étale if for any commutative A-algebra C the mapping space $\mathbb{R}\underline{Hom}_{A-\mathbf{Alg}}(B, C)$ is 0-truncated (i.e. equivalent to a discrete space).*

   – *The morphism f is called thh-étale if it is formally thh-étale and of finite presentation (as a morphism of commutative S-algebras).*

   • *A family of morphisms $\{f_i : A \longrightarrow A_i\}_{i \in I}$ in $\mathbf{S} - \mathbf{Alg}$ is called a (formal) thh-étale covering if it satisfies the following two conditions.*

   – *Each morphism $A \longrightarrow A_i$ is (formally) thh-étale.*

   – *There exists a finite subset $J \subset I$ such that the family of inverse image functors*

$$\{\mathbb{L}f_j^* : \mathbf{Ho}(A - \mathbf{Mod}) \longrightarrow \mathbf{Ho}(\mathbf{A_j} - \mathbf{Mod})\}_{j \in J}$$

   is conservative (i.e. a morphism in $\mathbf{Ho}(A - \mathbf{Mod})$ is an isomorphism if and only if its images by all $\mathbb{L}f_j^*$ are isomorphisms).

It is easy to check that (formal) *thh*-étale coverings define a model topology on the model category $\mathbf{S} - \mathbf{Aff}$, call the *(formal) thh-étale topology*. The model category $\mathbf{S} - \mathbf{Aff}$ together with these topologies will be called the *brave new thh-étale site* and the *brave new formal thh-étale site*, denoted by $(\mathbf{S} - \mathbf{Aff}, \text{thh-ét})$ and $(\mathbf{S} - \mathbf{Aff}, \text{fthh-ét})$, respectively. An equivalent way of stating the formal *thh*-étaleness condition for $A \to B$ is to say that the natural map $B \to S^1 \otimes^{\mathbb{L}} B$ in $\mathbf{Ho}(A - \mathbf{Alg})$ is an isomorphism, or equivalently (by [MSV]), that the canonical map $B \to \mathrm{THH}(B/A, B)$ is an isomorphism in $\mathbf{Ho}(A - \mathbf{Alg})$, where THH denotes the topological Hochschild cohomology spectrum

(see e.g. [EKMM, §IX]). This equivalent characterization follows from the observation that, for any morphism $f : B \to C$ of commutative A-algebras, one has an isomorphism (in $\mathbf{Ho(SSet)}$) of mapping spaces:

$$\mathrm{Map}_{B-\mathbf{Alg}}(\mathrm{THH}(B/A, B), C) \simeq \Omega_f \mathrm{Map}_{A-\mathbf{Alg}}(B, C),$$

where Ω_f denotes the loop space at f. Therefore, the canonical map $B \to \mathrm{THH}(B/A, B)$ is an isomorphism in $\mathbf{Ho}(A - \mathbf{Alg})$ iff for any such f, $\Omega_f \mathrm{Map}_{A-\mathbf{Alg}}(B, C)$ is contractible, i.e. iff the simplicial set $\mathrm{Map}_{A-\mathbf{Alg}}(B, C) \simeq \mathbb{R}\underline{Hom}_{A-\mathbf{Alg}}(B, C)$ is 0-truncated.

This explains the name of this topology and, since as observed in [MC-Min] the Goodwillie derivative of THH is the suspension of the topological André-Quillen spectrum TAQ (where, for any B-module M, $\mathrm{TAQ}(B/A; M)$ is defined as the derived internal Hom from $\mathbb{L}\Omega_{B/A}$ to M in the model category of B-modules) also shows that (formal) thh-étale morphisms are (formal) étale morphisms. Therefore the identity functor induces continuous morphisms of model sites

$$(\mathbf{S} - \mathbf{Aff}, \text{thh-ét}) \longrightarrow (\mathbf{S} - \mathbf{Aff}, \text{ét}) \qquad (\mathbf{S} - \mathbf{Aff}, \text{fthh-ét}) \longrightarrow (\mathbf{S} - \mathbf{Aff}, \text{fét}).$$

We refer to [MC-Min] for more details on the notion of thh-étale morphisms.

2.3 Standard topologies

Standard model topologies on $\mathbf{S} - \mathbf{Aff}$ are obvious extensions of usual Grothendieck topologies on affine schemes. They are defined in the following way.

Let τ be one of the usual Grothendieck topologies on affine schemes (i.e. Zariski, Nisnevich, étale or faithfully flat).

Definition 2.3.1 *A family of morphisms of commutative S-algebras $\{A \longrightarrow B_i\}_{i \in I}$ is a* standard τ-covering *(also called* strong τ-covering*) if it satisfies the following two conditions.*

- *The induced family of morphisms of schemes $\{Spec\, \pi_0(B_i) \longrightarrow Spec\, \pi_0(A)\}_{i \in I}$ is a τ-covering of affine schemes.*

- *For any $i \in I$ the natural morphism of $\pi_0(B_i)$-modules*

$$\pi_*(A) \otimes_{\pi_0(A)} \pi_0(B_i) \longrightarrow \pi_*(B_i)$$

is an isomorphism.

Its easy to check that this defines a model topology τ^s on $\mathbf{S} - \mathbf{Aff}$, called the *standard τ-topology*. The model site $(\mathbf{S} - \mathbf{Aff}, \tau^s)$ may be called the *brave new standard-τ site*. The importance of standard topologies is that all τ^s-coverings of commutative S-algebras of the form HR comes from usual τ-coverings of the scheme $Spec\, R$. Its behavior is therefore very close to the geometric intuition one gets in Algebraic Geometry.

Finally, let us also mention the *semi-standard* (or *semi-strong*) model topologies. A family of morphisms of commutative S-algebras $\{A \longrightarrow B_i\}_{i \in I}$ is a *semi-standard τ-covering* (also called *semi-strong τ-covering*) if the induced family of morphism of commutative graded rings $\{\pi_*(A) \longrightarrow \pi_*(B_i)\}_{i \in I}$ is a τ-covering. This also defines a model topology τ^{ss} on $\mathbf{S} - \mathbf{Aff}$.

Both the standard and semi-standard type model sites (and $\mathbf{S}$-stacks over them, see Section 3) could be of some interest in the study of geometry over even, periodic S-algebras (e.g. for elliptic spectra as in [AHS]).

3   S-stacks and geometric S-stacks

Let (M, τ) be a *model site* (i.e. a model category M endowed with a model topology τ in the sense of [To-Ve 1]). Associated to it one has a model category of *prestacks* $M^\wedge$ and of *stacks* $M^{\sim, \tau}$. For details concerning these model categories we refer to [To-Ve 1, §4], and for the sake of brevity we only recall the following facts.

- The model category $M^\wedge$ is a left Bousfield localization of the model category $SSet^{M^{op}}$, of simplicial presheaves on M together with the projective levelwise model structure. The local objects for this Bousfield localization are precisely the simplicial presheaves $F : M^{op} \longrightarrow SSet$ which are equivalences preserving.

- The model category $M^{\sim, \tau}$ is a left Bousfield localization ([Hi, §3]) of the model category of prestacks $M^\wedge$, and the localization (left Quillen) functor from $M^\wedge$ to $M^{\sim, \tau}$ preserves (up to equivalences) finite homotopy limits (i.e. homotopy pull-backs). The local objects for this Bousfield localization are the simplicial presheaves $F : M^{op} \longrightarrow SSet$ which satisfy the following two conditions.

  - The functor F preserves equivalences (i.e. is a local object in $M^\wedge$).
  - For any τ-hypercover $U_* \longrightarrow X$ in the model site (M, τ) ([To-Ve 1, §4.4]), the induced morphism
  $$F(X) \longrightarrow F(U_*)$$
  is an equivalence.

  There is an *associated stack* functor $a : \mathbf{Ho}(M^\wedge) \to \mathbf{Ho}(M^{\sim, \tau})$ right adjoint to the inclusion $\mathbf{Ho}(M^{\sim, \tau}) \hookrightarrow \mathbf{Ho}(M^\wedge)$.

- There is a homotopical variant [2]
$$\mathbb{R}\underline{h} : \mathbf{Ho}(M) \hookrightarrow \mathbf{Ho}(M^\wedge)$$
of the *Yoneda embedding* ([To-Ve 1, §4.2]).

Specializing to our present situation, where $M = \mathbf{S} - \mathbf{Aff}$, we have one model category $\mathbf{S} - \mathbf{Aff}^\wedge$ of prestacks and zounds of model categories stacks

$$\mathbf{S} - \mathbf{Aff}^{\sim, \mathrm{Zar}}, \qquad \mathbf{S} - \mathbf{Aff}^{\sim, \text{ét}}, \qquad \mathbf{S} - \mathbf{Aff}^{\sim, \text{thh-ét}},$$
$$\mathbf{S} - \mathbf{Aff}^{\sim, \mathrm{fZar}}, \qquad \mathbf{S} - \mathbf{Aff}^{\sim, \text{fét}}, \qquad \mathbf{S} - \mathbf{Aff}^{\sim, \text{fthh-ét}},$$
$$\mathbf{S} - \mathbf{Aff}^{\sim, \mathrm{Zar}^s}, \qquad \mathbf{S} - \mathbf{Aff}^{\sim, \text{ét}^s}, \qquad \mathbf{S} - \mathbf{Aff}^{\sim, \text{ffqc}^s},$$
$$\dots etc \dots$$

These model categories come with right Quillen functors (the morphism of change of sites)

$$\mathbf{S} - \mathbf{Aff}^{\sim, \text{ét}} \longrightarrow \mathbf{S} - \mathbf{Aff}^{\sim, \text{thh-ét}} \longrightarrow \mathbf{S} - \mathbf{Aff}^{\sim, \mathrm{Zar}} \longrightarrow \mathbf{S} - \mathbf{Aff}^\wedge$$

$$\mathbf{S} - \mathbf{Aff}^{\sim, \text{fét}} \longrightarrow \mathbf{S} - \mathbf{Aff}^{\sim, \text{fthh-ét}} \longrightarrow \mathbf{S} - \mathbf{Aff}^{\sim, \mathrm{fZar}} \longrightarrow \mathbf{S} - \mathbf{Aff}^\wedge$$

$$\mathbf{S} - \mathbf{Aff}^{\sim, \text{ét}} \longrightarrow \mathbf{S} - \mathbf{Aff}^{\sim, \text{fét}} \longrightarrow \mathbf{S} - \mathbf{Aff}^{\sim, \mathrm{fZar}} \longrightarrow \mathbf{S} - \mathbf{Aff}^\wedge$$

$$\mathbf{S} - \mathbf{Aff}^{\sim, \text{ét}^s} \longrightarrow \mathbf{S} - \mathbf{Aff}^{\sim, \text{ét}} \longrightarrow \mathbf{S} - \mathbf{Aff}^{\sim, \mathrm{Zar}} \longrightarrow \mathbf{S} - \mathbf{Aff}^\wedge$$
$$\dots etc \dots$$

which allow to compare the various topologies on $\mathbf{S} - \mathbf{Aff}$.

---

[2]If $x \in M$, $\mathbb{R}\underline{h}(x)$ essentially sends $y \in M$ to the mapping space $\mathrm{Map}_M(y, x)$.

Definition 3.0.2 *Let τ be a model topology on the model site* $\mathbf{S} - \mathbf{Aff}$.

- *The model category of* $\mathbf{S}$-stacks for the topology τ is $\mathbf{S} - \mathbf{Aff}^{\sim,\tau}$.

- *A simplicial presheaf* $F \in SPr(\mathbf{S} - \mathbf{Aff})$ *is called an S-stack if it is a local object in* $\mathbf{S} - \mathbf{Aff}^{\sim,\tau}$ *(i.e. preserves equivalences and satisfies the descent property for τ-hypercovers).*

- *Objects in the homotopy category* $\mathbf{Ho}(\mathbf{S} - \mathbf{Aff}^{\sim,\tau})$ *will simply be called* $\mathbf{S}$-stacks *(without referring, unless it is necessary, to the underlying topology).*

The category of $\mathbf{S}$-stacks, being the homotopy category of a model category, has all kind of homotopy limits and colimits. Moreover, one can show that it has internal Hom's. Actually, the model category of $\mathbf{S}$-stacks is a *model topos* in the sense of [To-Ve 1, §3.8] (see also [To-Ve 4]), and therefore behaves very much in the same way as a category sheaves (but in a homotopical sense). In practice this is very useful as it allows to use a lot of usual properties of simplicial sets in the context of $\mathbf{S}$-stacks (in the same way as a lot of usual properties of sets are true in any topos).

The Eilenberg-Mac Lane functor H from commutative rings to commutative S-algebras induces left Quillen functors

$$H_! : \mathbf{Aff}^{\sim,\tau_0} \longrightarrow \mathbf{S} - \mathbf{Aff}^{\sim,\tau},$$

where τ_0 is one of the standard topologies on affine schemes (e.g. Zar, ét, ffqc,...), and τ is one of its possible extension to the model category $\mathbf{S} - \mathbf{Aff}$ (e.g. Zar can be extended to Zar^s or to Zar, etc.). Here, $\mathbf{Aff}^{\sim,\tau}$ is the usual model category of simplicial presheaves on the Grothendieck site $(\mathbf{Aff}, \tau)$ (with the projective model structure [Bl]). By deriving on the left one gets a functor

$$\mathbb{L}H_! : \mathbf{Ho}(\mathbf{Aff}^{\sim,\tau_0}) \longrightarrow \mathbf{Ho}(\mathbf{S} - \mathbf{Aff}^{\sim,\tau}).$$

Therefore, our category of $\mathbf{S}$-stacks receives a functor from the homotopy category of simplicial presheaves. In particular, sheaves on affine schemes (and in particular the category of schemes itself), and also 1-truncated simplicial presheaves (and in particular the homotopy category of algebraic stacks) can be all viewed as examples of $\mathbf{S}$-stacks. However, one should be careful that the functor $\mathbb{L}H_!$ has no reason to be fully faithful in general, though this is the case for all the standard extensions (but not semi-standard) described in §2.3 (the reason for this is that all covering families of some HR are in fact induced from covering families of affine schemes. In particular the restriction functor from $\mathbf{S} - \mathbf{Aff}^{\sim,\tau} \longrightarrow \mathbf{Aff}^{\sim,\tau_0}$ will preserve local equivalences.).

If instead of requiring descent with respect to all τ-hypercovers, we only require descent with respect to those τ-hypercovers which arise as homotopy nerves of τ-covers, we obtain the following weaker notion of stack.

Definition 3.0.3 *Let τ be a model topology on* $\mathbf{S} - \mathbf{Aff}$. *A simplicial presheaf* $F : \mathbf{S} - \mathbf{Aff}^{op} \to \mathbf{SSet}$ *is said to be a Čech $\mathbf{S}$-stack with respect to τ if it preserves equivalences and satisfies the following Čech descent condition. For any τ-cover $\mathcal{U} = \{U_i \to X\}$, denoting by $\check{C}(\mathcal{U})_*$ its homotopy nerve, the canonical map*

$$F(X) \longrightarrow \mathrm{holim}F(\check{C}(\mathcal{U})_*)$$

is an isomorphism in $\mathbf{Ho}(\mathbf{SSet})$.

This weaker notion of stacks is the one used recently by J. Lurie in [Lu] and has also appeared for stacks over Grothendieck sites in [DHI].

Note that, similarly to the case of **S**-stacks, there is a model category $\mathbf{S} - \mathbf{Aff}_{\check{C}}^{\sim,\tau}$ of Čech **S**-stacks, defined as the left Bousfield localization of $\mathbf{S} - \mathbf{Aff}^{\wedge}$ with respect to all the Čech-nerves, whose homotopy category is equivalent to the full subcategory of $\mathbf{Ho}(\mathbf{S} - \mathbf{Aff}^{\wedge})$ consisting of Čech **S**-stacks.

In general the inclusion of **S**-stacks into Čech **S**-stacks is proper; however, one can prove (by adapting [H-S, Prop. 6.1] to the context of model sites), that given any n-truncated (equivalence preserving) simplicial presheaf F on $\mathbf{S} - \mathbf{Aff}$, F is an **S**-stack iff it is a Čech **S**-stack (regardless of the model topology τ). The reader might wish to read Appendix A of [DHI] for more comparison results between stacks and Čech stacks over usual Grothendieck sites.

3.1 Some descent theory

With the notations above, one can compose the Yoneda embedding $\mathbb{R}\underline{h} : \mathbf{Ho}(\mathbf{S} - \mathbf{Alg})^{op} \to \mathbf{Ho}(\mathbf{S} - \mathbf{Aff}^{\wedge})$ with the associated stack functor $a : \mathbf{Ho}(\mathbf{S} - \mathbf{Aff}^{\wedge}) \to \mathbf{Ho}(\mathbf{S} - \mathbf{Aff}^{\sim,\tau})$ and obtain the derived $Spec$ functor

$$\mathbb{R}Spec : \mathbf{Ho}(\mathbf{S} - \mathbf{Alg})^{op} = \mathbf{Ho}(\mathbf{S} - \mathbf{Aff}) \longrightarrow \mathbf{Ho}(\mathbf{S} - \mathbf{Aff}^{\sim,\tau}),$$

for any model topology τ on $\mathbf{S} - \mathbf{Aff}$.

Definition 3.1.1 *The topology τ is* sub-canonical *(respectively,* Čech-subcanonical*) if for any* $A \in \mathbf{S} - \mathbf{Alg}$, $\mathbb{R}\underline{h}_A$ *is an* **S**-*stack (resp., a* Čech **S**-*stack).*

Note that τ is sub-canonical iff the functor $\mathbb{R}Spec$ is fully faithful, and that sub-canonical implies Čech-subcanonical.

Knowing whether a given model topology τ is sub-canonical or not is known as the *descent problem for τ*, and in our opinion is a crucial question. At present, we do not know if all the model topologies presented in the previous Section are sub-canonical, and it might be that some of them are not. The following lemma gives examples of sub-canonical topologies.

Lemma 3.1.2 *The (semi-)standard Zariski, Nisnevich, étale and flat model topologies of §2.3 are all sub-canonical.*

Sketch of proof: Let τ be one of these topologies, A be a commutative **S**-algebra, and $A \longrightarrow B_*$ be a τ-hypercover ([To-Ve 1, §4.4]). Using the fact that $\pi_*(B_n)$ is flat over $\pi_*(A)$ for any n, one can check that the cosimplicial $\pi_*(A)$-algebra $\pi_*(B_*)$ is again a τ-hypercover of commutative rings. By usual descent theory for affine schemes this implies that the cohomology groups of the total complex of $[n] \mapsto \pi_*(B_n)$ vanish except for $H^0(\pi_*(B_*)) \simeq \pi_*(A)$. This implies that the spectral sequence for the holim

$$H^p([n] \mapsto \pi_q(B_n)) \Rightarrow \pi_{p-q}(\mathrm{holim}B_*)$$

degenerates at E_2 and that $\pi_*(A) \longrightarrow \pi_*(\mathrm{holim}B_*)$ is an isomorphism. $\qquad\square$

Concerning the brave new Zariski topology one has the following partial result.

Lemma 3.1.3 *Let* $\{A \longrightarrow A_i\}_{i \in I}$ *be a finite Zariski covering family of commutative S-algebras. Let* $A \longrightarrow B = \vee_i A_i$ *be the coproduct morphism. Let* $A \longrightarrow B_*$ *be the cosimplicial commutative A-algebra defined by*

$$B_n := \underbrace{B \wedge_A^{\mathbb{L}} B \wedge_A^{\mathbb{L}} \cdots \wedge_A^{\mathbb{L}} B}_{(n+1)\ times}$$

(i.e. homotopy co-nerve of the morphism $A \longrightarrow B$*). Then the induced morphism*

$$A \longrightarrow \mathrm{holim}_{n \in \Delta} B_n$$

is an equivalence.

Sketch of proof: By definition of Zariski open immersion it is not hard to see that the cosimplicial commutative A-algebra B_* is m-coskeletal, where m is the cardinality of I. This means the following: let $i_m : \Delta_{\leq m} \longrightarrow \Delta$ be the inclusion functor form the full sub-category of objects $[i]$ with $i \leq m$. Then, one has an equivalence of commutative A-algebras $B_* \simeq \mathbb{R}(i_m)_* i_m^*(B_*)$ (here $(i_m^*, \mathbb{R}(i_m)_*)$ is the derived adjunction between Δ-diagrams and $\Delta_{\leq m}$-diagrams). From this one deduces easily that

$$\operatorname{holim}_{n \in \Delta} B_n \simeq \operatorname{holim}_{n \in \Delta_{\leq m}} B_n.$$

In particular, $\operatorname{holim}_{n \in \Delta} B_n$ is in fact a finite homotopy limit and therefore will commute with the base change from A to B, i.e.

$$(\operatorname{holim}_{n \in \Delta} B_n) \wedge_A^{\mathbb{L}} B \simeq \operatorname{holim}_{n \in \Delta}(B_n \wedge^{\mathbb{L}} B).$$

Now, as the functor $\mathbf{Ho}(A - \mathbf{Mod}) \longrightarrow \mathbf{Ho}(B - \mathbf{Mod})$ is conservative (since the family $\{A \longrightarrow A_i\}_{i \in I}$ is a Zariski covering), one can replace A by B and the A_i by $B \wedge_A^{\mathbb{L}} A_i$, and in particular one can suppose that $A \longrightarrow B$ has a section. But, it is well known that any morphism $A \longrightarrow B$ which has a section is such that $A \simeq \operatorname{holim}_n B_n$ (the section can in fact be used in order to construct a retraction). $\square$

Corollary 3.1.4 *The Zariski topology on* $\mathbf{S} - \mathbf{Aff}$ *is Čech subcanonical.*

The results of the next sections, though stated for $\mathbf{S}$-stacks will also be correct by replacing "$\mathbf{S}$-stack" with "Čech $\mathbf{S}$-stack", and "subcanonical" with "Čech subcanonical".

3.2 The S-stack of perfect modules

Let τ be a model topology on $\mathbf{S} - \mathbf{Aff}$. One defines the $\mathbf{S}$-prestack $\underline{\mathsf{Perf}}$ of perfect modules in the following way. For any commutative S-algebra A, we consider the category $\mathsf{Perf}(A)$, whose objects are perfect and cofibrant A-modules, and whose morphisms are equivalences of A-modules. The pull back functors define a pseudo-functor

$$
\begin{array}{cccc}
\mathsf{Perf}: & \mathbf{S} - \mathbf{Alg} & \longrightarrow & \mathbf{Cat} \\
& A & \longmapsto & \mathsf{Perf}(A) \\
& (A \to B) & \longmapsto & (- \wedge_A B : \mathsf{Perf}(A) \to \mathsf{Perf}(B)).
\end{array}
$$

Making this pseudo-functor into a strict functor from $\mathbf{S} - \mathbf{Alg}$ to $\mathbf{Cat}$ ([May, Th. 3.4]), and applying the classifying space functor $\mathbf{Cat} \to \mathbf{SSet}$, we get a simplicial presheaf denoted by $\underline{\mathsf{Perf}}$.

The following theorem relies on the so called strictification theorem ([To-Ve 1, A.3.2]), and its proof will appear in [To-Ve 2].

Theorem 3.2.1 *The object* $\underline{\mathsf{Perf}}$ *is an* $\mathbf{S}$*-stack (i.e. satisfies the descent condition for all* τ*-hypercovers) iff the model topology* τ *is subcanonical.*

Another way to state Theorem 3.2.1 is by saying that τ is subcanonical iff, for any commutative S-algebra A, the natural morphism

$$\underline{Hom}_{\mathbf{S}-\mathbf{Aff}^{\sim,\tau}}(Spec\,A, \underline{\mathsf{Perf}}) \simeq \underline{\mathsf{Perf}}(A) \longrightarrow \mathbb{R}\underline{Hom}_{\mathbf{S}-\mathbf{Aff}^{\sim,\tau}}(Spec\,A, \underline{\mathsf{Perf}})$$

is an equivalence of simplicial sets.

The S-stack of perfect complexes is a brave new analog of the stack of vector bundles, and is of fundamental importance in brave new algebraic geometry.

3.3 Geometric S-stacks

In this paragraph we will work with a fixed sub-canonical model topology τ on the model site $\mathbf{S} - \mathbf{Aff}$. We will define the notion of *geometric* $\mathbf{S}$-*stack*, which roughly speaking are *quotients* of affine $\mathbf{S}$-stacks by a *smooth affine groupoid*. They will be brave new generalizations of Artin algebraic stacks (see [La-Mo]). In order to state the precise definition, one first needs a notion of smoothness for morphisms of commutative S-algebras.

For any perfect S-module M one has the (derived) free commutative S-algebra over M, $S \longrightarrow \mathbb{L}F_S(M)$. For any commutative S-algebra A, one gets a morphism

$$A \longrightarrow A \wedge_S^{\mathbb{L}} \mathbb{L}F_S(M) \simeq \mathbb{L}F_A(A \wedge_S^{\mathbb{L}} M).$$

Any morphism $A \longrightarrow B$ in $\mathbf{Ho}(A - \mathbf{Alg})$ which is isomorphic to such a morphism will be called a *perfect morphism* of commutative S-algebras (and we will also say that B is a *perfect commutative A-algebra*).

Definition 3.3.1 *A morphism of commutative S-algebras $f : A \longrightarrow B$ is called* smooth *if it satisfies the following two conditions.*

- *The A-algebra B is finitely presented.*

- *There exists an étale covering family $\{v_i : B \to B_i'\}_{i \in I}$ and, for any $i \in I$, a homotopy commutative square of commutative S-algebras*

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & B \\
{\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v_i} \\
A' & \xrightarrow{\ f_i'\ } & B_i',
\end{array}
$$

   where f_i' is a perfect morphism, and u is an étale morphism.

One checks easily that smooth morphisms are stable by compositions and homotopy base changes. Furthermore, any étale morphism is smooth, and therefore so is any Zariski open immersion.

Assumption: At this point we will assume that the notion of smooth morphisms is *local* with respect to the chosen model topology τ.

This assumption will insure that the notion of geometric S-stack, to be defined below, behaves well.

Some terminology:

- Let us come back to our homotopy category $\mathbf{Ho}(\mathbf{S} - \mathbf{Aff}^{\sim,\tau})$ of $\mathbf{S}$-stacks, and the Yoneda embedding (or derived $Spec$)

$$\mathbb{R}Spec : \mathbf{Ho}(\mathbf{S} - \mathbf{Alg})^{op} \longrightarrow \mathbf{Ho}(\mathbf{S} - \mathbf{Aff}^{\sim,\tau}).$$

   The essential image of $\mathbb{R}Spec$ is called the category of *affine* $\mathbf{S}$-*stacks*, which is therefore anti-equivalent to the homotopy category of commutative S-algebras. We will also call affine $\mathbf{S}$-stack any object in $\mathbf{S} - \mathbf{Aff}^{\sim,\tau}$ whose image in $\mathbf{Ho}(\mathbf{S} - \mathbf{Aff}^{\sim,\tau})$ is an affine $\mathbf{S}$-stack. Clearly, affine $\mathbf{S}$-stacks are stable by homotopy limits (indeed $\operatorname{holim}_i(\mathbb{R}Spec\, A_i) \simeq \mathbb{R}Spec\,(\operatorname{hocolim}_i A_i))$.

- A morphism of affine **S**-stacks is called *smooth* (over S) (resp. étale, a Zariski open immersions . . .) if the corresponding one in $\mathbf{Ho}(\mathbf{S} - \mathbf{Alg})$ is so.

- A *Segal groupoid object* in $\mathbf{S} - \mathbf{Aff}^{\sim,\tau}$ is a simplicial object

$$X_* : \Delta^{op} \longrightarrow \mathbf{S} - \mathbf{Aff}^{\sim,\tau}$$

    which satisfies the following two conditions.

    – For any $n \geq 1$, the n-th Segal morphism

$$X_n \longrightarrow \underbrace{X_1 \times^h_{X_0} X_1 \times^h_{X_0} \ldots X_1}_{n \text{ times}}$$

    is an equivalence (in the model category $\mathbf{S} - \mathbf{Aff}^{\sim,\tau}$ of **S**-stacks). When this condition is satisfied, it is well known that one can define a composition law (well defined up to homotopy)

$$\mu : X_1 \times^h_{X_0} X_1 \longrightarrow X_1.$$

    – The induced morphism

$$(\mu, pr_2) : X_1 \times^h_{X_0} X_1 \longrightarrow X_1 \times^h_{X_0} X_1$$

    is an equivalence (i.e. the composition law is invertible up to homotopy).

- For any simplicial object $X_* : \Delta^{op} \longrightarrow \mathbf{S} - \mathbf{Aff}^{\sim,\tau}$, we will denote by $|X_*|$ the homotopy colimit of X_* in the model category $\mathbf{S} - \mathbf{Aff}^{\sim,\tau}$.

We are now ready to define geometric **S**-stacks.

Definition 3.3.2 *An* **S**-*stack F is called* geometric *if it is equivalent to some $|X_*|$, where X_* is a Segal groupoid in* $\mathbf{S} - \mathbf{Aff}^{\sim,\tau}$ *satisfying the following two additional conditions.*

- *The* **S**-*stacks X_0 and X_1 are affine* **S**-*stacks.*

- *The morphism $d_0 : X_1 \longrightarrow X_0$ is a smooth morphism of affine* **S**-*stacks.*

The theory of geometric **S**-stacks can then be pursued along the same lines as the theory of algebraic stacks (as done in [La-Mo]). For example, one can define the notions of quasi-coherent and perfect modules on a geometric **S**-stack, K-theory of a geometric **S**-stack (using perfect modules on it), higher geometric **S**-stacks (such as 2-geometric **S**-stacks), etc. We refer the reader to [To-Ve 2] for details.

We will finish this paragraph with the definition of the *tangent* **S**-*stack* and its main properties.

First of all, one defines a commutative S-algebra $S[\varepsilon] := S \vee S$, which is the trivial extension of S by S. The S-algebra $S[\varepsilon]$ can be thought as the *brave new algebra of dual numbers*, i.e. the analog of $\mathbb{Z}[\varepsilon]$. For any commutative S-algebra A, one has $A[\varepsilon] := A \wedge^{\mathbb{L}}_S S[\varepsilon] \simeq A \vee A$, the commutative A-algebra of dual numbers over A.

For any **S**-stack $F \in \mathbf{S} - \mathbf{Aff}^{\sim,\tau}$, one defines the *tangent* **S**-*stack of F* as

$$
\begin{array}{rccc}
TF : & \mathbf{S} - \mathbf{Alg} & \longrightarrow & SSet \\
& A & \mapsto & F(A[\varepsilon]).
\end{array}
$$

20

The tangent **S**-stack TF comes equipped with a natural projection $p : TF \longrightarrow F$. One first notice that if F is a geometric **S**-stack (over any base A), then so is TF. Furthermore, the homotopy fibers of the projection p are *linear* **S**-stacks in the following sense. Let A be a commutative S-algebra and $x : \mathbb{R}Spec\, A \longrightarrow F$ be a morphism of **S**-stacks, i.e. an A-point of F. One considers the homotopy pull back

$$
\begin{array}{ccc}
F_x & \longrightarrow & TF \\
\downarrow & & \downarrow \\
\mathbb{R}Spec\, A & \xrightarrow{\quad x \quad} & F.
\end{array}
$$

Then, one can show that there exists an A-module M, such that F_x is equivalent (as a stack over $\mathbb{R}Spec\, A$) to $\mathbb{R}Spec(\mathbb{L}F_A(M))$. In other words, one has a natural equivalence

$$
F_x(B) \simeq \mathbb{R}\underline{Hom}_{A-\mathbf{Mod}}(M, B)
$$

for any commutative A-algebra B. The A-module M is called the *cotangent complex* of F at the point x, and denoted by $\mathbb{L}\Omega_{F,x}$. Its derived dual A-module $D(\mathbb{L}\Omega_{F,x})$ is called the *tangent space* of F at x.

4 Some examples of geometric S-stacks

In this last Section we present two examples of geometric **S**-stacks. The first one arises from a classification problem in Algebraic Topology, whereas the second one is directly related to topological modular forms.

The first of these examples (see §4.2) shows that moduli spaces in Algebraic Topology are not only discrete homotopy types (as e.g. in [B-D-G]), but have some additional rich geometric structures very similar to the moduli spaces one studies in Algebraic Geometry. It seems to us one of the most simple non trivial example of brave new moduli stack, and a relevant test for the whole theory.

The second example (see §4.3) seems to us much more intriguing, and might give new insights on the construction and the study of topological modular forms. We think that this research direction is definitely worth being investigated in the future, and therefore we present a key open question that could be the starting point of such an investigation.

We will work with a fixed subcanonical model topology τ on $\mathbf{S} - \mathbf{Aff}$.

4.1 The brave new group scheme $\mathbb{R}\underline{\mathbf{Aut}}(\mathbf{M})$

We fix a perfect S-module M, and we are going to define a *group* **S**-*stack* $\mathbb{R}\underline{Aut}(M)$, of auto-equivalences of M. This group **S**-stack will be a generalization of the group scheme GL_n, since it will be shown to be an *affine* and *smooth* group **S**-stack. Like many algebraic stacks in Algebraic Geometry are quotients of affine schemes by GL_n, our example of a geometric **S**-stack in §4.2 will be a quotient of an affine **S**-stack by $\mathbb{R}\underline{Aut}(M)$ for some S-module M.

For any commutative S-algebra A, one first defines

$$
\mathbb{R}\underline{End}(M)(A) := \mathbb{R}\underline{Hom}_{A-\mathbf{Mod}}(A \wedge^{\mathbb{L}} M, A \wedge^{\mathbb{L}} M).
$$

Using for example the Dwyer-Kan simplicial localization techniques ([D-K1, D-K2]), one can make $A \mapsto \mathbb{R}\underline{End}(M)(A)$ into a functor from $\mathbf{S} - \mathbf{Alg}$ to the category **SMon** of simplicial monoids

$$
\begin{array}{cccc}
\mathbb{R}\underline{End}(M) : & \mathbf{S} - \mathbf{Alg} & \longrightarrow & \mathbf{SMon} \\
& A & \longmapsto & \mathbb{R}\underline{End}(M)(A).
\end{array}
$$

This defines $\mathbb{R}\underline{End}(M)$ as a monoid object in $\mathbf{S} - \mathbf{Aff}^{\sim,\tau}$. As its underlying object in $\mathbf{S} - \mathbf{Aff}^{\sim,\tau}$ is an $\mathbf{S}$-stack (for example using Theorem 3.2.1), we will say that $\mathbb{R}\underline{End}(M)$ is a *monoid* $\mathbf{S}$-*stack*.

Lemma 4.1.1 *The* $\mathbf{S}$-*stack* $\mathbb{R}\underline{End}(M)$ *is affine and the structural morphism* $\mathbb{R}\underline{End}(M) \longrightarrow \mathbb{R}Spec\, S$ *is perfect (hence smooth).*

Proof: This is clear as

$$\mathbb{R}\underline{End}(M) \simeq \mathbb{R}Spec\,(\mathbb{L}F_S(M \wedge_S^{\mathbb{L}} D(M))).$$

$\square$

For any commutative S-algebra A, one defines $\mathbb{R}\underline{Aut}(M)(A)$ to be the sub-monoid of $\mathbb{R}\underline{End}(M)(A)$ consisting of auto-equivalences. In other words, $\mathbb{R}\underline{Aut}(M)(A)$ is defined by the following homotopy pull-back diagram in $\mathbf{SSet}$

$$
\begin{array}{ccc}
\mathbb{R}\underline{Aut}(M)(A) & \longrightarrow & \mathbb{R}\underline{End}(M)(A) \\
\downarrow & & \downarrow \\
[M \wedge^{\mathbb{L}} A, M \wedge^{\mathbb{L}} A]' & \hookrightarrow & [M \wedge^{\mathbb{L}} A, M \wedge^{\mathbb{L}} A]
\end{array}
$$

where $[-,-]'$ is the subset of isomorphisms in $\mathbf{Ho}(\mathbf{SSet})$. This defines a functor

$$
\begin{array}{cccc}
\mathbb{R}\underline{Aut}(M): & \mathbf{S} - \mathbf{Alg} & \longrightarrow & \mathbf{SMon} \\
& A & \longmapsto & \mathbb{R}\underline{Aut}(M)(A).
\end{array}
$$

Once again the underlying object in $\mathbf{S} - \mathbf{Aff}^{\sim,\tau}$ is an $\mathbf{S}$-stack, and therefore $\mathbb{R}\underline{Aut}(M)$ is a monoid $\mathbf{S}$-stack. Furthermore, the monoid law on $\mathbb{R}\underline{Aut}(M)$ is invertible up to homotopy, and we will therefore say that $\mathbb{R}\underline{Aut}(M)$ is a *group* $\mathbf{S}$-*stack*.

Lemma 4.1.2 *The* $\mathbf{S}$-*stack* $\mathbb{R}\underline{Aut}(M)$ *is affine and the structural morphism* $\mathbb{R}\underline{Aut}(M) \longrightarrow \mathbb{R}Spec\, S$ *is smooth. In other words,* $\mathbb{R}\underline{Aut}(M)$ *is an* affine *and* smooth *group* $\mathbf{S}$-*stack*.

Proof: The following proof is inspired by the proof of [EGA-I, I.9.6.4]. Let B be the commutative S-algebra $\mathbb{L}F_S(M \wedge_S^{\mathbb{L}} D(M))$ corresponding to the affine $\mathbf{S}$-stack $\mathbb{R}\underline{End}(M)$. There exists a universal endomorphism of B-modules

$$u : M \wedge_S^{\mathbb{L}} B \longrightarrow M \wedge_S^{\mathbb{L}} B$$

such that for any commutative B-algebra C, the endomorphism

$$u \wedge_B^{\mathbb{L}} id_C : M \wedge_S^{\mathbb{L}} C \longrightarrow M \wedge_S^{\mathbb{L}} C$$

is equal (in $\mathbf{Ho}(B - \mathbf{Mod})$) to the corresponding point in

$$\mathbb{R}\underline{End}(M)(C) \simeq \mathbb{R}\underline{Hom}_{\mathbf{S}-\mathbf{Alg}}(B, C).$$

Consider now the homotopy cofiber $K \in \mathbf{Ho}(B - \mathbf{Mod})$ of the universal endomorphism u. Clearly, K is a perfect B-module, and one can therefore consider the open Zariski immersion $B \longrightarrow B_K$ (Lemma 2.1.6). It is easy to check by construction that

$$\mathbb{R}\underline{Aut}(M) \simeq \mathbb{R}Spec\, B_K,$$

which proves that $\mathbb{R}\underline{Aut}(M)$ is an affine $\mathbf{S}$-stack. Finally, as the morphism $\mathbb{R}Spec\, B_K \longrightarrow \mathbb{R}Spec\, B$ is smooth (being a Zariski open immersion), one sees (using the fact that $\mathbb{R}Spec\, B$ is perfect hence smooth) that $\mathbb{R}\underline{Aut}(M) \longrightarrow \mathbb{R}Spec\, S$ is also smooth. $\square$

4.2 Moduli of algebra structures

In this paragraph, we fix a perfect S-module M. We will define an **S**-stack $\underline{Ass}_M$, classifying associative and unital algebras whose underlying module is M.

For any commutative S-algebra A, we have the category $A - \mathbf{Ass}$, of associative and unital A-algebras (i.e. associative monoids in the monoidal category $(A - \mathbf{Mod}, \wedge_A)$); these are new versions of the old A_∞-ring spectra. The category $A - \mathbf{Ass}$ has a model category structure for which fibrations and equivalences are detected on the underlying objects in $A - \mathbf{Mod}$. We denote by $A - \mathbf{Ass}_M^{\mathrm{cof}}$ the subcategory of $A - \mathbf{Ass}$ whose objects are cofibrant objects B such that there exists a τ-covering family $\{A \longrightarrow A_i\}_{i \in I}$ such that each A_i-module $B \wedge_A^{\mathbb{L}} A_i$ is equivalent to $M \wedge_A^{\mathbb{L}} A_i$ (we say that the underlying A-module of B is τ-*locally equivalent to* M), and whose morphisms are equivalences of A-algebras. The base change functors define a lax functor

$$
\begin{array}{rccc}
Ass_M : & \mathbf{S} - \mathbf{Alg} & \longrightarrow & \mathbf{Cat} \\
 & A & \mapsto & A - \mathbf{Ass}_M^{\mathrm{cof}} \\
 & (A \to B) & \mapsto & - \wedge_A B.
\end{array}
$$

Strictifying this functor ([May, Th. 3.4]) and then applying the classifying space functor, one gets a simplicial presheaf

$$
\begin{array}{rccc}
\underline{Ass}_M : & \mathbf{S} - \mathbf{Alg} & \longrightarrow & \mathbf{SSet} \\
 & A & \mapsto & B(A - \mathbf{Ass}_M^{\mathrm{cof}}).
\end{array}
$$

For the following theorem, let us recall that for any commutative S-algebra A, any associative and unital A-algebra B and any B-bimodule M, one has an A-module of A-derivations $Der_A(B, M)$ from B to M. This can be derived on the left (in the model category of associative and unital A-algebras !) to $\mathbb{L}Der_A(B, M)$.

Theorem 4.2.1 *Let τ be a subcanonical model topology on $\mathbf{S} - \mathbf{Aff}$, and M be a perfect S-module.*

1. *The object $\underline{Ass}_M \in \mathbf{S} - \mathbf{Aff}^{\sim, \tau}$ is an $\mathbf{S}$-stack.*

2. *The $\mathbf{S}$-stack $\underline{Ass}_M$ is geometric.*

Sketch of proof: Point (1) can be proved with the same techniques used in Theorem 3.2.1 and will not be proved here. We refer to [To-Ve 2] for details.

Let us prove part (2) which is in fact a corollary of one of the main result of C. Rezk thesis [Re].

Let us start by considering the full sub-$\mathbf{S}$-stack of $\underline{Perf}$ (see §3.2) consisting of perfect modules which are τ-locally equivalent to M. By the result of Dwyer and Kan [D-K3, 2.3], this $\mathbf{S}$-stack is clearly equivalent as an object in $\mathbf{S} - \mathbf{Aff}^{\sim, \tau}$ to $B\mathbb{R}\underline{Aut}(M)$, the classifying simplicial presheaf of the group $\mathbf{S}$-stack $\mathbb{R}\underline{Aut}(M)$. Forgetting the algebra structure gives a morphism of $\mathbf{S}$-stacks

$$
f : \underline{Ass}_M \longrightarrow B\mathbb{R}\underline{Aut}(M).
$$

Using the techniques of equivariant stacks developed in [Ka-Pa-To1] (or more precisely their straight-forward extensions to the present context of $\mathbf{S}$-stacks), one sees that the $\mathbf{S}$-stack $\underline{Ass}_M$ is equivalent to the quotient $\mathbf{S}$-stack

$$
[X/\mathbb{R}\underline{Aut}(M)],
$$

where X is the homotopy fiber of the morphism f and $\mathbb{R}\underline{Aut}(M)$ acts on X. By Lemma 4.1.2, $\mathbb{R}\underline{Aut}(M)$ is an affine smooth group $\mathbf{S}$-stack, so we only need to show that X is an affine $\mathbf{S}$-stack (because the classifying Segal groupoid for the action of $\mathbb{R}\underline{Aut}(M)$ on X will then satisfies the conditions of Definition 3.3.2).

Using [Re, Thm. 1.1.5], one sees that the homotopy fiber X is equivalent to the **S**-stack

$$\mathbb{R}\underline{Hom}_{\mathrm{Oper}}(\mathcal{ASS}, \underline{\mathcal{E}nd}(M)) : A \longmapsto \mathbb{R}\underline{Hom}_{\mathrm{Oper}}(\mathcal{ASS}, \mathcal{E}nd(M \wedge_S^{\mathbb{L}} A)),$$

where $\mathbb{R}\underline{Hom}_{\mathrm{Oper}}(\mathcal{ASS}, \mathcal{E}nd(M \wedge_S^{\mathbb{L}} A))$ is the derived Hom (or mapping space) of unital operad morphisms from the final operad $\mathcal{ASS}$ (classifying associative and unital algebras) to the endomorphisms operad $\mathcal{E}nd(M \wedge_S^{\mathbb{L}} A)$ of the A-module $M \wedge_S^{\mathbb{L}} A$ (here operads are in the symmetric monoidal category **S** of S-modules). This means that, for any commutative S-algebra A, there is an equivalence

$$X(A) \simeq \mathbb{R}\underline{Hom}_{\mathrm{Oper}}(\mathcal{ASS}, \underline{\mathcal{E}nd}(M))(A),$$

functorial in A.

Now, writing the operad $\mathcal{ASS}$ as a homotopy colimit

$$\mathcal{ASS} \simeq \mathrm{hocolim}_{n \in \Delta^{op}} \mathcal{O}_n,$$

where each $\mathcal{O}_n$ is a free operad, one sees that

$$X \simeq \mathrm{holim}_{n \in \Delta} \mathbb{R}\underline{Hom}_{\mathrm{Oper}}(\mathcal{O}_n, \underline{\mathcal{E}nd}(M)).$$

Since affine **S**-stacks are stable under homotopy limits, it is therefore enough to check that the **S**-stack $\mathbb{R}\underline{Hom}_{\mathrm{Oper}}(\mathcal{O}, \underline{\mathcal{E}nd}(M))$ is affine for any free operad $\mathcal{O}$. But, saying that an operad $\mathcal{O}$ is free means that there is a family $\{P_m\}_{m>0}$ of S-modules, and functorial (in $A \in S - \mathbf{Alg}$) equivalences

$$\mathbb{R}\underline{Hom}_{\mathrm{Oper}}(\mathcal{O}, \underline{\mathcal{E}nd}(M))(A) \simeq \prod_m \mathbb{R}\underline{Hom}_{S-\mathbf{Mod}}(P_m \wedge_S^{\mathbb{L}} M^{\wedge^{\mathbb{L}} m} \wedge_S^{\mathbb{L}} D(M), A),$$

where the funny notation $M^{\wedge^{\mathbb{L}} m}$ stands for the derived smash product $M \wedge^{\mathbb{L}} \cdots \wedge^{\mathbb{L}} M$ of M with itself m times. So it is enough to show that for any S-module P, the (pre)stack

$$A \longmapsto \mathbb{R}\underline{Hom}_{S-\mathbf{Mod}}(P \wedge_S^{\mathbb{L}} M^{\wedge^{\mathbb{L}} m} \wedge_S^{\mathbb{L}} D(M), A)$$

is affine. But this is clear since this stack is equivalent to $\mathbb{R}Spec\, B$ where B is the derived free commutative S-algebra

$$B := \mathbb{L}F_S(P \wedge_S^{\mathbb{L}} M^{\wedge^{\mathbb{L}} n} \wedge_S^{\mathbb{L}} D(M)).$$

This implies that X is an affine **S**-stack and completes the proof. □

Remark 4.2.2 The relationship between the tangent space of $\underline{Ass}_M$ at a point $x : \mathbb{R}Spec\, A \longrightarrow \underline{Ass}_M$ (corresponding to an associative A-algebra whose underlying A-module is τ-locally equivalent to $M \wedge_S^{\mathbb{L}} A$) and the suspension $\mathbb{L}Der_A(B, B)[1]$ of the A-module of derived A-derivations of the associative A-algebra B into the B-bimodule B, will be investigated elsewhere.

Theorem 4.2.1 has also generalizations when one consider algebra structures over a given operad (for example commutative algebra structures). It can also be enhanced by considering categorical structures such as A_∞-categorical structures, as explained in [To-Ve 3]; the corresponding moduli space gives an example of a 2-*geometric* **S**-stack.

4.3 Topological modular forms and geometric S-stacks

In this final section we will be working with the standard étale topology on $\mathbf{S} - \mathbf{Aff}$. The corresponding model site will be denoted by $(\mathbf{S} - \mathbf{Aff}, \text{ét}^s)$, and its model category of stacks by $\mathbf{S} - \mathbf{Aff}^{\sim, \text{ét}^s}$. We recall that the topology ét^s is known to be subcanonical (see Lem. 3.1.2).

As explained right after Def. 3.0.2, the Eilenberg-MacLane spectrum construction gives rise to a fully faithful functor

$$\mathbb{L}H_! : \mathbf{Ho}(\mathbf{Aff}^{\sim, \text{ét}}) \longrightarrow \mathbf{Ho}(\mathbf{S} - \mathbf{Aff}^{\sim, \text{ét}^s}),$$

where the left hand side is the homotopy category of simplicial presheaves on the usual étale site of affine schemes. This functor has a right adjoint, called the *truncation functor*

$$h^0 := H^* : \mathbf{Ho}(\mathbf{S} - \mathbf{Aff}^{\sim, \text{ét}^s}) \longrightarrow \mathbf{Ho}(\mathbf{Aff}^{\sim, \text{ét}}),$$

simply given by composing a simplicial presheaf $F : \mathbf{S} - \mathbf{Aff}^{op} \longrightarrow \mathbf{SSet}$ with the functor $H : \mathbf{Aff} \longrightarrow \mathbf{S} - \mathbf{Aff}$.

Let us denote by $\overline{\mathcal{E}}$ the moduli stack of generalized elliptic curves with integral geometric fibers, which is the standard compactification of the moduli stack of elliptic curves by adding the nodal curves at infinity (see e.g. [Del-Rap, IV], where it is denoted by $\mathcal{M}_{(1)}$); recall that $\overline{\mathcal{E}}$ is a Deligne-Mumford stack, proper and smooth over $Spec\,\mathbb{Z}$ ([Del-Rap, Prop. 2.2]).

As shown by recent works of M. Hopkins, H. Miller, P. Goerss, N. Strickland, C. Rezk and M. Ando, there exists a natural presheaf of commutative S-algebras on the small étale site $\overline{\mathcal{E}}_{\text{ét}}$ of $\overline{\mathcal{E}}$. We will denote this presheaf by *tmf*. Recall that by construction, if $U = Spec\,A \longrightarrow \overline{\mathcal{E}}$ is an étale morphism, corresponding to an elliptic curve E over the ring A, then $tmf(U)$ is the (connective) elliptic cohomology theory associated to the formal group of E (in particular, one has $\pi_0(tmf(U)) = A$). Recall also that the (derived) global sections $\mathbb{R}\Gamma(\overline{\mathcal{E}}, tmf)$, form a commutative S-algebra, well defined in $\mathbf{Ho}(\mathbf{S} - \mathbf{Alg})$, called the *spectrum of topological modular forms*, and denoted by tmf.

Let $U \longrightarrow \overline{\mathcal{E}}$ be a surjective étale morphism with U an affine scheme, and let us consider its nerve

$$
\begin{array}{cccc}
U_* : & \Delta^{op} & \longrightarrow & \mathbf{Aff} \\
& [n] & \mapsto & U_n := \underbrace{U \times_{\overline{\mathcal{E}}} U \times_{\overline{\mathcal{E}}} \cdots \times_{\overline{\mathcal{E}}} U}_{n \; times}.
\end{array}
$$

This is a simplicial object in $\overline{\mathcal{E}}_{\text{ét}}$, and by applying *tmf* we obtain a co-simplicial object in $\mathbf{S} - \mathbf{Alg}$

$$
\begin{array}{cccc}
tmf(U_*) : & \Delta & \longrightarrow & \mathbf{S} - \mathbf{Alg} \\
& [n] & \mapsto & tmf(U_n).
\end{array}
$$

Taking $\mathbb{R}Spec$ (§3.1) of this diagram we obtain a simplicial object in the model category $\mathbf{S} - \mathbf{Aff}^{\sim, \text{ét}^s}$

$$
\begin{array}{cccc}
\mathbb{R}Spec\,(tmf(U_*)) : & \Delta^{op} & \longrightarrow & \mathbf{S} - \mathbf{Aff}^{\sim, \text{ét}^s} \\
& [n] & \mapsto & \mathbb{R}Spec\,(tmf(U_n)).
\end{array}
$$

The homotopy colimit of this diagram will be denoted by

$$\overline{\mathcal{E}}_{\mathbf{S}} := \text{hocolim}_{n \in \Delta^{op}} \mathbb{R}Spec\,(tmf(U_*)) \in \mathbf{Ho}(\mathbf{S} - \mathbf{Aff}^{\sim, \text{ét}^s}).$$

The following result is just a remark as there is essentially nothing to prove; however, we prefer to state it as a theorem to emphasize its importance.

Theorem 4.3.1 *The stack $\overline{\mathcal{E}}_{\mathbf{S}}$ defined above is a geometric $\mathbf{S}$-stack. Furthermore, there exists a natural isomorphism in $\mathbf{Ho}(\mathbf{Aff}^{\sim, \text{ét}})$*

$$h^0(\overline{\mathcal{E}}_{\mathbf{S}}) \simeq \overline{\mathcal{E}}.$$

Proof: To prove that $\overline{\mathcal{E}}_{\mathbf{S}}$ is geometric, it is enough to check that the simplicial object $\mathbb{R}Spec\,(tmf(U_*))$ is a Segal groupoid satisfying the conditions of Def. 3.3.2. For this, recall that for any morphism $U = Spec\,B \to V = Spec\,A$ in $\overline{\mathcal{E}}_{\text{ét}}$, the natural morphism

$$\pi_*(tmf(V)) \otimes_{\pi_0(tmf(V))} \pi_0(tmf(U)) \simeq \pi_*(tmf(V)) \otimes_A B \longrightarrow \pi_*(tmf(U))$$

is an isomorphism. This shows that the functor

$$\mathbb{R}Spec\,(tmf(-)) : \overline{\mathcal{E}}_{\text{ét}} \longrightarrow \mathbf{S} - \mathbf{Aff}^{\sim,\text{ét}^s}$$

preserves homotopy fiber products and therefore sends Segal groupoid objects to Segal groupoid objects. This shows in particular that $\mathbb{R}Spec\,(tmf(U_*))$ is a Segal groupoid object. The same fact also shows that for any morphism $U = Spec\,B \to V = Spec\,A$ in $\overline{\mathcal{E}}_{\text{ét}}$, the induced map $tmf(V) \longrightarrow tmf(U)$ is a strong étale morphism in the sense of Def. 2.3.1, and therefore is an étale and thus smooth morphism. This implies that $\mathbb{R}Spec\,(tmf(U_*))$ satisfies the conditions of 3.3.2 and therefore shows that $\overline{\mathcal{E}}_{\mathbf{S}}$ is indeed a geometric $\mathbf{S}$-stack.

The truncation functor h^0 clearly commutes with homotopy colimits, and therefore

$$h^0(\overline{\mathcal{E}}_{\mathbf{S}}) \simeq \text{hocolim}_{n \in \Delta^{op}} h^0(\mathbb{R}Spec\,(tmf(U_n))) \in \mathbf{Ho}(\mathbf{Aff}^{\sim,\text{ét}}).$$

Furthermore, for any connective affine $\mathbf{S}$-stack $\mathbb{R}Spec\,A$ one has a natural isomorphism $h^0(\mathbb{R}Spec\,A) \simeq Spec\,\pi_0(A)$. Therefore, one sees immediately that there is a natural isomorphism of simplicial objects in $\mathbf{Aff}^{\sim,\text{ét}}$

$$h^0(\mathbb{R}Spec\,(tmf(U_*))) \simeq U_*.$$

Therefore, we get

$$h^0(\overline{\mathcal{E}}_{\mathbf{S}}) \simeq \text{hocolim}_{n \in \Delta^{op}} h^0(\mathbb{R}Spec\,(tmf(U_n))) \simeq \text{hocolim}_{n \in \Delta^{op}} U_n \simeq \overline{\mathcal{E}},$$

as U_* is the nerve of an étale covering of $\overline{\mathcal{E}}$. $\qquad\qquad\square$

Remark 4.3.2 The proof of Theorem 4.3.1 shows not only that $\overline{\mathcal{E}}_{\mathbf{S}}$ is geometric, but also that it is a *strong Deligne-Mumford geometric $\mathbf{S}$-stack,* in the sense that one can replace the word *smooth* by *strongly étale* in Definition 3.3.2. See [To-Ve 2] for further details.

Theorem 4.3.1 tells us that the presheaf of topological modular forms *tmf* provides a natural geometric $\mathbf{S}$-stack $\overline{\mathcal{E}}_{\mathbf{S}}$ whose truncation is the usual stack of elliptic curves $\overline{\mathcal{E}}$. Furthermore, as the small strong étale topoi of $\overline{\mathcal{E}}_{\mathbf{S}}$ and $\overline{\mathcal{E}}$ coincides (this is a general fact about strong étale model topologies, see [To-Ve 2]), we see that

$$\mathsf{tmf} := \mathbb{R}\Gamma(\overline{\mathcal{E}}, tmf) \simeq \mathbb{R}\Gamma(\overline{\mathcal{E}}_{\mathbf{S}}, \mathcal{O}),$$

and therefore that topological modular forms can be simply interpreted as *functions on the geometric $\mathbf{S}$-stack $\overline{\mathcal{E}}_{\mathbf{S}}$.* Of course, our construction of $\overline{\mathcal{E}}_{\mathbf{S}}$ has essentially been done to make this true, so this is not a surprise. However, we have gained a bit from the conceptual poit of view: since after all $\overline{\mathcal{E}}$ is a moduli stack, now that we know the existence of the geometric $\mathbf{S}$-stack $\overline{\mathcal{E}}_{\mathbf{S}}$ we can ask for a *modular interpretation* of it, or in other words for a direct geometric description of the corresponding simplicial presheaf on $\mathbf{S} - \mathbf{Aff}$. An answer to this question not only would provide a direct construction of tmf, but would also give a conceptual interpretation of it in a geometric language closer the usual notion of modular forms.

Question 4.3.3 *Find a modular interpretation of the $\mathbf{S}$-stack $\overline{\mathcal{E}}_{\mathbf{S}}$.*

Essentially, we ask which are the brave new "objects" that the **S**-stack $\overline{\mathcal{E}}_{\mathbf{S}}$ classifies. Of course we do not know the answer to this question, though some progress are being made by J. Lurie and the authors. It seems that the **S**-stack $\overline{\mathcal{E}}_{\mathbf{S}}$ itself is not really the right object to look at, and one should rather consider the non-connective version of it (defined using the non-connective version of *tmf*) for which a modular interpretation seems much more accessible. Though this modular interpretation is still conjectural and not completely achieved, it does use some very interesting notions of *brave new abelian varieties*, *brave new formal groups* and their geometry. We think that the achievement of such a program could be the starting point of a rather new and deep interaction between stable homotopy theory and algebraic geometry, involving many new questions and objects, but probably also new insights on classical objects of algebraic topology.

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