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Luenberger Observers For Switching Discrete-Time Linear Systems

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State estimation is considered for a class of switching discrete-time linear systems. The switching is assumed to be unknown among the various system modes associated with different known matrices. The proposed scheme relies on the combination of the estimation of the system mode with the application of a Luenberger-like observer whose gain is a function of the estimated mode. In the absence of noises, the estimate of the mode can be chosen among the ones that are consistent with the measurements and the stability of the estimation error is ensured under suitable conditions on the observer gains. Such conditions can be expressed by means of Linear Matrix Inequalities (LMIs). The presence of bounded disturbances is also taken explicitly into consideration. In this situation, a novel method based on a minimum-distance criterion is proposed in order to estimate the system mode. Also in this case the error of the resulting estimator is proved to be exponentially bounded.

1 Introduction

In a number of engineering problems, it is often desirable to obtain reliable estimates of the state variables for control or diagnostic purposes. Therefore, the synthesis of state observers has been of considerable interest in classical system theory from the pioneering work of Luenberger (1969). Recently, a special attention has been gained by hybrid systems, which allow one to describe a large class of plants with continuous dynamics, finite-state automata, and logic decisions. The topic of this paper is the design of state estimators for a class of hybrid systems having a discrete-time linear form that may switch from a matrix configuration to another one in a known finite set (the index denoting such a configuration being called the *mode* or the *discrete state*).

In the literature, among the various approaches to state estimation for switching systems, quite a diffused paradigm is based on the idea of using a bank of filters (each of which is tuned on a specific model) to track all the possible changes of the discrete state (see, e.g., (Bar-Shalom and Li, 1993)). When a probabilistic description of the switching is supposed to be available, the switching is modelled by means of a hidden finite-state Markov chain. In such framework, a major concern is the reduction of the computational burden (see, e.g., (Zhang, 2000); (Doucet and Andrieu, 2001); (Elliott et al., 2005)).

Different approaches have been proposed in the literature to address the state estimation problem when a probabilistic model of the switching is not available. In this context, an attempt of extending the Luenberger observer was proposed by Alessandri and Coletta (2001), with the limitation of assuming to know the switching times and modes exactly. In (Balluchi et al., 2002), an approach was presented for the estimation of both the continuous state and the discrete one for a particular class of hybrid systems for which the evolution of the discrete state is governed by a hidden finite-state machine and a discrete output is available. Mode estimation was specifically addressed by Del Vecchio et al. (2006) under the assumption that the continuous state variables are known exactly. Juloski et al. (2002) developed an observer for bimodal switching linear systems with no knowledge of the switching time. For linear dynamic systems

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where only the measurement equations may switch, Babaali et al. (2004) proposed an observer that results from the on-line solution of a nonlinear least-squares problem with local convergence properties for the estimation error. Moreover, receding-horizon state estimation for various classes of switching discrete-time linear systems was considered by Ferrari-Trecate et al. (2002); Alessandri et al. (2005); Pina and Botto (2006).

The design of estimators for switching systems rises the problem of determining the switching sequence on the basis of the observations as well as the need of introducing suitable mode-observability notions. For unforced noise-free switching linear systems such an issue was first addressed systematically by Vidal et al. (2002), under the assumption of a minimum dwell time between consecutive switches. More recent advances on this topic have been developed in (Babaali and Egerstedt, 2004), where arbitrary switching sequences were considered. In (Alessandri et al., 2005), such results were extended to comply with the presence of bounded disturbances that corrupt the dynamics and the measures. It is important to point out that, as shown in these papers, the possibility of exactly reconstructing the sequence of discrete states depends on the actual value of the continuous state of the system and, even in the absence of noises, there will always exist continuous states for which such a task turns out to be impossible.

In the first part of this paper, an estimation scheme for noise-free switching discrete-time linear systems is proposed that is based on the combination of a Luenberger-like observer with a mode estimator. The gain of the observer is selected in correspondence of the estimate of the switching mode. The possibility of committing an error in the estimation of the discrete state is explicitly taken into account and dealt with. This results in the development of novel conditions that the observer gains have to satisfy in order to ensure the exponential stability of the estimation error and that can be integrated into the LMI formulation of the whole problem. In the second part of the paper, the proposed observation scheme is generalized to take into account the presence of *unknown but bounded* disturbances acting on the system and measurement equations. Towards this end, a novel method for the estimation of the system mode in the noisy case is proposed that is based on a minimum-distance criterion. Theoretical results are provided that establish when such a method leads to the exact identification of the true switching sequence. Based on such results, the estimation error of the proposed observer is shown to be exponentially bounded.

The main contributions of the paper are the following: i) in the noise-free case, the inevitable errors in the estimation of the discrete state are dealt with by integrating suitable conditions into the formulation of the problem; ii) in the presence of disturbances, a minimum-distance criterion for the estimation of the discrete state is developed. With respect to previous results (specifically (Alessandri et al., 2005)), the proposed estimation scheme turns out to be more computationally efficient and does not require an exact knowledge of the form of the sets to which the system and measurement noises belong (indeed, it can be always applied regardless of the form of such sets). Moreover, the stability of the estimation error is ensured for a broader class of systems thanks to the novel conditions taken into account in the design of the observer.

Before concluding this section, let us introduce some notations and basic definitions. Given a generic vector v, $\|v\|$ denotes the Euclidean norm of v and, given a positive definite matrix P, $\|v\|_P$ denotes the weighted norm of v, $\|v\|_P \stackrel{\triangle}{=} (v^\top P v)^{1/2}$. Given a generic sequence $\{z_t;\, t=0,1,\ldots\}$ and two time instants $t_1 \leq t_2$, we define $\mathbf{z}_{t_1}^{t_2} \stackrel{\triangle}{=} \mathrm{col}\,(z_{t_1}, z_{t_1+1}, \ldots, z_{t_2})$. Furthermore, given two sequences $\mathbf{z}$ and $\mathbf{z}'$, let us denote by $\mathbf{z} \otimes \mathbf{z}'$ the sequence obtained from the concatenation of $\mathbf{z}$ and $\mathbf{z}'$, i.e., $\mathbf{z} \otimes \mathbf{z}' \stackrel{\triangle}{=} \mathrm{col}\,(\mathbf{z}, \mathbf{z}')$. For a symmetric positive or negative definite matrix D, $\sigma_{\min}\,(D)$ and $\sigma_{\max}\,(D)$ are the minimum and maximum eigenvalues of D, respectively. The norm of a matrix B is $\|\,B\,\| \stackrel{\triangle}{=} \sqrt{\sigma_{\max}\,(B^\top B)}$. Given a generic matrix M, we denote by $\mathrm{span}(M)$ the linear space generated by a linear combination of the columns of M.

2 Design of the observer in the absence of noises

Let us consider a class of switching discrete-time linear systems described by

$$\begin{aligned} x_{t+1} &= A(\lambda_t)\, x_t \\ y_t &= C(\lambda_t)\, x_t \end{aligned} \tag{1}$$

where $t = 0, 1, \ldots$ is the time instant, $x_t \in \mathbb{R}^n$ is the continuous state vector (the initial state x_0 is unknown), $y_t \in \mathbb{R}^m$ is the measurement vector, and $\lambda_t \in \Lambda \overset{\triangle}{=} \{1, 2, \ldots, M\}$ is the discrete state. $A(\lambda)$ and $C(\lambda)$, $\lambda \in \Lambda$, are $n \times n$ and $m \times n$ matrices, respectively.

If we assume to perfectly know λ_t, an observer for (1) can be given the form of a Luenberger observer, i.e.,

$$\hat{x}_{t+1} = A(\lambda_t)\, \hat{x}_t + L(\lambda_t)\, [y_t - C(\lambda_t)\, \hat{x}_t] \tag{2}$$

where $t = 0, 1, \ldots$, $\hat{x}_t$ is the estimate of x_t, $\hat{x}_0$ is chosen "a priori," and $L(\lambda_t)$ is an $n \times m$ matrix called the observer gain at the time t. We require that the gain $L(\lambda)$ is associated with the couple $(A(\lambda), C(\lambda))$, $\lambda \in \Lambda$.

Under the knowledge of λ_t, the dynamics of the estimation error $e_t \overset{\triangle}{=} x_t - \hat{x}_t$ behaves like a switching system and a common Lyapunov function can be searched in order to ensure the stability of the estimation error (Alessandri and Coletta, 2001). If the switching mode λ_t is not known exactly, the design of a Luenberger observer turns out to be much more difficult. In this case, at any time t one can estimate the discrete state λ_t on the basis of the observation of the output of the system over a certain interval "around" the current time t. The estimate $\hat{\lambda}_t$ can be used in (2) instead of the true value λ_t. Then the observer can be obtained as

$$\hat{x}_{t+1} = A(\hat{\lambda}_t)\, \hat{x}_t + L(\hat{\lambda}_t)\, \Big[y_t - C(\hat{\lambda}_t)\, \hat{x}_t \Big] \,. \tag{3}$$

In the following, a possible approach for the choice of the estimate $\hat{\lambda}_t$ will be proposed that ensures the convergence of the estimation error under suitable assumptions.

2.1 *Mode estimation and observability issues*

In order to try to identify the discrete state λ_t, a first very simple idea would consist in considering as possible estimates of λ_t only the discrete states $\hat{\lambda}_t$ such that $y_t = C(\hat{\lambda}_t)\, x$ for some $x \in \mathbb{R}^n$. Of course, this would not lead to a reliable estimate $\hat{\lambda}_t$. Along the lines of (Alessandri et al., 2005), a possible strategy is that of trying to identify the discrete state λ_t on the basis of the observations vector $\mathbf{y}_{t-\alpha}^{t+\omega}$ over an interval of the form $[t - \alpha, t + \omega]$. Of course, this causes a delay equal to ω in the computation of $\hat{\lambda}_t$ and so of $\hat{x}_{t+1}$. It is important to remark that a certain delay is unavoidable in order to obtain a reliable information on the discrete state λ_t. Indeed (as shown by Babaali and Egerstedt (2004); Alessandri et al. (2005)) unless the number of measures available at each time instant is at least equal to the number of state variables (i.e., $m \geq n$), it may not be possible to detect switches that occur in the last or in the first instants of an observation window.

Let us consider a generic sequence $\boldsymbol{\lambda} \in \Lambda^N$ of N discrete states, and denote by $\lambda^{(i)}$ the i-th element of such a sequence, i.e., $\boldsymbol{\lambda} \overset{\triangle}{=} \mathrm{col}\left[\lambda^{(1)}, \ldots, \lambda^{(N)}\right]$. We shall define the observability matrix associated with $\boldsymbol{\lambda}$

as

$$F(\boldsymbol{\lambda}) \triangleq \begin{bmatrix} C(\lambda^{(1)}) \\ C(\lambda^{(2)})A(\lambda^{(1)}) \\ \vdots \\ C(\lambda^{(N-1)})A(\lambda^{(N-2)})\cdots A(\lambda^{(1)}) \\ C(\lambda^{(N)})A(\lambda^{(N-1)})\cdots A(\lambda^{(1)}) \end{bmatrix}.$$

Moreover, let $\Phi(\boldsymbol{\lambda}) \triangleq A(\lambda^{(N)})\cdots A(\lambda^{(1)})$ be the transition matrix associated with $\boldsymbol{\lambda}$.

Furthermore, let us denote by $\mathcal{Y}(\boldsymbol{\lambda})$ the set of all the possible vectors of observations associated with the switching pattern $\boldsymbol{\lambda}$, i.e.,

$$\mathcal{Y}(\boldsymbol{\lambda}) \triangleq \left\{\mathbf{y} \in \mathbb{R}^{mN} : \mathbf{y} = F(\boldsymbol{\lambda})\tilde{x}\,,\ \tilde{x} \in \mathbb{R}^n\right\}.$$

Of course $\mathcal{Y}(\boldsymbol{\lambda})$ is the linear space generated by the columns of $F(\boldsymbol{\lambda})$.

Let us now consider observation sequences of length $\alpha+1+\omega$ where α and ω are positive integers. We define as $\mathcal{Y}^{\alpha,\omega}(\lambda)$ the set of all the observation sequences that can be generated when the switching pattern has the form $\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+$ (for some $\boldsymbol{\lambda}_- \in \Lambda^\alpha$ and $\boldsymbol{\lambda}_+ \in \Lambda^\omega$). Using the notations introduced so far, we have

$$\begin{aligned}\mathcal{Y}^{\alpha,\omega}(\lambda) &\triangleq \left\{\mathbf{y} \in \mathbb{R}^{m(\alpha+1+\omega)} : \mathbf{y} = F(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)\tilde{x}\,, \quad \tilde{x} \in \mathbb{R}^n,\ \boldsymbol{\lambda}_- \in \Lambda^\alpha,\ \boldsymbol{\lambda}_+ \in \Lambda^\omega\right\} \\ &= \bigcup_{\boldsymbol{\lambda}_- \in \Lambda^\alpha, \boldsymbol{\lambda}_+ \in \Lambda^\omega} \mathcal{Y}(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)\,.\end{aligned}$$

Moreover, given an observation sequence $\mathbf{y} \in \mathbb{R}^{m(\alpha+1+\omega)}$, let us denote by $\Lambda^{\alpha,\omega}(\mathbf{y})$ the set of all the discrete states consistent with the observations vector $\mathbf{y}$, defined as

$$\Lambda^{\alpha,\omega}(\mathbf{y}) \triangleq \{\lambda \in \Lambda : \quad \mathbf{y} \in \mathcal{Y}^{\alpha,\omega}(\lambda)\}\ .$$

In the following, we shall consider as possible estimates of λ_t only the discrete states $\hat{\lambda}_t$ belonging to the set of feasible discrete states

$$\Lambda^{\alpha,\omega}(\mathbf{y}_{t-\alpha}^{t+\omega}) \triangleq \left\{\lambda \in \Lambda : \quad \mathbf{y}_{t-\alpha}^{t+\omega} \in \mathcal{Y}^{\alpha,\omega}(\lambda)\right\}\ .$$

Remark 1 In order to define the sets $\mathcal{Y}^{\alpha,\omega}(\lambda)$ and $\Lambda^{\alpha,\omega}(\mathbf{y})$, no assumptions have been made on the evolution of the discrete state. It is important to remark that the proposed approach is well-suited to taking into account possible information on the evolution of the discrete state (e.g., when it is governed by a hidden finite-state machine). With this respect, at every time instant, instead of considering all the possible switching patterns belonging to $\Lambda^{\alpha+\omega+1}$, one could consider a restricted set $\mathcal{S}_t^{\alpha,\omega} \subseteq \Lambda^{\alpha+\omega+1}$ of all the admissible switching patterns, i.e., of all the switching patterns consistent with the a-priori knowledge of the law governing the evolution of the discrete state. This would lead to the introduction of a new time-varying set $\mathcal{Y}_t^{\alpha,\omega}(\lambda)$, that is,

$$\mathcal{Y}_t^{\alpha,\omega}(\lambda) \triangleq \bigcup_{\boldsymbol{\lambda}_- \in \Lambda^\alpha, \boldsymbol{\lambda}_+ \in \Lambda^\omega, (\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+) \in \mathcal{S}_t^{\alpha,\omega}} \mathcal{Y}(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)\,,$$

and to update the definition of the set $\Lambda^{\alpha,\omega}(\mathbf{y})$ accordingly. This would add no theoretical difficulty but some notational complication. Hence, not to complicate the presentation, in the following we shall always suppose the law governing the evolution of the discrete state to be completely unknown.

Of course, if no assumptions are made on system (1), it is quite possible that, at the generic time t, the cardinality of the set $\Lambda^{\alpha,\omega}(\mathbf{y}_{t-\alpha}^{t+\omega})$ is strictly greater than one. In this case, it is impossible to determine uniquely the current discrete state λ_t on the basis of the observations vector $\mathbf{y}_{t-\alpha}^{t+\omega}$. As shown by Babaali and Egerstedt (2004); Alessandri et al. (2005), the possibility of distinguishing between two different discrete states depends on the current continuous state. With this respect, let us define as $\mathcal{X}^{\alpha,\omega}(\lambda,\lambda')$ the set of all the continuous states x such that possible observation sequences $\mathbf{y}_{t-\alpha}^{t+\omega}$ exist that could be obtained if $x_t = x$ and $\lambda_t = \lambda$ but would be consistent also with $\lambda_t = \lambda'$ (even for values of x_t different from x). The set $\mathcal{X}^{\alpha,\omega}(\lambda,\lambda')$ can be determined as the set of all the continuous states x for which some $\tilde{x} \in \mathbb{R}^n$, some $\boldsymbol{\lambda}_- \in \Lambda^\alpha$, and some $\boldsymbol{\lambda}_+ \in \Lambda^\omega$ exist such that

(i) $x = \Phi(\boldsymbol{\lambda}_-)\tilde{x}$ (i.e., the state x may belong to the state trajectory);
(ii) $\lambda' \in \Lambda^{\alpha,\omega}(\mathbf{y})$ where $\mathbf{y} = F(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)\,\tilde{x}$ (i.e., the discrete state λ' is consistent with possible observation sequences associated with λ and x).

In other words, $\mathcal{X}^{\alpha,\omega}(\lambda,\lambda')$ is the set of states for which it could be impossible to uniquely determine if λ_t is either λ or λ'. From the above definition, $\mathcal{X}^{\alpha,\omega}(\lambda,\lambda')$ turns out to be the union of a finite number of linear subspaces of $\mathbb{R}^n$. The following elementary example should clarify the previous definition.

Example 1 Consider a simple linear switching system described by equation (1) with

$$\begin{aligned} A(1) &\triangleq \begin{bmatrix} 1 & 0 \\ -0.5 & 1 \end{bmatrix}, \quad A(2) \triangleq \begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}, \\ C(1) &\triangleq [-1 \;\; -2], \quad C(2) \triangleq [-1 \;\; -2]\,. \end{aligned} \tag{4}$$

Suppose that, at every time $t = 0, 1, \ldots$, we would like to determine the discrete state λ_t on the basis of the observations vector $\mathbf{y}_t^{t+2}$ (this corresponds to the choice $\alpha = 0$ and $\omega = 2$). Since we have

$$F(1,1,1) = F(1,1,2) = \begin{bmatrix} -1 & -2 \\ 0 & -2 \\ 1 & -2 \end{bmatrix}, \quad F(1,2,1) = F(1,2,2) = \begin{bmatrix} -1 & -2 \\ 0 & -2 \\ 2 & -2 \end{bmatrix},$$

$$F(2,1,1) = F(2,1,2) = \begin{bmatrix} -1 & -2 \\ 1 & -2 \\ 4 & -2 \end{bmatrix}, \quad F(2,2,1) = F(2,2,2) = \begin{bmatrix} -1 & -2 \\ 1 & -2 \\ 7 & -2 \end{bmatrix},$$

it is immediate to verify that it is impossible to distinguish between the two discrete states if and only if $x_t = [0 \;\; k]^\top$ for any $k \in \mathbb{R}$. Hence, in this case, $\mathcal{X}^{0,2}(1,2) = \mathcal{X}^{0,2}(2,1) = \{x = [0 \;\; k]^\top,\; k \in \mathbb{R}\}$. △

2.2 *Stability analysis of the estimation error*

By exploiting the definition of the sets $\mathcal{X}^{\alpha,\omega}(\lambda,\lambda')$, it is possible to give sufficient conditions that the gains of observer (3) have to satisfy in order to ensure the convergence of the estimation error. More specifically, the following theorem can be stated.

THEOREM 2.1 *Suppose that the gains $L(\lambda)$, $\lambda \in \Lambda$ satisfy the following conditions:*

(i) $[A(\lambda) - L(\lambda)\,C(\lambda)]^\top\, P\, [A(\lambda) - L(\lambda)\,C(\lambda)] - P < 0$, *for* $\lambda \in \Lambda$, *where* $P = P^\top > 0$;
(ii) $\{[A(\lambda) - A(\lambda')] - L(\lambda')\,[C(\lambda) - C(\lambda')]\}\,x = 0$, *for every* $x \in \mathcal{X}^{\alpha,\omega}(\lambda,\lambda')$ *and for every* $\lambda \neq \lambda'$.

Furthermore, suppose that, at any time $t = \alpha, \alpha+1, \ldots,$ *the estimate* $\hat{\lambda}_t$ *is chosen inside the set* $\Lambda^{\alpha,\omega}(\mathbf{y}_{t-\alpha}^{t+\omega})$. *Then observer (3) involves an estimation error* $e_t \triangleq x_t - \hat{x}_t$ *exponentially convergent to* *zero, i.e., there exist* $h > 0$ *and* $0 < \beta < 1$ *such that*

$$\|e_t\| \leq h\,\beta^{t-\alpha}\,\|e_\alpha\|\,, \quad t = \alpha, \alpha+1, \ldots\,. \tag{5}$$

□

The basic idea behind Theorem 2.1 is quite simple. Condition (i) is quite classical and ensures the existence of a quadratic Lyapunov function for the error dynamics; since from (1) and (3) we have

$$e_{t+1} = \Big[A(\hat{\lambda}_t) - L(\hat{\lambda}_t)C(\hat{\lambda}_t)\Big]\,e_t + \Big\{\Big[A(\lambda_t) - A(\hat{\lambda}_t)\Big] - L(\hat{\lambda}_t)\,\Big[C(\lambda_t) - C(\hat{\lambda}_t)\Big]\Big\}\,x_t\,,$$

for $t = \alpha, \alpha+1, \ldots$, it should be evident that condition (ii) leads to decoupling the error dynamics from that of the system.

Some remarks on the conditions of Theorem 2.1 are important. Let us first consider condition (i). As it is well known, necessary conditions for the Lyapunov inequalities to hold are that all the pairs $(A(\lambda), C(\lambda))$ are detectable. Note that, though each inequality in (i) separately admits a solution if and only if the pairs $(A(\lambda), C(\lambda))$ are detectable, indeed, in order to ensure stability, a more restrictive condition is required, i.e., the existence of a matrix P satisfying all the inequalities. In general, the existence of a common quadratic Lyapunov function turns out to be a quite restrictive requirement. Recently, Daafouz et al. (2002b) proposed less conservative conditions to ensure the stability of a switching system that are based on the use of switching quadratic Lyapunov functions. Later on (see Daafouz et al., 2002a), such conditions have been applied to the design of state-feedback controllers and Luenberger observers (under the assumption that the switching signal is available in real time). It is important to point out that Theorem 2.1 could be easily modified to take into account such results (this would just require the replacement of condition (i) with the one of Theorem 4 in (Daafouz et al., 2002a)). However, in order to focus on the effects of the unknown switches in the dynamics of the estimation error, we decided to avoid such complications. In the light of the above discussion, we believe this can be done without loss of generality.

Let us now consider condition (ii) of Theorem 2.1. First, recall that each set $\mathcal{X}^{\alpha,\omega}(\lambda,\lambda')$ is the union of a finite number, say $N_s(\lambda,\lambda')$, of linear subspaces of $\mathbb{R}^n$ and, consequently, can be written as

$$\mathcal{X}^{\alpha,\omega}(\lambda,\lambda') = \bigcup_{i=1}^{N_s(\lambda,\lambda')} \operatorname{span}\big(B_i(\lambda,\lambda')\big)$$

where the matrix $B_i(\lambda,\lambda')$ represents a basis of the i-th linear subspace composing $\mathcal{X}^{\alpha,\omega}(\lambda,\lambda')$. Thus, since each equality in condition (ii) depends affinely on the vector x, it has to be checked only for the bases $B_i(\lambda,\lambda')$ with $i = 1,\ldots,N_s(\lambda,\lambda')$. In other words, each equality in condition (ii) turns out to be equivalent to the $N_s(\lambda,\lambda')$ conditions

$$\big\{[A(\lambda) - A(\lambda')] - L(\lambda')[C(\lambda) - C(\lambda')]\big\}\,B_i(\lambda,\lambda') = 0 \tag{6}$$

for $i = 1,\ldots,N_s(\lambda,\lambda')$. In general, such conditions may be quite restrictive since they impose a hard constraint on the structure of the gains $L(\lambda)$. For the reader's convenience, this is illustrated by means of the following elementary example.

Example 2 Consider a simple linear switching system described by equation (1) with

$$A(1) \triangleq \begin{bmatrix} 0.5 & 2 \\ 0 & 1 \end{bmatrix}, \quad A(2) \triangleq \begin{bmatrix} 0.5 & 2 \\ 0 & 1 \end{bmatrix},$$
$$C(1) \triangleq \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad C(2) \triangleq \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}. \tag{7}$$

Suppose that, at every time $t = 0, 1, \dots$, we would like to determine the discrete state λ_t on the basis of the observations vector $\mathbf{y}_{t-1}^{t}$ (this corresponds to the choice $\alpha = 1$ and $\omega = 0$). In this case, we have

$$F(1,1) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0.5 & 2 \\ -0.5 & -1 \end{bmatrix}, \quad F(1,2) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ -0.5 & 0 \\ 0.5 & 1 \end{bmatrix}, \quad F(2,1) = \begin{bmatrix} -1 & 2 \\ 1 & -1 \\ 0.5 & 2 \\ -0.5 & -1 \end{bmatrix}, \quad F(2,2) = \begin{bmatrix} -1 & 2 \\ 1 & -1 \\ -0.5 & 0 \\ 0.5 & 1 \end{bmatrix}.$$

Thus, it can be seen that the sets of continuous states for which it is not possible to distinguish between the two discrete states are given by

$$\mathcal{X}^{1,0}(1,2) = \mathcal{X}^{1,0}(2,1) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \bigcup \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right).$$

As a consequence, in order for the gain $L(1)$ to fulfill condition (ii) of Theorem 2.1 it must be

$$L(1)\,[C(2) - C(1)] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0\,, \quad L(1)\,[C(2) - C(1)] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0\,.$$

It is immediate to verify that the first equality is always verified, while the second one is satisfied if and only if the gain $L(1)$ assumes the form

$$L(1) = \begin{bmatrix} a & a \\ c & c \end{bmatrix}$$

for any $a, c \in \mathbb{R}$. Analogous considerations can be made for the gain $L(2)$. △

As shown in the above example, generally speaking condition (ii) greatly reduces the degrees of freedom available in the choice of the gain matrices $L(\lambda)$. This may even lead to make the conditions of Theorem 2.1 globally unfeasible. However, it is important to remark that there are some special non-trivial cases in which condition (ii) holds regardless of the choice of the gains $L(\lambda)$. More specifically, condition (ii) holds if

(a) $\mathcal{X}^{\alpha,\omega}(\lambda, \lambda') = \{0\}$ for every $\lambda \neq \lambda'$; or
(b) $[A(\lambda) - A(\lambda')]\,x = 0$ and $[C(\lambda) - C(\lambda')]\,x = 0$ for every $x \in \mathcal{X}^{\alpha,\omega}(\lambda, \lambda')$ and for every $\lambda \neq \lambda'$.

Even if case (a) is actually a subcase of (b), for the sake of clarity, we prefer to consider such two cases separately.

First note that (a) corresponds to the complete observability of the discrete state λ_t: in this case, unless $x_t = 0$, the set of feasible discrete states $\Lambda^{\alpha,\omega}(\mathbf{y}_{t-\alpha}^{t+\omega})$ has always cardinality 1 and therefore the current discrete state λ_t can be determined uniquely on the basis of the observations vector $\mathbf{y}_{t-\alpha}^{t+\omega}$. As shown by Alessandri et al. (2005); Babaali and Egerstedt (2004), where the results presented in (Vidal et al., 2002) are extended, a necessary and sufficient condition for (a) to hold is that the rank of the joint observability matrix $[F(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)\ \ F(\boldsymbol{\lambda}'_- \otimes \lambda' \otimes \boldsymbol{\lambda}'_+)]$ is equal to $2n$ for every $\lambda \neq \lambda'$ and every $\boldsymbol{\lambda}_-, \boldsymbol{\lambda}'_- \in \Lambda^{\alpha}$ and $\boldsymbol{\lambda}_+, \boldsymbol{\lambda}'_+ \in \Lambda^{\omega}$.

With this respect, it is worth noting that the stability results by Alessandri et al. (2005) in the framework of receding-horizon estimation were derived under an assumption similar to (a) (see Theorem 2 in (Alessandri et al., 2005)). In the light of Theorem 2.1, the Luenberger-like estimation scheme proposed in this paper can be applied to a broader class of switching systems.

A situation for which condition (a) does not hold, but falling within case (b), is given by the simple system considered in Example 1, for which

$$[A(1) - A(2)] \begin{bmatrix} 0 \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad [C(1) - C(2)] \begin{bmatrix} 0 \\ k \end{bmatrix} = 0\,.$$

It is worth noting that Theorem 2.1 ensures the convergence of the estimation error regardless of the values of the estimates $\hat{\lambda}_t$, $t = \alpha, \alpha+1, \ldots,$ as long as they are chosen inside the sets $\Lambda^{\alpha,\omega}(\mathbf{y}_{t-\alpha}^{t+\omega})$. However, it should be clear that a sensible choice of such estimates could improve the performance of the proposed observer. With this respect, a reasonable criterion results from the minimum residual evaluation test, i.e.,

$$\hat{\lambda}_t = \arg \min_{\lambda \in \Lambda^{\alpha,\omega}(\mathbf{y}_{t-\alpha}^{t+\omega})} \|y_t - C(\lambda)\hat{x}_t\|^2 .$$

2.3 *An LMI-based approach for the synthesis of optimized observers*

As it is difficult to find a common Lyapunov function once the gains have been selected, it is preferable to simultaneously choose the matrices $L(\lambda)$ and P. This problem may be reduced to a simpler form that is well-suited to being solved by means of an LMI method. Likewise in Alessandri and Coletta (2001), using the Schur lemma and applying the change of variables $Y(\lambda) = P\,L(\lambda)$, each inequality in condition (i) of Theorem 2.1 turns out to be equivalent to

$$\begin{bmatrix} P & (P\,A(\lambda) - Y(\lambda)\,C(\lambda))^\top \\ P\,A(\lambda) - Y(\lambda)\,C(\lambda) & P \end{bmatrix} > 0 . \tag{8}$$

Moreover, recalling (6), each equality in condition (ii) of Theorem 2.1 can be rewritten as

$$\left\{P[A(\lambda) - A(\lambda')] - Y(\lambda')[C(\lambda) - C(\lambda')]\right\} B_i(\lambda, \lambda') = 0 \tag{9}$$

for $i = 1, \ldots, N_s(\lambda, \lambda')$.

By exploiting (8) and (9), observer (3) can be constructed by solving the following LMI problem.

Problem 1 *Find* $P = P^\top > 0$ *and* $Y(\lambda)$, $\lambda \in \Lambda$, *such that conditions (8) and (9) are satisfied for any* $\lambda, \lambda' \in \Lambda$ *and take the observer gains* $L(\lambda) = P^{-1}\,Y(\lambda)$. □

The satisfaction of the Lyapunov inequalities (8) and of the equalities (9) guarantees to get an exponentially stable error dynamics. In addition, an upper bound on a quadratic cost function of the estimation error can be found and, consequently, the gains of observer (3) may be selected so as to minimize it. To this end, consider the performance index

$$J = \lim_{N \to \infty} \sum_{t=\alpha}^{N} e_t^\top\, Q\, e_t \tag{10}$$

where the matrix $Q > 0$ can be arbitrarily chosen. An observer of the type (3) can be designed by finding the gain matrices such that the stability requirements be satisfied and an upper bound on (10) be minimized. To this aim, let Problem 1 be solvable, i.e., there exists a symmetric matrix $P > 0$ that satisfies inequalities (8). Then, given a symmetric weight matrix $Q > 0$, there exists a scalar $\mu > 0$ such that

$$[A(\lambda) - L(\lambda)C(\lambda)]^\top \frac{P}{\mu} [A(\lambda) - L(\lambda)C(\lambda)] - \frac{P}{\mu} < -Q\,, \quad \lambda \in \Lambda. \tag{11}$$

If we apply (11) at time t, we obtain

$$e_t^\top \left\{ \left[A(\hat{\lambda}_t) - L(\hat{\lambda}_t)C(\hat{\lambda}_t)\right]^\top \frac{P}{\mu} \left[A(\hat{\lambda}_t) - L(\hat{\lambda}_t)C(\hat{\lambda}_t)\right] - \frac{P}{\mu} \right\} e_t \le -e_t^\top Q\, e_t\,. \tag{12}$$

Note that, when condition (ii) of Theorem 2.1 holds (i.e., all the equalities (9) are satisfied), the error dynamics turns out to be

$$e_{t+1} = \left[A(\hat{\lambda}_t) - L(\hat{\lambda}_t)C(\hat{\lambda}_t)\right] e_t\,, \quad t = \alpha, \alpha+1, \ldots\,. \tag{13}$$

Thus, summing all the inequalities (12) from $t=\alpha$ to $t=N$ and applying repeatedly (13), we obtain

$$\sum_{t=\alpha}^{N} e_t^\top Q\, e_t \le e_\alpha^\top \frac{P}{\mu} e_\alpha - e_N^\top \left[A(\hat{\lambda}_N) - L(\hat{\lambda}_N)C(\hat{\lambda}_N)\right]^\top \frac{P}{\mu} \left[A(\hat{\lambda}_N) - L(\hat{\lambda}_N)C(\hat{\lambda}_N)\right] e_N\,.$$

The limit for $N \to \infty$ provides an upper bound on the cost functions as follows:

$$J = \lim_{N\to\infty} \sum_{t=\alpha}^{N} e_t^\top Q\, e_t \le e_\alpha^\top \frac{P}{\mu} e_\alpha\,.$$

This upper bound can be minimized by reducing the maximum eigenvalue of $\dfrac{P}{\mu}$. To this end, let us consider a scalar $\nu > 0$ such that $\nu I > \dfrac{P}{\mu}$. Moreover, using the Schur Lemma, note that (11) can be equivalently written as

$$\begin{bmatrix} P - \mu Q & (P\,A(\lambda) - Y(\lambda)\,C(\lambda))^\top \\ P\,A(\lambda) - Y(\lambda)\,C(\lambda) & P \end{bmatrix} > 0 \tag{14}$$

for $\lambda \in \Lambda$. Therefore, the observer can be designed by solving the following problem.

Problem 2 *Given a symmetric positive definite matrix Q, find $\nu > 0$, $\delta > 0$, $P = P^\top > 0$, and $Y(\lambda)$, $\lambda \in \Lambda$, that minimize ν under the constraints (14) for $\lambda \in \Lambda$, (9) for $\lambda, \lambda' \in \Lambda$, and*

$$\nu I - \frac{P}{\mu} > 0\,.$$

Then take the observer gains $L(\lambda) = P^{-1}\,Y(\lambda)$, for any $\lambda \in \Lambda$. □

Problem 2 can be solved by using LMI-based iterative optimization methods as the conditions are LMIs if either μ or ν is kept constant.

Example 1 (continued) Let us now consider once again the simple system described in Example 1. Furthermore, suppose that the weight matrix Q is chosen to be equal to I. By using the routines of the Matlab LMI Toolbox, the following solution of Problem 2 was obtained:

$$L(1) = \begin{bmatrix} 1.3596 \\ -1.8597 \end{bmatrix}, \quad L(2) = \begin{bmatrix} 4.0815 \\ -3.9012 \end{bmatrix}.$$

It is immediate to verify that such gains satisfy condition (i) of Theorem 3.2 with the Lyapunov

$$P = \begin{bmatrix} 212.2196 & 242.1431 \\ 242.1431 & 281.5651 \end{bmatrix}.$$

Since for the considered system condition (ii) is automatically verified, such gains involve an estimation error exponentially convergent to zero, provided that the estimates $\hat{\lambda}_t$ are chosen inside the sets $\Lambda^{\alpha,\omega}(\mathbf{y}_{t-\alpha}^{t+\omega})$, $\quad t = \alpha, \alpha+1, \ldots$. △

3 Design of the observer in the presence of bounded noises

It is natural to ask whether the estimation scheme proposed in the previous section can be generalized in order to take into account the presence of noises affecting the system and the measurement equations. With this respect, let us now suppose that system (1) is affected by noises, i.e., let us consider the noisy switching discrete-time linear system

$$\begin{aligned} x_{t+1} &= A(\lambda_t)\, x_t + w_t \\ y_t &= C(\lambda_t)\, x_t + v_t \end{aligned} \tag{15}$$

where $w_t \in \mathcal{W} \subset \mathbb{R}^n$ is the system noise vector and $v_t \in \mathcal{V} \subset \mathbb{R}^m$ is the measurement noise vector. We assume the statistics of w_t and v_t to be unknown.

First, it is important to remark that the definition of $\Lambda^{\alpha,\omega}(\mathbf{y})$ given in the previous section should be modified in order to take into account the presence of noises. Indeed, if the noise vectors are not identically null, in general the noisy observations vector $\mathbf{y}_{t-\alpha}^{t+\omega}$ does not belong to the set $\mathcal{Y}^{\alpha,\omega}(\lambda_t)$. However, if the noise vectors are "small," it is reasonable to think that $\mathbf{y}_{t-\alpha}^{t+\omega}$ is "close" (in some sense) to such a set. This simple intuition leads us to adopt a *minimum-distance criterion* for the estimation of the discrete state.

Towards this end, given a generic switching sequence $\boldsymbol{\lambda} \in \Lambda^N$, let us denote by $d(\mathbf{y}, \boldsymbol{\lambda})$ the distance between the observation sequence $\mathbf{y} \in \mathbb{R}^{mN}$ and the linear subspace $\mathcal{Y}(\boldsymbol{\lambda})$. Clearly, $d(\mathbf{y}, \boldsymbol{\lambda})$ can be obtained as

$$d(\mathbf{y}, \boldsymbol{\lambda}) = \| \, [I - \Pi(\boldsymbol{\lambda})] \, \mathbf{y} \, \|$$

where $\Pi(\boldsymbol{\lambda})$ is the matrix of the orthogonal projection on the linear subspace $\mathcal{Y}(\boldsymbol{\lambda})$. Then the distance $d^{\alpha,\omega}(\mathbf{y}, \lambda)$ between the observations vector $\mathbf{y}$ and the set $\mathcal{Y}^{\alpha,\omega}(\lambda)$ can be obtained as

$$d^{\alpha,\omega}(\mathbf{y}, \lambda) = \min_{\boldsymbol{\lambda}_- \in \Lambda^\alpha, \boldsymbol{\lambda}_+ \in \Lambda^\omega} d(\mathbf{y}, \boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+) \, .$$

The above-defined quantities allow one to define a new set $\bar{\Lambda}^{\alpha,\omega}(\mathbf{y})$ of "candidate" discrete states in the presence of noises that is made up of all the discrete states $\lambda \in \Lambda$ with minimum distance $d^{\alpha,\omega}(\mathbf{y}, \lambda)$, i.e.,

$$\bar{\Lambda}^{\alpha,\omega}(\mathbf{y}) \triangleq \left\{ \lambda \in \Lambda : \quad d^{\alpha,\omega}(\mathbf{y}, \lambda) \le d^{\alpha,\omega}(\mathbf{y}, \lambda') \, , \quad \forall \lambda' \in \Lambda \right\} \, .$$

At every time $t = \alpha, \alpha+1, \ldots$, the estimate $\hat{\lambda}_t$ of the discrete state λ_t is chosen inside the set $\bar{\Lambda}^{\alpha,\omega}(\mathbf{y}_{t-\alpha}^{t+\omega})$. It is important to note that such a criterion can be always applied regardless of the form of the sets $\mathcal{W}$ and $\mathcal{V}$ to which the system and measurement noises belong. Moreover, an exact knowledge of the form of such sets is not required.

By exploiting the foregoing definitions, in the noisy case the estimator can be obtained as

$$\begin{aligned} &\hat{x}_{t+1} = A(\hat{\lambda}_t)\, \hat{x}_t + L(\hat{\lambda}_t) \, \left[y_t - C(\hat{\lambda}_t)\, \hat{x}_t \right] \\ &\hat{\lambda}_t \in \bar{\Lambda}^{\alpha,\omega}(\mathbf{y}_{t-\alpha}^{t+\omega}) \end{aligned} \tag{16}$$

for $t = \alpha, \alpha+1, \ldots$.

In Section 2, it has been pointed out that, in the absence of noises, the possibility of distinguishing between the actual discrete state $\lambda_t = \lambda$ and another discrete state λ' on the basis of the observations vector $y_{t-\alpha}^{t+\omega}$ depends on the actual continuous state x_t. With this respect, the set $\mathcal{X}^{\alpha,\omega}(\lambda, \lambda')$ has been defined. Such a set turned out to be quite important in the design of the filter gains in order to ensure the convergence of the estimation error (see condition (ii) of Theorem 2.1). Of course, it would be interesting to know whether similar considerations hold also in the noisy case when the estimate $\hat{\lambda}_t$ is chosen inside the set $\bar{\Lambda}_t^{\alpha,\omega}$ (i.e., according to a minimum-distance criterion). Towards this end, a new set $\bar{\mathcal{X}}^{\alpha,\omega}(\lambda, \lambda')$ can be defined that represents the generalization of the set $\mathcal{X}^{\alpha,\omega}(\lambda, \lambda')$. This set is made up of all the continuous states x such that possible observation sequences $\mathbf{y}$ exist that could be obtained if $x_t = x$ and $\lambda_t = \lambda$ but the distance $d^{\alpha,\omega}(\mathbf{y}, \lambda')$ is minimal.

In order to define formally the set $\bar{\mathcal{X}}^{\alpha,\omega}(\lambda, \lambda')$, some preliminary definitions are needed. Given a generic sequence $\boldsymbol{\lambda} \in \Lambda^N$ of N discrete states, i.e., $\boldsymbol{\lambda} \triangleq \operatorname{col}\left[\lambda^{(1)}, \ldots, \lambda^{(N)}\right]$, let us define

$$H(\boldsymbol{\lambda}) \triangleq \begin{bmatrix} 0 & \cdots & 0 \\ C(\lambda^{(2)}) & \cdots & 0 \\ C(\lambda^{(3)})A(\lambda^{(2)}) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ C(\lambda^{(N)}) \prod_{i=1}^{N-2} A(\lambda^{(N-i)}) & \cdots & C(\lambda^{(N)}) \end{bmatrix}$$

and

$$\Gamma(\boldsymbol{\lambda}) \triangleq \left[\prod_{i=1}^{N-1} A(\lambda^{(N-i)}) \ldots A(\lambda^{(1)}) \; I \right].$$

By exploiting such definitions, the set $\bar{\mathcal{X}}^{\alpha,\omega}(\lambda, \lambda')$ can be determined as the set of all the continuous states x for which some $\tilde{x} \in \mathbb{R}^n$, some $\boldsymbol{\lambda}_- \in \Lambda^\alpha$, some $\boldsymbol{\lambda}_+ \in \Lambda^\omega$, some $\mathbf{w}_- \in \mathcal{W}^\alpha$, some $\mathbf{w}_+ \in \mathcal{W}^\omega$, and some $\mathbf{v} \in \mathcal{V}^{\alpha+1+\omega}$ exist such that

(i) $x = \Phi(\boldsymbol{\lambda}_-)\tilde{x} + \Gamma(\boldsymbol{\lambda}_-)\mathbf{w}_-$ (i.e., the state x may belong to the state trajectory);

(ii) $\lambda' \in \bar{\Lambda}^{\alpha,\omega}(\mathbf{y})$ where $\mathbf{y} = F(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)\,\tilde{x} + H(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)\,(\mathbf{w}_- \otimes \mathbf{w}_+) + \mathbf{v}$ (i.e., the discrete state λ' may be a candidate discrete state for possible observation sequences associated with λ and x).

In other words, $\bar{\mathcal{X}}^{\alpha,\omega}(\lambda, \lambda')$ is the set of continuous states for which the minimum-distance criterion may lead to choose as possible estimate λ' instead of λ.

The following technical lemma gives a characterization of such a set as a function of the set $\mathcal{X}^{\alpha,\omega}(\lambda, \lambda')$.

Lemma 3.1 *Suppose that the sets $\mathcal{W}$ and $\mathcal{V}$ are bounded. Then, for any $\lambda, \lambda' \in \Lambda$, each vector $x \in \bar{\mathcal{X}}^{\alpha,\omega}(\lambda, \lambda')$ can be written as*

$$x = x^h + x^b \tag{17}$$

where $x^h \in \mathcal{X}^{\alpha,\omega}(\lambda, \lambda')$ and x^b is norm-bounded, i.e., there exists a suitable constant $k(\lambda, \lambda')$ (see the proof) such that $\|x^b\| \leq k(\lambda, \lambda')$. □

Lemma 3.1 ensures that, as long as the current state $x_t = x$ is "far enough" from the set $\mathcal{X}^{\alpha,\omega}(\lambda, \lambda')$, then, even in the presence of bounded noises, it is possible to distinguish between the discrete states λ and λ' on the basis of the observations vector $\mathbf{y}_{t-\alpha}^{t+\omega}$ and according to a minimum-distance criterion. In the light of such a result, the following theorem can be stated.

Theorem 3.2 *Suppose that the sets $\mathcal{W}$ and $\mathcal{V}$ are bounded. Furthermore, suppose that the gains $L(\lambda)$, $\lambda \in \Lambda$ satisfy conditions (i) and (ii) of Theorem 2.1. Then observer (16) involves an estimation error that can be upper bounded as*

$$\|e_t\| \leq h\,\beta^{t-\alpha}\,\|e_\alpha\| + \frac{1-\beta^{t-\alpha}}{1-\beta}\,\gamma\,, \quad t = \alpha, \alpha+1, \ldots\,, \tag{18}$$

for some $0 < \beta < 1$, $h > 0$, and $\gamma > 0$ (see the proof). □

Note that, since $\beta < 1$, the upper bound on the estimation error given in Theorem 3.2 converges exponentially to the asymptotic value $\gamma/(1-\beta)$.

Of course, in this case, the presence of disturbances makes it impossible to achieve the convergence to zero of the estimation error. However, Theorem 3.2 ensures that, under suitable assumptions, the proposed estimation scheme always provides an asymptotically bounded estimation error. Moreover, it is important to point out that the asymptotic upper bound $\gamma/(1-\beta)$ depends on the "amplitude" of the noises (i.e., on the sets $\mathcal{W}$ and $\mathcal{V}$): the smaller the noises the smaller is the upper bound. When the noises are identically null (i.e., $\mathcal{W} = \{0\}$ and $\mathcal{V} = \{0\}$), it turns out that $\gamma = 0$ and so, in this particular case, Theorem 3.2 provides the same convergence result of Theorem 2.1.

4 Simulation results

In this section, a simulation example is given to illustrate the effectiveness of the proposed approach to state estimation for switching linear systems. Let us consider the discretized equations of an undamped oscillator that may switch between two different frequencies

$$A(1) = \begin{bmatrix} \cos(\omega_1\Delta) & -\omega_1 \sin(\omega_1\Delta) \\ \dfrac{1}{\omega_1}\sin(\omega_1\Delta) & \cos(\omega_1\Delta) \end{bmatrix}, \quad A(2) = \begin{bmatrix} \cos(\omega_2\Delta) & -\omega_2 \sin(\omega_1\Delta) \\ \dfrac{1}{\omega_2}\sin(\omega_2\Delta) & \cos(\omega_2\Delta) \end{bmatrix},$$
$$C(1) = C(2) = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{19}$$

where $\omega_1 = 1$, $\omega_2 = 2$, and the sampling time Δ is equal to 0.1. Note that both the system matrices have the two eigenvalues on the unit circle. However, due to the switching nature of the system, the trajectory of the continuous state may show a divergent behavior even in the absence of noises. We assume that the considered system has a minimum dwell time (i.e., the minimum number of steps between a switch and the next one) equal to 7. Moreover, we assume that x_0, w_t and v_t, $t = 0, 1, \ldots$, are independent random variables uniformly distributed in the sets $\mathcal{X}_0 = [-\bar{x}, \bar{x}]^2$, $\mathcal{W} = [-\bar{w}, \bar{w}]^2$, and $\mathcal{V} = [-\bar{v}, \bar{v}]$, respectively.

It is immediate to verify that, for the considered switching system, by choosing $\alpha = 1$ and $\omega = 2$ it is possible to ensure the complete observability of the discrete state since it turns out that $\mathcal{X}^{1,2}(1,2) = \mathcal{X}^{1,2}(2,1) = \{0\}$. Thus, in this case, condition (ii) of Theorem 2.1 is automatically verified.

In the following, for the sake of brevity, we shall refer to the estimator (16) (where the gains are chosen by solving Problem 2 with Q equal to the identity matrix) as the *Minimum-Distance Switching Luenberger Observer* (MDSLO). In order to evaluate the ability of the proposed estimation scheme to deal with unknown switches in the discrete state, we compare the proposed filter with the Luenberger switching observer obtained exploiting the exact knowledge of the discrete state instead of estimating it. Such an estimator corresponds to the one proposed by Alessandri and Coletta (2001) and will be called the *Switching Luenberger Observer with Perfect Information* (SLOPI).

In Fig. 1, the behaviors of the true values and the estimates of the two state components are shown for a randomly chosen simulation. In Figs. 2 and 3, the plots of the *Root Mean Square Errors* (RMSEs) (computed over 10^4 randomly chosen simulations) for the considered filters are shown in two different noise situations. Note that in both cases the proposed observer provides asymptotic performances similar to those of the SLOPI. This behavior is due to the divergence of the continuous state in most simulation runs. In fact, since in the considered framework we have $\mathcal{X}^{1,2}(1,2) = \mathcal{X}^{1,2}(2,1) = \{0\}$, in the presence of bounded disturbances the sets $\bar{\mathcal{X}}^{1,2}(1,2)$ and $\bar{\mathcal{X}}^{1,2}(2,1)$ turn out to be bounded (see Lemma 3.1). As a consequence, the divergent trajectories of the continuous state eventually escape the set $\bar{\mathcal{X}}^{1,2}(1,2) \cup \bar{\mathcal{X}}^{1,2}(2,1)$, thus making it possible to exactly identify the discrete state on the basis of the observations by means of the proposed minimum-distance criterion. As to the transient behavior, the higher the noises the slower is the convergence speed of the MDSLO. This is consistent with the fact that higher noises make it more difficult to determine the discrete state on the basis of the observations sequence.

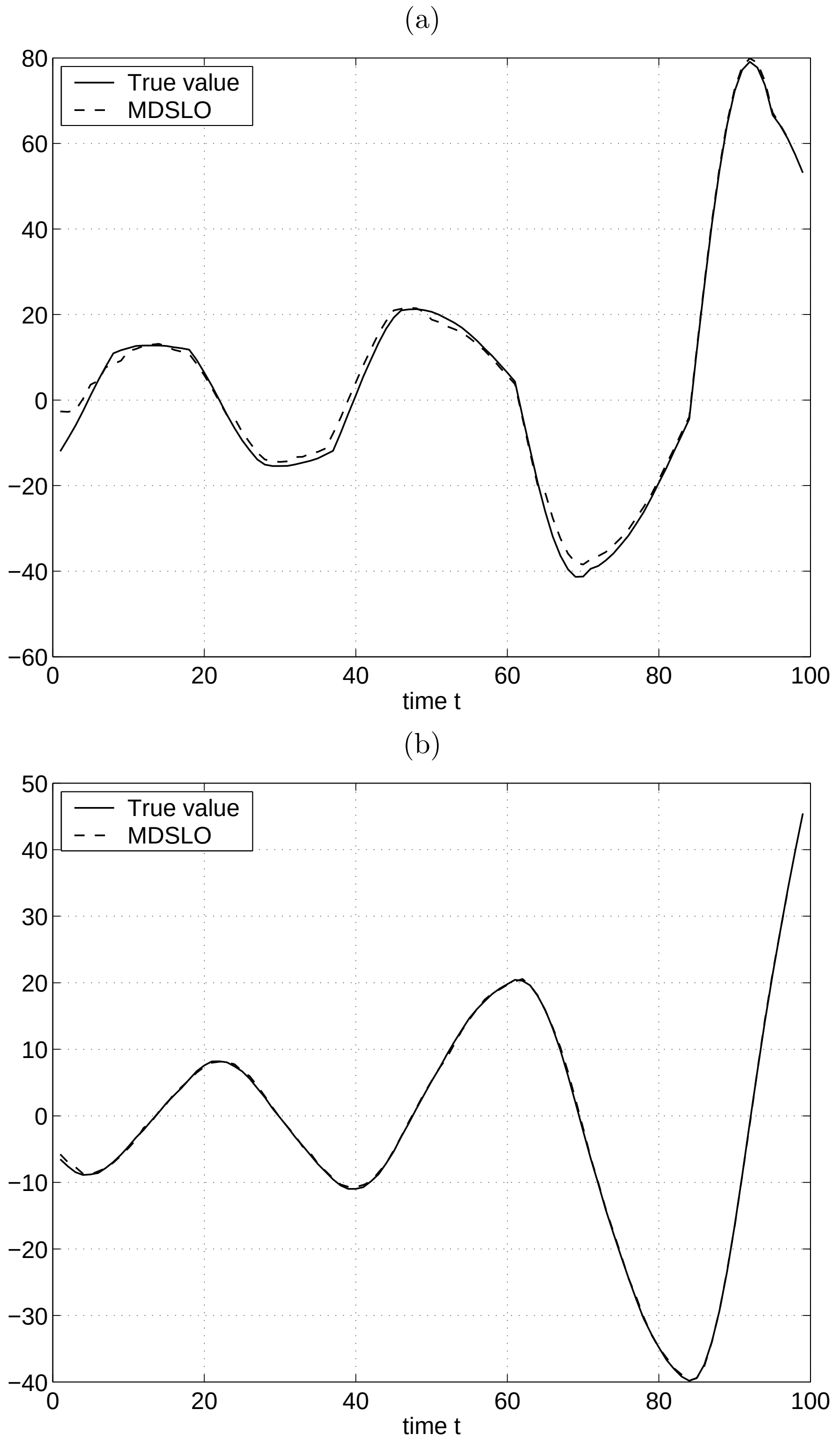

Figure 1. True values and estimates obtained with the MDSLO of the first (a) and the second (b) component of the state for a randomly chosen simulation with $\bar{x} = 10$, $\bar{w} = 0.1$, $\bar{v} = 0.1$.

5 Conclusions

The main topic of this paper is the problem of constructing observers for switching discrete-time linear systems where the law governing the evolution of the discrete state is supposed to be unknown. Conditions ensuring the stability of the error dynamics for these observers have been found and an LMI formulation for such conditions has been presented. The proposed estimation scheme has been suitably modified in order to provide an exponentially bounded estimation error in the presence of bounded disturbances affecting

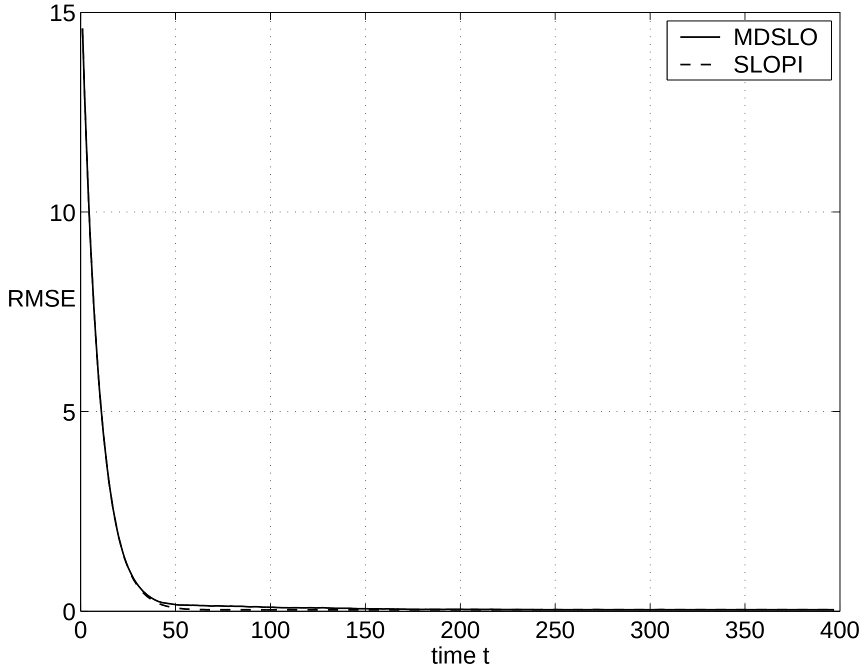

Figure 2. RMSEs of the considered filters for $\bar{x} = 10$, $\bar{w} = 0.01$, and $\bar{v} = 0.01$.

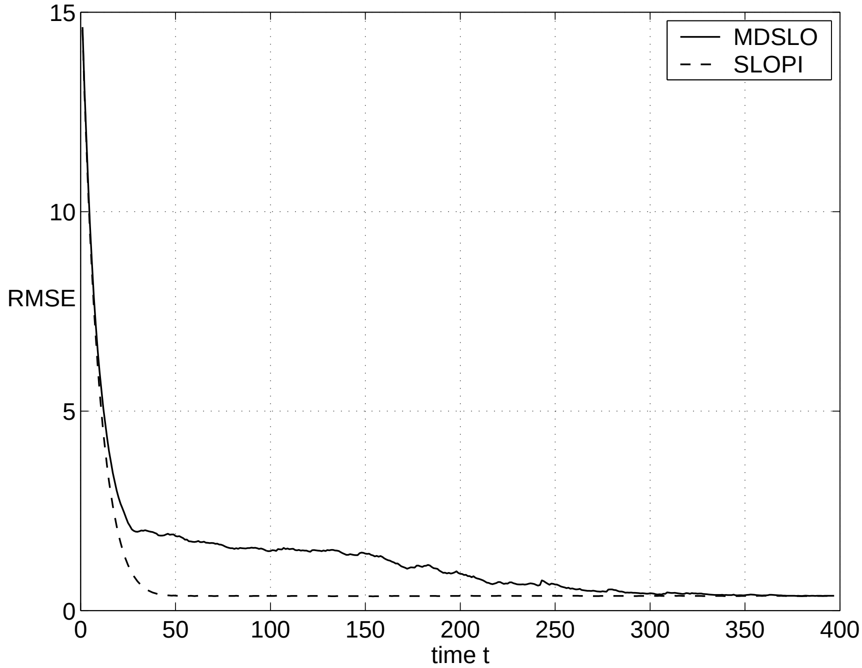

Figure 3. RMSEs of the considered filters for $\bar{x} = 10$, $\bar{w} = 0.1$, and $\bar{v} = 0.1$.

both the system and the measurement equations. With respect to previous results, the conditions ensuring the stability of the estimation error have been relaxed, thus making the approach applicable to a more general class of switching systems.

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Appendix A: Proofs

Proof of Theorem 2.1: Let us consider the error dynamics computed by means of (1) and (3)

$$e_{t+1} = \left[A(\hat{\lambda}_t) - L(\hat{\lambda}_t)\, C(\hat{\lambda}_t)\right]\, e_t + \left[A(\lambda_t) - A(\hat{\lambda}_t)\right] x_t - L(\hat{\lambda}_t)\, \left[C(\lambda_t) - C(\hat{\lambda}_t)\right]\, x_t, \quad t = 0, 1, \ldots \tag{A1}$$

There may be two cases, i.e., either $\hat{\lambda}_t = \lambda_t$ or $\hat{\lambda}_t \neq \lambda_t$. In the former case, (A1) yields

$$e_{t+1} = [A(\lambda_t) - L(\lambda_t)\, C(\lambda_t)]\; e_t, \quad t = 0, 1, \ldots \, . \tag{A2}$$

In the latter, since $\hat{\lambda}_t \in \Lambda^{\alpha,\omega}(\mathbf{y}_{t-\alpha}^{t+\omega})$, it is immediate to see that x_t must belong to $\mathcal{X}^{\alpha,\omega}(\lambda_t, \hat{\lambda}_t)$. Consequently, by applying condition (ii), we have

$$\left\{ \left[A(\lambda_t) - A(\hat{\lambda}_t) \right] - L(\hat{\lambda}_t) \left[C(\lambda_t) - C(\hat{\lambda}_t) \right] \right\} x_t = 0$$

and then, from (A1),

$$e_{t+1} = \left[A(\hat{\lambda}_t) - L(\hat{\lambda}_t)\, C(\hat{\lambda}_t) \right] e_t, \quad t = 0, 1, \ldots \; . \tag{A3}$$

By referring to either (A2) or (A3), let us consider the norm weighted by the positive definite matrix P, i.e., $\|e_t\|_P = (e_t^\top P\, e_t)^{1/2}$, and its induced matrix norm. If we define

$$\beta \triangleq \max_{\lambda \in \Lambda} \|A(\lambda) - L(\lambda) C(\lambda)\|_P \,,$$

we obtain $\|e_{t+1}\|_P \leq \beta \, \|e_t\|_P$ and so

$$\|e_t\|_P \leq \beta^{t-\alpha} \|e_\alpha\|_P \, .$$

Clearly, condition (i) implies $\beta < 1$. Then, in order to conclude this proof, it is sufficient to note that $\|e_\alpha\|_P \leq [\sigma_{\max}(P)]^{1/2} \|e_\alpha\|$ and $\|e_t\|_P \geq [\sigma_{\min}(P)]^{1/2} \|e_t\|$ and, consequently, to define $h \triangleq [\sigma_{\max}(P)/\sigma_{\min}(P)]^{1/2}$. ■

Proof of Lemma 3.1: Let us consider a continuous state x obtained as

$$x = \Phi(\boldsymbol{\lambda}_-)\, \tilde{x} + \Gamma(\boldsymbol{\lambda}_-)\, \mathbf{w}_- \tag{A4}$$

for some $\tilde{x} \in \mathbb{R}^n$, some $\boldsymbol{\lambda}_- \in \Lambda^\alpha$, and some $\mathbf{w}_- \in \mathcal{W}^\alpha$. By definition, in order for x to belong to the set $\bar{\mathcal{X}}^{\alpha,\omega}$, there must exist some $\boldsymbol{\lambda}_+ \in \Lambda^\omega$, some $\mathbf{w}_+ \in \mathcal{W}^\omega$, and some $\mathbf{v} \in \mathcal{V}^{\alpha+1+\omega}$ such that, for the observation sequence

$$\mathbf{y} = F(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)\, \tilde{x} + H(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)\, (\mathbf{w}_- \otimes \mathbf{w}_+) + \mathbf{v} \,,$$

it turns out that $\lambda' \in \bar{\Lambda}^{\alpha,\omega}(\mathbf{y})$ (i.e., $\mathbf{y}$ can lead to choose λ' as a candidate estimate of the discrete state).

Recalling the definition of the set $\bar{\Lambda}^{\alpha,\omega}(\mathbf{y})$, a discrete state $\lambda' \neq \lambda$ *may* belong to $\bar{\Lambda}^{\alpha,\omega}(\mathbf{y})$ only if

$$d^{\,\alpha,\omega}(\mathbf{y}, \lambda') \leq d^{\,\alpha,\omega}(\mathbf{y}, \lambda)$$

or, equivalently, only if there exist some $\boldsymbol{\lambda}'_- \in \Lambda^\alpha$ and some $\boldsymbol{\lambda}'_+ \in \Lambda^\omega$ such that

$$d(\mathbf{y}, \boldsymbol{\lambda}'_- \otimes \lambda' \otimes \boldsymbol{\lambda}'_+) \leq d(\mathbf{y}, \boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+) \, . \tag{A5}$$

First note that the distance $d(\mathbf{y}, \boldsymbol{\lambda}'_- \otimes \lambda' \otimes \boldsymbol{\lambda}'_+)$ can be written as

$$d(\mathbf{y}, \boldsymbol{\lambda}'_- \otimes \lambda' \otimes \boldsymbol{\lambda}'_+) = \left\| \left[I - \Pi(\boldsymbol{\lambda}'_- \otimes \lambda' \otimes \boldsymbol{\lambda}'_+) \right] \left[F(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)\, \tilde{x} + H(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)\, (\mathbf{w}_- \otimes \mathbf{w}_+) + \mathbf{v} \right] \right\| \, .$$

Thus, by using the reverse triangular inequality and by defining the quantity

$$\delta^{\,\alpha,\omega} \triangleq \sup_{\substack{\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \Lambda^{\alpha+1+\omega}; \\ \mathbf{w} \in \mathcal{W}^{\alpha+\omega};\, \mathbf{v} \in \mathcal{V}^{\alpha+1+\omega}}} \left\| \left[I - \Pi(\boldsymbol{\lambda}') \right] \left[H(\boldsymbol{\lambda}) \mathbf{w} + \mathbf{v} \right] \right\| ,$$

one can obtain the lower bound

$$\begin{aligned} d(\mathbf{y}, \boldsymbol{\lambda}'_- \otimes \lambda' \otimes \boldsymbol{\lambda}'_+) \geq & \left\| \left[I - \Pi(\boldsymbol{\lambda}'_- \otimes \lambda' \otimes \boldsymbol{\lambda}'_+)\right] F(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)\tilde{x} \right\| \\ & - \left\| \left[I - \Pi(\boldsymbol{\lambda}'_- \otimes \lambda' \otimes \boldsymbol{\lambda}'_+)\right] \left[H(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)\,(\mathbf{w}_- \otimes \mathbf{w}_+) + \mathbf{v}\right] \right\| \\ \geq & \left\| \left[I - \Pi(\boldsymbol{\lambda}'_- \otimes \lambda' \otimes \boldsymbol{\lambda}'_+)\right] F(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)\tilde{x} \right\| - \delta^{\,\alpha,\omega} . \end{aligned}$$

Note that the boundedness of the sets $\mathcal{W}$ and $\mathcal{V}$ ensures the finiteness of the scalar $\delta^{\,\alpha,\omega}$.

As to the distance $d(\mathbf{y}, \boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)$, since by definition the term $F(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)\tilde{x}$ belongs to the subspace $\mathcal{Y}(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)$, one can obtain the upper bound

$$d(\mathbf{y}, \boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+) = \left\| \left[I - \Pi(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)\right] \left[H(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)(\mathbf{w}_- \otimes \mathbf{w}_+) + \mathbf{v}\right] \right\| \leq \delta^{\,\alpha,\omega} .$$

As a consequence, inequality (A5) *may* hold only if the continuous state $\tilde{x}$ is such that

$$\left\| \left[I - \Pi(\boldsymbol{\lambda}'_- \otimes \lambda' \otimes \boldsymbol{\lambda}'_+)\right] F(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)\,\tilde{x} \right\| - \delta^{\,\alpha,\omega} \leq \delta^{\,\alpha,\omega}$$

or, equivalently,

$$\left\| \left[I - \Pi(\boldsymbol{\lambda}'_- \otimes \lambda' \otimes \boldsymbol{\lambda}'_+)\right] F(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)\,\tilde{x} \right\| \leq 2\,\delta^{\,\alpha,\omega} . \tag{A6}$$

Let us now decompose the state space into two orthogonal subspaces: the null space of $\left[I - \Pi(\boldsymbol{\lambda}'_- \otimes \lambda' \otimes \boldsymbol{\lambda}'_+)\right] F(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)$ defined as

$$\mathcal{N}\left(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+, \boldsymbol{\lambda}'_- \otimes \lambda' \otimes \boldsymbol{\lambda}'_+\right) \triangleq \left\{\tilde{x} \in \mathbb{R}^n : \quad \left[I - \Pi(\boldsymbol{\lambda}'_- \otimes \lambda' \otimes \boldsymbol{\lambda}'_+)\right] F(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)\,\tilde{x} = 0\right\}$$

and its orthogonal space $\mathcal{N}^{\perp}\left(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+, \boldsymbol{\lambda}'_- \otimes \lambda' \otimes \boldsymbol{\lambda}'_+\right)$ with respect to the Euclidean scalar product. By exploiting such a decomposition, the state vector $\tilde{x}$ can be written as

$$\tilde{x} = \tilde{x}^h + \tilde{x}^b \tag{A7}$$

where $\tilde{x}^h \in \mathcal{N}\left(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+, \boldsymbol{\lambda}'_- \otimes \lambda' \otimes \boldsymbol{\lambda}'_+\right)$ and $\tilde{x}^b \in \mathcal{N}^{\perp}\left(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+, \boldsymbol{\lambda}'_- \otimes \lambda' \otimes \boldsymbol{\lambda}'_+\right)$.

There are two possible cases.

(i) The matrix $\left[I - \Pi(\boldsymbol{\lambda}'_- \otimes \lambda' \otimes \boldsymbol{\lambda}'_+)\right] F(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)$ is null. Then the decomposition (A7) is always given by $\tilde{x}^h = \tilde{x}$ and $\tilde{x}^b = 0$. Note that this is a trivial case, in that inequality (A6) turns out to be satisfied for any $\tilde{x} \in \mathbb{R}^n$.

(ii) The matrix $\left[I - \Pi(\boldsymbol{\lambda}'_- \otimes \lambda' \otimes \boldsymbol{\lambda}'_+)\right] F(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)$ is not null. In this case, it is immediate to see that the term $\left\| \left[I - \Pi(\boldsymbol{\lambda}'_- \otimes \lambda' \otimes \boldsymbol{\lambda}'_+)\right] F(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)\,\tilde{x} \right\|$ can be lower bounded as

$$\left\| \left[I - \Pi(\boldsymbol{\lambda}'_- \otimes \lambda' \otimes \boldsymbol{\lambda}'_+)\right] F(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)\,\tilde{x} \right\| \geq \underline{\sigma}\left\{\left[I - \Pi(\boldsymbol{\lambda}'_- \otimes \lambda' \otimes \boldsymbol{\lambda}'_+)\right] F(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)\right\} \|\tilde{x}^b\|$$

where, given a matrix M, $\underline{\sigma}(M)$ denotes its minimum non-null singular value. Thus, inequality (A5) *may* hold only for continuous states $\tilde{x}$ of the form (A7), where the orthogonal component $\tilde{x}^b$ is such that

$$\|\tilde{x}^b\| \leq \frac{2\,\delta^{\,\alpha,\omega}}{\underline{\sigma}\left\{\left[I - \Pi(\boldsymbol{\lambda}'_- \otimes \lambda' \otimes \boldsymbol{\lambda}'_+)\right] F(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)\right\}} .$$

Recalling (17), each continuous state x belonging to the set $\bar{\mathcal{X}}^{\alpha,\omega}(\lambda, \lambda')$ can be written as

$$x = \Phi(\boldsymbol{\lambda}_-)\,\tilde{x}^h + \Phi(\boldsymbol{\lambda}_-)\,\tilde{x}^b + \Gamma(\boldsymbol{\lambda}_-)\,\mathbf{w}_- .$$

Now, by defining

$$x^h \triangleq \Phi(\boldsymbol{\lambda}_-)\,\tilde{x}^h\,,$$
$$x^b \triangleq \Phi(\boldsymbol{\lambda}_-)\,\tilde{x}^b + \Gamma(\boldsymbol{\lambda}_-)\,\mathbf{w}_-\,,$$

we obtain (17).

First note that $\tilde{x}_h$ belongs to the null space of $\left[I - \Pi(\boldsymbol{\lambda}'_- \otimes \lambda' \otimes \boldsymbol{\lambda}'_+)\right] F(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)$ and then the observation sequence $\mathbf{y}^h = F(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)\,\tilde{x}^h$ is such that $\lambda' \in \Lambda^{\alpha,\omega}(\mathbf{y}^h)$. As a consequence, recalling the definition of the set $\mathcal{X}^{\alpha,\omega}(\lambda,\lambda')$ given in Section 2, it turns out that $x^h \in \mathcal{X}^{\alpha,\omega}(\lambda,\lambda')$.

Let us now consider x^b. In case (i), since we have $\tilde{x}^b = 0$, by defining the constant

$$k^{(i)}(\lambda,\lambda') \triangleq \sup_{\boldsymbol{\lambda}_- \in \Lambda^\alpha,\, \mathbf{w}_- \in \mathcal{W}^\alpha} \left\{ \|\Gamma(\boldsymbol{\lambda}_-)\|\ \|\mathbf{w}_-\| \right\}$$

it turns out that $\|x^b\| \le k^{(i)}(\lambda,\lambda')$. As to case (ii), one can easily see that x^b can be upper bounded as

$$\|x^b\| \le \|\Phi(\boldsymbol{\lambda}_-)\|\ \frac{2\,\delta^{\alpha,\omega}}{\underline{\sigma}\left\{\left[I - \Pi(\boldsymbol{\lambda}'_- \otimes \lambda' \otimes \boldsymbol{\lambda}'_+)\right] F(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)\right\}} + \|\Gamma(\boldsymbol{\lambda}_-)\|\ \|\mathbf{w}_-\|\,.$$

Then it is immediate to verify that $\|x^b\| \le k^{(ii)}(\lambda,\lambda')$ with the constant $k^{(ii)}(\lambda,\lambda')$ defined as

$$k^{(ii)}(\lambda,\lambda') \triangleq \sup_{\boldsymbol{\lambda}_-,\, \boldsymbol{\lambda}'_-,\, \boldsymbol{\lambda}_+,\, \boldsymbol{\lambda}'_+,\, \mathbf{w}_-} \left\{ \|\Phi(\boldsymbol{\lambda}_-)\|\ \frac{2\,\delta^{\alpha,\omega}}{\underline{\sigma}\left\{\left[I - \Pi(\boldsymbol{\lambda}'_- \otimes \lambda' \otimes \boldsymbol{\lambda}'_+)\right] F(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)\right\}} + \|\Gamma(\boldsymbol{\lambda}_-)\|\ \|\mathbf{w}_-\| \right\}$$

where the supremum is extended to every $\mathbf{w}_- \in \mathcal{W}^\alpha$, every $\boldsymbol{\lambda}_-$, $\boldsymbol{\lambda}'_- \in \Lambda^\alpha$, and every $\boldsymbol{\lambda}_+$, $\boldsymbol{\lambda}'_+ \in \Lambda^\omega$ such that $\left[I - \Pi(\boldsymbol{\lambda}'_- \otimes \lambda' \otimes \boldsymbol{\lambda}'_+)\right] F(\boldsymbol{\lambda}_- \otimes \lambda \otimes \boldsymbol{\lambda}_+)$ is not null.

Note that the boundedness of the set $\mathcal{W}$ and the finiteness of the set Λ ensure that both $k^{(i)}(\lambda,\lambda')$ and $k^{(ii)}(\lambda,\lambda')$ are finite. In order to end the proof, it is sufficient to choose $k(\lambda,\lambda')$ as the maximum between $k^{(i)}(\lambda,\lambda')$ and $k^{(ii)}(\lambda,\lambda')$. ■

Proof of Theorem 3.2: The error dynamics computed by means of (15) and (16) is

$$\begin{aligned} e_{t+1} = & \left[A(\hat{\lambda}_t) - L(\hat{\lambda}_t)\,C(\hat{\lambda}_t)\right]\ e_t + \left[A(\lambda_t) - A(\hat{\lambda}_t)\right] x_t \\ & - L(\hat{\lambda}_t)\ \left[C(\lambda_t) - C(\hat{\lambda}_t)\right]\ x_t + w_t - L(\hat{\lambda}_t) v_t \end{aligned} \tag{A8}$$

for $t = 0, 1, \ldots$. There may be two cases, i.e., either $\hat{\lambda}_t = \lambda_t$ or $\hat{\lambda}_t \neq \lambda_t$. In the former case, (A8) yields

$$e_{t+1} = \left[A(\lambda_t) - L(\lambda_t)\,C(\lambda_t)\right]\ e_t + w_t - L(\hat{\lambda}_t) v_t\,.$$

In the latter, since $\hat{\lambda}_t$ belongs to $\bar{\Lambda}^{\alpha,\omega}(\mathbf{y}_{t-\alpha}^{t+\omega})$, the state x_t must belong to $\bar{\mathcal{X}}^{\alpha,\omega}(\lambda_t,\hat{\lambda}_t)$. Hence, in the light of Lemma 3.1, there exist a vector $x_t^h \in \mathcal{X}^{\alpha,\omega}(\lambda_t,\hat{\lambda}_t)$ and a vector x_t^b, with $\|x_t^b\| \le k(\lambda_t,\hat{\lambda}_t)$, such that $x_t = x_t^h + x_t^b$. Since condition (ii) implies

$$\left\{\left[A(\lambda_t) - A(\hat{\lambda}_t)\right] - L(\hat{\lambda}_t)\ \left[C(\lambda_t) - C(\hat{\lambda}_t)\right]\right\} \tilde{x}_t^h = 0\,,$$

in this case the error dynamics can be written as

$$e_{t+1} = \Big[A(\hat{\lambda}_t) - L(\hat{\lambda}_t)\, C(\hat{\lambda}_t)\Big]\; e_t + \Big\{ \Big[A(\lambda_t) - A(\hat{\lambda}_t)\Big] - L(\hat{\lambda}_t)\; \Big[C(\lambda_t) - C(\hat{\lambda}_t)\Big] \Big\}\, x_t^b + w_t - L(\hat{\lambda}_t) v_t \,.$$

Consider now the norm weighted by the positive definite matrix P, i.e., $\|e_t\|_P = (e_t^\top \, P \, e_t)^{1/2}$. If we define the constants β as in the proof of Theorem 2.1 and $\bar{\gamma}$ as

$$\bar{\gamma} = \max_{\lambda,\lambda',x^b,w,v} \left\| \Big\{ \big[A(\lambda) - A(\lambda')\big] - L(\lambda') \; \big[C(\lambda) - C(\lambda')\big] \Big\}\, x^b + w - L(\lambda') v \right\|_P$$

with $\lambda, \lambda' \in \Lambda$, $\|x^b\| \leq k(\lambda, \lambda')$, $w \in \mathcal{W}$, and $v \in \mathcal{V}$, then, by means of triangular inequality, we obtain $\|e_{t+1}\|_P \leq \beta \, \|e_t\|_P + \bar{\gamma}$ and so

$$\|e_t\|_P \leq \beta^{t-\alpha} \|e_\alpha\|_P + \frac{1 - \beta^{t-\alpha}}{1 - \beta}\, \bar{\gamma} \,.$$

Clearly, condition (i) implies $\beta < 1$ and Lemma 3.1 ensures that $\bar{\gamma} < +\infty$. Then, in order to conclude this proof, it is sufficient to note that $\|e_\alpha\|_P \leq [\sigma_{\max}(P)]^{1/2} \|e_\alpha\|$ and $\|e_t\|_P \geq [\sigma_{\min}(P)]^{1/2} \|e_t\|$ and, consequently, to define $h \triangleq [\sigma_{\max}(P)/\sigma_{\min}(P)]^{1/2}$ and $\gamma = \bar{\gamma}/[\sigma_{\min}(P)]^{1/2}$. ■