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Homotopical algebraic geometry I: topos theory

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Abstract

This is the first of a series of papers devoted to lay the foundations of Algebraic Geometry in homotopical and higher categorical contexts. In this first part we investigate a notion of *higher topos*.

For this, we use *S*-categories (i.e. simplicially enriched categories) as models for certain kind of ∞-categories, and we develop the notions of *S-topologies*, *S-sites* and *stacks* over them. We prove in particular, that for an *S*-category *T* endowed with an *S*-topology, there exists a model category of stacks over *T*, generalizing the model category structure on simplicial presheaves over a Grothendieck site of Joyal and Jardine. We also prove some analogs of the relations between topologies and localizing subcategories of the categories of presheaves, by proving that there exists a one-to-one correspondence between *S*-topologies on an *S*-category *T*, and certain *left exact Bousfield localizations* of the model category of pre-stacks on *T*. Based on the above results, we study the notion of *model topos* introduced by Rezk, and we relate it to our model categories of stacks over *S*-sites.

In the second part of the paper, we present a parallel theory where *S*-categories, *S*-topologies and *S*-sites are replaced by *model categories*, *model topologies* and *model sites*. We prove that a canonical way to pass from the theory of stacks over model sites to the theory of stacks over *S*-sites is provided by the simplicial localization construction of Dwyer and Kan. As an

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example of application, we propose a definition of *étale K-theory of ring spectra*, extending the étale K-theory of commutative rings.

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1. Introduction

This is the first part of a series of papers devoted to the foundations of Algebraic Geometry in homotopical and higher categorical contexts, the ultimate goal being a theory of *algebraic geometry over monoidal* ∞*-categories*, a higher categorical generalization of *algebraic geometry over monoidal categories* (as developed, for example, in [Del2,Del1,Ha]). We refer the reader to the Introduction of the research announcement [To-Ve 5] and to [To-Ve 4], where motivations and prospective applications (mainly to the so-called *derived moduli spaces* of [Ko,Ci-Ka1,Ci-Ka2]) are provided. These applications, together with the remaining required *monoidal* part of the theory, will be given in [To-Ve 6].

In the present work we investigate the required theory of *higher sheaves*, or equivalently *stacks*, as well as its associated notion of *higher topoi*.

1.1. Topologies, sheaves and topoi

As we will proceed by analogy, we will start by recalling some basic constructions and results from topos theory, in a way that is suited for our generalization. Our references for this overview are [SGA4-I,Sch,M-M]. Throughout this introduction we will neglect any kind of set theoretical issues, always assuming that categories are *small* when required.

Let us start with a category C and let us denote by $Pr(C)$ the category of presheaves of sets on C (i.e. $Pr(C) := Set^{C^{op}}$). If C is endowed with a Grothendieck topology τ, one can define the notion of τ*-local isomorphisms in* $Pr(C)$ by requiring injectivity and surjectivity only up to a τ-covering. We denote by Σ_τ the subcategory of $Pr(C)$ consisting of local isomorphisms. One possible way to define the category $Sh_\tau(C)$, of sheaves (of sets) on the Grothendieck site (C,τ), is by setting

$$Sh_\tau(C) := \Sigma_\tau^{-1} Pr(C),$$

where $\Sigma_\tau^{-1} Pr(C)$ denotes the *localization* of $Pr(C)$ along Σ_τ i.e. the category obtained from $Pr(C)$ by formally inverting the morphisms in Σ_τ (see [Sch, 19.1, 20.3.6(a)]). The main basic properties of the category $Sh_\tau(C)$ are collected in the following well known theorem.

Theorem 1.0.1. *Let (C, τ) be a Grothendieck site and $Sh_\tau(C)$ its category of sheaves as defined above.*

1. *The category $Sh_\tau(C)$ has all limits and colimits.*
2. *The natural localization morphism $a : Pr(C) \longrightarrow Sh_\tau(C)$ is left exact* (*i.e. commutes with finite limits*) *and has a fully faithful right adjoint $j : Sh_\tau(C) \longrightarrow Pr(C)$.*
3. *The category $Sh_\tau(C)$ is cartesian closed* (*i.e. has internal Hom-objects*).

Of course, the essential image of the functor $j : Sh_\tau(C) \longrightarrow Pr(C)$ is the usual subcategory of sheaves, i.e. of presheaves having descent with respect to τ-coverings, and the localization functor a becomes equivalent to the associated sheaf functor. The definition of $Sh_\tau(C)$ as $\Sigma_\tau^{-1}Pr(C)$ is therefore a way to define the category of sheaves without even mentioning what a sheaf is precisely.

In particular, Theorem 1.0.1 shows that the datum of a topology τ on C gives rise to an adjunction

$$a : Pr(C) \longrightarrow Sh_\tau(C), \quad Pr(C) \longleftarrow Sh_\tau(C) : j,$$

with j fully faithful and a left exact. Such an adjoint pair will be called an *exact localization* of the category $Pr(C)$. Another fundamental result in sheaf theory is the following:

Theorem 1.0.2. *The rule sending a Grothendieck topology τ on C to the exact localization*

$$a : Pr(C) \longrightarrow Sh_\tau(C), \quad Pr(C) \longleftarrow Sh_\tau(C) : j,$$

defines a bijective correspondence between the set of topologies on C and the set of (*equivalences classes*) *of exact localizations of the category $Pr(C)$. In particular, for a category T the following two conditions are equivalent*:

- *There exists a category C and a Grothendieck topology τ on C such that T is equivalent to $Sh_\tau(C)$.*
- *There exists a category C and a left exact localization*

$$a : Pr(C) \longrightarrow T, \qquad Pr(C) \longleftarrow T : j.$$

A category satisfying one the previous conditions is called a Grothendieck topos.

Finally, a famous theorem by Giraud ([SGA4-I, Exp. IV, Theoreme 1.2]) provides an internal characterization of Grothendieck topoi.

Theorem 1.0.3. (Giraud's Theorem). *A category T is a Grothendieck topos if and only if it satisfies the following conditions*:

1. *The category T is has a small set of strong generators.*
2. *The category T has small colimits.*
3. *Sums are disjoint in T* (*i.e. $x_j \times \coprod_i x_i x_k \simeq \emptyset$ for all $j \neq k$*).
4. *Colimits commute with pull backs.*
5. *Any equivalence relation is effective.*

The main results of this work are generalizations to a homotopical setting of the notions of topologies, sites and sheaves satisfying analogs of Theorems 1.0.1–1.0.3. We have chosen to use both the concept of *S-categories* (i.e. simplicially enriched categories) and of *model categories* as our versions of base categories carrying homotopical data. For both we have developed homotopical notions of *topologies*, *sites* and *sheaves*, and proved analogs of Theorems 1.0.1–1.0.3 which we will now describe in more details.

1.2. S-topologies, S-sites and stacks

Let T be a base *S*-category. We consider the category $SPr(T)$, of T^{op}-diagrams in the category *SSet* of simplicial sets. This category can be endowed with an objectwise model structure for which the equivalences are defined objectwise on T. This model category $SPr(T)$ will be called the *model category of pre-stacks on T*, and will be our higher analog of the category of presheaves of sets. The category $SPr(T)$ comes with a natural *Yoneda embedding* $L\underline{h} : T \longrightarrow SPr(T)$, a *up to homotopy analog* of the usual embedding of a category into the category of presheaves on it (see Corollary 2.4.3).

We now consider $\mathrm{Ho}(T)$, the category having the same objects as T but for which the sets of morphisms are the connected components of the simplicial sets of morphisms in T. Though it might be surprising at first sight, we define an *S-topology on the S-category T* to be simply a Grothendieck topology on the category $\mathrm{Ho}(T)$ (see Definition 3.1.1). A pair (T, τ), where T is an *S*-category and τ is an *S*-topology on T, will be called an *S-site*. Of course, when T is a usual category (i.e. all its simplicial sets of morphisms are discrete), an *S*-topology on T is nothing else than a Grothendieck topology on T. Therefore, a site is in particular an *S*-site, and our definitions are actual generalizations of the usual definitions of topologies and sites.

For the category of presheaves of sets on a Grothendieck site, we have already mentioned that the topology induces a notion of local isomorphisms. In the case where (T, τ) is an *S*-site we define a notion of *local equivalences* in $SPr(T)$ (see Definition 3.3.2). When T is a category, and therefore (T, τ) is a site in the usual sense, our notion of local equivalences specializes to the notion introduced by Illusie and later by Jardine ([Ja1]). Our first main theorem is a generalization of the existence of the local model category structure on the category of simplicial presheaves on a site (see [Ja1,Bl]).

Theorem 1.0.4 (*Theorem 3.4.1, Proposition 3.4.10 and Corollary 3.6.2*). *Let (T, τ) be an S-site.*

1. *There exists a model structure on the category $SPr(T)$, called the* local model structure, *for which the equivalences are the local equivalences. This new model category, denoted by $SPr_\tau(T)$, is furthermore the left Bousfield localization of the model category $SPr(T)$ of pre-stacks along the local equivalences.*
2. *The identity functor*

$$\mathrm{Id} : SPr(T) \longrightarrow SPr_\tau(T)$$

commutes with homotopy fibered products.

3. *The homotopy category* $\mathrm{Ho}(SPr_\tau(T))$ *is cartesian closed, or equivalently, it has internal Hom-objects.*

The model category $SPr_\tau(T)$ *is called the* model category of stacks on the S-site (T, τ).

This theorem is our *higher analog* of Theorem 1.0.1. Indeed, the existence of the local model structure formally implies the existence of *homotopy limits* and *homotopy colimits* in $SPr_\tau(T)$, which are homotopical generalizations of the notion of limits and colimits (see [Hi, Section 19]). Moreover, $SPr_\tau(T)$ being a left Bousfield localization of $SPr(T)$, the identity functor $\mathrm{Id} : SPr_\tau(T) \longrightarrow SPr(T)$ is a right Quillen functor and therefore induces an adjunction on the level of homotopy categories

$$a := \mathbb{L}\mathrm{Id} : \mathrm{Ho}(SPr(T)) \longrightarrow \mathrm{Ho}(SPr_\tau(T)),$$
$$\mathrm{Ho}(SPr(T)) \longleftarrow \mathrm{Ho}(SPr_\tau(T)) : j := \mathbb{R}\mathrm{Id}.$$

It is a general property of Bousfield localizations that the functor j is *fully faithful*, and Theorem 1.0.4(2) implies that the functor a is *homotopically left exact*, i.e. commutes with homotopy fibered products. Finally, part (3) of Theorem 1.0.4 is a homotopical analog of Theorem 1.0.1(3).

As in the case of sheaves on a site, it remains to characterize the essential image of the inclusion functor $j : \mathrm{Ho}(SPr_\tau(T)) \longrightarrow \mathrm{Ho}(SPr(T))$. One possible homotopy analog of the sheaf condition is the *hyperdescent property* for objects in $SPr(T)$ (see Definition 3.4.8). It is a corollary of our proof of the existence of the local model structure $SPr_\tau(T)$ that the essential image of the inclusion functor $j : \mathrm{Ho}(SPr_\tau(T)) \longrightarrow \mathrm{Ho}(SPr(T))$ is exactly the full subcategory of objects satisfying the hyperdescent condition (see Corollary 3.4.7). We call these objects *stacks* over the S-site (T, τ) (Definition 3.4.9). The functor $a : \mathrm{Ho}(SPr(T)) \longrightarrow \mathrm{Ho}(SPr_\tau(T))$ can then be identified with the *associated stack functor* (Definition 3.4.9).

Finally, we would like to mention that the model categories $SPr_\tau(T)$ are not in general Quillen equivalent to model categories of simplicial presheaves on some site. Therefore, Theorem 1.0.4 is a new result in the sense that neither its statement nor its proof can be reduced to previously known notions and results in the theory of simplicial presheaves.

1.3. Model topoi and S-topoi

Based on the previously described notions of S-sites and stacks, we develop a related theory of *topoi*. For this, note that Theorem 1.0.4 implies that an S-topology τ on an S-category T gives rise to the model category $SPr_\tau(T)$, which is a left Bousfield localization of the model category $SPr(T)$. This Bousfield localization has moreover the property that the identity functor $\mathrm{Id} : SPr(T) \longrightarrow SPr_\tau(T)$ preserves homotopy fibered products. We call such a localization a *left exact Bousfield localization* of $SPr(T)$ (see Definition 3.8.1). This notion is a homotopical analog of the notion of exact localization appearing in topos theory as reviewed before Theorem 1.0.2. The rule $\tau \mapsto SPr_\tau(T)$, defines a map from the set of S-topologies on a given S-category T to the set of left exact Bousfield localizations of the model category $SPr(T)$. The model category

$SPr_\tau(T)$ also possesses a natural *additional* property, called *t-completeness* which is a new feature of the homotopical context which does not have any counterpart in classical sheaf theory (see Definition 3.8.2). An object x in some model category M is called *n-truncated* if for any $y \in M$, the mapping space $Map_M(y,x)$ is an n-truncated simplicial set; an object in M is *truncated* if it is n-truncated for some $n \geqslant 0$. A model category M will then be called *t-complete* if truncated objects detect isomorphisms in $\mathrm{Ho}(M)$: a morphism $u : a \to b$ in $\mathrm{Ho}(M)$ is an isomorphism if and only if, for any truncated object x in $\mathrm{Ho}(M)$, the map $u^* : [b,x] \longrightarrow [a,x]$ is bijective.

The notion of t-completeness is very natural and very often satisfied as most of the equivalences in model categories are defined using isomorphisms on certain homotopy groups. The t-completeness assumption simply states that an object with trivial homotopy groups is homotopically trivial, which is a very natural and intuitive condition. The usefulness of this notion of t-completeness is explained by the following theorem, which is our analog of Theorem 1.0.2.

Theorem 1.0.5 (*Theorem 3.8.3 and Corollary 3.8.5*). *Let T be an S-category. The correspondence $\tau \mapsto SPr_\tau(T)$ induces a bijection between S-topologies on T and t-complete left exact Bousfield localizations of $SPr(T)$. In particular, for a model category M the following two conditions are equivalent*:

- *There exists an S-category T and an S-topology on T such that M is Quillen equivalent to $SPr_\tau(T)$.*
- *The model category M is t-complete and there exists an S-category T such that M is Quillen equivalent to a left exact Bousfield localization of $SPr(T)$.*

A model category satisfying one the previous conditions is called a t-complete model topos.

It is important to stress that there are t-complete model topoi which are *not* Quillen equivalent to any $SPr_\tau(C)$, for C a usual category (see Remark 3.8.7(1)). Therefore, Theorem 1.0.5 also shows the unavoidable relevance of considering topologies on general S-categories rather than only on usual categories. In other words, there is no way to reduce the theory developed in this paper to the theory of simplicial presheaves over Grothendieck sites as done in [Ja1,Jo1].

The above notion of *model topos* was suggested to us by Rezk, who defined a more general notion of *homotopy topos* (a model topos without the t-completeness assumption), which is a model category Quillen equivalent to an arbitrary left exact Bousfield localization of some $SPr(T)$ (see Definition 3.8.1). The relevance of Theorem 1.0.5 is that, on one hand it shows that the notion of S-topology we used is correct exactly because it classifies all (t-complete) left exact Bousfield localizations, and, on the other hand it provides an answer to a question raised by Rezk on which notion of topology could be the source of his homotopy topoi.

It is known that there exist model topoi which are not t-complete (see Remark 3.8.7), and therefore our notion of stacks over S-categories does not model *all* of Rezk's homotopy topoi. However, we are strongly convinced that Theorem 1.0.5 has a more general version, in which the t-completeness assumption is dropped, involving a corresponding notion of *hyper-topology* on S-categories as well as the associated notion of *hyper-stack* (see Remark 3.8.7).

Using the above notion of model topos, we also define the notion of *S-topos*. An *S*-topos is by definition an *S*-category which is equivalent, as an *S*-category, to some *LM*, for *M* a model topos (see Definition 3.8.8). Here we have denoted by *LM* the Dwyer–Kan simplicial localization of *M* with respect to the set of its weak equivalences (see the next paragraph for further explanations on the Dwyer-Kan localization).

1.4. S-Categories and model categories

Most of the *S*-categories one encounters in practice come from model categories via the Dwyer–Kan *simplicial localization*. The simplicial localization is a refined version of the Gabriel–Zisman localization of categories. It associates an *S*-category $L(C, S)$ to any category *C* equipped with a subcategory $S \subset C$ (see (Section 2.2)), such that the homotopy category $\mathrm{Ho}(L(C, S))$ is naturally equivalent to the Gabriel–Zisman localization $S^{-1}C$, but in general $L(C, S)$ contains non-trivial higher homotopical informations. The simplicial localization construction is particularly well behaved when applied to a model category *M* equipped with its subcategory of weak equivalences $W \subset M$: in fact, in this case, the *S*-category $LM := L(M, W)$ encodes the so-called *homotopy mapping spaces* of the model category *M* (see Section 2.2). We will show furthermore that the notions of *S*-topologies, *S*-sites and stacks previously described in this introduction, also have their analogs in the model category context, and that the simplicial localization construction allows one to pass from the theory over model categories to the theory over *S*-categories.

For a model[1] category *M*, we consider the category $SPr(M)$ of simplicial presheaves on *M*, together with its objectwise model structure. We define the model category $M^\wedge$ to be the left Bousfield localization of $SPr(M)$ along the set of equivalences in *M* (see Definition 4.1.4). In particular, unlike that of $SPr(M)$, the model structure of $M^\wedge$ takes into account the fact that *M* is not just a bare category but has an additional (model) structure. The model category $M^\wedge$ is called the *model category of pre-stacks on M*, and it is important to remark that its homotopy category can be identified with the full subcategory of $\mathrm{Ho}(SPr(M))$ consisting of functors $F : M^{op} \longrightarrow SSet$ sending equivalences in *M* to equivalences of simplicial sets. We construct a homotopical Yoneda-like functor

$$\underline{h} : M \longrightarrow M^\wedge,$$

roughly speaking by sending an object *x* to the simplicial presheaf $y \mapsto Map_M(y, x)$, where $Map_M(-, -)$ denotes the homotopy mapping space in the model category *M* (see Definition 4.2.5). An easy but fundamental result states that the functor $\underline{h}$ possesses a right derived functor

$$\mathbb{R}\underline{h} : \mathrm{Ho}(M) \longrightarrow \mathrm{Ho}(M^\wedge)$$

[1] Actually, in Section 4, all the constructions are given for the weaker notion of *pseudo-model categories* because we will need this increased flexibility in some present and future applications. However, the case of model categories will be enough for this introduction.

which is fully faithful (Theorem 4.2.3). This is a model category version of the Yoneda lemma.

We also define the notion of a *model pre-topology* on the model category M and show that this induces in a natural way a Grothendieck topology on the homotopy category $\mathrm{Ho}(M)$. A model category endowed with a model pre-topology will be called a *model site* (see Definition 4.3.1). For a model site (M, τ), we define a notion of *local equivalences* in the category of pre-stacks $M^{\wedge}$. The analog of Theorem 1.0.1 for model categories is then the following:

Theorem 1.0.6 (*Theorem 4.6.1*). *Let (M, τ) be a model site.*

1. *There exists a model structure on the category $M^{\wedge}$, called the local model structure, for which the equivalences are the local equivalences. This new model category, denoted by $M^{\sim,\tau}$, is furthermore the left Bousfield localization of the model category of pre-stacks $M^{\wedge}$ along the local equivalences.*
2. *The identity functor*

$$\mathrm{Id} : M^{\wedge} \longrightarrow M^{\sim,\tau}$$

commutes with homotopy fibered products.

3. *The homotopy category $\mathrm{Ho}(M^{\sim,\tau})$ is cartesian closed.*

The model category $M^{\sim,\tau}$ is called the model category of stacks *on the model site (M, τ).*

As for stacks over S-sites, there exists a notion of object satisfying a *hyperdescent condition* with respect to the topology τ, and we prove that $\mathrm{Ho}(M^{\sim,\tau})$ can be identified with the full subcategory of $\mathrm{Ho}(M^{\wedge})$ consisting of objects satisfying hyperdescent (see Definition 4.6.5).

Finally, we compare the two parallel constructions of stacks over S-sites and over model sites.

Theorem 1.0.7 (*Theorem 4.7.1*). *Let (M, τ) be a model site.*

(i) *The simplicial localization LM possesses an induced S-topology τ, and is naturally an S-site.*

(ii) *The two corresponding model categories of stacks $M^{\sim,\tau}$ and $SPr_{\tau}(LM)$ are naturally Quillen equivalent. In particular $M^{\sim,\tau}$ is a t-complete model topos.*

The previous comparison theorem finds its pertinence in the fact that the two approaches, stacks over model sites and stacks over S-sites, seem to possess their own advantages and disadvantages, depending of the situation and the goal that one wants to reach. On a computational level the theory of stacks over model sites seems to be better suited than that of stacks over S-sites. On the other hand, S-categories and S-sites are much more intrinsic than model categories and model sites, and this has already some consequences, e.g. at the level of functoriality properties of the categories of stacks. We are convinced that having the full picture, including the two approaches and the comparison Theorem 1.0.7, will be a very friendly setting for the purpose of several future applications.

1.5. A Giraud theorem for model topoi

Our version of Theorem 1.0.3 is on the model categories' side of the theory. The corresponding statement for S-categories would drive us too far away from the techniques used in this work, and will not be investigated here.

Theorem 1.0.8 (*Theorem 4.9.2*). *A combinatorial model category M is a model topos if and only if it satisfies the following conditions*:

1. *Homotopy coproducts are disjoints in M.*
2. *Homotopy colimits are stable under homotopy pullbacks.*
3. *All Segal equivalences relations are homotopy effective.*

The condition of being a combinatorial model category is a set theoretic condition on M (very often satisfied in practice), very similar to the condition of having a small set of generators (see Appendix A.2). Conditions (1) and (2) are straightforward homotopy theoretic analogs of conditions (3) and (4) of Theorem 1.0.3: we essentially replace pushouts, pullbacks and colimits by homotopy pushouts, homotopy pullbacks and homotopy colimits (see Definition 4.9.1). Finally, condition (3) of Theorem 1.0.8, spelled out in Definition 4.9.1(3) and (4), is a homotopical version of condition (5) of Giraud's theorem 1.0.3, where groupoids of equivalence relations are replaced by Segal groupoids and effectivity has to be understood homotopically.

The most important consequence of Theorem 1.0.8 is the following complete characterization of t-complete model topoi.

Corollary 1.0.9 (*Corollary 4.9.7*). *For a combinatorial model category M, the following two conditions are equivalent*:

(i) *There exists a small S-site (T, τ), such that M is Quillen equivalent to $SPr_\tau(T)$.*
(ii) *M is t-complete and satisfies the conditions of Theorem 1.0.8.*

1.6. A topological application: étale K-theory of commutative $\mathbb{S}$-algebras

As an example of application of our constructions, we give a definition of the *étale K-theory of* (*commutative*) *$\mathbb{S}$-algebras*, which is to algebraic K-theory of $\mathbb{S}$-algebras (as defined for example in [EKMM, Section VI]) what étale K-theory of rings is to algebraic K-theory of rings. For this, we use the notion of etale morphisms of $\mathbb{S}$-algebras introduced in [Min] (and in [To-Ve 5]) in order to define an *étale* pre-topology on the model category of commutative $\mathbb{S}$-algebras (see Definition 5.2.10). Associated to this model pre-topology, we have the model category of étale stacks $(Aff_{\mathbb{S}})^{\sim,\mathrm{et}}$; the functor K that maps an $\mathbb{S}$-algebra A to its algebraic K-theory space $K(A)$, defines an object $K \in (Aff_{\mathbb{S}})^{\sim,\acute{e}t}$. If $K_{\acute{e}t} \in (Aff_{\mathbb{S}})^{\sim,\acute{e}t}$ is an étale fibrant model for K, we define the space of étale K-theory of an $\mathbb{S}$-algebra A to be the simplicial set $K_{\acute{e}t}(A)$ (see Definition 5.3.1). Our general formalism also allows us to compare $K_{\acute{e}t}(Hk)$ with the usual definition of etale K-theory of a field k (see Corollary 5.3.3).

This definition of étale K-theory of $\mathbb{S}$-algebras gives a possible answer to a question raised by Rognes [Ro]. In the future, it might be used as a starting point to develop *étale localization techniques* in K-theory of $\mathbb{S}$-algebras, as Thomason's style étale descent theorem, analog of the Quillen-Lichtenbaum's conjecture, etc. For further applications of the general theory developed in this paper to algebraic geometry over commutative ring spectra, we refer the reader to [To-Ve 6,To-Ve 3].

1.7. Organization of the paper

The paper is organized in five sections and one appendix. In Section 2 we review the main definitions and results concerning S-categories. Most of the materials can be found in the original papers [D-K1,D-K2,DHK], with the possible exception of the last two subsections. In Section 3 we define the notion of S-topologies, S-sites, local equivalences and stacks over S-sites. This section contains the proofs of Theorems 1.0.4 and 1.0.5. We prove in particular the existence of the local model structure as well as internal *Hom*'s (or equivalently, stacks of morphisms). We also investigate here the relations between Rezk's model topoi and S-topologies. Section 4 is devoted to the theory of model topologies, model sites and stacks over them. As it follows a pattern very similar to the one followed in Section 3 (for S-categories), some details have been omitted. It also contains comparison results between the theory of stacks over S-sites and the theory of stacks over model sites, as well as the Giraud's style theorem for model topoi. In Section 5 we present one application of the theory to the notion of *étale K-theory of* $\mathbb{S}$*-algebras*. For this we review briefly the homotopy theory of $\mathbb{S}$-modules and $\mathbb{S}$-algebras, and we define an étale topology on the model category of commutative $\mathbb{S}$-algebra, which is an extension of the étale topology on affine schemes. Finally, we use our general formalism to define the étale K-theory space of a commutative $\mathbb{S}$-algebra.

Finally, in Appendix A we collected some definitions and conventions concerning model categories and the use of universes in this context.

1.8. Related works

There has been several recent works on (higher) stacks theory which use a simplicial and/or a model categorical approach (see [DHI, H-S, Hol, Ja2, S1, To2, To3]). The present work is strongly based on the same idea that simplicial presheaves are after all very good models for *stacks in* ∞*-groupoids*, and provide a powerful and rich theory. It may also be considered as a natural continuation of the foundational papers [Ja1,Jo1].

A notion of a topology on a 2-category, as well as a notion of *stack over a 2-site* has already been considered by R. Street in [Str], D. Bourne in [Bou], and more recently by Behrend in his work on DG-schemes [Be]. Using truncation functors (Section 3.7), a precise comparison with these approaches will appear in the second part of this work [To-Ve 6] (the reader is also referred to Remark 3.7.9).

We have already mentioned that the notion of model topos used in Section 3.8 essentially goes back to the unpublished manuscript [Re], though it was originally defined as left exact Bousfield localizations of model category of simplicial presheaves on some usual category, which is not enough as we have seen. A different, but similar, version of our Giraud's Theorem 4.9.2 appeared in [Re] as conjecture. The notion of S-topos introduced in Section 3.8 seems new, though more or less equivalent to the notion of model topos. However, we think that both theories of S-categories and of model categories reach here their limits, as it seems quite difficult to define a reasonable notion of geometric morphisms between model topoi or between S-topoi. This problem can be solved by using Segal categories of [H-S,P] in order to introduce a notion of *Segal topos* as explained in [To-Ve 1].

A notion relatively closed to the notion of Segal topos can also be found in [S2] where *Segal pre-topoi* are investigated and the question of the existence of a theory of Segal topoi is addressed.

Also closely related to our approach to model topoi is the notion of ∞-*topos* appeared in the recent preprint [Lu] by Lurie. The results of [Lu] are exposed in a rather different context, and are essentially disjoints from ours. Form example, the notion of topology is not considered in [Lu] and results of type 3.8.3, 3.8.5 or 4.9.7 do not appear in it. Also, the notion of stack used by Lurie is slightly different from ours (however the differences are quite subtle). An exception is Giraud's theorem which first appeared in [Lu] in the context of ∞-categories, and only later on in the last version of this work (February 2004) for model categories. These two works have been done independently, though we must mention that the first version of the present paper has been publicly available since July 2002 (an important part of it was announced in [To-Ve 5] which appeared on the web during October 2001), whereas [Lu] appeared in June 2003.

Let us also mention that Joyal [Jo2] has developed a theory of *quasi-categories*, which is expected to be equivalent to the theory of S-categories and of Segal categories, and for which he has defined a notion of *quasi-topos* very similar to the notion of Segal topos in [To-Ve 1]. The two approaches are expected to be equivalent. Also, the recent work of Cisinski [Cis] seems to be closely related to a notion of *hypertopology* we discuss in Remark 3.8.7(3).

Our definition of the étale topology for $\mathbb{S}$-algebras was strongly influenced by the content of [Min,MCM], and the definition of étale K-theory in the context of $\mathbb{S}$-algebras given in Section 5 was motivated by the note [Ro].

Notations and conventions. We will use the word *universe* in the sense of [SGA4-I, Exp. I, Appendice]. Universes will be denoted by $\mathbb{U} \in \mathbb{V} \in \mathbb{W} \ldots$. For any universe $\mathbb{U}$ we will assume that $\mathbb{N} \in \mathbb{U}$. The category of sets (resp. simplicial sets, resp. …) belonging to a universe $\mathbb{U}$ will be denoted by $Set_{\mathbb{U}}$ (resp. $SSet_{\mathbb{U}}$, resp. …). The objects of $Set_{\mathbb{U}}$ (resp. $SSet_{\mathbb{U}}$, resp. …) will be called $\mathbb{U}$-sets (resp. $\mathbb{U}$-simplicial sets, resp. …). We will use the expression $\mathbb{U}$-*small set* (resp. $\mathbb{U}$-*small simplicial set*, resp. …) to mean *a set isomorphic to a set in* $\mathbb{U}$ (resp. *a simplicial set isomorphic to a simplicial set in* $\mathbb{U}$, resp. …).

Our references for model categories are [Hi, Ho]. By definition, our model categories will always be *closed* model categories, will have all *small* limits and colimits and the functorial factorization property. The word *equivalence* will always mean *weak equivalence* and will refer to a model category structure.

The homotopy category of a model category M is $W^{-1}M$ (see [Ho, Definition 1.2.1]), where W is the subcategory of equivalences in M, and it will be denoted as $\mathrm{Ho}(M)$.

The sets of morphisms in $\mathrm{Ho}(M)$ will be denoted by $[-,-]_M$, or simply by $[-,-]$ when the reference to the model category M is clear. We will say that two objects in a model category M are equivalent if they are isomorphic in $\mathrm{Ho}(M)$. We say that two model categories are *Quillen equivalent* if they can be connected by a finite string of Quillen adjunctions each one being a Quillen equivalence.

The homotopy fibered product (see [Hi, Section 11] or [DHK, Chapter XIV]) of a diagram $x \longrightarrow z \longleftarrow y$ in a model category M will be denoted by $x \times^h_z y$. In the same way, the homotopy pushout of a diagram $x \longleftarrow z \longrightarrow y$ will be denoted by $x \coprod^h_z y$. When the model category M is a simplicial model category, its simplicial sets of morphisms will be denoted by $\underline{Hom}(-,-)$, and their derived functors by $\mathbb{R}\underline{Hom}$ (see [Ho, 1.3.2]).

For the notions of $\mathbb{U}$-cofibrantly generated, $\mathbb{U}$-combinatorial and $\mathbb{U}$-cellular model category, we refer to or to Appendix B, where the basic definitions and crucial properties are recalled in a way that is suitable for our needs.

As usual, the standard simplicial category will be denoted by Δ. For any simplicial object $F \in C^{\Delta^{op}}$ in a category C, we will use the notation $F_n := F([n])$. Similarly, for any co-simplicial object $F \in \mathcal{C}^{\Delta}$, we will use the notation $F_n := F([n])$.

For a Grothendieck site (C,τ) in a universe $\mathbb{U}$, we will denote by $Pr(C)$ the category of presheaves of $\mathbb{U}$-sets on C, $Pr(C) := C^{Set^{op}_{\mathbb{U}}}$. The subcategory of sheaves on (C,τ) will be denoted by $Sh_\tau(C)$, or simply by $Sh(C)$ if the topology τ is unambiguous.

2. Review of *S*-categories

In this first section we recall some facts concerning *S*-categories. The main references on the subject are [D-K1, D-K2, DHK], except for the material covered in the two final subsections for which it does not seem to exist any reference. The notion of *S*-category will be of fundamental importance in all this work, as it will replace the notion of usual category in our higher sheaf theory. In Section 3, we will define what an *S*-topology on an *S*-category is, and study the associated notion of stack.

We start by reviewing the definition of *S-category* and the Dwyer–Kan *simplicial localization* technique. We recall the existence of model categories of *diagrams* over *S*-categories, as well as their relations with the model categories of *restricted diagrams*. The new materials are presented in the last two subsections: here, we first prove a *Yoneda-like lemma* for *S*-categories and then introduce and study the notion of *comma S-category*.

2.1. The homotopy theory of S-categories

We refer to [Ke] for the basic notions of enriched category theory. We will be especially interested in the case where the enrichment takes place in the cartesian closed category *SSet* of simplicial sets.

Definition 2.1.1. An *S-category* T is a category enriched in *SSet*. A *morphism* of *S*-categories $T \to T'$ is a *SSet*-enriched functor.

More explicitly, an *S-category* T consists of the following data:

- A set $Ob(T)$ (whose elements are called the *objects of T*).
- For any pair of objects (x, y) of $Ob(T)$, a simplicial set $\underline{Hom}_T(x, y)$ (called the simplicial set of *morphisms from x to y*). A 0-simplex in $\underline{Hom}_T(x, y)$ will simply be called a *morphism* from x to y in T. The 1-simplices in $\underline{Hom}_T(x, y)$ will be called *homotopies*.
- For any triple of objects (x, y, z) in $Ob(T)$, a morphism of simplicial sets (called the *composition* morphism)

$$\underline{Hom}_T(x, y) \times \underline{Hom}_T(y, z) \longrightarrow \underline{Hom}_T(x, z).$$

- For any object $x \in Ob(T)$, a 0-simplex $\mathrm{Id}_x \in \underline{Hom}_T(x, x)_0$ (called the *identity* morphism at x).

These data are required to satisfy the usual associativity and unit axioms. A *morphism* between *S*-categories $f : T \longrightarrow T'$ consists of the following data:

- A map of sets $Ob(T) \longrightarrow Ob(T')$.
- For any two objects x and y in $Ob(T)$, a morphism of simplicial sets

$$\underline{Hom}_T(x, y) \longrightarrow \underline{Hom}_{T'}(f(x), f(y)),$$

  compatible with the composition and unit in an obvious way.

Morphisms of *S*-categories can be composed in the obvious way, thus giving rise to the category of *S*-categories.

Definition 2.1.2. The category of *S*-categories belonging to a universe $\mathbb{U}$, will be denoted by $S - Cat_{\mathbb{U}}$, or simply by $S - Cat$ if the universe $\mathbb{U}$ is clear from the context or irrelevant.

The natural inclusion functor $j : Set \longrightarrow SSet$, sending a set to the corresponding constant simplicial set, allows us to construct a natural inclusion $j : Cat \longrightarrow S - Cat$, and therefore to see any category as an *S*-category. Precisely, for a category C, $j(C)$ is the *S*-category with the same objects as T and whose simplicial set of morphism from x to y is just the constant simplicial set associated to the set $Hom_{\mathrm{c}}(x, y)$. In the following we will simply write C for $j(C)$.

Any *S*-category T has an *underlying category of 0-simplices* T_0; its set of objects is the same as that of T while the set of morphisms from x to y in T_0 is the set of 0-simplices of the simplicial set $\underline{Hom}_T(x, y)$. The construction $T \mapsto T_0$ defines a functor $S - Cat \longrightarrow Cat$ which is easily checked to be *right adjoint* to the inclusion $j : Cat \longrightarrow S - Cat$ mentioned above. This is completely analogous to (and actually, induced by) the adjunction between the constant simplicial set functor $\mathrm{c} : Set \longrightarrow SSet$ and the 0th level set functor $(-)_0 : SSet \longrightarrow Set$.

Any S-category T also has a *homotopy category*, denoted by $\mathrm{Ho}(T)$; its set of objects is the same as that of T, and the set of morphisms from x to y in $\mathrm{Ho}(T)$ is given by $\pi_0(\underline{Hom}_T(x, y))$, the set of connected components of the simplicial set of morphisms from x to y in T. The construction $T \mapsto \mathrm{Ho}(T)$ defines a functor $S-Cat \longrightarrow Cat$ which is easily checked to be *left adjoint* to the inclusion $j : Cat \longrightarrow S-Cat$. Again, this is completely analogous to (and actually, induced by) the adjunction between the constant simplicial set functor $\mathrm{c} : Set \longrightarrow SSet$ and the connected components' functor $\pi_0 : SSet \longrightarrow Set$.

Summarizing, we have the following two adjunction pairs (always ordered by writing the left adjoint on the left):

$$j : Cat \longrightarrow S-Cat, \quad Cat \longleftarrow S-Cat : (-)_0,$$

$$Ob(T_0) := Ob(T), \quad Hom_{T_0}(x, y) := \underline{Hom}_T(x, y)_0,$$

$$\mathrm{Ho}(-) : S-Cat \longrightarrow Cat, \quad S-Cat \longleftarrow Cat : j,$$

$$Ob(\mathrm{Ho}(T)) := Ob(T), \quad Hom_{\mathrm{Ho}(T)}(x, y) := \pi_0(\underline{Hom}_T(x, y)).$$

For an S-category T, the two associated categories T_0 and $\mathrm{Ho}(T)$ are related in the following way. There exist natural morphisms of S-categories

$$T_0 \xrightarrow{\ i\ } T \xrightarrow{\ p\ } \mathrm{Ho}(T),$$

which induce a functor $q : T_0 \longrightarrow \mathrm{Ho}(T)$. Being the underlying category of an S-category, the category T_0 has a natural notion of *homotopy* between morphisms. This induces an equivalence relation on the set of morphisms of T_0, by declaring two morphisms equivalent if there is a string of homotopies between them. This equivalence relation is furthermore compatible with composition. The category obtained from T_0 by passing to the quotient with respect to this equivalence relation is precisely $\mathrm{Ho}(T)$.

Definition 2.1.3. Let $f : T \longrightarrow T'$ be a morphism of S-categories.

1. The morphism f is *essentially surjective* if the induced functor $\mathrm{Ho}(f) : \mathrm{Ho}(T) \longrightarrow \mathrm{Ho}(T')$ is an essentially surjective functor of categories.
2. The *essential image* of f is the inverse image by the natural projection $T' \longrightarrow \mathrm{Ho}(T')$ of the essential image of $\mathrm{Ho}(f) : \mathrm{Ho}(T) \longrightarrow \mathrm{Ho}(T')$.
3. The morphism f is *fully faithful* if for any pair of objects x and y in T, the induced morphism $f_{x,y} : \underline{Hom}_T(x, y) \longrightarrow \underline{Hom}_{T'}(f(x), f(y))$ is an equivalence of simplicial sets.
4. The morphism f is an *equivalence* if it is essentially surjective and fully faithful.

The category obtained from $S-Cat$ by formally inverting the equivalences will be denoted by $\mathrm{Ho}(S-Cat)$. The set of morphisms in $\mathrm{Ho}(S-Cat)$ between two objects T and T' will simply be denoted by $[T,T']$.

Remark 2.1.4 (*DHK, Section XII-48*). contains the sketch of a proof that the category $S-Cat$ admits a model structure whose equivalences are exactly the ones defined above. It seems however that this proof is not complete, as the generating trivial cofibrations of [DHK, 48.5] fail to be equivalences. In his note [May2, Theorem 1.9], May informed us that he knows an alternative proof, but the reader will notice that the notion of fibrations used in [May2] is different from the one used in [DHK] and does not seem to be correct. We think however that the model structure described in [DHK] exists,[2] as we have the feeling that one could simply replace the wrong set of generating trivial cofibrations by the set of all trivial cofibrations between countable *S*-categories. The existence of this model structure would of course simplify some of our constructions, but it does not seem to be really unavoidable, and because of the lack of clear references we have decided not to use it at all. This will cause a "lower degree" of functoriality in some constructions, but will be enough for all our purposes.

Since the natural localization functor $SSet \longrightarrow \mathrm{Ho}(SSet)$ commutes with finite products, any category enriched in *SSet* gives rise to a category enriched in $\mathrm{Ho}(SSet)$. The $\mathrm{Ho}(SSet)$-enriched category associated to an *S*-category T will be denoted by $\underline{\mathrm{Ho}(T)}$, and has $\mathrm{Ho}(T)$ as underlying category. Furthermore, for any pair of objects x and y in $\mathrm{Ho}(T)$, one has $\underline{Hom}_{\mathrm{Ho}(T)}(x,y)=\underline{Hom}_T(x,y)$ considered as objects in $\mathrm{Ho}(SSet)$. Clearly, $T\mapsto \underline{\mathrm{Ho}}(T)$ defines a functor from $S-Cat$ to the category $\mathrm{Ho}(SSet)-Cat$ of $\mathrm{Ho}(SSet)$-enriched categories, and a morphism of *S*-categories is an equivalence if and only if the induced $\mathrm{Ho}(SSet)$-enriched functor is an $\mathrm{Ho}(SSet)$-enriched equivalence. Therefore, this construction induces a well-defined functor

$$\begin{array}{ccc}\mathrm{Ho}(S-Cat) & \longrightarrow & \mathrm{Ho}(\mathrm{Ho}(SSet)-Cat),\\ T & \mapsto & \underline{\mathrm{Ho}(T)},\end{array}$$

where $\mathrm{Ho}(\mathrm{Ho}(SSet)-Cat)$ is the localization of the category of $\mathrm{Ho}(SSet)$-enriched categories along $\mathrm{Ho}(SSet)$-enriched equivalences.

The previous construction allows one to define the notions of essentially surjective and fully faithful morphisms in $\mathrm{Ho}(S-Cat)$. Precisely, a morphism $f:T\longrightarrow T'$ in $\mathrm{Ho}(S-Cat)$ will be called essentially surjective (resp. fully faithful) if the corresponding $\mathrm{Ho}(SSet)$-enriched functor $\underline{\mathrm{Ho}(f)}:\underline{\mathrm{Ho}(T)}\longrightarrow \underline{\mathrm{Ho}(T')}$ is essentially surjective (resp. fully faithful) in the $\mathrm{Ho}(SSet)$-enriched sense.

Finally, for an *S*-category T and a property $\mathbf{P}$ of morphisms in $\mathrm{Ho}(T)$, we will often say that *a morphism* f *in* T satisfies the property $\mathbf{P}$ to mean that *the image of* f *in* $\mathrm{Ho}(T)$ *through the natural projection* $T\longrightarrow \mathrm{Ho}(T)$, *satisfies the property* $\mathbf{P}$. Recall that a *morphism* f in an *S*-category T is just an element in the zero simplex set of $\underline{Hom}_T(x,y)$ for some x and y in $Ob(T)$.

[2] Recent progresses have been made in this direction by J. Bergner (private communication).

2.2. Simplicial localization

Starting from a category C together with a subcategory $S \subset C$, Dwyer and Kan have defined in [D-K1] an S-category $\mathrm{L}(C,S)$, which is an enhanced version of the localized category $S^{-1}C$. It is an S-category with a diagram of morphisms in $S-Cat$ (viewing, according to our general conventions, any category as an S-category via the embedding $j : Cat \to S-Cat$)

$$C \xleftarrow{\;p\;} F_*C \xrightarrow{\;L\;} \mathrm{L}(C,S) \quad ,$$

where F_*C is the so-called *standard simplicial free resolution of the category* C, and in particular, the projection p is an equivalence of S-categories. Therefore, there exists a well-defined *localization morphism* in $\mathrm{Ho}(S-Cat)$

$$L : C \longrightarrow \mathrm{L}(C,S).$$

The construction $(C,S) \mapsto \mathrm{L}(C,S)$ is functorial in the pair (C,S) and it also extends naturally to the case where S is a sub-S-category of an S-category C (see [D-K1, Section 6]). Note also that by construction, if C belongs to a universe $\mathbb{U}$ so does $L(C,S)$.

Remark 2.2.1. (1) One can also check that the localization morphism L satisfies the following universal property. For each S-category T, let us denote by $[C,T]^S$ the subset of $[C,T] = Hom_{\mathrm{Ho}(S-Cat)}(C,T)$ consisting of morphisms for which the induced morphism $C \longrightarrow \mathrm{Ho}(T)$ sends morphisms of S into isomorphisms in $\mathrm{Ho}(T)$ (the reader will easily check that this property is well-defined). Then the localization morphism L is such that for any S-category T the induced map

$$L^* : [\mathrm{L}(C,S),T] \longrightarrow [C,T]$$

is injective and its image is $[C,T]^S$. This property characterizes the S-category $\mathrm{L}(C,S)$ as an object in the comma category $C/\mathrm{Ho}(S-Cat)$. This universal property will not be used in the rest of the paper, but we believe it makes the meaning of the simplicial localization more transparent.

(2) It is important to mention the fact that any S-category T is equivalent to some $L(C,S)$, for a category C with a subcategory $S \subset C$ (this is the *delocalization theorem* of [D-K2]). Furthermore, it is clear by the construction given in [D-K1] that, if T is $\mathbb{U}$-small, then so are C, S and $L(C,S)$.

Two fundamental properties of the functor $L : (C,S) \mapsto \mathrm{L}(C,S)$ are the following:

1. The localization morphism L induces a well-defined (up to a unique isomorphism) functor

$$\mathrm{Ho}(L) : C \simeq \mathrm{Ho}(F_*C) \longrightarrow \mathrm{Ho}(\mathrm{L}(C,S)),$$

that identifies $\mathrm{Ho}(\mathrm{L}(C,S))$ with the (usual Gabriel–Zisman) localization $S^{-1}C$.

2. Let M be a simplicial model category, $W \subset M$ its subcategory of equivalences and let $\mathrm{Int}(M)$ be the S-category of fibrant and cofibrant objects in M together with their simplicial sets of morphisms. The full (not simplicial) subcategory $M^{\mathrm{cf}} \subset M$ of fibrant and cofibrant objects in M has two natural morphisms in $S-Cat$

$$M \longleftarrow M^{\mathrm{cf}} \longrightarrow \mathrm{Int}(M),$$

which induce isomorphisms in $\mathrm{Ho}(S-Cat)$

$$\mathrm{L}(M, W) \simeq \mathrm{L}(M^{\mathrm{cf}}, W \cap M^{\mathrm{cf}})$$

$$\simeq \mathrm{L}(\mathrm{Int}(M), W \cap M^{\mathrm{cf}}) \simeq \mathrm{Int}(M).$$

In the same way, if M^{f} (resp. M^{c}) is the full subcategory of fibrant (resp. cofibrant) objects in M, the natural morphisms $M^{\mathrm{f}} \longrightarrow M$, $M^{\mathrm{c}} \longrightarrow M$ induce isomorphisms in $\mathrm{Ho}(S-Cat)$

$$\mathrm{L}(M^{\mathrm{f}}, W \cap M^{\mathrm{f}}) \simeq \mathrm{L}(M, W) \quad \mathrm{L}(M^{\mathrm{c}}, W \cap M^{\mathrm{c}}) \simeq \mathrm{L}(M, W).$$

Definition 2.2.2. If M is any model category, we set $LM := \mathrm{L}(M, W)$, where $W \subset M$ is the subcategory of equivalences in M.

The construction $M \mapsto LM$ is functorial, up to equivalences, for Quillen functors between model categories. To see this, let $f : M \longrightarrow N$ be a right Quillen functor. Then, the restriction to the category of fibrant objects $f : M^{\mathrm{f}} \longrightarrow N^{\mathrm{f}}$ preserves equivalences, and therefore induces a morphism of S-categories

$$Lf : LM^{\mathrm{f}} \longrightarrow LN^{\mathrm{f}}.$$

Using the natural isomorphisms $LM^{\mathrm{f}} \simeq LM$ and $LN^{\mathrm{f}} \simeq LN$ in $\mathrm{Ho}(S-Cat)$, one gets a well-defined morphism $Lf : LM \longrightarrow LN$. This is a morphism in the homotopy category $\mathrm{Ho}(S-Cat)$, and one checks immediately that $M \mapsto LM$ is a functor from the category of model categories (belonging to a fixed universe $\mathbb{U}$) with right Quillen functors, to $\mathrm{Ho}(S-Cat_{\mathbb{U}})$. The dual construction gives rise to a functor $M \mapsto LM$ from the category of model categories which belongs to a universe $\mathbb{U}$ and left Quillen functors to $\mathrm{Ho}(S-Cat_{\mathbb{U}})$.

The reader will check easily that if

$$f : M \longrightarrow N \qquad M \longleftarrow N : g$$

is a Quillen adjunction which is a Quillen equivalence, then the morphisms $Lf : LM \longrightarrow LN$ and $Lg : LN \longrightarrow LM$ are isomorphisms inverse to each others in $\mathrm{Ho}(S-Cat)$.

2.3. Model categories of diagrams

In this paragraph we discuss the notion of *pre-stack over an S-category* which is a generalization of the notion of presheaf of sets on a usual category.

2.3.1. Diagrams

Let T be any S-category in a universe $\mathbb{U}$, and M a simplicial model category which is $\mathbb{U}$-cofibrantly generated (see [Hi, 13.2] and Appendix A). Since M is simplicial, we may view it as an S-category, with the same set of objects as M and whose simplicial sets of morphisms are provided by the simplicial structure. Therefore, we may consider the category M^T, of morphisms of S-categories $F : T \longrightarrow M$. To be more precise, an *object* $F : T \longrightarrow M$ in M^T consists of the following data:

- A map $F : Ob(T) \longrightarrow Ob(M)$.
- For any pair of objects $(x, y) \in Ob(T) \times Ob(T)$, a morphism of simplicial sets

$$F_{x,y} : \underline{Hom}_T(x, y) \longrightarrow \underline{Hom}(F(x), F(y))$$

  (or equivalently, morphisms $F_{x,y} : \underline{Hom}_T(x, y) \otimes F(x) \longrightarrow F(y)$ in M) satisfying the obvious associativity and unit axioms.

A *morphism* from F to G in M^T consists of morphisms $H_x : F(x) \longrightarrow G(x)$ in M, for all $x \in Ob(T)$, such that the following diagram commutes in M:

$$\begin{array}{ccc} \underline{Hom}_T(x, y) \otimes F(x) & \xrightarrow{F_{x,y}} & F(y) \\ \Big\downarrow{\scriptstyle \mathrm{Id}\otimes H_x} & & \Big\downarrow{\scriptstyle H_y} \\ \underline{Hom}_T(x, y) \otimes G(x) & \xrightarrow[G_{x,y}]{} & G(y). \end{array}$$

One defines a model structure on M^T, by defining a morphism H to be a fibration (resp. an equivalence) if for all $x \in Ob(T)$, the induced morphism H_x is a fibration (resp. an equivalence) in M. It is known that these definitions make M^T into a simplicial model category which is again $\mathbb{U}$-cofibrantly generated (see [Hi, 13.10.17] and Appendix A). This model structure will be called the *projective model structure on* M^T. Equivalences and fibrations in M^T will be called *objectwise equivalences* and *objectwise fibrations*.

Let us suppose now that M is an *internal* model category (i.e. a symmetric monoidal model category for the direct product, in the sense of [Ho, Chapter 4]). The category

M^T is then naturally tensored and co-tensored over M. Indeed, the external product $A \otimes F \in M^T$ of $A \in M$ and $F \in M^T$, is simply defined by the formula $(A \otimes F)(x) := A \times F(x)$ for any $x \in Ob(T)$. For any x and y in $Ob(T)$, the transition morphisms of $A \otimes F$ are defined by

$$\begin{aligned}(A \otimes F)_{x,y} &:= A \times F_{x,y} : \underline{Hom}_T(x, y) \times A \times F(x)\\ &\simeq A \times \underline{Hom}_T(x, y) \times F(x) \longrightarrow A \times F(y).\end{aligned}$$

In the same way, the exponential $F^A \in M^T$ of F by A, is defined by $(F^A)(x) := F(x)^A$ for any x in $Ob(T)$.

With these definitions the model category M^T becomes a M-model category in the sense of [Ho, Definition 4.2.18]. When M is the model category of simplicial sets, this implies that $SSet^T$ has a natural structure of simplicial model category where exponential and external products are defined levelwise. In particular, for any $x \in Ob(T)$, the evaluation functor

$$\begin{aligned} j_x^* : M^T &\longrightarrow M,\\ F &\mapsto F(x),\end{aligned}$$

commutes with the geometric realization and total space functors of [Hi, Section 19.5]. As fibrant (resp. cofibrant) objects in M^T are also objectwise fibrant (resp. objectwise cofibrant), this easily implies that j_x^* commutes, up to an equivalence, with homotopy limits and homotopy colimits. One may also directly shows that j_x^* is indeed a left and right Quillen functor. Finally, if M is a proper model category, then so is M^T.

Let $f : T \longrightarrow T'$ be a morphism in $S - Cat_{\mathbb{U}}$. It gives rise to an adjunction

$$f_! : M^T \longrightarrow M^{T'} \qquad M^T \longleftarrow M^{T'} : f^*,$$

where f^* is defined by the usual formula $f^*(F)(x) := F(f(x))$, for any $F \in M^{T'}$ and any $x \in Ob(T)$, and $f_!$ is its left adjoint. The functor f^* is clearly a right Quillen functor, and therefore $(f_!, f^*)$ is a Quillen adjunction.

The following theorem is proved in [D-K2] when M is the category of simplicial sets; its proof generalizes immediately to our situation. As above, M is a simplicial $\mathbb{U}$-cofibrantly generated model category.

Theorem 2.3.1. *If $f : T \to T'$ is an equivalence of S-categories, then $(f_!, f^*)$ is a Quillen equivalence of model categories.*

Definition 2.3.2. Let $T \in S - Cat_{\mathbb{U}}$ be an S-category in $\mathbb{U}$, and M a $\mathbb{U}$-cofibrantly generated simplicial model category. The model category $Pr(T, M)$ of *pre-stacks on* T *with values in* M is defined as

$$Pr(T, M) := M^{T^{op}}.$$

We will simply write $SPr(T)$ for $Pr(T, SSet_{\mathbb{U}})$, and call it the model category of *pre-stacks on T*.

Theorem 2.3.1 implies that the model category $Pr(T, M)$, for a fixed M, is an invariant, up to Quillen equivalence, of the isomorphism class of T in $\mathrm{Ho}(S-Cat_{\mathbb{U}})$. In the same way, if $f : T \longrightarrow T'$ is a morphism in $\mathrm{Ho}(S-Cat_{\mathbb{U}})$, one can represent f by a string of morphisms in $S-Cat_{\mathbb{U}}$

$$T \xleftarrow{p_1} T_1 \xrightarrow{f_1} T_2 \xleftarrow{p_3} T_3 \xrightarrow{f_3} T_4 \quad \cdots \quad \xleftarrow{p_{2n-1}} T_{2n-1} \xrightarrow{f_{2n-1}} T',$$

where each p_i is an equivalence of S-categories. We deduce a diagram of right Quillen functors

$$Pr(T, M) \xrightarrow{p_1^*} Pr(T_1, M) \xleftarrow{f_1^*} Pr(T_2, M) \xrightarrow{p_3^*} Pr(T_3, M)$$
$$\cdots \qquad \xrightarrow{p_{2n-1}^*} Pr(T_{2n-1}, M) \xleftarrow{f_{2n-1}^*} Pr(T', M),$$

such that each p_i^* is a right adjoint of a Quillen equivalence. By definition, this diagram gives a *Quillen adjunction between $Pr(T, M)$ and $Pr(T', M)$, up to Quillen equivalences*, which can also be interpreted as a morphism in the category of model categories localized along Quillen equivalences. In particular, we obtain a well-defined morphism in $\mathrm{Ho}(S-Cat)$

$$\mathbb{R}f^* := (p_1^*)^{-1} \circ (f_1^*) \circ \ldots \circ (p_{2n-1}^*)^{-1} \circ (f_{2n-1}^*) : LPr(T', M) \longrightarrow LPr(T, M).$$

Using direct images (i.e. functors $(-)_!$) instead of inverse images, one also gets a morphism in the other direction

$$\mathbb{L}f_! := (f_{2n-1})_! \circ (p_{2n-1})_!^{-1} \circ \ldots \circ (f_1)_! \circ (p_1)_!^{-1} : LPr(T, M) \longrightarrow LPr(T', M)$$

(again well-defined in $\mathrm{Ho}(S-Cat)$). Passing to the associated $\mathrm{Ho}(SSet)$-enriched categories, one obtains a $\mathrm{Ho}(SSet)$-enriched adjunction

$$\mathbb{L}f_! : \underline{\mathrm{Ho}(Pr(T, M))} \longrightarrow \underline{\mathrm{Ho}(Pr(T', M))} \quad \underline{\mathrm{Ho}(Pr(T, M))} \longleftarrow \underline{\mathrm{Ho}(Pr(T', M))} : \mathbb{R}f^*.$$

The two $\mathrm{Ho}(SSet)$-enriched functors are well-defined up to a unique isomorphism. When M is fixed, the construction above defines a well-defined functor from the category $\mathrm{Ho}(S-Cat)$ to the homotopy category of $\mathrm{Ho}(SSet)$-enriched adjunctions.

2.3.2. Restricted diagrams

Let C be a $\mathbb{U}$-small S-category, $S \subset C$ a sub-S-category, and M a simplicial model category which is $\mathbb{U}$-cofibrantly generated. We will assume also that M is a $\mathbb{U}$-combinatorial or $\mathbb{U}$-cellular model category so that the left Bousfield localization techniques of [Hi, Chapter 4] can be applied to homotopically invert any $\mathbb{U}$-set of morphisms (see Appendix A).

We consider the model category M^C, of simplicial functors from C to M, endowed with its projective model structure. For any object $x \in C$, the evaluation functor $i_x^* : M^C \longrightarrow M$, defined by $i_x^*(F) := F(x)$, has a left adjoint $(i_x)_! : M \longrightarrow M^C$ which is a left Quillen functor. Let I be a $\mathbb{U}$-set of generating cofibrations in M. For any $f : A \longrightarrow B$ in I and any morphism $u : x \longrightarrow y$ in $S \subset C$, one consider the natural morphism in M^C

$$f \Box u : (i_y)_!(A) \coprod_{(i_x)_!(A)} (i_x)_!(B) \longrightarrow (i_y)_!(B).$$

Since M is a $\mathbb{U}$-combinatorial (or $\mathbb{U}$-cellular) model category, then so is M^C (see [Du2, i] and Appendix A). As the set of all $f\Box u$, for $f \in I$ and u a morphism in S, belongs to $\mathbb{U}$, the following definition is well posed.

Definition 2.3.3. The model category $M^{C,S}$ is the left Bousfield localization of M^c along the set of all morphisms $f\Box u$, where $f \in I$ and u is a morphism in S.

The model category $M^{C,S}$ will be called the *model category of restricted diagrams* from (C, S) to M.

Remark 2.3.4. If $M = SSet_{\mathbb{U}}$, we may take I to be the usual set of generating cofibrations

$$I = \left\{ f_n : \partial\Delta[n] \hookrightarrow \Delta[n] \mid n \geqslant 0 \right\}.$$

Since as it is easily checked, we have a canonical isomorphism $(i_x)_!(* = \Delta[0]) \simeq \underline{h}_x$ in $SSet^{(C,S)^{op}}$, for any $x \in C$, where $\underline{h}_x$ denotes the simplicial diagrams defined by $\underline{h}_x(y) := \underline{Hom}_T(y, x)$. Then, for any $u : x \to y$ in S, we have that the set of morphisms $f_n\Box u$ is exactly the set of augmented horns on the set of morphisms $\underline{h}_x \to \underline{h}_y$ (see [Hi, Section 4.3]). This implies that $SSet^{C,S}$ is simply the left Bousfield localization of $SSet^C$ along the set of morphisms $\underline{h}_x \to \underline{h}_y$ for any $x \to y$ in S.

By the general theory of left Bousfield localization of [Hi], the fibrant objects in the model category $M^{C,S}$ are the functors $F : C \longrightarrow M$ satisfying the following two conditions:

1. For any $x \in C$, $F(x)$ is a fibrant object in M (i.e. F is fibrant in M^C for the projective model structure).
2. For any morphism $u : x \longrightarrow y$ in S, the induced morphism $F_{x,y}(u) : F(x) \longrightarrow F(y)$ is an equivalence in M.

Now, let (F_*C, F_*S) be the canonical free resolution of (C, S) in $S-Cat_{\mathbb{U}}$ (see [D-K1]). Then, one has a diagram of pairs of S-categories

$$(C, S) \xleftarrow{\;p\;} (F_*C, F_*S) \xrightarrow{\;l\;} (F_*S)^{-1}(F_*C) = \mathrm{L}(C, S),$$

inducing a diagram of right Quillen functors

$$M^{C,S} \xrightarrow{\;p^*\;} M^{F_*C, F_*S} \xleftarrow{\;l^*\;} M^{\mathrm{L}(C,S)} .$$

The following result is proved in [D-K2] in the case where $M = SSet_{\mathbb{U}}$, and its proof generalizes easily to our situation.

Theorem 2.3.5. *The previously defined right Quillen functors p^* and l^* are Quillen equivalences. In particular, the two model categories $M^{\mathrm{L}(C,S)}$ and $M^{C,S}$ are Quillen equivalent.*

The model categories of restricted diagrams are functorial in the following sense. Let $f : C \longrightarrow D$ be a functor between two $\mathbb{U}$-small S-categories, and let $S \subset C$ and $T \subset D$ be two sub-S-categories such that $f(S) \subset T$. The functor f induces the usual adjunction on the categories of diagrams in M

$$f_! : M^{C,S} \longrightarrow M^{D,T}, \quad M^{C,S} \longleftarrow M^{D,T} : f^*.$$

The adjunction $(f_!, f^*)$ is a Quillen adjunction for the objectwise model structures. Furthermore, using the description of fibrant objects given above, it is clear that f^* sends fibrant objects in $M^{D,T}$ to fibrant objects in $M^{C,S}$. By the general formalism of left Bousfield localizations (see [Hi, Section 3]), this implies that $(f_!, f^*)$ is also a Quillen adjunction for the restricted model structures.

Corollary 2.3.6. *Let $f : (C, S) \longrightarrow (D, T)$ be as above. If the induced morphism of S-categories $Lf : \mathrm{L}(C, S) \longrightarrow \mathrm{L}(D, T)$ is an equivalence, then the Quillen adjunction $(f_!, f^*)$ is a Quillen equivalence between $M^{C,S}$* and $M^{D,T}$.

Proof. This is a consequence of Theorems 2.3.1 and 2.3.5. □

2.4. The Yoneda embedding

In this paragraph we define a Yoneda embedding for S-categories. To be precise it will be only defined as a morphism in $S-Cat$ for fibrant S-categories, i.e. for S-categories whose simplicial sets of morphisms are all fibrant; for arbitrary S-categories, the Yoneda embedding will only be defined as a morphism in the homotopy category $\mathrm{Ho}(S-Cat)$.

We fix T, a $\mathbb{U}$-small S-category. The category $SPr(T)$ (see Definition 2.3.2) is naturally enriched over *SSet* and the corresponding S-category will be denoted by $SPr(T)_s$. Note that $\mathrm{Int}(SPr(T))$ is a full sub-S-category of $SPr(T)_s$ (recall that $\mathrm{Int}(SPr(T))$ is the S-category of fibrant and cofibrant objects in the simplicial model category $SPr(T)$).

Recall the following *SSet*-enriched version of Yoneda lemma (e.g., [G-J, IX Lemma 1.2])

Proposition 2.4.1. *Let T be an S-category. For any object x in T, let us denote by $\underline{h}_x$ the object in $SPr(T)_s$ defined by $\underline{h}_x(y) := \underline{Hom}_T(y,x)$. Then, for any simplicial functor $F : T \to SSet$, there is a canonical isomorphism of simplicial sets*

$$F(x) \simeq \underline{Hom}_{SPr(T)_s}(\underline{h}_x, F)$$

which is functorial in the pair (F, x).

Then, for any $T \in S-Cat_{\mathbb{U}}$, one defines a morphism of S-categories $\underline{h} : T \longrightarrow SPr(T)_s$, by setting for $x \in Ob(T)$

$$\begin{array}{rcl} \underline{h}_x : T^{op} & \longrightarrow & SSet_{\mathbb{U}}, \\ y & \mapsto & \underline{Hom}_T(y,x). \end{array}$$

Note that Proposition 2.4.1 defines immediately $\underline{h}$ at the level of morphisms between simplicial *Hom*'s and shows that $\underline{h}$ is fully faithful (in a strong sense) as a morphism in $S-Cat_{\mathbb{V}}$. Now, the morphism $\underline{h}$ induces a functor between the associated homotopy categories that we will still denote by

$$\underline{h} : \mathrm{Ho}(T) \longrightarrow \mathrm{Ho}(SPr(T)_s).$$

Now, we want to compare $\mathrm{Ho}(SPr(T)_s)$ to $\mathrm{Ho}(SPr(T))$; note that the two $\mathrm{Ho}(-)$'s here have different meanings, as the first one refers to the homotopy category of an S-category while the second one to the homotopy category of a model category. By definition, in the set of morphisms between F and G in $\mathrm{Ho}(SPr(T)_s)$, simplicially homotopic maps in $Hom_{SPr(T)}(F,G) = \underline{Hom}_{SPr(T)_s}(F,G)_0$, give rise to the same element. Then, since simplicially homotopic maps in $Hom_{SPr(T)}(F,G)$ have the same image in $\mathrm{Ho}(SPr(T))$ (see, for example, [Hi, Corollary 10.4.5]), the identity functor induces a well-defined localization morphism

$$\mathrm{Ho}(SPr(T)_s) \longrightarrow \mathrm{Ho}(SPr(T)).$$

Composing this with the functor $\underline{h}$, one deduces a well-defined functor (denoted with the same symbol)

$$\underline{h} : \mathrm{Ho}(T) \longrightarrow \mathrm{Ho}(SPr(T)).$$

The following is a homotopy version of the enriched Yoneda lemma (i.e. a homotopy variation of Proposition 2.4.1).

Proposition 2.4.2. *For any object $F \in SPr(T)$ and any $x \in Ob(T)$, there exists an isomorphism in* $\mathrm{Ho}(SSet_{\mathbb{U}})$

$$F(x) \simeq \mathbb{R}\underline{Hom}(\underline{h}_x, F)$$

which is functorial in the pair $(F, x) \in \mathrm{Ho}(SPr(T)) \times \mathrm{Ho}(T)$. *In particular, the functor* $\underline{h} : \mathrm{Ho}(T) \longrightarrow \mathrm{Ho}(SPr(T))$ *is fully faithful.*

Proof. Using Proposition 2.4.1, since equivalences in $SPr(T)$ are defined objectwise, by taking a fibrant replacement of F, we may suppose that F is fibrant. Moreover, again by Proposition 2.4.1, the unique morphism $* \to \underline{h}_x$ has the right lifting property with respect to all trivial fibrations, hence $\underline{h}_x$ is a cofibrant object in $SPr(T)$. Therefore, for any fibrant object $F \in SPr(T)$, one has natural isomorphisms in $\mathrm{Ho}(SSet_{\mathbb{U}})$

$$F(x) \simeq \underline{Hom}(\underline{h}_x, F) \simeq \mathbb{R}\underline{Hom}(\underline{h}_x, F). \qquad \square$$

The following corollary is a refined version of Proposition 2.4.2.

Corollary 2.4.3. *Let T be an S-category in $\mathbb{U}$ with fibrant simplicial Hom-sets. Then, the morphism $\underline{h} : T \longrightarrow SPr(T)_s$ factors through* $\mathrm{Int}(SPr(T))$ *and the induced morphism* $\underline{h} : T \longrightarrow \mathrm{Int}(SPr(T))$ *in $S-Cat$ is fully faithful.*

Proof. The assumption on T implies that $\underline{h}_x$ is fibrant and cofibrant in $SPr(T)$, for any $x \in Ob(T)$ and therefore that $\underline{h}$ factors through $\mathrm{Int}(SPr(T)) \subset SPr(T)_s$. Finally, Proposition 2.4.2 immediately implies that $\underline{h}$ is fully faithful. Actually, this is already true for $\underline{h} : T \longrightarrow SPr(T)_s$, by Proposition 2.4.1, and hence this is true for our factorization since $\mathrm{Int}(SPr(T))$ is a full sub-S-category of $SPr(T)_s$. $\square$

In case T is an *arbitrary* S-category in $\mathbb{U}$ (possibly with non-fibrant simplicial Hom sets), one can consider a fibrant replacement $j : T \longrightarrow RT$, defined by applying the Kan Ex^{∞}-construction [G-J, III.4] to each simplicial set of morphisms in T, together with its Yoneda embedding

$$T \xrightarrow{j} RT \xrightarrow{\underline{h}} \mathrm{Int}(SPr(RT)).$$

When viewed in $\mathrm{Ho}(S-Cat_{\mathbb{V}})$, this induces a well-defined morphism

$$T \xrightarrow{j} RT \xrightarrow{\underline{h}} \mathrm{Int}(SPr(RT)) \simeq LSPr(RT).$$

Finally, composing with the isomorphism $\mathbb{L}j_! = (j^*)^{-1} : LSPr(RT) \simeq LSPr(T)$ of Theorem 2.3.1, one gets a morphism

$$L\underline{h} : T \longrightarrow LSPr(T).$$

This is a morphism in $\mathrm{Ho}(S-Cat_{\mathbb{V}})$, called the *S-Yoneda embedding of T*; when no confusion is possible, we will simply call it the Yoneda embedding of T. Now, Corollary 2.4.3 immediately implies that $L\underline{h}$ is fully faithful, and is indeed isomorphic to the morphism $\underline{h}$ defined above when T has fibrant simplicial Hom-sets.

Definition 2.4.4. Let T be an S-category. An object in $\mathrm{Ho}(SPr(T))$ is called *representable* if it belongs to the essential image (see Definition 2.1.3, 2.) of the functor $L\underline{h} : T \longrightarrow LSPr(T)$.

For any $T \in \mathrm{Ho}(S-Cat_{\mathbb{U}})$, the Yoneda embedding $L\underline{h} : T \longrightarrow LSPr(T)$ induces an isomorphism in $\mathrm{Ho}(S-Cat_{\mathbb{U}})$ between T and the full sub-S-category of $LSPr(T)$ consisting of representable objects.

Note that the functor induced on the level of homotopy categories

$$L\underline{h} : \mathrm{Ho}(T) \longrightarrow \mathrm{Ho}(LSPr(T)) = \mathrm{Ho}(SPr(T))$$

simply sends $x \in Ob(T)$ to the simplicial presheaf $\underline{h}_x \in \mathrm{Ho}(SPr(T))$.

2.5. Comma S-categories

In this subsection we will use the Yoneda embedding defined above, in order to define, for an S-category T and an object $x \in T$, the comma S-category T/x in a meaningful way.

Let T be an S-category in $\mathbb{U}$, and let us consider its (usual, enriched) Yoneda embedding

$$\underline{h} : T \longrightarrow SPr(T) := SSet_{\mathbb{U}}^{T^{op}}.$$

For any object $x \in Ob(T)$, we consider the comma category $SPr(T)/\underline{h}_x$, together with its natural induced model structure (i.e. the one created by the forgetful functor $SPr(T)/\underline{h}_x \to SPr(T)$, see [Ho, p. 5]). For any object $y \in Ob(T)$, and any morphism $u : \underline{h}_y \longrightarrow \underline{h}_x$, let $F_u \in SPr(T)/\underline{h}_x$ be a fibrant replacement of u. Since u is already a cofibrant object in $SPr(T)/\underline{h}_x$ (as we already observed in the proof of Proposition 2.4.2), the object F_u is then actually fibrant and cofibrant.

Definition 2.5.1. The *comma S-category* T/x is defined to be the full sub-S-category of $L(SPr(T)/\underline{h}_x)$ consisting of all objects F_u, for all u of the form $u : \underline{h}_y \to \underline{h}_x$, $y \in Ob(T)$.

Note that since T belongs to $\mathbb{U}$, so does the S-category T/x, for any object $x \in Ob(T)$.

There exists a natural morphism in $\mathrm{Ho}(S-Cat_{\mathbb{V}})$

$$T/x \longrightarrow L(SPr(T)/\underline{h}_x) \longrightarrow LSPr(T),$$

where the morphism on the right is induced by the forgetful functor $SPr(T)/\underline{h}_x \longrightarrow SPr(T)$. One checks immediately that the essential image of this morphism is contained in the essential image of the Yoneda embedding $L\underline{h} : T \longrightarrow LSPr(T)$. Therefore, there exists a natural factorization in $\mathrm{Ho}(S-Cat_{\mathbb{V}})$

$$\begin{array}{ccc} T/x & \longrightarrow & LSPr(T) \\ & {\scriptstyle j_x}\searrow \quad \nearrow{\scriptstyle L\underline{h}} & \\ & T & \end{array}$$

As the inclusion functor $\mathrm{Ho}(S-Cat_{\mathbb{U}}) \longrightarrow \mathrm{Ho}(S-Cat_{\mathbb{V}})$ is fully faithful (see Appendix A), this gives a well-defined morphism in $\mathrm{Ho}(S-Cat_{\mathbb{U}})$

$$j_x : T/x \longrightarrow T.$$

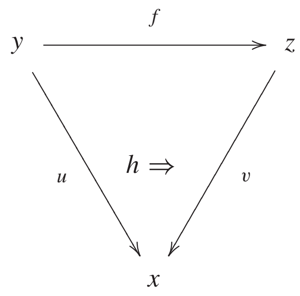
It is important to observe that the functor $\mathbb{R}(j_x)_! : \mathrm{Ho}(SPr(T/x)) \longrightarrow \mathrm{Ho}(SPr(T))$, induced by j_x is such that $\mathbb{R}(j_x)_!(*) \simeq \underline{h}_x$.

Up to a natural equivalence of categories, the homotopy category $\mathrm{Ho}(T/x)$ has the following explicit description. For the sake of simplicity we will assume that T is a fibrant S-category (i.e. all the simplicial sets $\underline{Hom}_T(x, y)$ of morphisms are fibrant). The objects of $\mathrm{Ho}(T/x)$ are simply pairs (y, u), consisting of an object $y \in Ob(T)$ and a 0-simplex $u \in \underline{Hom}_T(y, x)_0$ (i.e. a morphism $y \to x$ in the category T_0).

Let us consider two objects (y, u) and (z, v), and a pair (f, h), consisting of a 0-simplex $f \in \underline{Hom}_T(y, z)$ and a 1-simplex $h \in \underline{Hom}_T(y, x)^{\Delta^1}$ such that

$$\partial_0(h) = u \qquad \partial_1(h) = v \circ f.$$

We may represent diagramatically this situation as

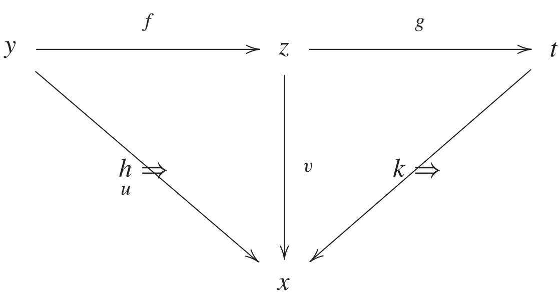


Two such pairs (f,h) and (g,k) are defined to be equivalent if there exist a 1-simplex $H \in \underline{Hom}_T(y,z)^{\Delta^1}$ and a 2-simplex $G \in \underline{Hom}_T(y,x)^{\Delta^2}$ such that

$$\partial_0(H) = f \quad \partial_1(H) = g \quad \partial_0(G) = h \quad \partial_1(G) = k \quad \partial_2(G) = v \circ H.$$

The set of morphisms in $\mathrm{Ho}(T/x)$ from (y,u) to (z,v) is then the set of equivalences classes of such pairs (f,h). In other words, the set of morphisms from (y,u) to (z,v) is the set of connected components of the homotopy fiber of $-\circ v : \underline{Hom}_T(y,z) \longrightarrow \underline{Hom}_T(y,x)$ at the point u.

Let $(f,h) : (y,u) \longrightarrow (z,v)$ and $(g,k) : (z,v) \longrightarrow (t,w)$ be two morphisms in $\mathrm{Ho}(T/x)$. The composition of (f,h) and (g,k) in $\mathrm{Ho}(T/x)$ is the class of $(g \circ f, k\dot{h})$, where $k\dot{h}$ is the concatenation of the 1-simplices h and $k \circ f$ in $\underline{Hom}_T(y,x)$. Pictorially, one composes the triangles as



As the concatenation of 1-simplices is well-defined, associative and unital up to homotopy, this gives a well-defined, associative and unital composition of morphisms in $\mathrm{Ho}(T/x)$.

Note that there is a natural projection $\mathrm{Ho}(T/x) \longrightarrow \mathrm{Ho}(T)/x$, which sends an object (y,u) to the object y together with the image of u in $\pi_0(\underline{Hom}_T(y,x)) = Hom_{\mathrm{Ho}(T)}(y,x)$. This functor is not an equivalence but is always full and essentially surjective. The composition functor $\mathrm{Ho}(T/x) \longrightarrow \mathrm{Ho}(T)/x \longrightarrow \mathrm{Ho}(T)$ is isomorphic to the functor induced by the natural morphism $T/x \longrightarrow T$.

3. Stacks over *S*-sites

This section is devoted to the definition of the notions of *S*-topologies, *S*-sites and stacks over them. We start by defining *S-topologies* on *S*-categories, generalizing the notion of Grothendieck topologies on usual categories and inducing an obvious notion of *S-site*. For an *S*-site *T*, we define a notion of *local equivalence* in the model category of pre-stacks $SPr(T)$, analogous to the notion of local isomorphism between presheaves on a given Grothendieck site. The first main result of this section is the existence of a model structure on $SPr(T)$, the *local model structure*, whose equivalences are exactly the local equivalences. This model structure is called the *model category of stacks*. To motivate this terminology we prove a criterion characterizing fibrant objects in the model category of stacks as objects satisfying a *hyperdescent* property with respect to the given *S*-topology, which is a homotopy analog of the usual descent or sheaf condition. We also investigate functoriality properties (i.e. inverse and direct images functors) of the model categories of stacks, as well as the very useful notion of *stack of morphisms* (i.e. internal *Hom*'s).

The second main result of this section is a correspondence between *S*-topologies on an *S*-category *T* and *t*-complete left Bousfield localizations of the model category of pre-stacks $SPr(T)$. Finally, we relate our definition of stacks over *S*-sites to the notion of *model topos* due to Rezk, and we conclude from our previous results that almost all model topoi are equivalent to a model category of stacks over an *S*-site.

3.1. S-topologies and S-sites

We refer to [SGA4-I, Exp. II] or [M-M] for the definition of a Grothendieck topology and for the associated sheaf theory.

Definition 3.1.1. An *S-topology* on an *S*-category *T* is a Grothendieck topology on the category $\mathrm{Ho}(T)$. An *S-site* (T, τ) is the datum of an *S*-category *T* together with an *S*-topology τ on *T*.

Remark 3.1.2. 1. It is important to remark that the notion of an *S*-topology on an *S*-category *T* only depends on the isomorphism class of $T \in Ho(S-Cat)$, since equivalent *S*-categories have equivalent homotopy categories.

2. From the point of view of higher category theory, *S*-categories are models for ∞-categories in which all *i*-arrows are *invertible* for all $i > 1$. Therefore, if one tries to define the notion of a topology on this kind of higher categories, the stability axiom will imply that all *i*-morphisms should be automatically coverings for $i > 1$. The datum of the topology should therefore only depends on isomorphism classes of 1-morphisms, or, in other words, on the homotopy category. This might give a more conceptual

explanation of Definition 3.1.1. See also Remark 3.8.7 for more on topologies on higher categories.

Let $T \in S - Cat_{\mathbb{U}}$ be a $\mathbb{U}$-small S-category and $SPr(T)$ its model category of pre-stacks. Given any pre-stack $F \in SPr(T)$, one can consider its associated presheaf of connected components

$$\begin{array}{rcl} T^{op} & \longrightarrow & Set_{\mathbb{U}}, \\ x & \mapsto & \pi_0(F(x)). \end{array}$$

The universal property of the homotopy category of T^{op} implies that there exists a unique factorization

$$\begin{array}{ccc} T^{op} & \longrightarrow & Set_{\mathbb{U}} \\ \downarrow & \nearrow_{\pi_0^{pr}(F)} & \\ \mathrm{Ho}(T)^{op} & & \end{array}$$

The construction $F \mapsto \pi_0^{pr}(F)$, being obviously functorial in F, induces a well-defined functor $SPr(T) \longrightarrow Set_{\mathbb{U}}^{\mathrm{Ho}(T)^{op}}$; but, since equivalences in $SPr(T)$ are defined objectwise, this induces a functor

$$\pi_0^{pr}(-) : \mathrm{Ho}(SPr(T)) \longrightarrow Set_{\mathbb{U}}^{\mathrm{Ho}(T)^{op}}.$$

Definition 3.1.3. Let (T, τ) be a $\mathbb{U}$-small S-site.

1. For any object $F \in SPr(T)$, the sheaf associated to the presheaf $\pi_0^{pr}(F)$ is denoted by $\pi_0^{\tau}(F)$ (or $\pi_0(F)$ if the topology τ is clear from the context). It is a sheaf on the site $(\mathrm{Ho}(T), \tau)$, and is called the *sheaf of connected components of F*.
2. A morphism $F \longrightarrow G$ in $\mathrm{Ho}(SPr(T))$ is called a *τ-covering* (or just a *covering* if the topology τ is clear from the context) if the induced morphism $\pi_0^{\tau}(F) \longrightarrow \pi_0^{\tau}(G)$ is an epimorphism of sheaves.
3. A morphism $F \longrightarrow G$ in $SPr(T)$ is called a *τ-covering* (or just a *covering* if the topology τ is unambiguous) if its image by the natural functor $SPr(T) \longrightarrow \mathrm{Ho}(SPr(T))$ is a τ-covering as defined in the previous item.

Clearly, for two objects x and y in T, any morphism $x \longrightarrow y$ such that the sieve generated by its image in $\mathrm{Ho}(T)$ is a covering sieve of y, induces a covering $\underline{h}_x \longrightarrow \underline{h}_y$.

More generally, one has the following characterization of coverings as *homotopy locally surjective* morphisms. This is the homotopy analog of the notion of epimorphism of stacks (see for example [La-Mo, Section, 1]), where one requires that any object in the target is locally isomorphic to the image of an object in the source.

Proposition 3.1.4. *A morphism $f : F \longrightarrow G$ in $SPr(T)$ is a covering if it has the following* homotopy local surjectivity *property. For any object $x \in Ob(T)$, and any morphism in* $\mathrm{Ho}(SPr(T))$, $\underline{h}_x \longrightarrow G$, *there exists a covering sieve R of x in* $\mathrm{Ho}(T)$, *such that for any morphism $u \to x$ in R there is a commutative diagram in* $\mathrm{Ho}(SPr(T))$:

$$\begin{array}{ccc} F & \longrightarrow & G \\ \uparrow & & \uparrow \\ \underline{h}_u & \longrightarrow & \underline{h}_x. \end{array}$$

In other words, f is a covering if and only if any object of G over x lifts locally and up to homotopy to an object of F.

Proof. First of all, let us observe that both the definition of a covering and the *homotopy local surjectivity property* hold true for the given $f : F \to G$ if and only if they hold true for $RF \to RG$, where $R(-)$ is a fibrant replacement functor in $SPr(T)$. Therefore, we may suppose both F and G fibrant. Now, by Mac Lane and Moerdirk [M-M, III.7, Corollary 6], $f : F \to G$ is a covering iff the induced map of presheaves $\pi_0^{pr}(F) \to \pi_0^{pr}(G)$ is locally surjective. But, by Yoneda $\pi_0^{pr}(H)(y) \simeq \pi_0(\underline{Hom}_{SPr(T)}(\underline{h}_y, H))$, for any $H \in SPr(T)$ and any object y in T. Since F and G are fibrant, we then have $\pi_0^{pr}(F)(y) \simeq Hom_{\mathrm{Ho}(SPr(T))}(\underline{h}_y, F)$ and $\pi_0^{pr}(G)(y) \simeq Hom_{\mathrm{Ho}(SPr(T))}(\underline{h}_y, G)$, for any object y in T. But then, the local surjectivity of $\pi_0^{pr}(F) \to \pi_0^{pr}(G)$ exactly translates to the *homotopy local surjectivity property* in the proposition and we conclude. □

Remark 3.1.5. If the morphism f of Proposition 3.1.4 is an objectwise fibration (i.e. for any $x \in T$, the morphism $F(x) \longrightarrow G(x)$ is a fibration of simplicial sets), then the homotopy local surjectivity property implies the local surjectivity property. This means that the diagrams

$$\begin{array}{ccc} F & \longrightarrow & G \\ \uparrow & & \uparrow \\ \underline{h}_u & \longrightarrow & \underline{h}_x \end{array}$$

of Proposition 3.1.4 can be chosen to be commutative in $SPr(T)$, and not only in $\mathrm{Ho}(SPr(T))$.

From this characterization one concludes easily that coverings have the following stability properties.

Corollary 3.1.6. 1. *A morphism in $SPr(T)$ which is a composition of coverings is a covering.*

2. *Let*

$$\begin{array}{ccc} F' & \xrightarrow{f'} & G' \\ \downarrow & & \downarrow \\ F & \xrightarrow[f]{} & G \end{array}$$

be a homotopy cartesian diagram in $SPr(T)$. If f is a covering so is f'.

3. *Let* $F \xrightarrow{u} G \xrightarrow{v} H$ *be two morphisms in $SPr(T)$. If the morphism $v \circ u$ is a covering then so is v.*

4. *Let*

$$\begin{array}{ccc} F' & \xrightarrow{f'} & G' \\ \downarrow & & \downarrow{\scriptstyle p} \\ F & \xrightarrow[f]{} & G \end{array}$$

be a homotopy cartesian diagram in $SPr(T)$. If p and f' are coverings then so is f.

Proof. Properties (1) and (3) follow immediately from Proposition 3.1.4, and (4) follows from (3). It remains to prove (2). Let us f and f' be as in (2) and let us consider a diagram

$$\begin{array}{ccc} & & \underline{h}_x \\ & & \downarrow \\ F' & \xrightarrow{f'} & G' \\ \downarrow & & \downarrow \\ F & \xrightarrow{f} & G \end{array}$$

As f is a covering, there exists a covering sieve R over $x \in \mathrm{Ho}(T)$, such that for any $u \to x$ in R, one has a commutative diagram

$$\begin{array}{ccc} \underline{h}_u & \longrightarrow & \underline{h}_x \\ \downarrow & & \downarrow \\ F & \xrightarrow{f} & G \end{array}$$

By the universal property of homotopy fibered products, the morphisms $\underline{h}_u \longrightarrow F$ and $\underline{h}_u \longrightarrow \underline{h}_x \longrightarrow G'$ are the two projections of a (non unique) morphism $\underline{h}_u \longrightarrow F'$. This gives, for all $u \to x$, the required liftings

$$\begin{array}{ccc} \underline{h}_u & \longrightarrow & \underline{h}_x \\ \downarrow & & \downarrow \\ F' & \xrightarrow{f'} & G'. \end{array} \qquad \Box$$

3.2. Simplicial objects and hypercovers

Let us now consider $sSPr(T) := SPr(T)^{\Delta^{op}}$, the category of simplicial objects in $SPr(T)$. Its objects will be denoted as

$$\begin{array}{rcl} F_* : \Delta^{op} & \longrightarrow & SPr(T) \\ [m] & \mapsto & F_m. \end{array}$$

As the category $SPr(T)$ has all kind of limits and colimits indexed in $\mathbb{U}$, the category $sSPr(T)$ has a natural structure of tensored and co-tensored category over $SSet_{\mathbb{U}}$ (see [G-J, Chapter II, Theorem 2.5]). The external product of $F_* \in sSPr(T)$ by $A \in SSet_{\mathbb{U}}$, denoted by $\underline{A} \otimes F_*$, is the simplicial object in $SPr(T)$ defined by

$$\begin{array}{rcl} \underline{A} \otimes F_* : \Delta^{op} & \longrightarrow & SPr(T), \\ [n] & \mapsto & \coprod_{A_n} F_n. \end{array}$$

The exponential (or co-tensor) of F_* by A, is denoted by $F_*^{\underline{A}}$ and is determined by the usual adjunction isomorphism

$$Hom(\underline{A} \otimes F_*, G_*) \simeq Hom(F_*, G_*^{\underline{A}}).$$

Notation. We will denote by $F_*^A \in SPr(T)$ the 0th level of the simplicial object $F_*^{\underline{A}} \in sSPr(T)$.

Explicitly, the object F_*^A is the *end* of the functor

$$\begin{array}{rcl}\Delta^{op} \times \Delta & \longrightarrow & SPr(T), \\ ([n],[m]) & \mapsto & \prod_{A_m} F_n.\end{array}$$

One checks immediately that for any $F_* \in sSPr(T)$, one has a natural isomorphism $F_*^{\Delta^n} \simeq F_n$.

We endow the category $sSPr(T)$ with its Reedy model structure (see [Ho, Theorem 5.2.5]). The equivalences in $sSPr(T)$ are the morphisms $F_* \longrightarrow G_*$ such that, for any n, the induced morphism $F_n \longrightarrow G_n$ is an equivalence in $SPr(T)$. The fibrations are the morphisms $F_* \longrightarrow G_*$ such that, for any $[n] \in \Delta$, the induced morphism

$$F_*^{\Delta^n} \simeq F_n \longrightarrow F_*^{\partial\Delta^n} \times_{G_*^{\partial\Delta^n}} G_*^{\Delta^n}$$

is a fibration in $SPr(T)$.

Given any simplicial set $A \in SSet_{\mathbb{U}}$, the functor

$$\begin{array}{rcl}sSPr(T) & \longrightarrow & SPr(T), \\ F_* & \mapsto & F_*^A\end{array}$$

is a right Quillen functor for the Reedy model structure on $sSPr(T)$ [Ho, Proposition 5.4.1]. Its right derived functor will be denoted by

$$\begin{array}{rcl}\mathrm{Ho}(sSPr(T)) & \longrightarrow & \mathrm{Ho}(SPr(T)), \\ F_* & \mapsto & F_*^{\mathbb{R}A}.\end{array}$$

For any object $F \in SPr(T)$, one can consider the constant simplicial object $c(F)_* \in sSPr(T)$ defined by $c(F)_n := F$ for all n. One the other hand, one can consider

$$\begin{array}{rcl}(RF)^{\Delta^*} : \Delta^{op} & \longrightarrow & SPr(T), \\ [n] & \mapsto & (RF)^{\Delta^n},\end{array}$$

where RF is a fibrant model for F in $SPr(T)$, and $(RF)^{\Delta^n}$ is the exponential object defined using the simplicial structure on $SPr(T)$. The object $(RF)^{\Delta^*}$ is a fibrant replacement of $c_*(F)$ in $sSPr(T)$. Furthermore, for any object $G \in SPr(T)$ and $A \in SSet_{\mathbb{U}}$, there exists a natural isomorphism in $SPr(T)$

$$(G^{\Delta^*})^A \simeq G^A.$$

This induces a natural isomorphism in $\mathrm{Ho}(SPr(T))$

$$(c(F)_*)^{\mathbb{R}A} \simeq ((RF)^{\Delta^*})^A \simeq (RF)^A.$$

However, we remark that $c(F)_*^A$ is not isomorphic to F^A as an object in $SPr(T)$.

Notation. For any $F \in SPr(T)$ and $A \in SSet_{\mathbb{U}}$, we will simply denote by $F^{\mathbb{R}A} \in \mathrm{Ho}(SPr(T))$ the object $c(F)_*^{\mathbb{R}A} \simeq (RF)^A$.

We let $\Delta_{\leqslant n}$ be the full subcategory of Δ consisting of objects $[p]$ with $p \leqslant n$, and denote by $s_n SPr(T)$ the category of functors $\Delta_{\leqslant n}^{op} \longrightarrow SPr(T)$. The natural inclusion $i_n : \Delta_{\leqslant n} \to \Delta$ induces a restriction functor

$$i_n^* : sSPr(T) \longrightarrow s_n SPr(T)$$

which has a right adjoint $(i_n)_* : s_n SPr(T) \longrightarrow sSPr(T)$, as well as a left adjoint $(i_n)_! : s_n SPr(T) \longrightarrow sSPr(T)$. The two adjunction morphisms induce isomorphisms $i_n^*(i_n)_* \simeq \mathrm{Id}$ and $i_n^*(i_n)_! \simeq \mathrm{Id}$: therefore both functors $(i_n)_*$ and $(i_n)_!$ are fully faithful.

Definition 3.2.1. Let $F_* \in sSPr(T)$ and $n \geqslant 0$.

1. One defines the *n*th *skeleton and n-coskeleton* of F_* as

$$Sk_n(F_*) := (i_n)_! i_n^*(F_*) \qquad Cosk_n(F_*) := (i_n)_* i_n^*(F_*).$$

2. The simplicial object F_* is called *n-bounded* if the adjunction morphism $F_* \longrightarrow Cosk_n(F_*)$ is an isomorphism.

It is important to note that F_*, $Cosk_n(F_*)$ and $Sk_n(F_*)$ all coincide in degrees $\leqslant n$

$$i_n^*(F_*) \simeq i_n^*(Cosk_n F_*) \simeq i_n^*(Sk_n F_*).$$

The adjunctions $(i_n^*, (i_n)_*)$ and $((i_n)_!, i_n^*)$ induce a natural adjunction isomorphism

$$Hom(Sk_n(F_*), G_*) \simeq Hom(F_*, Cosk_n(G_*)),$$

for any F_* and G_* in $sSPr(T)$ and any $n \geqslant 0$. As a special case, for any $A \in SSet_{\mathbb{U}}$, one has an isomorphism in $SPr(T)$

$$F_*^{Sk_n A} \simeq (Cosk_n\, F_*)^A.$$

As $Sk_n \Delta^{n+1} = \partial \Delta^{n+1}$, one gets natural isomorphisms

$$F_*^{\partial \Delta^{n+1}} \simeq Cosk_n(F_*)_{n+1}. \qquad (1)$$

Lemma 3.2.1. *The functor* $Cosk_n : sSPr(T) \longrightarrow sSPr(T)$ *is a right Quillen functor for the Reedy model structure on* $sSPr(T)$.

Proof. By adjunction, for any integer p with $p \leqslant n$, one has

$$(Cosk_n(F_*))^{\partial\Delta^p} \simeq F_*^{\partial\Delta^p} \qquad (Cosk_n(F_*))^{\Delta^p} \simeq F_*^{\Delta^p},$$

while, for $p > n+1$, one has

$$(Cosk_n(F_*))^{\partial\Delta^p} \simeq (Cosk_n(F_*))^{\Delta^p}.$$

Finally, for $p = n+1$ one has

$$(Cosk_n(F_*))^{\partial\Delta^{n+1}} \simeq F_*^{\partial\Delta^{n+1}} \qquad (Cosk_n(F_*))^{\Delta^{n+1}} \simeq F_*^{\partial\Delta^{n+1}}.$$

Using these formulas and the definition of Reedy fibrations in $sSPr(T)$ one checks immediately that the functor $Cosk_n$ preserves fibrations and trivial fibrations. As it is a right adjoint (its left adjoint being Sk_n), this implies that $Cosk_n$ is a right Quillen functor. □

The previous lemma allows us to consider the right derived version of the coskeleton functor

$$\mathbb{R}Cosk_n : \mathrm{Ho}(sSPr(T)) \longrightarrow \mathrm{Ho}(sSPr(T)).$$

It comes with a natural morphism $\mathrm{Id}_{\mathrm{Ho}(sSPr(T))} \longrightarrow \mathbb{R}Cosk_n(F)$, induced by the adjunction morphism $\mathrm{Id}_{sSPr(T)} \longrightarrow (i_n)_* i_n^*$. There exist obvious *relative* notions of the functors Sk_n and $Cosk_n$ whose formulations are left to the reader. Let us only mention that the relative derived coskeleton of a morphism $F_* \longrightarrow G_*$ in $sSPr(T)$ may be defined by the following homotopy cartesian square in $SPr(T)$:

$$\begin{array}{ccc} \mathbb{R}Cosk_n(F_*/G_*) & \longrightarrow & G_* \\ \downarrow & & \downarrow \\ \mathbb{R}Cosk_n(F_*) & \longrightarrow & \mathbb{R}Cosk_n(G_*). \end{array}$$

The functor $\mathbb{R}Cosk_0(-/c(G)_*)$, relative to a constant diagram $c(G)_*$, where $G \in SPr(T)$, has the following interpretation in terms of *derived nerves*. For any morphism $F_* \longrightarrow c_*(G)$ in $sSPr(T)$, with $c_*(G)$ the constant simplicial diagram with value G, we consider the induced morphism $f : F_0 \longrightarrow G$ in $\mathrm{Ho}(SPr(T))$. Let us represent this morphism by a fibration in $SPr(T)$, and let us consider its usual nerve $N(f)$:

$$\begin{array}{rcl} N(f) : \Delta^{op} & \longrightarrow & SPr(T), \\ [n] & \mapsto & \underbrace{F_0 \times_G F_0 \times_G \ldots \times_G F_0}_{\text{n times}}. \end{array}$$

The nerve $N(f)$ *is naturally augmented over* G*, and therefore is an object of* $sSPr(T)/c_*(G)$. Then, there is a natural isomorphism in $\mathrm{Ho}(sSPr(T)/c_*(G))$

$$\mathbb{R}Cosk_0(F_*/c_*(G)) \simeq N(f).$$

Definition 3.2.3. Let (T, τ) be a $\mathbb{U}$-small *S*-site.

1. A morphism in $sSPr(T)$

$$F_* \longrightarrow G_*$$

is called a τ-*hypercover* (or just a *hypercover* if the topology τ is unambiguous) if for any *n*, the induced morphism

$$F_*^{\mathbb{R}\Delta^n} \simeq F_n \longrightarrow F_*^{\mathbb{R}\partial\Delta^n} \times^h_{G_*^{\mathbb{R}\partial\Delta^n}} G_*^{\mathbb{R}\Delta^n}$$

is a covering in $\mathrm{Ho}(SPr(T))$ (see Definition 3.1.3(2)).

2. A morphism in $\mathrm{Ho}(sSPr(T))$

$$F_* \longrightarrow G_*$$

is called a τ-*hypercover* (or just a *hypercover* if the topology τ is unambiguous) if one of its representatives in $sSPr(T)$ is a τ-hypercover.

Using isomorphisms (1), Definition 3.2.3 may also be stated as follows. A morphism $f : F_* \longrightarrow G_*$ is a τ-hypercover *if and only if* for any $n \geqslant 0$ the induced morphism

$$F_n \longrightarrow \mathbb{R}Cosk_{n-1}(F_*/G_*)_n$$

is a covering in $\mathrm{Ho}(SPr(T))$.

Note also that in Definition 3.2.3(2), if one of the representatives of *f* is a hypercover, then so is any representative. Being a hypercover is therefore a property of morphisms in $\mathrm{Ho}(sSPr(T))$.

3.3. Local equivalences

Throughout this subsection, we fix a $\mathbb{U}$-small S-site (T, τ).

Let x be an object in T. The topology on $\mathrm{Ho}(T)$ induces a natural topology on the comma category $\mathrm{Ho}(T)/x$. We define a Grothendieck topology on $\mathrm{Ho}(T/x)$ by pulling back the topology of $\mathrm{Ho}(T)/x$ through the natural projection $\mathrm{Ho}(T/x) \longrightarrow \mathrm{Ho}(T)/x$. By this, we mean that a sieve S over an object $y \in Ho(T/x)$, is defined to be a covering sieve if and only if (the sieve generated by) its image in $\mathrm{Ho}(T)$ is a τ-covering sieve of the object $y \in \mathrm{Ho}(T)/x$. The reader will check easily that this indeed defines a topology on $\mathrm{Ho}(T/x)$, and therefore an S-topology on T/x. This topology will still be denoted by τ.

Definition 3.3.1. The S-site $(T/x, \tau)$ will be called the *comma S-site* of (T, τ) over x.

Let $F \in SPr(T)$, $x \in Ob(T)$ and $s \in \pi_0(F(x))$ be represented by a morphism $s : \underline{h}_x \longrightarrow F$ in $\mathrm{Ho}(SPr(T))$ (see 2.4.2). By pulling-back this morphism along the natural morphism $j_x : T/x \longrightarrow T$, one gets a morphism in $\mathrm{Ho}(SPr(T/x))$

$$s : j_x^*(\underline{h}_x) \longrightarrow j_x^*(F).$$

By definition of the comma category T/x, it is immediate that $j_x^*(\underline{h}_x)$ has a natural global point $* \longrightarrow j_x^*(\underline{h}_x)$ in $\mathrm{Ho}(SPr(T/x))$. Note that the morphism $* \longrightarrow j_x^*(\underline{h}_x)$ is also induced by adjunction from the identity of $\underline{h}_x \simeq \mathbb{R}(j_x)_!(*)$. Therefore we obtain a global point of $j_x^*(F)$

$$s : * \longrightarrow j_x^*(\underline{h}_x) \longrightarrow j_x^*(F).$$

Definition 3.3.2. Let $F \in SPr(T)$ and $x \in Ob(T)$.

1. For any integer $n > 0$, the sheaf $\pi_n(F, s)$ is defined as

$$\pi_n(F, s) := \pi_0(j_x^*(F)^{\mathbb{R}\Delta^n} \times_{j_x^*(F)^{\mathbb{R}\partial\Delta^n}} *).$$

   It is a sheaf on the site $(\mathrm{Ho}(T/x), \tau)$ called the *nth homotopy sheaf* of F *pointed* at s.
2. A morphism $f : F \longrightarrow G$ in $SPr(T)$ is called a *π_*-equivalence* or, equivalently, a *local equivalence*, if the following two conditions are satisfied:
   (a) The induced morphism $\pi_0(F) \longrightarrow \pi_0(G)$ is an isomorphism of sheaves on $\mathrm{Ho}(T)$.
   (b) For any object $x \in Ob(T)$, any section $s \in \pi_0(F(x))$ and any integer $n > 0$, the induced morphism $\pi_n(F, s) \longrightarrow \pi_n(G, f(s))$ is an isomorphism of sheaves on $\mathrm{Ho}(T/x)$.
3. A morphism in $\mathrm{Ho}(SPr(T))$ is a π_*-equivalence if one of its representatives in $SPr(T)$ is a π_*-equivalence.

Obviously, an equivalence in the model category $SPr(T)$ is always a π_*-equivalence for any topology τ on T. Indeed, an equivalence in $SPr(T)$ induces isomorphisms between the homotopy presheaves which are the homotopy sheaves for the trivial topology.

Note also that in Definition 3.3.2(3), if a representative of f is a π_*-equivalence then so is any of its representatives. Therefore, being a π_*-equivalence is actually a property of morphisms in $\mathrm{Ho}(SPr(T))$.

The following characterization of π_*-equivalences is interesting as it does not involve any base point.

Lemma 3.3.3. *A morphism $f : F \longrightarrow G$ in $SPr(T)$ is a π_*-equivalence if and only if for any $n \geqslant 0$ the induced morphism*

$$F^{\mathbb{R}\Delta^n} \longrightarrow F^{\mathbb{R}\partial\Delta^n} \times^h_{G^{\mathbb{R}\partial\Delta^n}} G^{\mathbb{R}\Delta^n}$$

is a covering.

In other words, $f : F \longrightarrow G$ is a π_*-equivalence if and only if it is a τ-hypercover when considered as a morphism of constant simplicial objects in $SPr(T)$.

Proof. Without loss of generality, we can assume that f is a fibration between fibrant objects in the model category $SPr(T)$. This means that for any $x \in Ob(T)$, the induced morphism $f : F(x) \longrightarrow G(x)$ is a fibration between fibrant simplicial sets. In particular, the morphism

$$F^{\mathbb{R}\Delta^n} \longrightarrow F^{\mathbb{R}\partial\Delta^n} \times^h_{G^{\mathbb{R}\partial\Delta^n}} G^{\mathbb{R}\Delta^n}$$

in $\mathrm{Ho}(SPr(T))$ is represented by the morphism in $SPr(T)$

$$F^{\Delta^n} \longrightarrow F^{\partial\Delta^n} \times_{G^{\partial\Delta^n}} G^{\Delta^n}.$$

This morphism is furthermore an objectwise fibration, and therefore the local lifting property of τ-coverings (see Proposition 3.1.4) holds not only in $\mathrm{Ho}(SPr(T))$ but in $SPr(T)$ (see Remark 3.1.5). Hence, f is a hypercover if and only if it satisfies the following local lifting property.

For any $x \in \mathrm{Ho}(T)$, and any morphism in $SPr(T)$

$$\underline{h}_x \longrightarrow F^{\partial\Delta^n} \times_{G^{\partial\Delta^n}} G^{\Delta^n},$$

there exists a covering sieve R of x and, for any $u \to x$ in R, a commutative diagram in $SPr(T)$

$$\begin{array}{ccc} F^{\Delta^n} & \longrightarrow & F^{\partial\Delta^n} \times_{G^{\partial\Delta^n}} G^{\Delta^n} \\ \uparrow & & \uparrow \\ \underline{h}_u & \longrightarrow & \underline{h}_x. \end{array}$$

By adjunction, this is equivalent to the following condition. For any object $x \in Ob(T)$ and any commutative diagram in $SSet_{\mathbb{U}}$

$$\begin{array}{ccc} F(x) & \longrightarrow & G(x) \\ \uparrow & & \uparrow \\ \partial\Delta^n & \longrightarrow & \Delta^n \end{array}$$

there exists a covering sieve R of x in $\mathrm{Ho}(T)$ such that for any morphism $u \to x$ in T, whose image belongs to R, there is a commutative diagram in $SSet_{\mathbb{U}}$

$$\begin{array}{ccc} F(u) & \longrightarrow & G(u) \\ \uparrow & \nwarrow & \uparrow \\ F(x) & \longrightarrow & G(x) \\ \uparrow & & \uparrow \\ \partial\Delta^n & \longrightarrow & \Delta^n \end{array}$$

By definition of the homotopy sheaves, this last condition is easily seen to be equivalent to being a π_*-equivalence (the details are left to the reader, who might also wish to consult [Ja1, Theorem 1.12]). □

Corollary 3.3.4. *Let $f : F \longrightarrow G$ be a morphism in $SPr(T)$ and $G' \longrightarrow G$ be a covering. Then, if the induced morphism*

$$f' : F \times^h_G G' \longrightarrow G'$$

is a π_*-equivalence, then so is f.

Proof. Apply Lemma 3.3.3 and Proposition 3.1.6(2). □

Corollary 3.3.5. *Let $f : F \longrightarrow G$ be a π_*-equivalence in $SPr(T)$ and $G' \longrightarrow G$ be an objectwise fibration. Then, the induced morphism*

$$f' : F \times_G G' \longrightarrow G'$$

is a π_*-equivalence.

Proof. This follows from Corollary 3.3.4 since $SPr(T)$ is a proper model category. □

Let x be an object in T and $f : F \to G$ be a morphism in $\mathrm{Ho}(SPr(T))$. For any morphism $s : \underline{h}_x \longrightarrow G$ in $\mathrm{Ho}(SPr(T))$, let us define $F_s \in \mathrm{Ho}(SPr(T/x))$ by the following homotopy cartesian square in $SPr(T/x)$;

$$\begin{array}{ccc} j_x^*(F) & \xrightarrow{j_x^*(f)} & j_x^*(G) \\ \uparrow & & \uparrow \\ F_s & \longrightarrow & \bullet \end{array}$$

where the morphism $* \longrightarrow j_x^*(G)$ is adjoint to the morphism $s : \mathbb{R}(j_x)_!(*) \simeq \underline{h}_x \longrightarrow G$.

Corollary 3.3.6. *Let $f : F \longrightarrow G$ be a morphism in $SPr(T)$. With the same notations as above, the morphism f is a π_*-equivalence if and only for any $s : \underline{h}_x \longrightarrow G$ in $\mathrm{Ho}(SPr(T))$, the induced morphism $F_s \longrightarrow *$ is a π_*-equivalence in $\mathrm{Ho}(SPr(T/x))$.*

Proof. By Lemma 3.3.3 it is enough to show that the morphism f is a covering if and only if all the $F_s \longrightarrow *$ are coverings in $\mathrm{Ho}(SPr(T/x))$. The *only if* part follows from Proposition 3.1.6(2), so it is enough to show that if all the $F_s \longrightarrow *$ are coverings then f is a covering.

Given $s : \underline{h}_x \longrightarrow G$ in $\mathrm{Ho}(SPr(T))$, let us prove that it lifts locally to F. By adjunction, s corresponds to a morphism $* \longrightarrow j_x^*(G)$. As the corresponding morphism $F_s \longrightarrow *$ is a covering, there exists a covering sieve R of $*$ in $\mathrm{Ho}(T/x)$ and, for each $u \to *$ in R, a commutative diagram in $\mathrm{Ho}(SPr(T/x))$

$$\begin{array}{ccc} j_x^*(F) & \longrightarrow & j_x^*(G) \\ \uparrow & & \uparrow \\ \underline{h}_u & \longrightarrow & * \end{array}$$

By adjunction, this commutative diagram induces a commutative diagram in $\mathrm{Ho}(SPr(T))$

$$\begin{array}{ccc} F & \longrightarrow & G \\ \uparrow & & \uparrow \\ \mathbb{R}(j_x)_!(\underline{h}_u) & \longrightarrow & \underline{h}_x \end{array}$$

But $\mathbb{R}(j_x)_!(\underline{h}_u) \simeq \underline{h}_{j_x(u)}$, and by definition of the induced topology on $\mathrm{Ho}(T/x)$, the morphisms in $(j_x)(R)$ form a covering sieve of x. Therefore, the commutative diagram above shows that the morphism s lifts locally to F. $\square$

We end this paragraph by describing the behaviour of π_*-equivalences under homotopy push-outs.

Proposition 3.3.7. *Let $f : F \longrightarrow G$ be a π_*-equivalence in $SPr(T)$ and $F \longrightarrow F'$ be an objectwise cofibration* (*i.e. a monomorphism*). *Then, the induced morphism*

$$f' : F' \longrightarrow F' \coprod_F G$$

is a π_*-equivalence.

Proof. It is essentially the same proof as that of [Ja1, Proposition 2.2]. $\square$

3.4. The local model structure

Throughout this subsection, we fix a $\mathbb{U}$-small S-site (T, τ).

The main purpose of this paragraph is to prove the following theorem which is a generalization of the existence of the local projective model structure on the category of simplicial presheaves on a Grothendieck site (see for example [Bl,H-S, Section 5]). The proof we present here is based on some arguments found in [S1,H-S,DHI], (as well as on some hints from V. Hinich) and uses the Bousfield localization techniques of [Hi], but does not assume the results of [Bl,Ja1].

Theorem 3.4.1. *Let (T, τ) be an S-site. There exists a closed model structure on $SPr(T)$, called the local projective model structure, for which the equivalences are the π_*-equivalences and the cofibrations are the cofibrations for the projective model structure on $SPr(T)$. Furthermore, the local projective model structure is $\mathbb{U}$-cofibrantly generated and proper. The category $SPr(T)$ together with its local projective model structure will be denoted by $SPr_\tau(T)$.*

Proof. We are going to apply the existence theorem for left Bousfield localizations [Hi, Theorem 4.1.1] to the objectwise model structure $SPr(T)$ along a certain $\mathbb{U}$-small set H of morphisms. The main point will be to check that equivalences in this localized model structure are exactly π_*-equivalences.

3.4.1. Definition of the set H

As the S-category T is $\mathbb{U}$-small, the set

$$E(T) := \coprod_{n\in\mathbb{N}} \coprod_{(x,y)\in Ob(T)^2} \underline{Hom}_T(x,y)_n,$$

of all simplices in all simplicial set of morphisms of T is also $\mathbb{U}$-small. We denote by α a $\mathbb{U}$-small cardinal bigger than the cardinal of $E(T)$ and than $\aleph_0$. Finally, we let β be a $\mathbb{U}$-small cardinal with $\beta > 2^\alpha$.

The size of a simplicial presheaf $F \in SPr(T)$ is by definition the cardinality of the set

$$\coprod_{n\in\mathbb{N}} \coprod_{x\in Ob(T)} F_n(x).$$

We will denote it by $\mathrm{Card}(F)$.

For an object $x \in Ob(T)$ we consider a fibrant replacement $\underline{h}_x \hookrightarrow R(\underline{h}_x)$ as well as the simplicial object it defines $R(\underline{h}_x)^{\Delta^*} \in sSPr(T)$. Note that as $\underline{h}_x$ is a cofibrant object, so is $R(\underline{h}_x)$. We define a subset $\mathscr{H}_\beta(x)$ of objects in $sSPr(T)/R(\underline{h}_x)^{\Delta^*}$ in the following way. We consider the following two conditions.

1. The morphism $F_* \longrightarrow R(\underline{h}_x)^{\Delta^*} \in \mathrm{Ho}(sSPr(T))$ is a hypercover.
2. For all $n \geqslant 0$, one has $\mathrm{Card}(F_n) < \beta$. Furthermore, for each $n \geqslant 0$, F_n is isomorphic in $\mathrm{Ho}(SPr(T))$ to a coproduct of representable objects

$$F_n \simeq \coprod_{u\in I_n} \underline{h}_u.$$

We define $\mathscr{H}_\beta(x)$ to be a set of representatives in $sSPr(T)/R(\underline{h}_x)^{\Delta^*}$, for the isomorphism classes of objects $F_* \in sSPr(T)/R(\underline{h}_x)^{\Delta^*}$ which satisfy conditions (1) and (2) above. Note that condition (2) insures that $\mathscr{H}_\beta(x)$ is a $\mathbb{U}$-small set for any $x \in Ob(T)$.

Now, for any $x \in Ob(T)$, any $F_* \in \mathscr{H}_\beta(x)$ we consider its geometric realization $|F_*|$ in $SPr(T)$, together with its natural adjunction morphism $|F_*| \longrightarrow R(\underline{h}_x)$ (see [Hi, 19.5.1]). Note that $|F_*|$ is naturally equivalent to the homotopy colimit of the diagram $[n] \mapsto F_n$. Indeed, for any $y \in Ob(T)$, $|F_*|(y)$ is naturally isomorphic to diagonal of the bi-simplicial set $F_*(y)$ (see [Hi, 16.10.6]). We define the set H to be the union of all the $\mathscr{H}_\beta(x)$'s when x varies in $Ob(T)$. In other words, H consists of all morphisms

$$|F_*| \longrightarrow R(\underline{h}_x),$$

for all $x \in Ob(T)$ and all $F_* \in \mathscr{H}_\beta(x)$. Clearly, the set H is $\mathbb{U}$-small, so one can apply Theorem A.2.2 or A.2.4 to the objectwise model category $SPr(T)$ and the set of morphisms H. We let $\mathrm{L}_H SPr(T)$ be the left Bousfield localization of $SPr(T)$ along the set of morphisms H. We are going to show that equivalences in $\mathrm{L}_H SPr(T)$ are exactly π_*-equivalences. This will clearly implies the existence of the local model structure of 3.4.1.

3.4.2. The morphisms in H are π_*-equivalences

The main point in the proof is the following lemma.

Lemma 3.4.2. *For any object $x \in Ob(T)$ and any hypercover $F_* \longrightarrow R(\underline{h}_x)^{\Delta^*}$, the natural morphism in* $\mathrm{Ho}(SPr(T))$

$$\mathrm{hocolim}_{[n]\in\Delta^n}(F_n) \longrightarrow R(\underline{h}_x) \simeq \underline{h}_x$$

is a π_*-equivalence.

Proof. By applying the base change functor $j_x^* : \mathrm{Ho}(SPr(T)) \longrightarrow \mathrm{Ho}(SPr(T/x))$ one gets a natural morphism $j_x^*(\mathrm{hocolim}_{[n]\in\Delta^n}(F_n)) \longrightarrow j_x^*(\underline{h}_x)$. By definition of the homotopy sheaves one sees that it is enough to show that the homotopy fiber of this morphism at the natural point $* \longrightarrow j_x^*\underline{h}_x$ is π_*-contractible (see Corollary 3.3.6). In other words, one can always assume that x is a final object in T, or in other words that $\underline{h}_x \simeq *$ (this reduction is not necessary but simplifies notations). We can also assume that F_* is fibrant as an object in $sSPr(T)$, so $Cosk_n(F_*) \simeq \mathbb{R}Cosk_n(F_*)$. We will simply denote by $|G_*|$ the homotopy colimit of a simplicial diagram G_* in $SPr(T)$.

Step 1: Let us first assume that F_* is a 0-bounded hypercover. Recall that this means that for any $n > 0$ one has $F_n \simeq F_*^{\mathbb{R}\partial\Delta^n}$, or in other words that F_* is the nerve of the covering $F_0 \longrightarrow *$. Therefore, we can suppose that F_0 is fibrant in $SPr(T)$, and that $F_n = F_0^n$ (the face and degeneracy morphisms being induced by the various projections and diagonals). As $F_0 \longrightarrow *$ is a covering, one can find a covering sieve R of $*$ such that for any object $u \to *$ in S, there exists a commutative diagram

$$\begin{array}{ccc} F_0 & \longrightarrow & * \\ \uparrow & \nearrow & \\ \underline{h}_u & & \end{array}$$

Furthermore, as π_*-equivalences are local for the topology τ (see Corollary 3.3.4), it is enough to prove that for any such u, the nerve of the morphism

$$F_0 \times \underline{h}_u \longrightarrow \underline{h}_u$$

is a π_*-equivalence. We can therefore assume that the morphism $F_0 \longrightarrow *$ admits a section. But then, for any object $x \in Ob(T)$, $|F_*|(x) \in \mathrm{Ho}(SSet_{\mathbb{U}})$ is the geometric realization of the nerve of a morphism of simplicial sets which has a section, and therefore is contractible. This proves Lemma 3.4.2 for 0-bounded hypercovers.

Step 2: Let us now assume that F_* is $(n+1)$-bounded for some integer $n > 0$ (see Definition 3.2.1), and let us consider the morphism

$$F_* \longrightarrow Cosk_n\, F_*.$$

For any integer p, and any simplicial set $K \in SSet_{\mathbb{U}}$, there is a co-cartesian square of simplicial sets

$$\begin{array}{ccc} Sk_p K & \longrightarrow & Sk_{p+1} K \\ \uparrow & & \uparrow \\ \coprod_{K^{\partial\Delta^{p+1}}} \partial\Delta^{p+1} & \longrightarrow & \coprod_{K_{p+1}} \Delta^{p+1} \end{array}$$

This induces a cartesian square in $SPr(T)$

$$\begin{array}{ccc} F_*^{Sk_{p+1}K} & \longrightarrow & F_*^{Sk_p K} \\ \downarrow & & \downarrow \\ \prod_{K_{p+1}} F_{p+1} & \longrightarrow & \prod_{K^{\partial\Delta^{p+1}}} F_*^{\partial\Delta^{p+1}} \end{array}$$

As F_* is fibrant for the Reedy structure and a hypercover, each bottom horizontal morphism is a fibration which is again a covering. This shows by induction and by Proposition 3.1.6(1), that $F_*^{Sk_{p+i}K} \longrightarrow F_*^{Sk_p K}$ is a covering and a fibration for any $i > 0$. But, since we have

$$(Cosk_n\, F_*)^K \simeq F_*^{Sk_n K},$$

we easily conclude that for any $K \in SSet_{\mathbb{U}}$ such that $K = Sk_p K$ for some p, the natural morphism

$$F_*^K \longrightarrow (Cosk_n\, F_*)^K$$

is again a fibration and a covering. In particular, taking $K = \Delta^p$, one finds that the natural morphism

$$F_p \longrightarrow (Cosk_n F_*)_p$$

is a fibration and a covering.

Let $U_{*,*}$ be the bi-simplicial object such that $U_{p,*}$ is the nerve of the morphism $F_p \longrightarrow (Cosk_n\, F_*)_p$. It fits into a commutative diagram of bi-simplicial objects

$$\begin{array}{ccc} F_* & \longrightarrow & Cosk_n F_* \\ \downarrow & \nearrow & \\ U_{*,*}, & & \end{array}$$

where F_* and $Cosk_n\, F_*$ are considered as constant in the second spot. Furthermore, for any p, $U_{p,*} \longrightarrow (Cosk_n\, F_*)_p$ is a 0-truncated hypercover. Therefore, by Step 1, we deduce that

$$|diag(U_{*,*})| \simeq \operatorname{hocolim}_p\ \operatorname{hocolim}_q(U_{p,q}) \longrightarrow |Cosk_n\, F_*|$$

is a π_*-equivalence.

Now, let $U_* := diag(U_{*,*})$ be the diagonal of $U_{*,*}$. It fits into a commutative diagram

$$\begin{array}{ccc} F_* & \stackrel{\pi}{\longrightarrow} & Cosk_n F_* \\ {\scriptstyle f}\downarrow & \nearrow {\scriptstyle \phi} & \\ U_*. & & \end{array}$$

We are going to construct a morphism $U_* \longrightarrow F_*$ that will be a retract of f compatible with the two projections π and ϕ (i.e. construct a retraction of ϕ on π).

The above diagram consists clearly of isomorphisms in degrees $p \leqslant n$, showing that π is a retract of ϕ is degrees $p \leqslant n$. As F_* is $(n+1)$-bounded, to extend this retraction to the whole ϕ, it is enough to define a morphism $U_{n+1} \longrightarrow F_{n+1}$ which is equalized by all the face morphisms $F_{n+1} \longrightarrow F_n$. But, by definition

$$U_{n+1} = \underbrace{F_*^{\Delta^{n+1}} \times_{F_*^{\partial\Delta^{n+1}}} F_*^{\Delta^{n+1}} \times \ldots \times_{F_*^{\partial\Delta^{n+1}}} F_*^{\Delta^{n+1}}}_{\text{(n+1) times}},$$

and so any of the natural projections $U_{n+1} \longrightarrow F_{n+1}$ to one of these factors will produce the required extension.

In conclusion, the morphism $F_* \longrightarrow Cosk_n\, F_*$ is a retract of $U_* \longrightarrow Cosk_n\, F_*$, which itself induces a π_*-equivalence on the homotopy colimits. As π_*-equivalences are stable by retracts, this shows that the induced morphism $|F_*| \longrightarrow |Cosk_n\, F_*|$ is also a π_*-equivalence. Therefore, by induction on n and Step 1, this implies that $|F_*| \longrightarrow *$ is a π_*-equivalence.

Step 3: Finally, for a general hypercover F_*, the ith homotopy presheaf of $|F_*|$ only depends on the nth coskeleton of F_* for $i < n$ (as the $(n-1)$-skeleton of $|F_*|$ and $|Cosk_n F_*|$ coincide). In particular, the ith homotopy sheaf of $|F_*|$ only depends on $\mathbb{R}Cosk_n(F_*)$ for $i < n$. Therefore one can always suppose that $F_* = Cosk_n F_*$ for some integer n and apply Step 2.

Lemma 3.4.2 is proved. □

Now, let $f : F \longrightarrow G$ be any H-local equivalence (i.e. an equivalence in $\mathrm{L}_H SPr(T)$), and let us prove that it is a π_*-equivalence. By definition of H-local equivalences, the induced morphism on the H-local models

$$\mathrm{L}_H f : \mathrm{L}_H F \longrightarrow \mathrm{L}_H G$$

is an objectwise equivalence, and in particular a π_*-equivalence. By considering the commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{f} & G \\ \downarrow & & \downarrow \\ \mathrm{L}_H F & \xrightarrow[\mathrm{L}_H f]{} & \mathrm{L}_H G, \end{array}$$

one sees that it is enough to show that the localization morphisms $F \longrightarrow \mathrm{L}_H F$ and $G \longrightarrow \mathrm{L}_H G$ are π_*-equivalences. But the functor L_H can be defined via the small object argument applied to the set of augmented horns on H, $\overline{\Lambda(H)}$ (see [Hi, Section 4.3]). In the present situation, the morphisms in $\overline{\Lambda(H)}$ are either trivial cofibrations in $SPr(T)$ or projective cofibrations which are isomorphic in $\mathrm{Ho}(SPr(T))$ to

$$\Delta^n \otimes |F_*| \coprod_{\partial\Delta^n \otimes |F_*|}^{h} \partial\Delta^n \otimes R(\underline{h}_x) \longrightarrow \Delta^n \otimes R(\underline{h}_x).$$

By Proposition 3.3.7 and Lemma 3.4.2, these morphisms are π_*-equivalences, and therefore all morphisms in $\overline{\Lambda(H)}$ are projective cofibrations and π_*-equivalences. As π_*-equivalences are also stable by filtered colimits, another application of Proposition 3.3.7 shows that relative cell complexes on $\overline{\Lambda(H)}$ are again π_*-equivalences. This shows that the localization morphisms $F \longrightarrow \mathrm{L}_H F$ are always π_*-equivalences, and finish the proof that H-local equivalences are π_*-equivalences.

3.4.3. π_*-Equivalences are H-local equivalences

To conclude the proof of Theorem 3.4.1, we are left to show that π_*-equivalences are H-local equivalences.

Recall that we denoted by α a $\mathbb{U}$-small cardinal bigger than $\aleph_0$ and than the cardinality of the set $E(T)$ of all simplices in all simplicial set of morphisms in T. Recall also that β is a $\mathbb{U}$-small cardinal with $\beta > 2^{\alpha}$.

Lemma 3.4.3. *Let $f : F \longrightarrow G$ be a morphism in $SPr(T)$ which is a π_*-equivalence and an objectwise fibration between fibrant objects. Then, for any object $x \in Ob(T)$ and any morphism $R(\underline{h}_x) \longrightarrow G$, there exists an $F_* \in \mathscr{H}_\beta(x)$ and a commutative diagram in $SPr(T)$*

$$\begin{array}{ccc} F & \longrightarrow & G \\ \uparrow & & \uparrow \\ |F_*| & \longrightarrow & R(\underline{h}_x). \end{array}$$

Proof. By adjunction, it is equivalent to find a commutative diagram in $sSPr(T)$

$$\begin{array}{ccc} F^{\Delta^*} & \longrightarrow & G^{\Delta^*} \\ \uparrow & & \uparrow \\ F_* & \longrightarrow & R(\underline{h}_x)^{\Delta^*} \end{array}$$

with $F_* \in \mathscr{H}_\beta(x)$. We will define F_* inductively. Let us suppose we have constructed $F(n)_* \in sSPr(T)/R(\underline{h}_x)^{\Delta^*}$, with a commutative diagram

$$\begin{array}{ccc} F^{\Delta^*} & \longrightarrow & G^{\Delta^*} \\ \uparrow & & \uparrow \\ F(n)_* & \underset{p_n}{\longrightarrow} & R(\underline{h}_x)^{\Delta^*}, \end{array}$$

such that $Sk_n F(n)_* = F(n)_*$, and p_n is a Reedy fibration and a hypercover in degrees $i \leqslant n$. By the latter condition we mean that

$$F(n)_i \longrightarrow F(n)^{\partial\Delta^i} \times_{R(\underline{h}_x)^{\partial\Delta^i}} R(\underline{h}_x)^{\Delta^i}$$

is an objectwise fibration and a covering for any $i \leqslant n$ (we do not require p_n to be a Reedy fibration). We also assume that $\mathrm{Card}(F(n)_m) < \beta$ for any m. We need the following (technical) factorization result with control on the cardinality.

Lemma 3.4.4. *Let* $f : F \longrightarrow G$ *be a morphism in* $SPr(T)$ *such that* $\mathrm{Card}(F)$ *and* $\mathrm{Card}(G)$ *are both strictly smaller than* β*. Then, there exists a factorization in* $SPr(T)$

$$F \stackrel{i}{\longrightarrow} RF \stackrel{p}{\longrightarrow} G \quad ,$$

with i *a trivial cofibration,* p *a fibration, and* $\mathrm{Card}(RF) < \beta$.

Proof. We use the standard small object argument in order to produce such a factorization (see [Ho, Section 2.1.2]). The trivial cofibrations in $SPr(T)$ are generated by the set of morphisms

$$\Lambda^{n,k} \otimes \underline{h}_x \longrightarrow \Delta^n \otimes \underline{h}_x,$$

for all $x \in Ob(T)$ and all $n \in \mathbb{N}$, $0 \leqslant k \leqslant n$. This set is clearly of cardinality smaller than $\aleph_0.\alpha$, and therefore is strictly smaller than β. Furthermore, for any of these generating trivial cofibrations, the set of all commutative diagrams

$$\begin{array}{ccc} F & \longrightarrow & G \\ \uparrow & & \uparrow \\ \Lambda^{n,k} \otimes \underline{h}_x & \longrightarrow & \Delta^n \otimes \underline{h}_x \end{array}$$

is in bijective correspondence with the set of all commutative diagrams

$$\begin{array}{ccc} F(x) & \longrightarrow & G(x) \\ \uparrow & & \uparrow \\ \Lambda^{n,k} & \longrightarrow & \Delta^n \end{array}$$

By the assumptions made on F and G, this set is therefore of cardinality strictly smaller than β. Furthermore, by the choice of β, it is clear that $\mathrm{Card}(A \otimes \underline{h}_x) \leqslant \alpha < \beta$ for any finite simplicial set A. Therefore, the push-out

$$\begin{array}{ccc} F & \longrightarrow & F_1 \\ \uparrow & & \uparrow \\ \coprod_I \Lambda^{n,k} \otimes \underline{h}_x & \longrightarrow & \coprod_I \Delta^n \otimes \underline{h}_x \end{array}$$

where I consists of all objects $x \in Ob(T)$ and commutative diagrams

$$\begin{array}{ccc} F & \longrightarrow & G \\ \uparrow & & \uparrow \\ \Lambda^{n,k} \otimes \underline{h}_x & \longrightarrow & \Delta^n \otimes \underline{h}_x \end{array}$$

is such that

$$\mathrm{Card}(F_1) \leqslant \mathrm{Card}(F) + \mathrm{Card}\left(\coprod_I \Delta^n \otimes \underline{h}_x\right) < \beta + \mathrm{Card}(I).\alpha.$$

But $\mathrm{Card}(I) < \alpha.\beta$, and therefore one has $\mathrm{Card}(F_1) < \beta$. As the factorization $F \longrightarrow RF \longrightarrow G$ is obtained after a numerable number of such push-outs constructions (see [Ho, Theorem 2.1.14])

$$F \longrightarrow F_1 \longrightarrow \cdots \longrightarrow F_n \longrightarrow \cdots \longrightarrow RF = colim_i F_i,$$

we conclude that $\mathrm{Card}(RF) < \beta$. The proof of Lemma 3.4.4 is achieved. □

Let us come back to the proof of Lemma 3.4.3. We consider the following diagram:

$$\begin{array}{ccc} F^{\Delta^{n+1}} & \longrightarrow & F^{\partial\Delta^{n+1}} \times_{G^{\partial\Delta^{n+1}}} G^{\Delta^{n+1}} \\ & & \uparrow \\ & & F(n)_*^{\partial\Delta^{n+1}} \times_{R(\underline{h}_x)^{\partial\Delta^{n+1}}} R(\underline{h}_x)^{\Delta^{n+1}} \end{array}$$

By Lemma 3.4.4, we can suppose that $\mathrm{Card}(R(\underline{h}_x)) < \beta$. Therefore, by induction on n

$$\mathrm{Card}(F(n)_*^{\partial\Delta^{n+1}} \times_{R(\underline{h}_x)^{\partial\Delta^{n+1}}} R(\underline{h}_x)^{\Delta^{n+1}}) < \beta.$$

This implies that there exists a $\mathbb{U}$-small set J of objects in T, with $\mathrm{Card}(J) < \beta$, and a covering

$$\coprod_{z \in J} \underline{h}_z \longrightarrow F(n)_*^{\partial\Delta^{n+1}} \times_{R(\underline{h}_x)^{\partial\Delta^{n+1}}} R(\underline{h}_x)^{\Delta^{n+1}}.$$

Now, by considering the induced diagram

$$\begin{array}{ccc} F^{\Delta^{n+1}} & \longrightarrow & F^{\partial\Delta^{n+1}} \times_{G^{\partial\Delta^{n+1}}} G^{\Delta^{n+1}} \\ & & \uparrow \\ & & \coprod_{z \in J} \underline{h}_z \end{array}$$

and using the fact that the top horizontal morphism is a covering, one sees that there exists, for all $z \in J$, a covering sieve S_z of $z \in \mathrm{Ho}(T)$, and a commutative diagram

$$\begin{array}{ccc} F^{\Delta^{n+1}} & \longrightarrow & F^{\partial\Delta^{n+1}} \times_{G^{\partial\Delta^{n+1}}} G^{\Delta^{n+1}} \\ \uparrow & & \uparrow \\ \coprod_{z \in J, (u \to z) \in S_z} \underline{h}_u & \longrightarrow & \coprod_{z \in J} \underline{h}_z \end{array}$$

Clearly, one has

$$\mathrm{Card}\left(\coprod_{z \in J, (u \to z) \in S_z} \underline{h}_u\right) \leqslant \mathrm{Card}(J).2^{\alpha}.\alpha < \beta.$$

We now consider the commutative diagram

$$\begin{array}{ccc} F^{\Delta^{n+1}} & \longrightarrow & F^{\partial\Delta^{n+1}} \times_{G^{\partial\Delta^{n+1}}} G^{\Delta^{n+1}} \\ \uparrow & & \uparrow \\ \coprod_{z\in J,(u\to z)\in S_z} \underline{h}_u & \longrightarrow & F(n)_*^{\partial\Delta^{n+1}} \times_{R(\underline{h}_x)^{\partial\Delta^{n+1}}} R(\underline{h}_x)^{\Delta^{n+1}} \end{array}$$

Lemma 3.4.4 implies the existence of an object $H(n+1) \in SPr(T)$, with $\mathrm{Card}(H(n+1)) < \beta$, and a factorization

$$\coprod_{z\in J,(u\to z)\in S_z} \underline{h}_u \longrightarrow H(n+1) \longrightarrow F(n)_*^{\partial\Delta^{n+1}} \times_{R(\underline{h}_x)^{\partial\Delta^{n+1}}} R(\underline{h}_x)^{\Delta^{n+1}}$$

into an objectwise trivial cofibration followed by a fibration in $SPr(T)$. Since the morphism

$$F^{\Delta^{n+1}} \longrightarrow F^{\partial\Delta^{n+1}} \times_{G^{\partial\Delta^{n+1}}} G^{\Delta^{n+1}}$$

is an objectwise fibration, there exists a commutative diagram in $SPr(T)$

$$\begin{array}{ccccc} F^{\Delta^{n+1}} & & \longrightarrow & & F^{\partial\Delta^{n+1}} \times_{G^{\partial\Delta^{n+1}}} G^{\Delta^{n+1}} \\ \uparrow & \nwarrow & & & \uparrow \\ \coprod_{z\in J,(u\to z)\in S_z} \underline{h}_u & \longrightarrow & H(n+1) & \longrightarrow & F(n)_*^{\partial\Delta^{n+1}} \times_{R(\underline{h}_x)^{\partial\Delta^{n+1}}} R(\underline{h}_x)^{\Delta^{n+1}}. \end{array}$$

We define $F(n+1)_p := F(n)_p$ for any $p < n+1$, and $F(n+1)_{n+1}$ to be the coproduct of $H(n+1)$ together with $L_{n+1}F$, the $(n+1)$th latching space of $F(n)$. The face morphisms $F(n+1)_{n+1} \longrightarrow F(n)_n$ are defined as the identity on $L_{n+1}F(n)$ and via the $(n+1)$ natural projections (corresponding to the face inclusions $\Delta^n \subset \partial\Delta^{n+1}$)

$$F(n)^{\partial\Delta^{n+1}} \longrightarrow F(n)^{\Delta^n} = F(n)_n$$

on the factor $H(n+1)$. Then, by adjunction, one has a natural commutative diagram in $s_{n+1}SPr(T)$

$$\begin{array}{ccc} F^{\Delta^*} & \longrightarrow & G^{\Delta^*} \\ \uparrow & & \uparrow \\ F(n+1)_* & \xrightarrow[p_{n+1}]{} & R(\underline{h}_x)^{\Delta^*}, \end{array}$$

which extends via the functor $(i_{n+1})_!$ to the required diagram in $sSPr(T)$. It is clear by construction, that p_{n+1} is a Reedy fibration and a hypercover in degrees $i \leqslant n+1$ and that its nth skeleton is p_n. Therefore, by defining F_* to be the limit of the $F(n)$'s, the natural morphism $F_* \longrightarrow R(\underline{h}_x)^{\Delta^*}$ is a hypercover. It is also clear by construction that F_* satisfies condition (2) defining the set $\mathscr{H}_\beta(x)$. □

We are now ready to finish the proof that π_*-equivalences are H-local equivalences. Let $f : F \longrightarrow G$ be a π_*-equivalence; we can clearly assume f to be an objectwise fibration between fibrant objects. Furthermore, as H-local equivalences are already known to be π_*-equivalences, we can also suppose that f is a H-local fibration between H-local objects. We are going to prove that f is in fact an objectwise equivalence.

Let

$$\begin{array}{ccc} F & \xrightarrow{f} & G \\ \uparrow & & \uparrow \\ \partial\Delta^n \otimes \underline{h}_x & \longrightarrow & \Delta^n \otimes \underline{h}_x \end{array}$$

be a commutative diagram in $SPr(T)$. We need to show that there exist a lifting $\Delta^n \otimes \underline{h}_x \longrightarrow F$. By adjunction, this is equivalent to showing that the natural morphism

$$\underline{h}_x \longrightarrow F^{\partial\Delta^n} \times_{G^{\partial\Delta^n}} G^{\Delta^n}$$

lifts to a morphism $\underline{h}_x \longrightarrow F^{\Delta^n}$.

As F and G are objectwise fibrant, the previous morphism factors through

$$\underline{h}_x \longrightarrow R(\underline{h}_x) \longrightarrow F^{\partial\Delta^n} \times_{G^{\partial\Delta^n}} G^{\Delta^n}$$

An application of Lemma 3.4.3 to the morphism

$$F^{\Delta^n} \longrightarrow F^{\partial\Delta^n} \times_{G^{\partial\Delta^n}} G^{\Delta^n}$$

which satisfies the required hypothesis, shows that there exists an $F_* \in \mathscr{H}_\beta(x)$ and a commutative diagram

$$\begin{array}{ccc} F^{\Delta^n} & \longrightarrow & F^{\partial\Delta^n} \times_{G^{\partial\Delta^n}} G^{\Delta^n} \\ \uparrow & & \uparrow \\ |F_*| & \longrightarrow & R(\underline{h}_x) \end{array}$$

By adjunction, this commutative diagram yields a commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{f} & G \\ \uparrow & & \uparrow \\ \Delta^n \otimes |F_*| \coprod_{\partial\Delta^n \otimes |F_*|} \partial\Delta^n \otimes R(\underline{h}_x) & \longrightarrow & \Delta^n \otimes R(\underline{h}_x). \end{array}$$

The horizontal bottom morphism is an H-local equivalence by definition, and therefore a lifting $\Delta^n \otimes R(\underline{h}_x) \longrightarrow F$ exists in the homotopy category $\mathrm{Ho}(\mathrm{L}_H SPr(T))$. But, as f is a H-local fibration, F and G are H-local objects and $R(\underline{h}_x)$ is cofibrant, this lifting can be represented in $SPr(T)$ by a commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{f} & G \\ & \nwarrow & \uparrow \\ & & \Delta^n \otimes R(\underline{h}_x) \end{array}$$

Composing with $\underline{h}_x \longrightarrow R(\underline{h}_x)$, we obtain the required lifting. This implies that π_*-equivalences are H-local equivalences, and completes the proof of the existence of the local model structure.

By construction, $SPr_\tau(T)$ is the left Bousfield localization of $SPr(T)$ along the set of morphisms H: this implies that it is a $\mathbb{U}$-cellular and $\mathbb{U}$-combinatorial model category. In particular, it is $\mathbb{U}$-cofibrantly generated. Finally, properness of $SPr_\tau(T)$ follows from Corollary 3.3.5 and Proposition 3.3.7.

This concludes the proof of Theorem 3.4.1. □

Let us keep the notations introduced in the proof of Theorem 3.4.1. We choose a $\mathbb{U}$-small cardinal β as in the proof and consider, for any object $x \in Ob(T)$, the subset of hypercovers $\mathscr{H}_\beta(x)$.

Corollary 3.4.5. *The model category* $SPr_\tau(T)$ *is the left Bousfield localization of* $SPr(T)$ *with respect to the set of morphisms*

$$\{|F_*| \longrightarrow \underline{h}_x \mid x \in Ob(T),\ F_* \in \mathscr{H}_\beta(x)\}.$$

Proof. This is exactly the way we proved Theorem 3.4.1. □

Remark 3.4.6. It is worthwhile emphasizing that the proof of Theorem 3.4.1 shows actually a bit more than what's in its statement. In fact, the argument proves both Theorem 3.4.1 and Corollary 3.4.5, in that it gives *two descriptions* of the same model category $SPr_\tau(T)$: one as the left Bousfield localization of $SPr(T)$ with respect to *local equivalences* and the other as the left Bousfield localization of the same $SPr(T)$ but this time with respect to *hypercovers* (more precisely, with respect to the set of morphisms defined in the statement of Corollary 3.4.5).

In the special case where (T,τ) is a usual Grothendieck site (i.e. when T is a category), the following corollary was announced in [Du1] and proved in [DHI].

Corollary 3.4.7. *An object* $F \in SPr_\tau(T)$ *is fibrant if and only if it is objectwise fibrant and for any object* $x \in Ob(T)$ *and any* $H_* \in \mathscr{H}_\beta(x)$, *the natural morphism*

$$F(x) \simeq \mathbb{R}\underline{Hom}(\underline{h}_x, F) \longrightarrow \mathbb{R}\underline{Hom}(|H_*|, F)$$

is an isomorphism in $\mathrm{Ho}(SSet)$.

Proof. This follows from Theorem 3.4.1 and from the explicit description of fibrant objects in a left Bousfield localization (see [Hi, Theorem 4.1.1]). □

The previous corollary is more often described in the following way. For any $H_* \in \mathscr{H}_\beta(x)$ and any $n \geqslant 0$, H_n is equivalent to a coproduct of representables

$$H_n \simeq \coprod_{i \in I_n} \underline{h}_{u_i}$$

Therefore, for any $H_* \in \mathscr{H}_\beta(x)$ and any fibrant object F in $SPr(T)$, the simplicial set $\mathbb{R}\underline{Hom}(|H_*|, F)$ is naturally equivalent to the homotopy limit of the cosimplicial diagram in $SSet$

$$[n] \mapsto \prod_{i \in I_n} F(u_i)$$

Then, Corollary 3.4.7 states that an object $F \in SPr(T)$ is fibrant if and only if, for any $x \in Ob(T)$, $F(x)$ is fibrant, and the natural morphism

$$F(x) \longrightarrow \operatorname{holim}_{[n]\in\Delta} \left(\prod_{i\in I_n} F(u_i) \right)$$

is an equivalence of simplicial sets, for any $H_* \in \mathscr{H}_\beta(x)$.

Definition 3.4.8. 1. A hypercover $H_* \longrightarrow \underline{h}_x$ is said to be *semi-representable* if for any $n \geqslant 0$, H_n is isomorphic in $\mathrm{Ho}(SPr(T))$ to a coproduct of representable objects

$$H_n \simeq \coprod_{u\in I_n} \underline{h}_u.$$

2. An object $F \in SPr(T)$ is said to *have hyperdescent* if, for any object $x \in Ob(T)$ and any semi-representable hypercover $H_* \longrightarrow \underline{h}_x$, the induced morphism

$$F(x) \simeq \mathbb{R}\,\underline{Hom}(\underline{h}_x, F) \longrightarrow \mathbb{R}\,\underline{Hom}(|H_*|, F)$$

is an isomorphism in $\mathrm{Ho}(SSet_{\mathbb{U}})$.

An immediate consequence of the proof of Theorem 3.4.1 is that an object $F \in SPr(T)$ has hyperdescent with respect to all hypercover $H_* \in \mathscr{H}_\beta(x)$ *if and only* if it has hyperdescent with respect to all semi-representable hypercovers.

From now on we will adopt the following terminology and notations.

Definition 3.4.9. Let (T, τ) be an S-site in $\mathbb{U}$.

1. A *stack* on the site (T, τ) is a pre-stack $F \in SPr(T)$ which satisfies the hyperdescent condition of Definition 3.4.8.
2. The model category $SPr_\tau(T)$ is also called the *model category of stacks* on the S-site (T, τ). The category $\mathrm{Ho}(SPr(T))$ (resp. $\mathrm{Ho}(SPr_\tau(T))$) is called the *homotopy category of pre-stacks*, and (resp. the *homotopy category of stacks*). Objects of $\mathrm{Ho}(SPr(T))$ (resp. $\mathrm{Ho}(SPr_\tau(T))$) will simply be called *pre-stacks* on T (resp., *stacks* on (T, τ)). The functor $a : \mathrm{Ho}(SPr(T)) \longrightarrow \mathrm{Ho}(SPr_\tau(T))$ will be called the *associated stack functor*.
3. The topology τ is said to be *sub-canonical* if for any $x \in Ob(T)$, the pre-stack $\underline{h}_x \in \mathrm{Ho}(SPr(T))$ is a stack (in other words, if the Yoneda embedding $L\underline{h}$: $\mathrm{Ho}(T) \longrightarrow \mathrm{Ho}(SPr(T))$ factors through the subcategory of stacks).
4. For pre-stacks F and G on T, we will denote by $\mathbb{R}\underline{Hom}(F, G) \in \mathrm{Ho}(SSet_{\mathbb{U}})$ (resp. by $\mathbb{R}_\tau\underline{Hom}(F, G) \in \mathrm{Ho}(SSet_{\mathbb{U}})$) the derived *Hom*-simplicial set computed in the simplicial model category $SPr(T)$ (resp. $SPr_\tau(T)$).

Let us explain why, given Definition 3.4.9(1), we also call the objects in $\mathrm{Ho}(SPr_\tau(T))$ stacks (Definition 3.4.9(2)). As $SPr_\tau(T)$ is a left Bousfield localization of $SPr(T)$, the identity functor $SPr(T) \longrightarrow SPr_\tau(T)$ is left Quillen, and its right adjoint (which is still the identity functor) induces a fully faithful functor

$$j : \mathrm{Ho}(SPr_\tau(T)) \longrightarrow \mathrm{Ho}(SPr(T))$$

Furthermore, the essential image of this inclusion functor is exactly the full subcategory consisting of objects having the hyperdescent property; in other words, the essential image of j is the full subcategory of $\mathrm{Ho}(SPr(T))$ consisting of stacks. We will often identify $\mathrm{Ho}(SPr_\tau(T))$ with its essential image via j (which is equivalent to $\mathrm{Ho}(SPr_\tau(T))$). The left adjoint

$$a : \mathrm{Ho}(SPr(T)) \longrightarrow \mathrm{Ho}(SPr_\tau(T))$$

to the inclusion j, is a left inverse to j. Note that $F \in \mathrm{Ho}(SPr(T))$ is a stack iff the canonical adjunction map $F \to ja(F)$ (which we will write as $F \to a(F)$ taking into account our identification) is an isomorphism in $\mathrm{Ho}(SPr(T))$.

As explained in the Introduction, this situation is the analog for stacks over *S*-sites of the usual picture for sheaves over Grothendieck sites. In particular, this gives a *sheaf-like* description of objects of $\mathrm{Ho}(SPr_\tau(T))$, via the hyperdescent property. However, this description is not as useful as one might at first think, though it allows to prove easily that some adjunctions are Quillen adjunctions (see for example, [DHI, 7.1], [To2,To3, Proposition 2.2.2, Proposition 2.9]) or to check that an *S*-topology is sub-canonical.

We will finish this paragraph with the following proposition.

Proposition 3.4.10. 1. *Let F and G be two pre-stacks on T. If G is a stack, then the natural morphism*

$$\mathbb{R}\underline{Hom}(F, G) \longrightarrow \mathbb{R}_\tau\underline{Hom}(F, G)$$

is an isomorphism in $\mathrm{Ho}(SSet)$.

2. *The functor* $\mathrm{Id} : SPr(T) \longrightarrow SPr_\tau(T)$ *preserves homotopy fibered products.*

Proof. Condition (1) follows formally from Corollary 3.4.5. To prove (2) it is enough to show that π_*-equivalences are stable under pull-backs along objectwise fibrations, and this follows from Corollary 3.3.5. □

Remark 3.4.11. *If M* is *any* left proper $\mathbb{U}$-combinatorial or $\mathbb{U}$-cellular (see Appendix A) simplicial model category, one can also define the local projective model structure on $Pr(T, M) := M^{T^{op}}$ as the left Bousfield localization of the objectwise model structure, obtained by *inverting hypercovers*. This allows one to consider the model category of stacks on the *S*-site (T, τ) *with values in M*. Moreover, in many cases (e.g., symmetric spectra [HSS], simplicial abelian groups, simplicial groups, etc.) the local equivalences also have a description in terms of some appropriately defined π_*-equivalences. We will not pursue this here as it is a purely formal exercise to adapt the proof of Theorem 3.4.1 to these situations.

In many cases these model categories of stacks with values in M may also be described by performing the constructions defining M *directly* in the model category $SPr_{\tau}(T)$. More precisely, one can consider e.g. the categories of symmetric spectra, abelian group objects, group objects etc., in $SPr(T)$, and use some general results to provide these categories with model structures. For reasonable model categories M both approaches give Quillen equivalent model categories (e.g. for group objects in $SPr_{\tau}(T)$, and *stacks of simplicial groups* on (T,τ)). The reader might wish to consult [Bek] in which a very general approach to these considerations is proposed.

3.5. Functoriality

Let (T,τ) and (T',τ') be two $\mathbb{U}$-small S-sites and $f : T \longrightarrow T'$ a morphism of S-categories. As we saw in Section 2.3.1 before Theorem 2.3.1, the morphism f induces a Quillen adjunction on the model categories of pre-stacks

$$f_! : SPr(T) \longrightarrow SPr(T') \qquad SPr(T) \longleftarrow SPr(T') : f^*.$$

Definition 3.5.1. We say that the morphism f is *continuous* (with respect to the topologies τ and τ') if the functor $f^* : SPr(T') \longrightarrow SPr(T)$ preserves the subcategories of stacks.

As the model categories of stacks $SPr_{\tau}(T)$ and $SPr_{\tau'}(T)$ are left Bousfield localizations of $SPr(T)$ and $SPr(T')$, respectively, the general machinery of [Hi] implies that f is continuous if and only if the adjunction $(f_!, f^*)$ induces a Quillen adjunction

$$f_! : SPr_{\tau}(T) \longrightarrow SPr_{\tau}(T') \qquad SPr_{\tau}(T) \longleftarrow SPr_{\tau'}(T') : f^*$$

between the model category of stacks.

Recall from the proof of Theorem 3.4.1 that we have defined the sets of distinguished hypercovers $\mathcal{H}_{\beta}(x)$, for any object $x \in T$. These distinguished hypercovers detect continuous functors, as shown in the following proposition.

Proposition 3.5.2. *The morphism f is continuous if and only if, for any $x \in Ob(T)$ and any $H_* \in \mathcal{H}_{\beta}(x)$, the induced morphism*

$$\mathbb{L}f_!(|H_*|) \longrightarrow \mathbb{L}f_!(\underline{h}_x) \simeq \underline{h}_{f(x)}$$

is an isomorphism in $\mathrm{Ho}(SPr_{\tau'}(T'))$.

Proof. This follows immediately by adjunction, from Corollary 3.4.7. □

3.6. Injective model structure and stacks of morphisms

The goal of this paragraph is to present an injective version of the local model structure on $SPr(T)$ for which cofibrations are monomorphisms, and to use it in order to construct *stacks of morphisms*. Equivalently, we will show that the injective model category of stacks over an S-site possesses derived internal Hom's, and as a consequence the homotopy category of stacks $\mathrm{Ho}(SPr_{\tau}(T))$ is *cartesian closed* (in the usual sense of [ML, Chapter IV, Section 10]). These stacks of morphisms will be important especially for applications to Derived Algebraic Geometry (see [To-Ve 4, 6]), since many of the *moduli stacks* are defined as stacks of morphisms to a certain *classifying stack* (for example, the stack of vector bundles on a scheme).

Before going into details, let us observe that in general, as explained in [H-S, Section 11], the projective model structure on $SPr_{\tau}(T)$ is not an *internal model category,* i.e. is not a closed symmetric monoidal model category for the direct product [Ho, Definition 4.2.6], and therefore the internal *Hom*'s of the category $SPr_{\tau}(T)$ are not compatible with the model structure. This prevents one from defining derived internal *Hom*'s in the usual way (i.e. by applying the internal *Hom*'s of $SPr(T)$ to fibrant models for the targets and cofibrant models for the sources). One way to solve this problem is to work with another model category which is internal and Quillen equivalent to $SPr(T)$. The canonical choice is to use an *injective model structure on* $SPr(T)$, analogous to the one described in [Ja1].

Proposition 3.6.1. *Let* (T,τ) *be an* S*-site in* $\mathbb{U}$. *Then there exists a simplicial closed model structure on the category* $SPr(T)$, *called the* local injective model structure, and denoted by $SPr_{\mathrm{inj},\tau}(T)$ *where the cofibrations are the monomorphisms and the equivalences are the local equivalences. Moreover, the local injective model structure on* $SPr(T)$ *is proper and internal.*[3]

Proof. The proof is essentially the same as the proof of our Theorem 3.4.1. The starting point is the objectwise injective model structure $SPr_{\mathrm{inj}}(T)$, for which equivalences and cofibrations are defined objectwise. The existence of this model structure can be proved by the same cardinality argument as in the case where T is a usual category (see [Ja1]). The model category $SPr_{\mathrm{inj}}(T)$ is clearly proper, $\mathbb{U}$-cellular and $\mathbb{U}$-combinatorial, so one can apply the localization techniques of [Hi]. We define the model category $SPr_{\mathrm{inj},\tau}(T)$ as the left Bousfield localization of $SPr_{\mathrm{inj}}(T)$ along the set of hypercovers H defined in the proof of Theorem 3.4.1. Note that the identity functor $SPr_{\mathrm{inj},\tau}(T) \longrightarrow SPr_{\tau}(T)$ is the right adjoint of a Quillen equivalence. From this and Theorem 3.4.1 we deduce that equivalences in $SPr_{\mathrm{inj},\tau}(T)$ are exactly the local equivalences of Definition 3.3.2. This proves the existence of the model category $SPr_{\mathrm{inj},\tau}(T)$. The fact that it is proper follows easily from the fact the model category $SSet$ is proper and from the description of equivalences in $SPr_{\mathrm{inj},\tau}(T)$ as π_*-equivalences. It only re-

[3] Recall once again that a model category is said to be *internal* if it is a monoidal model category (in the sense of [Ho, Definition 4.2.6]) for the monoidal structure given by the direct product.

mains to show that $SPr_{\mathrm{inj},\tau}(T)$ is internal. But, as cofibrations are the monomorphisms this follows easily from the fact that finite products preserves local equivalences. □

As the equivalences in $SPr_{\mathrm{inj},\tau}(T)$ and $SPr_{\tau}(T)$ are the same, the corresponding homotopy categories coincide

$$\mathrm{Ho}(SPr_{\mathrm{inj},\tau}(T)) = \mathrm{Ho}(SPr_{\tau}(T)).$$

Since the homotopy category of an internal model category is known to be cartesian closed, Proposition 3.6.1 implies the following corollary.

Corollary 3.6.2. *For any S-site T in $\mathbb{U}$, the homotopy category of stacks* $\mathrm{Ho}(SPr_{\tau}(T))$ *is cartesian closed.*

Proof. Apply [Ho, Theorem 4.3.2] to the symmetric monoidal model category $SPr_{\mathrm{inj},\tau}(T)$, with the monoidal structure given by the direct product. □

Definition 3.6.3. 1. The internal *Hom*'s of the category $\mathrm{Ho}(SPr_{\tau}(T))$ will be denoted by

$$\mathbb{R}_{\tau}\,\underline{\mathscr{H}om}(-,-) : \mathrm{Ho}(SPr_{\tau}(T)) \times \mathrm{Ho}(SPr_{\tau}(T)) \longrightarrow \mathrm{Ho}(SPr_{\tau}(T)).$$

2. Let (T,τ) be an S-site in $\mathbb{U}$, and F, G be stacks in $\mathrm{Ho}(SPr_{\tau}(T))$. The *stack of morphisms* from F to G is defined to be the stack

$$\mathbb{R}_{\tau}\underline{\mathscr{H}om}(F,G) \in \mathrm{Ho}(SPr_{\tau}(T)).$$

Explicitly, we have for any pair of stacks F and G

$$\mathbb{R}_{\tau}\,\underline{\mathscr{H}om}(F,G) \simeq \underline{\mathscr{H}om}(F, R_{\mathrm{inj}}G),$$

where R_{inj} is the fibrant replacement functor in the objectwise injective model category $SPr_{\mathrm{inj}}(T)$, and $\underline{\mathscr{H}om}$ is the internal *Hom* functor of the category $SPr(T)$. In fact, if G is a stack, then both $R_{\mathrm{inj}}G$ and $\underline{\mathscr{H}om}(F, R_{\mathrm{inj}}G)$ are stacks.

Actually, Proposition 3.6.1 gives more than the cartesian closedness of $\mathrm{Ho}(SPr_{\tau}(T))$. Indeed, one can consider the full sub-category $SPr_{\mathrm{inj},\tau}(T)^{\mathrm{f}}$ of fibrant objects in $SPr_{\mathrm{inj},\tau}(T)$. As any object is cofibrant in $SPr_{\mathrm{inj},\tau}(T)$, for any two objects F and G in $SPr_{\mathrm{inj},\tau}(T)^{\mathrm{f}}$ the internal Hom $\underline{\mathscr{H}om}(F,G)$ is also a fibrant object and therefore lives in $SPr_{\mathrm{inj},\tau}(T)^{\mathrm{f}}$. This shows in particular that $SPr_{\mathrm{inj},\tau}(T)^{\mathrm{f}}$ becomes cartesian closed for the direct product, and therefore one can associate to it a natural $SPr_{\mathrm{inj},\tau}(T)^{\mathrm{f}}$-enriched category $\underline{SPr_{\mathrm{inj},\tau}(T)^{\mathrm{f}}}$. Precisely, the set of object of $\underline{SPr_{\mathrm{inj},\tau}(T)^{\mathrm{f}}}$ is the set of fibrant objects in $SPr_{\mathrm{inj},\tau}(T)$, and for two such objects F and G the object of morphisms is $\underline{\mathscr{H}om}(F,G)$.

The $SPr_{\mathrm{inj},\tau}(T)^{\mathrm{f}}$-enriched category $\underline{SPr_{\mathrm{inj},\tau}(T)^{\mathrm{f}}}$ yields in fact a *up-to-equivalence* $SPr_{\mathrm{inj},\tau}(T)^{\mathrm{f}}$*-enrichment of the S-category* $LSPr_\tau(T)$. Indeed, as $SPr_\tau(T)$ and $SPr_{\mathrm{inj},\tau}(T)$ has the same simplicial localizations (because they are the same categories with the same notion of equivalence), one has a natural equivalence of S-categories

$$LSPr_\tau(T) = LSPr_{\mathrm{inj},\tau}(T) \simeq Int(SPr_{\mathrm{inj},\tau}(T)).$$

Recall that the S-category $Int(SPr_{\mathrm{inj},\tau}(T))$ consists of fibrant objects in $SPr_{\mathrm{inj},\tau}(T)$ and their simplicial Hom-sets. In other words the *SSet*-enriched category $Int(SPr_{\mathrm{inj},\tau}(T))$ is obtained from the $SPr_{\mathrm{inj},\tau}(T)^{\mathrm{f}}$-enriched category $\underline{SPr_{\mathrm{inj},\tau}(T)^{\mathrm{f}}}$ by applying the global section functor $\Gamma : SPr_{\mathrm{inj},\tau}(T) \longrightarrow SSet$. In conclusion, one has a triple

$$(LSPr_\tau(T), \underline{SPr_{\mathrm{inj},\tau}(T)^{\mathrm{f}}}, \alpha),$$

where α is an isomorphism in $\mathrm{Ho}(S-Cat)$ between $LSPr_\tau(T)$ and the underlying S-category of $\underline{SPr_{\mathrm{inj},\tau}(T)^{\mathrm{f}}}$. This triple is what we refer to as an *up-to-equivalence* $SPr_{\mathrm{inj},\tau}(T)^{\mathrm{f}}$-enrichment of $LSPr_\tau(T)$. For example, the $SPr_{\mathrm{inj},\tau}(T)^{\mathrm{f}}$-enriched functor

$$\underline{\mathscr{H}om} : (\underline{SPr_{\mathrm{inj},\tau}(T)^{\mathrm{f}}})^{op} \times \underline{SPr_{\mathrm{inj},\tau}(T)^{\mathrm{f}}} \longrightarrow \underline{SPr_{\mathrm{inj},\tau}(T)^{\mathrm{f}}}$$

gives rise to a well-defined morphism in $\mathrm{Ho}(S-cat)$

$$\mathbb{R}_\tau\underline{\mathscr{H}om} : LSPr_\tau(T)^{op} \times LSPr_\tau(T) \longrightarrow LSPr_\tau(T),$$

lifting the internal Hom-structure on the homotopy category $\mathrm{Ho}(SPr_\tau(T))$.

Remark 3.6.4. This last structure is at first sight more subtle than the cartesian closedness of the homotopy category $Ho(SPr_\tau(T))$, as $\underline{SPr_{\mathrm{inj},\tau}(T)^{\mathrm{f}}}$ encodes strictly associative and unital compositions between stacks of morphisms, which are only described by $\mathrm{Ho}(SPr_\tau(T))$ as up-to-homotopy associative and unital compositions. This looks like comparing the notions of simplicial monoids (i.e. monoids in *SSet*) and up-to-homotopy simplicial monoids (i.e. monoids in $\mathrm{Ho}(SSet)$), and the former is well known to be the *right notion*. However, we would like to mention that we think that the S-category alone $LSPr_\tau(T) \in \mathrm{Ho}(S-Cat)$, together with the fact that $\mathrm{Ho}(SPr_\tau(T))$ is cartesian closed, completely determines its up-to-equivalence $SPr_{\mathrm{inj},\tau}(T)^{\mathrm{f}}$-enrichment. In other words, the structure

$$(LSPr_\tau(T), \underline{SPr_{\mathrm{inj},\tau}(T)^{\mathrm{f}}}, \alpha)$$

only depends, up to an adequate notion of equivalence, on the S-category $LSPr_\tau(T)$. Unfortunately, investigating this question would drive us way too far from our purpose, as we think the right context to treat it is the general theory of *symmetric monoidal S-categories*, as briefly exposed in [To4, Section 5.1].

3.7. Truncated stacks and truncation functors

We start by recalling some very general definition of truncated objects in model categories.

Definition 3.7.1. 1. Let $n \geqslant 0$. An object $x \in \mathrm{Ho}(M)$ is called *n-truncated* if for any $y \in \mathrm{Ho}(M)$, the mapping space $Map_M(y, x) \in \mathrm{Ho}(SSet)$ is n-truncated.

2. An object $x \in \mathrm{Ho}(M)$ is called *truncated* if it is n-truncated for some integer $n \geqslant 0$.

Clearly, a simplicial set X is n-truncated in the sense above if and only it is n-truncated in the classical sense (i.e. if for any base point $x \in X$, $\pi_i(X, x) = 0$ for all $i > n$).

We now fix an S-site (T, τ) in $\mathbb{U}$, and we consider the corresponding model category of stacks $SPr_\tau(T)$.

Definition 3.7.2. Let $n \geqslant 0$ be an integer. A morphism $f : F \longrightarrow G$ in $SPr_\tau(T)$ is a $\pi_{\leqslant n}$*-equivalence* (or a *local n-equivalence*) if the following two conditions are satisfied:

1. The induced morphism $\pi_0(F) \longrightarrow \pi_0(G)$ is an isomorphism of sheaves on $\mathrm{Ho}(T)$.

2. For any object $x \in Ob(T)$, any section $s \in \pi_0(F(x))$ and any integer i such that $n \geqslant i > 0$, the induced morphism $\pi_i(F, s) \longrightarrow \pi_i(G, f(s))$ is an isomorphism of sheaves on $\mathrm{Ho}(T/x)$.

Theorem 3.7.3. *There exists a closed model structure on $SPr(T)$, called the n-truncated local projective model structure, for which the equivalences are the $\pi_{\leqslant n}$-equivalences and the cofibrations are the cofibrations for the projective model structure on $SPr(T)$. Furthermore the n-local projective model structure is $\mathbb{U}$-cofibrantly generated and proper.*

The category $SPr(T)$ together with its n-truncated local projective model structure will be denoted by $SPr_\tau^{\leqslant n}(T)$.

Proof. The proof is essentially a corollary of Theorem 3.4.1. Let J (resp., I) be a $\mathbb{U}$-small set of generating trivial cofibrations (resp., generating cofibrations) for the model category $SPr_\tau(T)$. Let J' be the set of morphisms $\partial\Delta^i \otimes \underline{h}_x \longrightarrow \Delta^i \otimes \underline{h}_x$, for all $i > n$ and all $x \in Ob(T)$. We define $J(n) = J \cup J'$. Finally, let $W(n)$ be the set of $\pi_{\leqslant n}$-equivalences. It is easy (and left to the reader) to prove that [Ho, Theorem 2.1.19] can be applied to the sets $W(n)$, I and $J(n)$. $\square$

Corollary 3.7.4. *The model category $SPr_\tau^{\leqslant n}(T)$ is the left Bousfield localization of $SPr_\tau(T)$ with respect to the morphisms $\partial\Delta^i \otimes \underline{h}_x \longrightarrow \Delta^i \otimes \underline{h}_x$, for all $i > n$ and all $x \in Ob(T)$.*

Proof. This follows immediately from the explicit description of the set $J(n)$ of generating cofibrations given in the proof of Theorem 3.7.3 above. $\square$

Note that Corollaries 3.4.5 and 3.7.4 also imply that $SPr_{\tau}^{\leqslant n}(T)$ is a left Bousfield localization of $SPr(T)$.

For the next corollary, an object $F \in SPr(T)$ is called *objectwise n-truncated* if for any $x \in Ob(T)$, the simplicial set $F(x)$ is n-truncated (i.e. for any base point $s \in F(x)_0$, one has $\pi_i(F(x), s) = 0$ for $i > n$).

Corollary 3.7.5. *An object $F \in SPr_{\tau}^{\leqslant n}(T)$ is fibrant if and only if it is objectwise fibrant, satisfies the hyperdescent condition* (*see Definition* 3.4.8) *and is objectwise n-truncated*.

Proof. This again follows formally from the explicit description of the set $J(n)$ of generating cofibrations given in the proof of Theorem 3.7.3. $\square$

From the previous corollaries we deduce that the identity functor $\mathrm{Id} : SPr_{\tau}(T) \longrightarrow SPr_{\tau}^{\leqslant n}(T)$ is a left Quillen functor, which then induces an adjunction on the homotopy categories

$$t_{\leqslant n} := \mathbb{L}\mathrm{Id} : \mathrm{Ho}(SPr_{\tau}(T)) \longrightarrow \mathrm{Ho}(SPr_{\tau}^{\leqslant n}(T))$$
$$\mathrm{Ho}(SPr_{\tau}(T)) \longleftarrow \mathrm{Ho}(SPr_{\tau}^{\leqslant n}(T)) : j_n := \mathbb{R}\mathrm{Id}.$$

Note however that the functor

$$t_{\leqslant n} : \mathbb{L}\mathrm{Id} : \mathrm{Ho}(SPr_{\tau}(T)) \longrightarrow \mathrm{Ho}(SPr_{\tau}^{\leqslant n}(T))$$

does not preserves homotopy fibered products in general. Finally, j_n is fully faithful and a characterization of its essential image is given in the following lemma.

Lemma 3.7.6. *Let $F \in SPr_{\tau}(T)$ and $n \geqslant 0$. The following conditions are equivalent.*

1. *F is an n-truncated object in the model category $SPr_{\tau}(T)$* (*in the sense of Definition* 3.7.1).
2. *For any $x \in Ob(T)$ and any base point $s \in F(x)$, one has $\pi_i(F, s) = 0$ for any $i > n$.*
3. *The adjunction morphism $F \longrightarrow j_n t_{\leqslant n}(F)$ is an isomorphism in* $\mathrm{Ho}(SPr_{\tau}(T))$.

Proof. The three conditions are invariant under isomorphisms in $\mathrm{Ho}(SPr_{\tau}(T))$; we can therefore always assume that F is fibrant in $SPr_{\tau}(T)$.

To prove that (1) $\Rightarrow$ (2), it is enough to observe that $\mathbb{R}_{\tau}\underline{Hom}(\underline{h}_x, F) \simeq F(x)$. Conversely, let us suppose that (2) holds and let $j : F \longrightarrow RF$ be a fibrant replacement in $SPr_{\tau}^{\leqslant n}(T)$. The hypothesis on F and Corollary 3.7.5 imply that j is a π_*-equivalence, thus showing that we can assume F to be fibrant in $SPr_{\tau}^{\leqslant n}(T)$, and by Corollary 3.7.5 again, that F can be also assumed to be objectwise n-truncated. In particular, the natural morphism $F^{\Delta^i} \longrightarrow F^{\partial\Delta^i}$ is an objectwise trivial fibration for any $i > n$. Therefore, one has for any $i > n$,

$$\mathbb{R}_{\tau}\underline{Hom}(G, F)^{\mathbb{R}\partial\Delta^i} \simeq \mathbb{R}\underline{Hom}_{\tau}(G, F^{\partial\Delta^i}) \simeq \mathbb{R}\underline{Hom}_{\tau}(G, F^{\Delta^i}) \simeq \mathbb{R}_{\tau}\underline{Hom}(G, F)^{\mathbb{R}\Delta^i}.$$

This implies that $\mathbb{R}_{\tau}\underline{Hom}(G, F)$ is n-truncated for any $G \in SPr_{\tau}(T)$. This proves the equivalence between (1) and (2).

For any $F \in \mathrm{Ho}(SPr_{\tau}(T))$, the adjunction morphism $F \longrightarrow j_n t_{\leqslant n}(F)$ is represented in $SPr(T)$ by a fibrant resolution $j : F \longrightarrow RF$ in the model category $SPr_{\tau}^{\leqslant n}(T)$. If F satisfies condition (2), we have already seen that j is a π_*-equivalence, and therefore that (3) is satisfied. Conversely, by Corollary 3.7.5, RF always satisfies condition (2) and then (3) $\Rightarrow$ (2). □

In the rest of the paper we will systematically use Lemma 3.7.6 and the functor j_n to identify the homotopy category $\mathrm{Ho}(SPr_{\tau}^{\leqslant n}(T))$ with the full subcategory of $\mathrm{Ho}(SPr_{\tau}(T))$ consisting of n-truncated objects. We will therefore never specify the functor j_n. With this convention, the functor $t_{\leqslant n}$ becomes an endofunctor

$$t_{\leqslant n} : \mathrm{Ho}(SPr_{\tau}(T)) \longrightarrow \mathrm{Ho}(SPr_{\tau}(T)),$$

called the *nth truncation functor*. There is an adjunction morphism $\mathrm{Id} \longrightarrow t_{\leqslant n}$, and for any $F \in \mathrm{Ho}(SPr_{\tau}(T))$, the morphism $F \longrightarrow t_{\leqslant n}(F)$ is universal among morphisms from F to an n-truncated object. More precisely, for any n-truncated object $G \in \mathrm{Ho}(SPr_{\tau}(T))$, the natural morphism

$$\mathbb{R}_{\tau}\underline{Hom}(t_{\leqslant n}(F), G) \longrightarrow \mathbb{R}_{\tau}\underline{Hom}(F, G)$$

is an isomorphism in $\mathrm{Ho}(SSet)$.

Definition 3.7.7. The *nth truncation functor* is the functor previously defined

$$t_{\leqslant n} : \mathrm{Ho}(SPr_{\tau}(T)) \longrightarrow \mathrm{Ho}(SPr_{\tau}(T)).$$

The essential image of $t_{\leqslant n}$ is called the subcategory of *n-truncated stacks*.

Note that the essential image of $t_{\leqslant n}$ is by construction equivalent to the category $\mathrm{Ho}(SPr_{\tau}^{\leqslant n}(T))$.

The following proposition gives a complete characterization of the category of 0-truncated stacks and of the 0th truncation functor $t_{\leqslant 0}$.

Proposition 3.7.8. *The functor* $\pi_0^{pr} : SPr(T) \longrightarrow Pr(\mathrm{Ho}(T))$ *induces an equivalence of categories*

$$\mathrm{Ho}(SPr_{\tau}^{\leqslant 0}(T)) \simeq Sh_{\tau}(\mathrm{Ho}(T))$$

where $Sh_\tau(\mathrm{Ho}(T))$ *denotes the category of sheaves of sets on the usual Grothendieck site* $(\mathrm{Ho}(T), \tau)$.

Proof. Let us first suppose that the topology τ is trivial. In this case, we define a quasi-inverse functor as follows. By considering sets as constant simplicial sets, we obtain an embedding $Pr(\mathrm{Ho}(T)) \subset SPr(\mathrm{Ho}(T))$ that we compose with the pullback $p^* : SPr(\mathrm{Ho}(T)) \longrightarrow SPr(T)$ along the natural projection $p : T \longrightarrow \mathrm{Ho}(T)$. It is quite clear that $F \mapsto \pi_0^{pr}(F)$ and $F \mapsto p^*(F)$ induce two functors, inverse of each others

$$\pi_0^{pr} : \mathrm{Ho}(SPr^{\leqslant 0}(T)) \simeq Pr(\mathrm{Ho}(T)) : p^*.$$

In the general case, we use Corollary 3.4.5. We need to show that a presheaf $F \in Pr(\mathrm{Ho}(T))$ is a sheaf for the topology τ if and only if the corresponding object $p^*(F)$ has the hyperdescent property. This last step is left to the reader as an exercise. □

Remark 3.7.9. 1. The previous proposition implies, in particular, that the homotopy category of stacks $\mathrm{Ho}(SPr_\tau(T))$ always contains the category of sheaves on the site $(\mathrm{Ho}(T), \tau)$ as the full subcategory of 0-truncated objects. Again, we will not mention explicitly the functor $p^* : Sh_\tau(\mathrm{Ho}(T)) \longrightarrow \mathrm{Ho}(SPr_\tau(T))$ and identify $Sh_\tau(\mathrm{Ho}(T))$ with the full subcategory of $\mathrm{Ho}(SPr_\tau(T))$ consisting of 0-truncated objects.

2. Proposition 3.7.8 is actually just the 0th stage of a series of similar results involving higher truncations. In fact Proposition 3.7.8 can be generalized to a Quillen equivalence between $SPr_\tau^{\leqslant n}(T)$ and a certain model category of presheaves of n-groupoids on the $(n+1)$-category $t_{\leqslant n}(T)$ obtained from T by applying the n-th fundamental groupoid functor to its simplicial sets of morphisms (see [H-S, Section 2, p. 28]). We will not investigate these results further in this paper.

3.8. Model topoi

Let M be any $\mathbb{U}$-cellular [Hi, Section 14.1] or $\mathbb{U}$-combinatorial [Sm,Du2, Definition 2.1] left proper model category (see also Appendix A). Let us recall from Theorem A.2.2 and A.2.4 that for any $\mathbb{U}$-set of morphisms S in M, the left Bousfield localization $\mathrm{L}_S M$ exists. It is a model category, whose underlying category is still M, whose cofibrations are those of M and whose equivalences are the so-called S-local equivalences [Hi, Section 3.4]. A left Bousfield localization of M is any model category of the form $\mathrm{L}_S M$, for a $\mathbb{U}$-small set S of morphisms in M.

The following definition is a slight modification of the a notion communicated to us by Rezk [Re]. It is a model categorical analog of the notion of topos defined as a reflexive subcategory of the category of presheaves with an exact localization functor (see for example [Sch, Chapter 20]).

Definition 3.8.1. 1. If T is an S-category, a *left exact Bousfield localization* of $SPr(T)$ is a left Bousfield localization $\mathrm{L}_S SPr(T)$ of $SPr(T)$, such that the identity functor $\mathrm{Id} : SPr(T) \longrightarrow \mathrm{L}_S SPr(T)$ preserves homotopy fiber products.

2. A $\mathbb{U}$-*model topos* is a model category in $\mathbb{V}$ which is Quillen equivalent to a left exact Bousfield localization of $SPr(T)$ for some $T \in S-Cat_{\mathbb{U}}$.

For 2, recall our convention throughout the paper, according to which two model categories are Quillen equivalent if they can be connected by a finite *chain* of Quillen equivalences, regardless of their direction. We will also need the following general definitions related to the notion of truncated objects in a model category (see Remark 3.8.7 for some comments on it).

Definition 3.8.2. Let M be any model category.

We say that M is *t-complete* if truncated objects detect isomorphisms in $\mathrm{Ho}(M)$ i.e. if a morphism $u : a \to b$ in $\mathrm{Ho}(M)$ is an isomorphism if and only if, for any truncated object x in $\mathrm{Ho}(M)$, the map $u^* : [b, x] \longrightarrow [a, x]$ is bijective.

A $\mathbb{U}$-model topos is *t-complete* if its underlying model category is t-complete.

The next theorem shows that given an S-category T, t-complete left exact Bousfield localizations of $SPr(T)$ correspond exactly to simplicial topologies on T. It should be considered as a homotopy analog of the correspondence for usual Grothendieck topologies as described e.g. in [Sch, Theorem 20.3.7].

Theorem 3.8.3. *Let T be a $\mathbb{U}$-small S-category. There exists a bijective correspondence between S-topologies on T and left exact Bousfield localizations of $SPr(T)$ which are t-complete.*

Proof.

Let $\mathcal{T}(T)$ be the set of S-topologies on T, which by definition is also the set of Grothendieck topologies on $\mathrm{Ho}(T)$. Let $\mathcal{B}(T)$ be the set of left exact Bousfield localizations of $SPr(T)$, and $\mathcal{B}_t(T) \subset \mathcal{B}(T)$ the subset of those which are t-complete. We are first going to define maps $\phi : \mathcal{T}(T) \to \mathcal{B}_t(T)$ and $\psi : \mathcal{B}_t(T) \to \mathcal{T}(T)$,

The map $\phi : \mathcal{T}(T) \to \mathcal{B}_t(T)$.

Let $\tau \in \mathcal{T}(T)$ be an S-topology on T. According to Corollary 3.4.5 and Proposition 3.4.10(2), $SPr_\tau(T)$ is a left exact Bousfield localization of $SPr(T)$. We are going to show that $SPr_\tau(T)$ is also t-complete. We know by Lemma 3.7.6, that an object $F \in \mathrm{Ho}(SPr_\tau(T))$ is n-truncated if and only if $F \simeq t_{\leqslant n}(F)$. Therefore, if a morphism $f : F \longrightarrow G$ satisfies condition (3) of Definition 3.8.2, one has

$$[t_{\leqslant n}(F), H] \simeq [F, H] \simeq [G, H] \simeq [t_{\leqslant n}(G), H]$$

for any n-truncated object $H \in \mathrm{Ho}(SPr_\tau(T))$. This implies that for any n, the induced morphism $t_{\leqslant n}(F) \longrightarrow t_{\leqslant n}(G)$ is an isomorphism in $\mathrm{Ho}(SPr_\tau^{\leqslant n}(T))$, and hence in $\mathrm{Ho}(SPr_\tau(T))$. In other words, f is an $\pi_{\leqslant n}$-equivalence for any n, and hence a π_*-equivalence. This shows that the model category $SPr_\tau(T)$ is a t-complete model category and allows us to define the map $\phi : \mathscr{T}(T) \longrightarrow \mathscr{B}_t(T)$ by the formula $\phi(\tau) = SPr_\tau(T)$.

The map $\psi : \mathscr{B}_t(T) \to \mathscr{T}(T)$.

Let $\mathrm{L}_S SPr(T) \in \mathscr{B}_t(T)$, and let us consider the derived Quillen adjunction given by the identity functor $\mathrm{Id} : SPr(T) \longrightarrow \mathrm{L}_S SPr(T)$

$$a := \mathbb{L}\mathrm{Id} : \mathrm{Ho}(SPr(T)) \longrightarrow \mathrm{Ho}(\mathrm{L}_S SPr(T))$$
$$\mathrm{Ho}(SPr(T)) \longleftarrow \mathrm{Ho}(\mathrm{L}_S SPr(T)) : \mathbb{R}\mathrm{Id} =: i.$$

The reader should note that the above functor a is not equal a priori to the associated stack functor of Definition 3.4.9(5), as no S-topology on T has been given yet. We know that j is fully faithful and identifies $\mathrm{Ho}(\mathrm{L}_S SPr(T))$ with the full subcategory of $\mathrm{Ho}(SPr(T))$ consisting of S-local objects (see [Hi, Definition 3.2.41(a); Theorem 4.1.1(2)]).

We consider the full subcategory $\mathrm{Ho}_{\leqslant 0}(\mathrm{L}_S SPr(T))$ (resp. $\mathrm{Ho}_{\leqslant 0}(SPr(T))$) of Ho $(\mathrm{L}_S SPr(T))$ (resp. of $\mathrm{Ho}(SPr(T))$) consisting of 0-truncated objects. Note that in general, an object x in a model category is 0-truncated if and only if for any $n \geqslant 1$, the natural morphism $x^{\mathbb{R}\Delta^n} \longrightarrow x^{\mathbb{R}\partial\Delta^n}$ is an equivalence. As both a and i preserve homotopy fiber products, they also preserve 0-truncated objects. Therefore we have an induced adjunction

$$a_0 : \mathrm{Ho}_{\leqslant 0}(SPr(T)) \longrightarrow \mathrm{Ho}_{\leqslant 0}(\mathrm{L}_S SPr(T)),$$
$$\mathrm{Ho}_{\leqslant 0}(SPr(T)) \longleftarrow \mathrm{Ho}_{\leqslant 0}(\mathrm{L}_S SPr(T)) : i_0.$$

Now, the functor $\pi_0^{pr} : \mathrm{Ho}(SPr(T)) \longrightarrow Set^{\mathrm{Ho}(T)^{op}}$ induces an equivalence of categories

$$\mathrm{Ho}_{\leqslant 0}(SPr(T)) \simeq Set^{\mathrm{Ho}(T)^{op}} =: Pr(\mathrm{Ho}(T)),$$

and so the adjunction (a_0, i_0) is in fact equivalent to an adjunction

$$a_0 : Pr(\mathrm{Ho}(T)) \longrightarrow \mathrm{Ho}_{\leqslant 0}(\mathrm{L}_S SPr(T)), \quad Pr(\mathrm{Ho}(T)) \longleftarrow \mathrm{Ho}_{\leqslant 0}(\mathrm{L}_S SPr(T)) : i_0,$$

where, of course, the functor i_0 is still fully faithful and the functor a_0 is exact. By [Sch, Theorem 20.3.7], there exists then a unique Grothendieck topology τ on $\mathrm{Ho}(T)$ such that the essential image of i_0 is exactly the full subcategory of sheaves on $\mathrm{Ho}(T)$ for the topology τ. The functor a_0 is then equivalent to the associated sheaf functor. Thus, we define $\psi : \mathscr{B}_t(T) \longrightarrow \mathscr{T}(T)$ by the formula $\psi(\mathrm{L}_S SPr(T)) := \tau \in \mathscr{T}(T)$.

Proof of $\phi\circ\psi = \mathrm{Id}$.

Let $\mathrm{L}_S SPr(T) \in \mathscr{B}_t(T)$ be a left exact Bousfield localization of $SPr(T)$ and $\tau = \psi(\mathrm{L}_S SPr(T))$ the corresponding topology on T. We need to prove that the set of S-local equivalences equal the set of π_*-equivalences. Recall that we have denoted by

$$a := \mathbb{L}\mathrm{Id} : \mathrm{Ho}(SPr(T)) \longrightarrow \mathrm{Ho}(\mathrm{L}_S SPr(T))$$
$$\mathrm{Ho}(SPr(T)) \longleftarrow \mathrm{Ho}(\mathrm{L}_S SPr(T)) : \mathbb{R}\mathrm{Id} =: i,$$

the adjunction induced by the identity functor $\mathrm{Id} : L_S SPr(T) \longrightarrow SPr(T)$.

Let us first prove that S-local equivalences are π_*-equivalences. Equivalently, we need to prove that for any morphism $f : F \longrightarrow G$ which is an equivalence in $\mathrm{L}_S SPr(T)$, f is an hypercover in $SPr_\tau(T)$. For this we may assume that F and G are both objectwise fibrant objects. As the identity functor $\mathrm{Id} : SPr(T) \longrightarrow SPr_\tau(T)$ preserves homotopy fiber products, the induced morphism

$$F^{\Delta^n} \longrightarrow F^{\partial\Delta^n} \times_{G^{\partial\Delta^n}} G^{\Delta^n}$$

is still an S-local equivalence. Using this fact and Lemma 3.3.3, one sees that it is enough to show that f is a covering in a $SPr_\tau(T)$.

Recall that the topology τ is defined in such a way that the associated sheaf to a presheaf of sets E on $\mathrm{Ho}(T)$ is $i_0 a_0(E)$ (where the adjunction (a_0, i_0) is the one considered above in the definition of the map ψ). It is therefore enough to prove that the induced morphism $a_0(\pi_0^{pr}(F)) \longrightarrow a_0(\pi_0^{pr}(G))$ is an isomorphism.[4]

Lemma 3.8.4. *For any* $F \in \mathrm{Ho}(SPr(T))$, *one has*

$$a_0(\pi_0^{pr}(F)) \simeq a_0\pi_0^{pr}(ia(F)).$$

Proof. This immediately follows from the adjunctions (a, i) and (a_0, i_0), and the fact that π_0^{pr} is isomorphic to the 0-th truncation functor $t_{\leqslant 0}$ on $\mathrm{Ho}(SPr(T))$. □

As f is an S-local equivalence, the morphism $ia(F) \longrightarrow ia(G)$ is an isomorphism in $\mathrm{Ho}(SPr(T))$, and therefore the same is true for

$$a_0(\pi_0^{pr}(F)) \simeq a_0\pi_0^{pr}(ia(F)) \longrightarrow a_0\pi_0^{pr}(ia(G)) \simeq a_0(\pi_0^{pr}(G)).$$

We have thus shown that the S-local equivalences are π_*-equivalences. Conversely, to show that π_*-equivalences are S-local equivalences it is enough to show that for any $x \in Ob(T)$ and any hypercover $F_* \longrightarrow \underline{h}_x$ in $SPr_\tau(T)$, the natural morphism

$$ia(|F_*|) \longrightarrow ia(\underline{h}_x)$$

[4] Recall that $\pi_0^{pr}(F)$ is a presheaf of sets on $\mathrm{Ho}(T)$, that is considered via the projection $p : T \longrightarrow \mathrm{Ho}(T)$ as a presheaf of discrete simplicial sets on T, and therefore as an object in $SPr(T)$.

is an isomorphism in $\mathrm{Ho}(SPr(T))$ (see Corollary 3.4.5). As a preserves homotopy fibered products, one has $(ia(G))^{\mathbb{R}K} \simeq ia(G^{\mathbb{R}K})$, for any $G \in \mathrm{Ho}(SPr(T))$ and any finite simplicial set K (here $(-)^{\mathbb{R}K}$ is computed in the model category $SPr(T)$). Therefore, for any n, one has, by t-completeness,

$$t_{\leqslant n}(ia(|F_*|)) \simeq t_{\leqslant n}(ia(|\mathbb{R}Cosk_n F_*|)).$$

This shows that one can assume that $F_* = \mathbb{R}Cosk_n(F_*/\underline{h}_x)$, for some n (i.e. that $F_* \longrightarrow \underline{h}_x$ is relatively n-bounded). Furthermore, the same argument as in the proof of Theorem 3.4.1, but relative to $\underline{h}_x$, shows that, by induction, one can assume $n = 0$. In other words, one can assume that F_* is the derived nerve of a covering $F_0 \longrightarrow \underline{h}_x$ (which will be assumed to be an objectwise fibration).

By the left exactness property of a and i, the object $ia(|F_*|)$ is isomorphic in $\mathrm{Ho}(SPr(T))$ to the geometric realization of the derived nerve of $ia(F_0) \longrightarrow ia(\underline{h}_x)$. This implies that for any $y \in Ob(T)$, the morphism $ia(|F_*|)(y) \longrightarrow ia(\underline{h}_x)(y)$ is isomorphic in $\mathrm{Ho}(SSet)$ to the geometric realization of the nerve of a fibration between simplicial sets. It is well known that such a morphism is isomorphic in $\mathrm{Ho}(SSet)$ to an inclusion of connected components. Therefore it is enough to show that the morphism

$$\pi_0^{pr}(ia(|F_*|)) \longrightarrow \pi_0^{pr}(ia(\underline{h}_x))$$

induces an isomorphism on the associated sheaves. By Lemma 3.8.4, this is equivalent to showing that the morphism

$$i_0 a_0 \pi_0^{pr}(ia(|F_*|)) \longrightarrow i_0 a_0 \pi_0^{pr}(ia(\underline{h}_x))$$

is an isomorphism of presheaves of sets on $\mathrm{Ho}(T)$. This morphism is also isomorphic to

$$i_0 a_0(\pi_0^{pr}(|F_*|)) \longrightarrow i_0 a_0 \pi_0^{pr}(\underline{h}_x)$$

whose left-hand side is the sheaf associated to the co-equalizer of the two projections

$$pr_1, pr_2 : \pi_0^{pr}(F_0) \times_{\pi_0^{pr}(\underline{h}_x)} \pi_0^{pr}(F_0) \longrightarrow \pi_0^{pr}(\underline{h}_x),$$

whereas the right-hand side is the sheaf associated to $\pi_0^{pr}(\underline{h}_x)$. To conclude the proof, it is enough to notice that $\pi_0^{pr}(F_0) \longrightarrow \pi_0^{pr}(\underline{h}_x)$ induces an epimorphism of sheaves (because F_* is a hypercover) and that epimorphisms of sheaves are always effective (see [SGA4-I, Exp. II, Theoreme 4.8]).

Proof of $\psi \circ \phi = \mathrm{Id}$.

Let τ be a topology on T. By definition of the maps ψ and ϕ, to prove that $\psi \circ \phi = \mathrm{Id}$, it is equivalent to show that the functor $\pi_0^{pr} : \mathrm{Ho}(SPr_\tau(T)) \longrightarrow Pr(\mathrm{Ho}(T))$, when restricted to the full subcategory of 0-truncated objects in $\mathrm{Ho}(SPr_\tau(T))$, induces an equivalence to the category of sheaves on the site $(\mathrm{Ho}(T), \tau)$. But this follows from Proposition 3.7.8. □

Corollary 3.8.5. *Let M be a model category in* $\mathbb{U}$. *The following conditions are equivalent*:

1. *The model category M is a t-complete* $\mathbb{U}$*-model topos*.
2. *The model category M is t-complete and there exists a* $\mathbb{U}$*-small category C and a subcategory* $S \subset C$, *such that M is Quillen equivalent to a left exact Bousfield localization of* $M^{C,S}$ (see Definition 2.3.3).
3. *There exists a* $\mathbb{U}$*-small S-site* (T, τ) *such that M is Quillen equivalent to* $SPr_\tau(T)$.

Proof. The equivalence of (2) and (3) follows immediately from Theorem 2.3.5 and the *delocalization theorem* [D-K2, Theorem 2.5], while (1) and (3) are equivalent by Theorem 3.8.3. □

The previous results imply in particular the following interesting rigidity property for *S*-groupoids.

Corollary 3.8.6. *Let T be a* $\mathbb{U}$*-small S-category such that* $\mathrm{Ho}(T)$ *is a groupoid* (*i.e. every morphism in T is invertible up to homotopy*). *Then, there is no non-trivial t-complete left exact Bousfield localization of* $SPr(T)$.

Proof. In fact, there is no non-trivial topology on a groupoid, and therefore there is no non-trivial *S*-topology on *T*. □

Remark 3.8.7. 1. There exist *t*-complete $\mathbb{U}$-model topoi which are not Quillen equivalent to some $SPr_\tau(T)$, for T a $\mathbb{U}$-small category. Indeed, when T is a category, the model category $SPr_\tau(T)$ is such that any object is a homotopy colimits of 0-truncated objects (this is because representable objects are 0-truncated). It is not difficult to see that this last property is not satisfied when T is a general *S*-category. For example, let $T = BK(\mathbb{Z}, 1)$ be the *S*-category with a unique object and the simplicial monoid $K(\mathbb{Z}, 1)$ as simplicial set of endomorphisms. Then, $SPr(T)$ is the model category of simplicial sets together with an action of $K(\mathbb{Z}, 1)$, and 0-truncated objects in $SPr(T)$ are all equivalent to discrete simplicial set with a trivial action of $K(\mathbb{Z}, 1)$. Therefore any homotopy colimit of such will be a simplicial set with a trivial action by $K(\mathbb{Z}, 1)$. However, the action of $K(\mathbb{Z}, 1)$ on itself by left translations is *not* equivalent to a trivial one.

2. As observed by Lurie, there are examples of left exact Bousfield localization of $SPr(T)$ which are *not* of the form $SPr_\tau(T)$. To see this, let (T, τ) be a Grothendieck site and consider the left Bousfield localization $\mathrm{L}_{\mathrm{cov}}\, SPr(T)$ of $SPr(T)$ along only those hypercovers which are nerves of coverings (obviously, not all hypercovers are of this kind). Now, an example due to Simpson shows that there are Grothendieck sites (T, τ) such that $\mathrm{L}_{\mathrm{cov}}\, SPr(T)$ is not the same as $SPr_\tau(T)$ (see for example [DHI, Example

(A.10)]). However, $L_{cov}\, SPr(T)$ is a left exact Bousfield localization of $SPr(T)$, and the topology it induces on T via the procedure used in the proof of Theorem 3.8.3, coincides with τ. Of course, the point here is that $L_{cov} SPr(T)$ is *not* a *t*-complete model category. This shows that one cannot omit the hypothesis of *t*-completeness in Theorem 3.8.3.

3. Though the hypothesis of *t*-completeness in Theorem 3.8.3 is quite natural, and allows for a clean explanation in terms of *S*-topologies, it could be interesting to look for a similar comparison result without such an assumption. One way to proceed would be to introduce a notion of *hyper-topology* on a category (or more generally on an *S*-category), a notion which was suggested to us by some independent remarks of Hinich, Joyal and Simpson. A hyper-topology on a category would be essentially the same thing as a topology with the difference that one specifies directly the hypercovers and not only the coverings; the conditions it should satisfy are analogous to the conditions imposed on the family of coverings in the usual definition of a Grothendieck (pre)topology. The main point here is that for a given Grothendieck site (T, τ), the two hyper-topologies defined using *all* τ-hypercovers on one side or only *bounded* τ*-coverings* on the other side, will not be equivalent in general. It seems reasonable to us that our Theorem 3.8.3 can be generalized to a correspondence between hyper-topologies on T and arbitrary left exact Bousfield localizations of $SPr(T)$. This notion of hypertopology seems to be closely related to Cisinki's results in [Cis].

4. Theorem 3.8.3 suggests also a way to think of *higher topologies* on *n*-categories (and of *higher topoi*) for $n \geqslant 1$ as appropriate *left exact localizations*. In this case, the explicit notion of higher topology (that one has to reconstruct e.g. assuming the Theorem still holds for higher categories), will obviously depend on more then the associated homotopy category. For example, for the case of 2-categories, as opposed to the case when all *i*-morphisms are invertible for $i > 1$ (see Remark 3.1.2), a topology should give rise to some kind of topologies on the various categories of 1-morphisms and these topologies should satisfy some compatibility condition with respect to the composition.

We finish this paragraph with the following definition.

Definition 3.8.8. An $\mathbb{U}-S$*-topos* is an *S*-category which is isomorphic in $\mathrm{Ho}(S-Cat)$ to some $LSPr_\tau(T)$, for (T, τ) a $\mathbb{U}$-small *S*-site.

4. Stacks over pseudo-model categories

In this section we define the notion of a *model pre-topology* on a model category and the notion of *stacks* on such *model sites*. A model pre-topology is a homotopy variation of the usual notion of a Grothendieck pre-topology and it reduces to the latter when the model structure is trivial (i.e. when equivalences are isomorphisms and any map is a fibration and a cofibration). We develop the theory in the slightly more general context of *pseudo-model categories*, i.e of full subcategories of model categories that are closed under equivalences and homotopy pull-backs (see Definition 4.1.1). We have chosen to work in this more general context because in some applications we will need to use subcategories of model categories defined by *homotopy invariant conditions* but not necessarily closed under small limits and/or colimits (e.g., certain subcategories of objects of *finite presentation*). The reader is however strongly encouraged to cancel everywhere the word *pseudo*- in the following and to restore it only when interested in some application that requires such a degree of generality (as for example, the problem of defining étale K-theory on the pseudo-model category of connective commutative $\mathbb{S}$-algebras, see Proposition 5.1.2). On the other hand, the theory itself presents no additional difficulty, except possibly for the linguistic one.

4.1. Model categories of pre-stacks on a pseudo-model category

In this subsection we will define the (model)category of pre-stacks on a *pseudo-model* category which is essentially a category with weak equivalences that admits a nice embedding into a model category.

Definition 4.1.1. A $\mathbb{U}$-small *pseudo-model category* is a triple $(C, S, \imath)$ where C is a $\mathbb{U}$-small category, $S \subset C$ is a subcategory of C and $\imath : C \to M$ is a functor to a model $\mathbb{U}$-category M satisfying the following four conditions:

1. The functor $\imath$ is fully faithful.
2. One has $\imath(S) = W \cap \imath(C)$, where W is the set of weak equivalences in the model category M.
3. The category C is closed under equivalences in M, i.e. if $x \to y$ is an equivalence in M and x (resp. y) is in the image of $\imath$, then so is y (resp. x).
4. The category C is closed under homotopy pullbacks in M.

The localization $S^{-1}C$ will be called the *homotopy category* of (C, S) and often denoted by $\mathrm{Ho}(C, S)$ or simply $\mathrm{Ho}(C)$ when the choice of S is unambiguous.

Condition (4) of the previous definition can be precised as follows. Denoting by $\mathrm{Ho}(\imath) : S^{-1}C \to \mathrm{Ho}(M)$ the functor induced by $\imath$ (due to (2).), which is fully faithful due to (1) and (3), the image of $\mathrm{Ho}(\imath)$, that coincides with its essential image, is closed under homotopy pullbacks.

Note also that because of condition (3) of Definition 4.1.1, the functor $\imath$ is an isomorphism from C to its essential image in M. Hence we will most of the time identify C with its image $\imath(C)$ in the model category M; therefore an object $x \in C$ will be called *fibrant* (respectively, *cofibrant*) in C if $\imath(x)$ is fibrant (resp. cofibrant) in M. Moreover, we will sometimes call the maps in S simply equivalences.

Conditions (3) and (4) imply in particular that for any diagram

$$\begin{array}{ccc} x & \xrightarrow{p} & y \\ & & \uparrow \\ & & z \end{array}$$

of fibrant objects in C, such that p is a fibration, the fibered product $x \times_z y$ exists. Indeed, this fibered product exists in the ambient model category M, and being equivalent to the homotopy fibered product, it also belongs to C by conditions (3) and (4).

Remark 4.1.2. 1. Being a pseudo-model category is not a self-dual property, in the sense that if M is a pseudo-model category, then M^{op} is not pseudo-model in general. Objects satisfying Definition 4.1.1 should be called more correctly *right pseudo-model categories* and the dual definition (i.e. closure by homotopy push-outs) should deserve the name of *left pseudo-model category*. However, to simplify the terminology, we fix once for all Definition 4.1.1 as it is stated.

2. Note that if M is a model category with weak equivalences W, the triple (M, W, Id_M) is a pseudo-model category. Moreover, a pseudo-model category is *essentially* a model category. In fact, conditions (1)–(3) imply that C satisfies conditions (1), (2) and (4) of the definition of a *model structure* in the sense of [Ho, Definition 1.1.3]. However, C is not exactly a model category in general, since it is not required to be complete and co-complete (see [Ho, Definition 1.1.4]), and the lifting property (3) of [Ho, Definition 1.1.3] is not necessarily satisfied.

3. If C is a complete and co-complete category and S consists of all isomorphisms in C, then $(C, S, \mathrm{Id}_{\mathrm{c}})$ is a *trivial* pseudo-model category, where we consider on C the trivial model structure with equivalences consisting of all isomorphisms and any map being a fibration (and a cofibration). If C is not necessarily complete and co-complete but has finite limits, then we may view it as a *trivial* pseudo-model category by replacing it with its essential image in $Pr(C)$ or $SPr(C)$, endowed with the trivial model structures, and taking S to be all the isomorphisms.

Example 4.1.3. 1. Let k be a commutative ring and $M := Ch(k)^{op}$ the opposite model category of unbounded chain complexes of k-modules (see [Ho, Definition 2.3.3]). The full subcategory $C \hookrightarrow M$ of *homologically positive* objects (i.e. objects $P_\bullet$ such that $H_i(P_\bullet) = 0$ for $i < 0$) is a pseudo-model category.

2. Let k be a commutative ring (respectively, a field of characteristic zero) and let $M := (E_\infty - \mathrm{Alg}_k)^{op}$ (respectively, $M = \mathrm{CDGA}_k^{op}$) be the opposite model category of E_∞-algebras over the category of unbounded cochain complexes of k-modules (resp., the opposite model category of commutative and unital differential graded k-algebras in non-positive degrees) which belong to $\mathbb{U}$ (see for example [Hin] for a description of these model structures). We say that an object A of M is *finitely presented* if for any filtered direct diagram $C : J \to M^{op}$, with $J \in \mathbb{U}$, the natural map

$$\operatorname*{hocolim}_{j \in J} \mathrm{Map}_{M^{op}}(A, C_j) \longrightarrow \mathrm{Map}_{M^{op}}\left(A, \operatorname*{hocolim}_{j \in J} C_j\right)$$

is an equivalence of simplicial sets. Here $\mathrm{Map}_{M^{op}}(-,-)$ denotes the mapping spaces (or function complexes) in the model category M^{op} (see [Ho, Section 5.4]). The reader will check that the full subcategory $C \hookrightarrow M$ of finitely presented objects is a pseudo-model category.

3. Let A be a commutative $\mathbb{S}$-algebra as defined in [EKMM, Chapter 2, Section 3]. Let M be the opposite category of the comma model category of commutative $\mathbb{S}$-algebras under A: an object in M is then a map of commutative $\mathbb{S}$-algebras $A \to B$. Then, the full subcategory $C \hookrightarrow M$ consisting of finitely presented A-algebras (see the previous example or Definition 5.2.1) is a pseudo-model category. The full subcategory $C \hookrightarrow M$ consisting of *étale maps* $A \to B$ (see Definition 5.2.3) is also a pseudo-model category. This pseudo-model category will be called the *small étale site* over A.

4. Let X be a scheme and $C(X, \mathcal{O})$ be the category of unbounded cochain complexes of $\mathcal{O}$-modules. There exists a model structure on $C(X, \mathcal{O})$ where the equivalences are the local quasi-isomorphisms. Then, the full subcategory of $C(X, \mathcal{O})$ consisting of *perfect complexes* is a pseudo-model category.

Recall from Section 2.3.2 that for any category C in $\mathbb{U}$ and any subcategory $S \subset C$, we have defined (Definition 2.3.3) the model category $SSet_{\mathbb{U}}^{C,S}$ of *restricted diagrams* on (C, S) of simplicial sets. Below, we will consider restricted diagrams on (C^{op}, S^{op}), where $(C, S, \imath)$ is a pseudo-model category.

Definition 4.1.4. 1. Let (C, S) be a category with a distinguished subset of morphisms. The model category $SSet_{\mathbb{U}}^{C^{op},S^{op}}$, of restricted diagrams of simplicial sets on (C^{op}, S^{op}) will be denoted by $(C, S)^{\wedge}$ and called the *model category of pre-stacks on* (C, S) (note that if $(C, S, \imath)$ is a pseudo-model category, $(C, S)^{\wedge}$ does not depend on $\imath$).

2. Let $(C, S, \imath)$ be a pseudo-model category and let C^{c} (resp. C^{f}, resp. C^{cf}) be the full subcategory of C consisting of cofibrant (resp. fibrant, resp cofibrant and fibrant) objects, and $S^{\mathrm{c}} := C^{\mathrm{c}} \cap S$ (resp. $S^{\mathrm{f}} := C^{\mathrm{f}} \cap S$, resp. $S^{\mathrm{cf}} := C^{\mathrm{cf}} \cap S$). We will denote by $((C, S)^{\mathrm{c}})^{\wedge}$ (resp. $((C, S)^{\mathrm{f}})^{\wedge}$, resp. $((C, S)^{\mathrm{cf}})^{\wedge}$) the model category of restricted diagrams of $\mathbb{U}$-simplicial sets on $(C^{\mathrm{c}}, S^{\mathrm{c}})^{op}$ (resp. on $(C^{\mathrm{f}}, S^{\mathrm{f}})^{op}$, resp. on $(C^{\mathrm{cf}}, S^{\mathrm{cf}})^{op}$).

Objects of $(C, S)^{\wedge}$ are simply functors $F : C^{op} \longrightarrow SSet_{\mathbb{U}}$ and, as observed in Section 2.3.2, F is fibrant in $(C, S)^{\wedge}$ if and only if it is objectwise fibrant and preserves equivalences.

The category $(C, S)^{\wedge}$ is naturally tensored and co-tensored over $SSet_{\mathbb{U}}$, with external products and exponential objects defined objectwise. This makes $(C, S)^{\wedge}$ into a *simplicial closed model category*. This model category is furthermore left proper, $\mathbb{U}$-cellular and $\mathbb{U}$-combinatorial (see [Du2,Hi, Chapter 14] and Appendix A). The derived simplicial *Hom*'s of the model category $(C, S)^{\wedge}$ will be denoted by

$$\mathbb{R}_w \underline{Hom}(-,-) : \mathrm{Ho}((C, S)^{\wedge})^{op} \times \mathrm{Ho}((C, S)^{\wedge}) \longrightarrow \mathrm{Ho}((C, S)^{\wedge}).$$

The derived simplicial *Hom*'s of the model categories $((C, S)^{\mathrm{c}})^{\wedge}$, $((C, S)^{\mathrm{f}})^{\wedge}$ and $((C, S)^{\mathrm{cf}})^{\wedge}$, will be denoted similarly by

$$\mathbb{R}_{w,\mathrm{c}} \underline{Hom}(-,-), \quad \mathbb{R}_{w,\mathrm{f}} \underline{Hom}(-,-), \quad \mathbb{R}_{w,\mathrm{cf}} \underline{Hom}(-,-).$$

For an object $x \in C$, the evaluation functor $j_x^* : (C, S)^\wedge \longrightarrow SSet_{\mathbb{U}}$ is a right Quillen functor. Its left adjoint is denoted by $(j_x)_! : SSet_{\mathbb{U}} \longrightarrow (C, S)^\wedge$. We note that there is a canonical isomorphism $h_x \simeq (j_x)_!(*)$ in $(C, S)^\wedge$, where $h_x : C^{op} \longrightarrow SSet_{\mathbb{U}}$ sends an object $y \in C$ to the constant simplicial set $Hom(y, x)$. More generally, for any $A \in SSet_{\mathbb{U}}$, one has $(j_x)_!(A) \simeq A \otimes h_x$.

As $(C, S)^\wedge$ is a left Bousfield localization of $SPr(C)$, the identity functor $\mathrm{Id} : SPr(C) \longrightarrow (C, S)^\wedge$ is left Quillen. In particular, homotopy colimits of diagrams in $(C, S)^\wedge$ can be computed in the objectwise model category $SPr(C)$. On the contrary, homotopy limits in $(C, S)^\wedge$ are not computed in the objectwise model structure; moreover, the identity functor $\mathrm{Id} : (C, S)^\wedge \longrightarrow SPr(C)$ does not preserve homotopy fibered products in general.

As explained in Section 2.3.2 (before Corollary 2.3.6), if (C, S) and (C', S') are categories with distinguished subsets of morphisms (e.g., pseudo-model categories) and $f : C \to C'$ is a functor sending S into S', then we have a direct and inverse image Quillen adjunction

$$f_! : (C, S)^\wedge \longrightarrow (C, S')^\wedge, \quad (C, S)^\wedge \longleftarrow (C', S')^\wedge : f^*.$$

In particular, if $(C, S, \imath)$ is a pseudo-model category, we may consider the inclusions

$$(C^{\mathrm{c}}, S^{\mathrm{c}}) \subset (C, S), \quad (C^{\mathrm{f}}, S^{\mathrm{f}}) \subset (C, S), \quad (C^{\mathrm{cf}}, S^{\mathrm{cf}}) \subset (C, S).$$

As a consequence of Theorem 2.3.5 (or by a direct check), we get

Proposition 4.1.5. *Let $(C, S, \imath)$ be a pseudo-model category. The natural inclusions*

$$i_{\mathrm{c}} : (C, S)^{\mathrm{c}} \hookrightarrow (C, S), \quad i_{\mathrm{f}} : (C, S)^{\mathrm{f}} \hookrightarrow (C, S), \quad i_{\mathrm{cf}} : (C, S)^{\mathrm{cf}} \hookrightarrow (C, S),$$

induce right Quillen equivalences

$$i_{\mathrm{c}}^* : (C, S)^\wedge \simeq ((C, S)^{\mathrm{c}})^\wedge, \quad i_{\mathrm{f}}^* : (C, S)^\wedge \simeq ((C, S)^{\mathrm{f}})^\wedge, \quad i_{\mathrm{cf}}^* : (C, S)^\wedge \simeq ((C, S)^{\mathrm{cf}})^\wedge.$$

These equivalences are furthermore compatible with derived simplicial Hom, in the sense that there exist natural isomorphisms

$$\mathbb{R}_{w,\mathrm{c}}\, \underline{Hom}(\mathbb{R}(i_{\mathrm{c}})^*(-), \mathbb{R}(i_{\mathrm{c}})^*(-)) \simeq \mathbb{R}_w\, \underline{Hom}(-, -),$$

$$\mathbb{R}_{w,\mathrm{f}}\, \underline{Hom}(\mathbb{R}(i_{\mathrm{f}})^*(-), \mathbb{R}(i_{\mathrm{f}})^*(-)) \simeq \mathbb{R}_w \underline{Hom}(-, -),$$

$$\mathbb{R}_{w,\mathrm{cf}}\, \underline{Hom}(\mathbb{R}(i_{\mathrm{cf}})^*(-), \mathbb{R}(i_{\mathrm{cf}})^*(-)) \simeq \mathbb{R}_w \underline{Hom}(-, -).$$

4.2. The Yoneda embedding of a pseudo-model category

Let us fix a pseudo-model category $(C, S, \imath : C \to M)$. Throughout this subsection we will also fix a *cofibrant resolution functor* $(\Gamma : M \longrightarrow M^{\Delta}, i)$ in the model category M (see [Hi, 17.1.3, (1)]). This means that for any object $x \in M$, $\Gamma(x)$ is a co-simplicial object in M, which is cofibrant for the Reedy model structure on M^{Δ}, together with a natural equivalence $i(x) : \Gamma(x) \longrightarrow c^*(x)$, $c^*(x)$ being the constant co-simplicial object in M at x. Let us remark that when the model category M is simplicial, one can use the standard cofibrant resolution functor $\Gamma(x) := \Delta^* \otimes Q(x)$, where Q is a cofibrant replacement functor in M.

We define the functor $\underline{h} : C \longrightarrow SPr(C)$, by sending each $x \in C$ to the simplicial presheaf

$$\begin{array}{rcl} \underline{h}_x : M^{op} & \longrightarrow & SSet_{\mathbb{U}}, \\ y & \mapsto & Hom_M(\Gamma(y), x), \end{array}$$

where, to be more explicit, the presheaf of n-simplices of $\underline{h}_x$ is given by the formula

$$(\underline{h}_x)_n(-) := Hom_M(\Gamma(-)_n, x).$$

Note that for any $y \in M$, $\Gamma(y)_n \to y$ is an equivalence in M, therefore $y \in C$ implies that $\Gamma(y) \in C^{\Delta}$ (since C is a pseudo-model category).

We warn the reader that the two functors h and $\underline{h}$ from C to $(C, S)^{\wedge}$ are different and should not be confused. For any $x \in C$, h_x is a presheaf of discrete simplicial sets (i.e. a presheaf of sets) whereas $\underline{h}_x$ is an actual simplicial presheaf. The natural equivalence $i(-) : \Gamma(-) \longrightarrow c^*(-)$ induces a morphism in $(C, S)^{\wedge}$

$$h_x = Hom(c^*(-), x) \longrightarrow Hom(\Gamma(-), x) = \underline{h}_x,$$

which is functorial in $x \in M$.

If, for a moment we denote by $\underline{h}^C : C \longrightarrow (C, S)^{\wedge}$ and by $\underline{h}^M : M \longrightarrow (M, W)^{\wedge}$ the functor defined for the pseudo-model categories $(C, S, \imath)$ and (M, W, Id), respectively, we have a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\underline{h}^C} & (C, S)^{\wedge} \\ {\scriptstyle \imath}\downarrow & & \uparrow{\scriptstyle \imath^*} \\ M & \xrightarrow{\underline{h}^M} & M^{\wedge} \end{array}$$

where $\imath^*$ is the restriction, right Quillen functor.

Lemma 4.2.1. *Both functors* $\underline{h} : C \longrightarrow SPr(C)$ *and* $\underline{h}^C : C \longrightarrow (C, S)^{\wedge}$ *preserves fibrant objects and equivalences between them.*

Proof. The statement for $\underline{h} : M \longrightarrow SPr(M)$ follows from the standard properties of mapping spaces, see [Ho, Section 5.4] or [Hi, Proposition 18.1.3, Theorem 18.8.7]. The statement for $\underline{h} : M \longrightarrow M^{\wedge}$ follows from the previous one and from [Hi, Theorem 18.8.7(2)]. Finally, the statements for $\underline{h} : C \longrightarrow SPr(C)$ and $\underline{h}^C : C \longrightarrow (C, S)^{\wedge}$ follow from the previous ones for M and from the commutativity of diagram (red), since $\imath^*$ is right Quillen. $\square$

Lemma 4.2.1 enables us to define a right derived functor of $\underline{h}$ as

$$
\begin{array}{rcl}
\mathbb{R}\underline{h} : S^{-1}C & \longrightarrow & \mathrm{Ho}((C, S)^{\wedge}), \\
x & \mapsto & (\underline{h} \circ R \circ \imath)(x).
\end{array}
$$

where R denotes a fibrant replacement functor in M and we implicitly used the fact that $R\imath(x)$ is still in C for $x \in C$. Also note that, by definition of $(C, S)^{\wedge}$, the functor $h : C \longrightarrow (C, S)^{\wedge}$ preserves equivalences, hence induces a functor $\mathrm{Ho}(h) : S^{-1}C \longrightarrow \mathrm{Ho}((C, S)^{\wedge})$.

The reader should notice that if (Γ', i') is another cofibrant resolution functor in M, then the two derived functor $\mathbb{R}\underline{h}$ and $\mathbb{R}\underline{h'}$ obtained using, respectively, Γ and Γ', are naturally isomorphic. Therefore, our construction does not depend on the choice of Γ once one moves to the homotopy category.

Lemma 4.2.2. *The functors* $\mathrm{Ho}(h)$ *and* $\mathbb{R}\underline{h}$ *from* $S^{-1}C$ *to* $\mathrm{Ho}((C, S)^{\wedge})$ *are canonically isomorphic. More precisely, if R be a fibrant replacement functor in M, then the natural equivalence* $i(-) : \Gamma(-) \longrightarrow c^*(-)$ *induces, for any* $x \in C$, *an equivalence in* $(C, S)^{\wedge}$ (*hence a fibrant replacement, by Lemma* 4.2.1)

$$
h_x = Hom(-, x) \longrightarrow Hom(\Gamma(-), R(x)) = \underline{h}_{R(x)}.
$$

Proof.

First we show that if x is a fibrant and cofibrant object in C, then the natural morphism $h_x \longrightarrow \underline{h}_x$ is an equivalence in $((C, S)^{\mathrm{c}})^{\wedge}$. To see this, let $x \longrightarrow x_*$ be a simplicial resolution of x in M, hence in C (see [Hi, 17.1.2]). We consider the following two simplicial presheaves:

$$
\begin{array}{rcl}
h_{x_*} : (C^{\mathrm{c}})^{op} & \longrightarrow & SSet_{\mathbb{U}}, \\
y & \mapsto & Hom(y, x_*),
\end{array}
$$

$$
\begin{array}{rcl}
\underline{h}_{x_*} : (C^{\mathrm{c}})^{op} & \longrightarrow & SSet_{\mathbb{U}}, \\
y & \mapsto & \mathrm{diag}(Hom(\Gamma(y), x_*)).
\end{array}
$$

The augmentation $\Gamma(-) \longrightarrow c(-)$ and co-augmentation $x \longrightarrow x_*$ induce a commutative diagram in $((C, S)^{\mathrm{cf}})^{\wedge}$

$$\begin{array}{ccc} h_x & \xrightarrow{a} & \underline{h}_x \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle d} \\ h_{x_*} & \xrightarrow{c} & \underline{h}_{x_*} \end{array}$$

By the properties of mapping spaces (see [Ho, Section 5.4]), both morphisms c and d are equivalences in $SPr(C^{\mathrm{c}})$. Furthermore, the morphism $h_x \longrightarrow h_{x_*}$ is isomorphic in $\mathrm{Ho}(SPr(C^{\mathrm{c}}))$ to the induced morphism $h_x \longrightarrow \mathrm{hocolim}_{[n]\in\Delta} h_{x_n}$. As each morphism $h_x \longrightarrow h_{x_n}$ is an equivalence in $((C, S)^{\mathrm{c}})^{\wedge}$, this implies that d is an equivalence in $((C, S)^{\mathrm{c}})^{\wedge}$. We deduce from this that also the natural morphism $h_x \longrightarrow \underline{h}_x$ is an equivalence in $((C, S)^{\mathrm{c}})^{\wedge}$. Let us show how this implies that for any $x \in C$, the natural morphism $h_x \longrightarrow \underline{h}_{Rx}$ is an equivalence in $(C, S)^{\wedge}$.

Since for any equivalence $z \to z'$ in C, the induced map $h_z \to h_{z'}$ is an equivalence in $(C, S)^{\wedge}$ (see Remark 2.3.4), it is enough to show that, for any $x \in C$, the canonical map $h_{Rx} \longrightarrow \underline{h}_{Rx}$ is an equivalence. By the Yoneda lemma for $\mathrm{Ho}((C, S)^{\wedge})$, it is enough to show that the induced map $Hom_{\mathrm{Ho}((C,S)^{\wedge})}(\underline{h}_{Rx}, F) \to Hom_{\mathrm{Ho}((C,S)^{\wedge})}(h_{Rx}, F)$ is a bijection for any $F \in \mathrm{Ho}((C, S)^{\wedge})$. Now,

$$Hom_{\mathrm{Ho}((C,S)^{\wedge})}(G, F) \simeq \pi_0(\mathbb{R}_w\, \underline{Hom}(G, F))$$

for any G and F in $(C, S)^{\wedge}$, hence it is enough to show that we have an induced equivalence of simplicial sets

$$\mathbb{R}_w\, \underline{Hom}(\underline{h}_{Rx}, F)) \simeq \mathbb{R}_w\, \underline{Hom}(h_x, F).$$

By the properties of mapping spaces (see [Ho, Section 5.4]), if Q denotes a cofibrant replacement functor in M, the map $\underline{h}_{Rx} \longrightarrow \underline{h}_{QRx}$ is an equivalence in $(C, S)^{\wedge}$; therefore, if we denote by $(-)_{\mathrm{c}}$ th restriction to C^{c}, we have an equivalence of simplicial sets

$$\mathbb{R}_w\, \underline{Hom}((\underline{h}_{Rx}, F)) \simeq \mathbb{R}_{w,\mathrm{c}}\, \underline{Hom}((\underline{h}_{QRx})_{\mathrm{c}}, F_{\mathrm{c}}).$$

Since $QR(x)$ is fibrant and cofibrant, we have already proved that

$$\mathbb{R}_{w,\mathrm{c}}\, \underline{Hom}((\underline{h}_{QRx})_{\mathrm{c}}, F_{\mathrm{c}}) \longrightarrow \mathbb{R}_{w,\mathrm{c}}\, \underline{Hom}((h_x)_{\mathrm{c}}, F_{\mathrm{c}})$$

is an equivalence of simplicial sets and we conclude since $\mathbb{R}_{w,\mathrm{c}}\, \underline{Hom}((h_x)_{\mathrm{c}}, F_{\mathrm{c}})) \simeq \mathbb{R}_w\, \underline{Hom}(h_x, F)$ by Proposition 4.1.5. $\square$

The main result of this subsection is the following theorem.

Theorem 4.2.3. *If* $(C, S, \imath : C \to M)$ *is a pseudo-model category, the functor* $\mathbb{R}\underline{h}$: $S^{-1}C \longrightarrow \mathrm{Ho}((C, S)^{\wedge})$ *is fully faithful.*

Proof. We will identify C as a full subcategory of M and $S^{-1}C$ as a full subcategory of $\mathrm{Ho}(M)$ using $\imath$. For any x and y in $S^{-1}C$, letting R be a fibrant replacement functor in M, one has

$$Hom_{S^{-1}C}(x, y) \simeq \pi_0(Hom_M(\Gamma(x), R(y))$$

since $\mathrm{Ho}(\imath)$ is fully faithful and $Hom(\Gamma(-), R(-))$ is a homotopy mapping complex in M (see [Ho, 5.4]). As $(C, S, \imath)$ is a pseudo-model category, we have $Hom_M(\Gamma(x)), R(y)) = Hom_{\mathrm{c}}(\Gamma(x), R(y))$. But, by definition of $\underline{h}$ and the enriched Yoneda lemma in $(C, S)^{\wedge}$, we have isomorphisms of simplicial sets

$$Hom_{\mathrm{c}}(\Gamma(x), R(y)) \simeq \underline{h}_{R(y)}(x) \simeq \underline{Hom}_{(C,S)^{\wedge}}(h_x, \underline{h}_{R(y)}).$$

Now, h_x is cofibrant in $(C, S)^{\wedge}$ and, by Lemma 4.2.1, $\underline{h}_{R(y)}$ is fibrant in $(C, S)^{\wedge}$, so that

$$\pi_0(\underline{Hom}_{(C,S)^{\wedge}}(h_x, \underline{h}_{R(y)})) \simeq Hom_{\mathrm{Ho}((C,S)^{\wedge})}(h_x, \underline{h}_{R(y)})$$

since $(C, S)^{\wedge}$ is a simplicial model category. Finally, by Lemma 4.2.2 we have

$$Hom_{\mathrm{Ho}((C,S)^{\wedge})}(h_x, \underline{h}_{R(y)}) \simeq Hom_{\mathrm{Ho}((C,S)^{\wedge})}(\mathbb{R}\underline{h}_x, \mathbb{R}\underline{h}_y)$$

showing that $\mathbb{R}\underline{h}$ is fully faithful. □

Corollary 4.2.4. *For any* $x \in C$ *and any* $F \in SPr(C)$, *there is an isomorphism in* $\mathrm{Ho}(SSet)$

$$\mathbb{R}_w \underline{Hom}_{(C,S)^{\wedge}}(\underline{h}_x, F) \simeq F(x).$$

Definition 4.2.5. For any pseudo-model category $(C, S, \imath)$ which is $\mathbb{U}$-small, the fully faithful embedding

$$\mathbb{R}\underline{h} : \mathrm{Ho}(C, S) \longrightarrow \mathrm{Ho}((C, S)^{\wedge})$$

is called the *Yoneda embedding* of $(C, S, \imath)$.

Remark 4.2.6. 1. According to Definition 4.2.5, the Yoneda embedding of a pseudo-model category a priori depends on the embedding $\imath : C \hookrightarrow M$. However, it will be shown in 4.7.3 that it *only* depends on the pair (C, S).

2. The Yoneda embedding for (pseudo-)model categories is one of the key ingredients used in [To-Ve 2] to prove that, for a large class of Waldhausen categories, the K-theory only depends on the Dwyer–Kan simplicial localization (though it is known to depend on strictly more than the usual localization).

4.3. Model pre-topologies and pseudo-model sites

Definition 4.3.1. A *model pre-topology* τ on a $\mathbb{U}$-small pseudo-model category $(C, S, \imath)$, is the datum for any object $x \in C$, of a set $Cov_{\tau}(x)$ of subsets of objects in $\mathrm{Ho}(C, S)/x$, called *τ-covering families* of x, satisfying the following three conditions.

1. (*Stability*) For all $x \in C$ and any isomorphism $y \to x$ in $\mathrm{Ho}(C, S)$, the one-element set $\{y \to x\}$ is in $Cov_{\tau}(x)$.
2. (*Composition*) If $\{u_i \to x\}_{i \in I} \in Cov_{\tau}(x)$, and for any $i \in I$, $\{v_{ij} \to u_i\}_{j \in J_i} \in Cov-\tau(u_i)$, the family $\{v_{ij} \to x\}_{i \in I, j \in J_i}$ is in $Cov_{\tau}(x)$.
3. (*Homotopy base change*) Assume the two previous conditions hold. For any $\{u_i \to x\}_{i \in I} \in Cov_{\tau}(x)$, and any morphism in $\mathrm{Ho}(C, S)$, $y \to x$, the family $\{u_i \times^h_x y \to y\}_{i \in I}$ is in $Cov_{\tau}(y)$.

A $\mathbb{U}$-small pseudo-model category $(C, S, \imath)$ together with a model pre-topology τ will be called a $\mathbb{U}$-small *pseudo-model site*.

Remark 4.3.2. 1. Note that in the third condition (*Homotopy base-change*) we used the homotopy fibered product of diagrams $x \longrightarrow z \longleftarrow y$ in $\mathrm{Ho}(M)$. By this we mean the homotopy fibered product of a lift (up to equivalence) of this diagram to M. This is a well-defined object in $\mathrm{Ho}(M)$ but only up to a *non-canonical* isomorphism in $\mathrm{Ho}(M)$ (in particular it is not functorially defined). However, condition (3) of the previous definition still makes sense because we assumed the two previous conditions (1) and (2) hold.

2. When the pseudo-model structure on (C, S) is trivial as in Remark 4.1.2 2, a model pre-topology on (C, S) is the same thing as a Grothendieck pre-topology on the category C as defined in [SGA4-I, Exp. II]. Indeed, in this case we have a canonical identification $\mathrm{Ho}(C, S) = C$ under which homotopy fibered products correspond to fibered products.

Let $(C, S, \imath; \tau)$ be a $\mathbb{U}$-small pseudo-model site and $\mathrm{Ho}(C, S) = S^{-1}C$ the homotopy category of (C, S). A sieve R in $\mathrm{Ho}(C, S)$ over an object $x \in \mathrm{Ho}(C, S)$ will be called a *τ-covering sieve* if it contains a τ-covering family.

Lemma 4.3.3. *For any $\mathbb{U}$-small pseudo-model site $(C, S, \imath; \tau)$, the τ-covering sieves form a Grothendieck topology on* $\mathrm{Ho}(C, S)$.

Proof. The stability and composition axioms of Definition 4.3.1 clearly imply conditions (i′) and (iii′) of [M-M, Chapter III, Section 2, Definition 2]. It is also clear that if $u : y \to x$ is any morphism in $\mathrm{Ho}(C, S)$, and if R is a sieve on x which contains a τ-covering family $\{u_i \to x\}_{i \in I}$, then the pull-back sieve $u^*(R)$ contains the family

$\{u_i \times^h_x y \rightarrow y\}_{i\in I}$. Therefore, the homotopy base change axiom of Definition 4.3.1 implies condition (ii') of [M-M, Chapter III, Section 2, Definition 2]. $\Box$

The previous lemma shows that any ($\mathbb{U}$-small) pseudo-model site $(C, S, \iota\ \tau)$ gives rise to a ($\mathbb{U}$-small) S-site $(L(C, S), \tau)$, where $\mathrm{L}(C, S)$ is the Dwyer–Kan localization of C with respect to S and τ is the Grothendieck topology on $\mathrm{Ho}(\mathrm{L}(C, S)) = \mathrm{Ho}(C, S)$ defined by τ-covering sieves. We will say that the S-topology τ on $\mathrm{L}(C, S)$ is *generated* by the pre-topology τ on (C, S).

Conversely, a topology on $\mathrm{Ho}(C, S)$ induces a model pre-topology on the pseudo-model category (C, S, ι) in the following way. A subset of objects $\{u_i \rightarrow x\}_{i\in I}$ in $\mathrm{Ho}(C, S)/x$ is defined to be a τ-covering family if the sieve it generates is a covering sieve (for the given topology on $\mathrm{Ho}(C, S)$).

Lemma 4.3.4. *Let (C, S, ι) be a $\mathbb{U}$-small pseudo-model category and let τ be a Grothendieck topology on $\mathrm{Ho}(C, S)$. Then, the τ-covering families in $\mathrm{Ho}(C, S)$ defined above form a model pre-topology on (C, S, ι), called the* induced *model pre-topology*.

Proof. Conditions (1) and (2) of Definition 4.3.1 are clearly satisfied and it only remains to check condition (3). For this, let us recall that the homotopy fibered products have the following semi-universal property in $\mathrm{Ho}(C, S)$. For any commutative diagram in $\mathrm{Ho}(C, S)$

$$\begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & & \downarrow \\ z & \longrightarrow & t, \end{array}$$

there exists a morphism $x \rightarrow z \times^h_t y$ compatible with the two projections to z and y. Using this property one sees that for any subset of objects $\{u_i \rightarrow x\}_{i\in I}$ in $\mathrm{Ho}(C, S)/x$, and any morphism $u : y \rightarrow x$, the sieve over y generated by the family $\{u_i \times^h_x y \rightarrow y\}_{i\in I}$ coincides with the pull-back by u of the sieve generated by $\{u_i \rightarrow x\}_{i\in I}$. Therefore, the base change axiom (ii′) of [M-M, Chapter III, Section 2, Definition 2] implies the homotopy base change property (3) of Definition 4.3.1. $\Box$

Lemmas 4.3.3 and 4.3.4 show that model pre-topologies on a pseudo-model category (C, S) are essentially the same as Grothendieck topologies on $\mathrm{Ho}(C, S)$, and therefore the same thing as S-topologies on the S-category $\mathrm{L}(C, S)$. As in the usual case (i.e. for the trivial model structure on (C, S)) the two above constructions are not exactly mutually inverse but we have the following

Proposition 4.3.5. *Let (C, S, ι) be a pseudo-model category. The rule assigning to a model pre-topology τ on (C, S, ι) the S-topology on $\mathrm{L}(C, S)$ generated by τ and the rule assigning to an S-topology on $\mathrm{L}(C, S)$ the induced model pre-topology on (C, S, ι),* *induce a bijection*

$$\left\{ \begin{array}{c} \text{Saturated model} \\ \text{pre-topologies on } (C, S, \iota) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} S\text{-topologies} \\ \text{on } \mathrm{L}(C, S) \end{array} \right\}$$

where we call a model pretopology τ saturated if any family of morphisms in $\mathrm{Ho}(C, S)/x$ *that contains a τ-covering family for x is again a τ-covering family for x.*

Proof. Straightforward from Lemma 4.3.3 and 4.3.4. □

Example 4.3.6. 1. *Topological spaces.* Let us take as $C = M$ the model category of $\mathbb{U}$-topological spaces, *Top*, with $S = W$ consisting of the usual weak equivalences. We define a model pre-topology τ in the following way. A family of morphism in $\mathrm{Ho}(Top)$, $\{X_i \to X\}_{i \in I}$, $I \in \mathbb{U}$, is defined to be in $Cov_\tau(X)$ if the induced map $\coprod_{i \in I} \pi_0(X_i) \longrightarrow \pi_0(X)$ is surjective. The reader will check easily that this defines a topology on *Top* in the sense of Definition 4.3.1.

2. *Strong model pre-topologies for E_∞-algebras over k.* Let k be a commutative ring (respectively, a field of characteristic zero) and let $C = M := (E_\infty - \mathrm{Alg}_k)^{op}$ (resp. $C = M := (\mathrm{CDGA}_{\leq 0;\, k})^{op}$) be the opposite model category of E_∞-algebras over the category of (unbounded) complexes of k-modules (resp., the opposite model category of commutative and unital differential graded k-algebras in negative degrees) which belong to $\mathbb{U}$; see for example [Bo-Gu, n] for a description of these model structures. Let τ be one of the usual topologies defined on k-schemes (e.g. Zariski, Nisnevich, étale, ffpf or ffqc). Let us define the *strong topology* τ_{str} on M in the sense of Definition 4.3.1, as follows. A family of morphisms in $\mathrm{Ho}(M^{op})$, $\{B \to A_i\}_{i \in I}$, $I \in \mathbb{U}$, is defined to be in $Cov_{\tau_{\mathrm{str}}}(B)$ if it satisfies the two following conditions.

- The induced family of morphisms of affine k-schemes $\{Spec\, H^0(A_i) \to Spec\, H^0(B)\}_{i \in I}$ is a τ-covering.
- For any $i \in I$, one has $H^*(A_i) \simeq H^*(B) \otimes_{H^0(B)} H^0(A_i)$.

In the case of negatively graded commutative differential graded algebras over a field of characteristic zero, the strong étale topology $(\acute{e}t)_{\mathrm{str}}$ has been considered in [Be]. We will use these kind of model pre-topologies in [To-Ve 6] to give another approach to the theory of DG-schemes of [ck1, Ci-Ka2] (or, more generally, to the theory of E_∞-schemes, when the base ring is not a field of characteristic zero) by viewing them as *geometric stacks over the category of complexes of k-modules*.

3. *Semi-strong model pre-topologies for E_∞-algebras over k.* With the same notations as in the previous example, we define the *semi-strong topology* τ_{sstr} on M by stipulating that a family of morphisms in $\mathrm{Ho}(M^{op})$, $\{B \to A_i\}_{i \in I}$, $I \in \mathbb{U}$, is in $Cov_{\tau_{\mathrm{sstr}}}(B)$ if the induced family of morphisms of affine k-schemes

$$\{Spec\, H^*(A_i) \to Spec\, H^*(B)\}_{i \in I}$$

is a τ-covering.

4. *The $Tor_{\geqslant 0}$ model pre-topology for E_∞-algebras over k.* Let k be a commutative ring and $C = M := (E_\infty - \mathrm{Alg}_k)^{op}$ be the opposite model category of E_∞-algebras

over the category of (unbounded) complexes of *k*-modules which belong to $\mathbb{U}$. For any E_∞-algebra *A*, we denote by Mod_A the model category of *A*-modules (see [Hin] or [Sp]). We define the *positive Tor-dimension* pre-topology, $Tor_{\geqslant 0}$, on *M*, as follows. A family of morphisms in $\mathrm{Ho}(M^{op})$, $\{f_i : B \to A_i\}_{i\in I}$, $I \in \mathbb{U}$, is defined to be in $Cov_{Tor_{\geqslant 0}}(B)$ if it satisfies the two following conditions:

- For any $i \in I$, the derived base-change functor $\mathbb{L}f_i^* = - \otimes^{\mathbb{L}}_B A_i : \mathrm{Ho}(\mathrm{Mod}_B) \longrightarrow \mathrm{Ho}(\mathrm{Mod}_{A_i})$ preserves the subcategories of *positive modules* (i.e. of modules *P* such that $H^i(P) = 0$ for any $i \leqslant 0$).
- The family of derived base-change functors

$$\{\mathbb{L}f_i^* : \mathrm{Ho}(\mathrm{Mod}_B) \longrightarrow \mathrm{Ho}(\mathrm{Mod}_{A_i})\}_{i\in I}$$

  is conservative (i.e. a morphism in $\mathrm{Ho}(\mathrm{Mod}_B)$ is an isomorphism if and only if, for any $i \in I$, its image in $\mathrm{Ho}(\mathrm{Mod}_{A_i})$ is an isomorphism).

This positive *Tor*-dimension pre-topology is particularly relevant in interpreting *higher tannakian duality* ([To1]) as a part of algebraic geometry over the category of unbounded complexes of *k*-modules. We will come back on this in [To-Ve 6].

We fix a model pre-topology τ on a pseudo-model category (C, S, ι) and consider the pseudo-model site $(C, S, \iota; \tau)$. The induced Grothendieck topology on $\mathrm{Ho}(C, S)$ described in the previous paragraphs will still be denoted by τ.

Let $F \in (C, S)^\wedge$ be a pre-stack on the pseudo-model site $(C, S, \iota; \tau)$, and let $F \to RF$ be a fibrant replacement of *F* in $(C, S)^\wedge$. We may consider the presheaf of connected components of *RF*, defined as

$$\begin{array}{rcl} \pi_0^{pr}(RF) : C^{op} & \longrightarrow & Set, \\ x & \mapsto & \pi_0(RF(x)). \end{array}$$

Since any other fibrant model of *F* in $(C, S)^\wedge$ is actually objectwise equivalent to *RF*, the presheaf $\pi_0^{pr}(RF)$ is well-defined up to a unique isomorphism. This defines a functor

$$\begin{array}{rcl} \pi_0^{eq} : (C, S)^\wedge & \longrightarrow & Pr(C), \\ F & \mapsto & \pi_0^{pr}(RF). \end{array}$$

As *RF* is fibrant, it sends equivalences in *C* to equivalences of simplicial sets, hence the presheaf $\pi_0^{eq}(F)$ always sends equivalences in *C* to isomorphisms, so it factors through $\mathrm{Ho}(C, S)^{op}$. Again, this defines a functor

$$\begin{array}{rcl} \pi_0^{eq} : (C, S)^\wedge & \longrightarrow & Pr(\mathrm{Ho}(C, S)), \\ F & \mapsto & \pi_0^{eq}(F). \end{array}$$

Finally, if $F \longrightarrow G$ is an equivalence in $(C, S)^\wedge$, the induced morphism $RF \longrightarrow RG$ is an objectwise equivalence, and therefore the induced morphism $\pi_0^{eq}(F) \longrightarrow \pi_0^{eq}(G)$

is an isomorphism of presheaves of sets. In other words, the functor π_0^{eq} factors through the homotopy category $\mathrm{Ho}((C,S)^\wedge)$ as

$$\pi_0^{eq} : \mathrm{Ho}((C,S)^\wedge) \longrightarrow Pr(\mathrm{Ho}(C,S)).$$

Definition 4.3.7. Let $(C,S,\iota;\tau)$ be a pseudo-model site in $\mathbb{U}$.

1. For any object $F \in (C,S)^\wedge$, the sheaf associated to the presheaf $\pi_0^{eq}(F)$ is denoted by $\pi_0^\tau(F)$ (or $\pi_0(F)$ if the topology τ is clear from the context). It is a usual sheaf on the site $(\mathrm{Ho}(C,S),\tau)$, and is called the *sheaf of connected components* of F;

2. A morphism $f : F \longrightarrow G$ in $\mathrm{Ho}((C,S)^\wedge)$ is called a *τ-covering* (or just a *covering* if the topology τ is clear from the context) if the induced morphism of presheaves $\pi_0^{eq}(F) \longrightarrow \pi_0^{eq}(G)$ induces an epimorphism of sheaves on $\mathrm{Ho}(C,S)$ for the topology τ;

3. A morphism $F \longrightarrow G$ in $(C,S)^\wedge$ is called a *τ-covering* (or just a *covering* if the topology τ is clear) if the induced morphism in $\mathrm{Ho}((C,S)^\wedge)$ is a *τ-covering* according to the previous definition.

Coverings in the model category $(C,S)^\wedge$ behave exactly as coverings in the model category of pre-stacks over an S-site (see Section 3.1). It is easy to check (Proposition 3.1.4) that a morphism $F \longrightarrow G$ between fibrant objects in $(C,S)^\wedge$ is a τ-covering iff for any object $x \in C$ and any morphism $h_x \longrightarrow G$ in $(C,S)^\wedge$, there exists a covering family $\{u_i \to x\}_{i\in I}$ in C (meaning that its image in $\mathrm{Ho}(C,S)$ is a τ-covering family), and for each $i \in I$, a commutative diagram in $\mathrm{Ho}((C,S)^\wedge)$

$$\begin{array}{ccc} F & \longrightarrow & G \\ \uparrow & & \uparrow \\ h_{u_i} & \longrightarrow & h_x \end{array}$$

Moreover, we have the following analog of Proposition 3.1.6.

Proposition 4.3.8. *Let $(C,S,\iota;\tau)$ be a pseudo-model site.*

1. *A morphism in $SPr(T)$ which is a composition of coverings is a covering.*
2. *Let*

$$\begin{array}{ccc} F' & \xrightarrow{f'} & G' \\ \downarrow & & \downarrow \\ F & \xrightarrow[f]{} & G \end{array}$$

be a homotopy cartesian diagram in $(C,S)^\wedge$. If f is a covering then so is f'.

3. *Let* $F \xrightarrow{u} G \xrightarrow{v} H$ *be two morphisms in* $(C, S)^{\wedge}$. *If the morphism* $v \circ u$ *is a covering then so is* v.
4. *Let*

$$\begin{array}{ccc} F' & \xrightarrow{f'} & G' \\ \downarrow & & \downarrow p \\ F & \xrightarrow[f]{} & G \end{array}$$

be a homotopy cartesian diagram in $(C, S)^{\wedge}$. *If* p *and* f' *are coverings then so is* f.

Proof. Easy exercise left to the reader. □

4.4. Simplicial objects and hypercovers

In this subsection we fix a pseudo-model site $(C, S, \iota; \tau)$ in $\mathbb{U}$ and keep the notations of Section 3.2, with $SPr(T)$ replaced by $(C, S)^{\wedge}$; more precisely we take $T = L(C^{op}, S^{op})$ (with the induced S-topology, see Proposition 4.3.5) and use Theorem 2.3.5 with $M = SSet$ to have definitions and results of Section 3.2 available for $(C, S)^{\wedge} = SSet_{\mathbb{U}}^{C^{op}, S^{op}}$.

We introduce a nice class of hypercovers that will be used in the proof of the existence of the local model structure; this class will replace our distinguished set of hypercovers H used in the proof of Theorem 3.4.1.

Definition 4.4.1. 1. An object $F \in (C, S)^{\wedge}$ is called *pseudo-representable* if it is a $\mathbb{U}$-small disjoint union of representable presheaves

$$F \simeq \coprod_{u \in I} h_u.$$

2. A morphism between pseudo-representable objects

$$f : \coprod_{u \in I} h_u \longrightarrow \coprod_{v \in J} h_v$$

is called a *pseudo-fibration* if for all $u \in I$, the corresponding projection

$$f \in \prod_{u \in I} \coprod_{v \in J} Hom(h_u, h_v) \longrightarrow \coprod_{v \in J} Hom(h_u, h_v) \simeq \coprod_{v \in J} Hom_{\mathrm{C}}(u, v)$$

is represented by a fibration in C.

Let

$$f : \coprod_{u \in I} h_u \longrightarrow \coprod_{v \in J} h_v$$

be a morphism between pseudo-representable objects, and for any $v \in J$ let I_v be the sub-set of I of components h_u which are sent to h_v. The morphism is called a *pseudo-covering* if for any $v \in J$, the family of morphisms

$$\{h_u \to h_v\}_{u \in I_v}$$

corresponds to a covering family in the pseudo-model site (C, S).

4. Let x be a fibrant object in C. A *pseudo-representable hypercover* of x is an object $F_* \longrightarrow h_x$ in $s(C,S)^\wedge/h_x$ such that for any integer $n \geqslant 0$ the induced morphism

$$F_n \longrightarrow F_*^{\partial\Delta^n} \times_{h_x^{\partial\Delta^n}} h_x^{\Delta^n}$$

is a pseudo-fibration and a pseudo-covering between pseudo-representable objects.

The first thing to check is that pseudo-representable hypercovers are hypercovers.

Lemma 4.4.2. *A pseudo-representable hypercover $F_* \longrightarrow h_x$ is a τ-hypercover* (*see Definition* 3.2.3).

Proof. It is enough to check that the natural morphism

$$F_*^{\partial\Delta^n} \times_{h_x^{\partial\Delta^n}} h_x^{\Delta^n} \longrightarrow F_*^{\mathbb{R}\partial\Delta^n} \times^h_{h_x^{\mathbb{R}\partial\Delta^n}} h_x^{\mathbb{R}\Delta^n}$$

is an isomorphism in $\mathrm{Ho}((C,S)^\wedge)$. But this follows from the fact that h preserves finite limits (when they exists) and the fact that (C,S) is a pseudo-model category. □

4.5. Local equivalences

This subsection is completely analogous (actually a bit easier, because the notion of comma site is completely harmless here) to Section 3.3.

Let $(C, S, \imath; \tau)$ be a $\mathbb{U}$-small pseudo-model site, and x be a fibrant object in C. The comma category $(C/x, S, \imath)$ is then endowed with its natural structure of a pseudo-model category. The underlying category is C/x, the category of objects over x. The equivalences S in C/x are simply the morphisms whose images in C are equivalences. Finally, the embedding $\imath : C \longrightarrow M$ induces an embedding $\imath : C/x \longrightarrow M/\imath(x)$. The comma category $M/\imath(x)$ is endowed with its natural model category structure (see [Ho, Section 1]). It is easy to check that $(C/x, S, \imath)$ is a pseudo-model category in the sense of Definition 4.1.1.

We define a model pre-topology, still denoted by τ, on the comma pseudo-model category $(C/x, S, \imath)$ by declaring that a family $\{u_i \to y\}_{i \in I}$ of objects in $\mathrm{Ho}((C/x, S))/y$ is a τ-covering family if its image family under the natural functor $\mathrm{Ho}((C/x, S))/y \longrightarrow \mathrm{Ho}((C, S))/y$ is a τ-covering family for y. As the object x is fibrant in (C, S) the forgetful functor $(C/x, S) \longrightarrow (C, S)$ preserves homotopy fibered products, and therefore one checks immediately that this defines a model pre-topology τ on $(C/x, S, \imath)$.

Definition 4.5.1. The pseudo-model site $(C/x, S, \imath; \tau)$ will be called the *comma pseudo-model site* of $(C, S, \imath; \tau)$ over the (fibrant) object x.

Remark 4.5.2. Note that in the case where $(C, S, \imath)$ is a right proper pseudo-model category, the hypothesis that x is fibrant is unnecessary.

For any object $x \in C$, the evaluation functor

$$\begin{array}{rccc} j_x^* : & (C, S)^\wedge & \longrightarrow & SSet_{\mathbb{U}}, \\ & F & \mapsto & F(x) \end{array}$$

has a left adjoint $(j_x)_!$. The adjunction

$$(j_x)_! : SSet_{\mathbb{U}} \longrightarrow (C, S)^\wedge \quad SSet_{\mathbb{U}} \longleftarrow (C, S)^\wedge : j_x^*$$

is clearly a Quillen adjunction.

Let $F \in (C, S)^\wedge$, x a fibrant object in (C, S) and $s \in \pi_0^{eq}(F(x))$ be represented by a morphism $s : h_x \longrightarrow F$ in $\mathrm{Ho}((C, S)^\wedge)$. By pulling-back this morphism through the functor

$$\mathbb{R}j_x^* : \mathrm{Ho}((C, S)^\wedge) \longrightarrow \mathrm{Ho}((C/x, S)^\wedge)$$

one gets a morphism in $\mathrm{Ho}((C/x, S)^\wedge)$

$$s : \mathbb{R}j_x^*(h_x) \longrightarrow \mathbb{R}j_x^*(F).$$

By definition of the comma pseudo-model category $(C/x, S)$, it is immediate that $\mathbb{R}j_x^*(h_x)$ has a natural global point $* \longrightarrow \mathbb{R}j_x^*(h_x)$ in $\mathrm{Ho}((C/x, S)^\wedge)$. Observe that the morphism $* \longrightarrow \mathbb{R}j_x^*(h_x)$ can also be seen as induced by adjunction from the identity of $h_x \simeq \mathbb{L}(j_x)_!(*)$. We therefore obtain a global point

$$s : * \longrightarrow \mathbb{R}j_x^*(h_x) \longrightarrow \mathbb{R}j_x^*(F).$$

Definition 4.5.3. 1. For an integer $n > 0$, the sheaf $\pi_n(F, s)$ is defined to be

$$\pi_n(F, s) := \pi_0(\mathbb{R}j_x^*(F)^{\mathbb{R}\Delta^n} \times_{\mathbb{R}j_x^*(F)^{\mathbb{R}\partial\Delta^n}} *).$$

It is a usual sheaf on the site $(\mathrm{Ho}(C/x, S), \tau)$ called the *nth homotopy sheaf* of F *pointed at s*.

A morphism $f : F \longrightarrow G$ in $(C, S)^{\wedge}$ is called a π_*-*equivalence* (or equivalently a *local equivalence*) if the following two conditions are satisfied:

1. The induced morphism $\pi_0(F) \longrightarrow \pi_0(G)$ is an isomorphism of sheaves on $\mathrm{Ho}(C, S)$;
2. For any fibrant object $x \in (C, S)$, any section $s \in \pi_0^{eq}(F(x))$ and any integer $n > 0$, the induced morphism $\pi_n(F, s) \longrightarrow \pi_n(G, f(s))$ is an isomorphism of sheaves on $\mathrm{Ho}(C/x, S)$.

As observed in Section 3.3, an equivalence in the model category $(C, S)^{\wedge}$ is always a π_*-equivalence, for any model pre-topology τ on (C, S).

The π_*-equivalences in $(C, S)^{\wedge}$ behave the same way as the π_*-equivalences in $SPr(T)$ (see Section 3.3). We will therefore state the following basic facts without repeating their proofs.

Lemma 4.5.4. *A morphism $f : F \longrightarrow G$ in $(C, S)^{\wedge}$ is a π_*-equivalence if and only if for any $n \geqslant 0$, the induced morphism*

$$F^{\mathbb{R}\Delta^n} \longrightarrow F^{\mathbb{R}\partial\Delta^n} \times^h_{G^{\mathbb{R}\partial\Delta^n}} G^{\mathbb{R}\Delta^n}$$

is a covering. In other words, f is a π_*-equivalence if and only if it is a τ-hypercover when considered as a morphism between constant simplicial objects in $(C, S)^{\wedge}$.

Corollary 4.5.5. *Let $f : F \longrightarrow G$ be a morphism in $(C, S)^{\wedge}$ and $G' \longrightarrow G$ be a covering. Then, if the induced morphism*

$$f' : F \times^h_G G' \longrightarrow G'$$

is a π_*-equivalence, so is f.

Let $f : F \longrightarrow G$ be a morphism in $(C, S)^{\wedge}$. For any fibrant object $x \in (C, S)$ and any morphism $s : h_x \longrightarrow G$ in $\mathrm{Ho}((C, S)^{\wedge})$, let us define $F_s \in \mathrm{Ho}(((C, S)/x)^{\wedge})$ via the following homotopy cartesian square:

$$\begin{array}{ccc} \mathbb{R}j_x^*(F) & \xrightarrow{\mathbb{R}j_x^*(f)} & \mathbb{R}j_x^*(G) \\ \uparrow & & \uparrow \\ F_s & \longrightarrow & * \end{array}$$

where the morphism $* \longrightarrow \mathbb{R}j_x^*(G)$ is adjoint to the morphism $s : \mathbb{L}(j_x)_!(*) \simeq h_x \longrightarrow G$.

Corollary 4.5.6. *Let $f : F \longrightarrow G$ be a morphism in $(C, S)^{\wedge}$. With the same notations as above, the morphism f is a π_*-equivalence if and only if for any $s : h_x \longrightarrow G$ in* $\mathrm{Ho}((C, S)^{\wedge})$, *the induced morphism $F_s \longrightarrow *$ is a π_*-equivalence in* $\mathrm{Ho}(((C, S)/x)^{\wedge})$.

Proposition 4.5.7. *Let $f : F \longrightarrow G$ be a π_*-equivalence in $(C, S)^{\wedge}$ and $F \longrightarrow F'$ be an objectwise cofibration (i.e. a monomorphism). Then, the induced morphism*

$$f' : F' \longrightarrow F' \coprod_{\mathrm{f}} G'$$

is a π_*-equivalence.

Proof. As $F \longrightarrow F'$ is an objectwise monomorphism, $F' \coprod_{\mathrm{f}} G'$ is a homotopy coproduct in $SPr(C)$, and therefore in $(C, S)^{\wedge}$. One can therefore replace F, G and F' by their fibrant models in $(C, S)^{\wedge}$ and suppose therefore that they preserve equivalences. The proof is then the same as in [Ja1, Proposition 2.2]. □

4.6. The local model structure

The following result is completely similar to Theorem 3.4.1, also as far as the proof is concerned. Therefore we will omit to repeat the complete proof below, only mentioning how to replace the set H used in the proof of Theorem 3.4.1.

Theorem 4.6.1. *Let $(C, S, \imath; \tau)$ be a pseudo-model site. There exists a closed model structure on $SPr(C)$, called the* local projective model structure, *for which the equivalences are the π_*-equivalences and the cofibrations are the cofibrations for the projective model structure on $(C, S)^{\wedge}$. Furthermore, the local projective model structure is $\mathbb{U}$-combinatorial and left proper.*

The category $SPr(C)$ together with its local projective model structure will be denoted by $(C, S)^{\sim,\tau}$.

Proof. It is essentially the same as the proof of 3.4.1. We will however give the set of morphism H that one needs to use. We choose α to be a $\mathbb{U}$-small cardinal which is bigger than the cardinality of the set of morphisms in C and than $\aleph_0$. Let β be a $\mathbb{U}$-small cardinal such that $\beta > 2^{\alpha}$.

For a fibrant object $x \in C$, we consider a set $H_{\beta}(x)$, of representatives of the set of isomorphism classes of objects $F_* \longrightarrow h_x$ in $s(C, S)^{\wedge}/h_x$ satisfying the following two conditions:

1. The morphism $F_* \longrightarrow h_x$ is a pseudo-representable hypercover in the sense of Definition 4.4.1.
2. For all $n \geqslant 0$, one has $\mathrm{Card}(F_n) < \beta$.

We set $H = \coprod_{x \in C^{\mathrm{f}}} H_{\beta}(x)$, which is clearly a $\mathbb{U}$-small set.

The main point of the proof is then to check that equivalences in the left Bousfield localization $L_H(C, S)^{\wedge}$ are exactly local equivalences. The argument follows exactly the main line of the proof of Theorem 3.4.1 and we leave details to the interested reader. □

The following corollaries and definitions are the same as the ones following Theorem 3.4.1.

Corollary 4.6.2. *The model category* $(C, S)^{\sim,\tau}$ *is the left Bousfield localization of* $(C, S)^{\wedge}$ *with respect to the set of morphisms*

$$\{|F_*| \longrightarrow h_x | x \in Ob(C^{\mathrm{f}}),\ F_* \in H_\beta(x)\}.$$

Proof. This is exactly the way we proved Theorem 4.6.1. □

Corollary 4.6.3. *An object* $F \in (C, S)^{\sim,\tau}$ *is fibrant if and only if it is objectwise fibrant, preserves equivalences and satisfies the following hyperdescent condition*:
– *For any fibrant object* $x \in C$ *and any* $H_* \in H_\beta(x)$, *the natural morphism*

$$F(x) \simeq \mathbb{R}_w \underline{Hom}(h_x, F) \longrightarrow \mathbb{R}_w \underline{Hom}(|H_*|, F)$$

is an isomorphism in $\mathrm{Ho}(SSet)$.

Proof. This follows from Corollary 4.6.2 and from the explicit description of fibrant objects in a left Bousfield localization (see [Hi, Theorem 4.1.1]). □

Remark 4.6.4. As we did in Remark 3.4.6, we would like to stress here that the proof of Theorem 4.6.1 (i.e. of Theorem 3.4.1) proves actually both Theorem 4.6.1 and Corollary 4.6.2, in that it gives *two descriptions* of the same model category $(C, S)^{\sim,\tau}$: one as the left Bousfield localization of $(C, S)^{\wedge}$ with respect to *local equivalences* and the other as the left Bousfield localization of the same $(C, S)^{\wedge}$ but this time with respect to *hypercovers* (precisely, with respect to the set of morphisms defined in the statement of Corollary 4.6.2).

Definition 4.6.5. An object $F \in (C, S)^{\wedge}$ is said to *have hyperdescent* (or τ-hyperdescent if the topology τ has to be reminded) if for any fibrant object $x \in C$ and any pseudo-representable hypercover $H_* \longrightarrow h_x$, the induced morphism

$$F(x) \simeq \mathbb{R}_w \underline{Hom}(h_x, F) \longrightarrow \mathbb{R}_w \underline{Hom}(|H_*|, F)$$

is an isomorphism in $\mathrm{Ho}(SSet_{\mathbb{U}})$.

From now on, we will adopt the following terminology and notations.

Definition 4.6.6. Let $(C, S, \iota; \tau)$ be a pseudo-model site in $\mathbb{U}$.

- A *stack* on $(C, S, \iota; \tau)$ is a pre-stack $F \in (C, S)^{\wedge}$ that has τ-hyperdescent (Definition 4.6.5).

- The model category $(C,S)^{\sim,\tau}$ is called the *model category of stacks* on the pseudo-model site $(C,S,\iota;\tau)$. The category $\mathrm{Ho}((C,S)^\wedge)$ (resp. $\mathrm{Ho}((C,S)^{\sim,\tau})$) is called the *homotopy category of pre-stacks* (resp. the *homotopy category of stacks*). Objects of $\mathrm{Ho}((C,S)^\wedge)$ (resp. $\mathrm{Ho}((C,S)^{\sim,\tau})$) will simply be called *pre-stacks* on (C,S,ι) (resp., *stacks* on $(C,S,\iota;\tau)$). The functor $a:\mathrm{Ho}((C,S)^\wedge)\longrightarrow \mathrm{Ho}((C,S)^{\sim,\tau})$ will be called the *associated stack functor*.
- The topology τ is said to be *sub-canonical* if for any $x\in C$ the pre-stack $\mathbb{R}\underline{h}_x\in \mathrm{Ho}((C,S)^\wedge)$ is a stack (in other words if the Yoneda embedding $\mathbb{R}\underline{h}_x:\mathrm{Ho}(C,S)\longrightarrow \mathrm{Ho}((C,S)^\wedge)$ factors through the subcategory of stacks).
- For pre-stacks F and G on $(C,S,\iota;\tau)$, we will denote by $\mathbb{R}_w\underline{Hom}(F,G)\in \mathrm{Ho}(SSet_{\mathbb{U}})$ (resp. by $\mathbb{R}_{w,\tau}\underline{Hom}(F,G)\in \mathrm{Ho}(SSet_{\mathbb{U}})$) the simplicial derived *Hom*-simplicial set computed in the simplicial model category $(C,S)^\wedge$ (resp. $(C,S)^{\sim,\tau}$).

As $(C,S)^{\sim,\tau}$ is a left Bousfield localization of $(C,S)^\wedge$, the identity functor $(C,S)^\wedge\longrightarrow (C,S)^{\sim,\tau}$ is left Quillen and its right adjoint (which is still the identity functor) induces by right derivation a fully faithful functor

$$j:\mathrm{Ho}((C,S)^{\sim,\tau})\longrightarrow \mathrm{Ho}((C,S)^\wedge).$$

Furthermore, the essential image of this inclusion functor is exactly the full subcategory consisting of objects having the hyperdescent property. The left adjoint

$$a:\mathrm{Ho}((C,S)^\wedge)\longrightarrow \mathrm{Ho}((C,S)^{\sim,\tau})$$

to the inclusion j, is a left inverse to j.

We will finish this paragraph by the following proposition.

Proposition 4.6.7. 1. *Let F and G be two pre-stacks on $(C,S,\iota;\tau)$. If G is a stack then the natural morphism*

$$\mathbb{R}_w\,\underline{Hom}(F,G)\longrightarrow \mathbb{R}_{w,\tau}\,\underline{Hom}(F,G)$$

is an isomorphism in $\mathrm{Ho}(SSet)$.

2. *The functor* $\mathrm{Id}:(C,S)^\wedge\longrightarrow (C,S)^{\sim,\tau}$ *preserves homotopy fibered products.*

Proof. Condition (1) follows formally from Corollary 4.6.2 while (2) follows from Corollary 4.5.5. □

4.7. Comparison between the S-theory and the pseudo-model theory

In this subsection, we fix a pseudo-model category (C,S,ι) in $\mathbb{U}$, together with a pre-topology τ on it. The natural induced topology on $\mathrm{Ho}(C,S)$ will be denoted again by τ. We let T be $L(C,S)$, the simplicial localization of (C,S) along the set S of its equivalences. As $\mathrm{Ho}(T)=\mathrm{Ho}(C,S)$ (though the two $\mathrm{Ho}(-)$'s here have different meanings), the topology τ may also be considered as an S-topology on T. Therefore, we have on one side a pseudo-model site $(C, S, \iota; \tau)$, and on the other side an S-site (T, τ), and we wish to compare the two corresponding model categories of stacks.

Theorem 4.7.1. *The two model categories* $(C, S)^{\sim,\tau}$ *and* $SPr_\tau(T)$ *are Quillen equivalent.*

Proof. By Theorem 2.3.5, the model categories of pre-stacks $SPr(T)$ and $(C, S)^\wedge$ are Quillen equivalent. Furthermore, it is quite clear that through this equivalence the notions of local equivalences in $SPr(T)$ and $(C, S)^\wedge$ coincide. As the local model structures are both left Bousfield localizations with respect to local equivalences, this shows that this Quillen equivalence between $(C, S)^\wedge$ and $SPr(T)$ induces a Quillen equivalence on the model categories of stacks. □

Then, Corollaries 3.6.2 and 3.8.5 imply the following

Corollary 4.7.2. 1. *The model category* $(C, S)^{\sim,\tau}$ *is a t-complete* $\mathbb{U}$*-model topos.*
2. *The homotopy category* $\mathrm{Ho}((C, S)^{\sim,\tau})$ *is internal.*
3. *There exists an isomorphism of S-categories in* $\mathrm{Ho}(S-Cat_{\mathbb{U}})$

$$LSPr_\tau(T) \simeq L(C, S)^{\sim,\tau}.$$

Now we want to compare the two Yoneda embeddings (the simplicial one and the pseudo-model one). To do this, let us suppose now that the topology τ is *sub-canonical* so that the two Yoneda embeddings factor through the embeddings of the homotopy categories of stacks:

$$\mathbb{R}\underline{h} : \mathrm{Ho}(C, S) \longrightarrow \mathrm{Ho}((C, S)^{\sim,\tau}),$$
$$L\underline{h} : \mathrm{Ho}(T) \longrightarrow \mathrm{Ho}(Int(SPr_\tau(T))) \simeq \mathrm{Ho}(SPr_\tau(T)).$$

One has $\mathrm{Ho}(C, S) = \mathrm{Ho}(T)$, and Corollary 4.7.2 gives an equivalence of categories between $\mathrm{Ho}(SPr_\tau(T))$ and $\mathrm{Ho}((C, S)^{\sim,\tau})$.

Corollary 4.7.3. *The following diagram commutes up to an isomorphism*:

$$\begin{array}{ccc} \mathrm{Ho}(C, S) & \xrightarrow{\mathbb{R}\underline{h}} & \mathrm{Ho}((C, S)^{\sim,\tau}) \\ \sim\downarrow & & \downarrow\sim \\ \mathrm{Ho}(T) & \xrightarrow[L\underline{h}]{} & \mathrm{Ho}(SPr_\tau(T)). \end{array}$$

Proof. This follows from the fact that for any $x \in M$, one has natural isomorphisms

$$[\mathbb{R}\underline{h}_x, F]_{\mathrm{Ho}((C,S)^{\sim,\tau})} \simeq F(x) \simeq [L\underline{h}_x, F]_{\mathrm{Ho}(SPr_\tau(T))}.$$

This implies that $\mathbb{R}\underline{h}_x$ and $L\underline{h}_x$ are naturally isomorphic as objects in $\mathrm{Ho}((C,S)^\wedge)$. □

4.8. Functoriality

In this subsection, we state and prove in detail the functoriality results and some useful criteria for continuous morphisms and continuous equivalences between pseudo-model sites, in such a way that the reader only interested in working with stacks over pseudo-model sites will find here a more or less self-contained treatment. However, at the end of the subsection and in occasionally scattered remarks, we will also mention the comparison between functoriality on pseudo-model sites and the corresponding functoriality on the associated Dwyer–Kan localization S-sites.

Recall from Section 4.1 (or Section 2.3.2 before Corollary 2.3.6) that if (C,S) and (C',S') are categories with a distinguished subset of morphisms (e.g., pseudo-model categories) and $f : C \to C'$ is a functor sending S into S', we have a Quillen adjunction

$$f_! : (C,S)^\wedge \longrightarrow (C,S')^\wedge, \quad (C,S)^\wedge \longleftarrow (C',S')^\wedge : f^*.$$

If $(C,S,\imath)$ is a pseudo-model category, by Proposition 4.1.5, we have in particular the following Quillen equivalences

$$i_{\mathrm{c}}^* : (C,S)^\wedge \simeq ((C,S)^{\mathrm{c}})^\wedge, \quad i_{\mathrm{f}}^* : (C,S)^\wedge \simeq ((C,S)^{\mathrm{f}})^\wedge$$

$$i_{\mathrm{cf}}^* : (C,S)^\wedge \simeq ((C,S)^{\mathrm{cf}})^\wedge,$$

which will be useful to establish functorial properties of the homotopy category $\mathrm{Ho}((C,S)^\wedge)$. Indeed, if $f : (C,S) \longrightarrow (C',S')$ is a functor *such that* $f(S^{\mathrm{cf}}) \subset S'$ (e.g. a left or right Quillen functor), then f induces well-defined functors

$$\mathbb{R}f^* : \mathrm{Ho}((C',S')^\wedge) \longrightarrow \mathrm{Ho}(((C,S)^{\mathrm{cf}})^\wedge) \simeq \mathrm{Ho}((C,S)^\wedge),$$

$$\mathbb{L}f_! : \mathrm{Ho}((C,S)^\wedge) \simeq \mathrm{Ho}(((C,S)^{\mathrm{cf}})^\wedge) \longrightarrow \mathrm{Ho}((C',S')^\wedge).$$

The (derived) *inverse image* functor $\mathbb{R}f^*$ is clearly right adjoint to the (derived) *direct image* functor $\mathbb{L}f_!$.

The reader should be warned that the direct and inverse image functors are not, in general, functorial in f. However, the following proposition ensures in many cases the functoriality of these constructions.

Proposition 4.8.1. *Let* (C, S), (C', S') *and* (C'', S'') *be pseudo-model categories and*

$$(C, S) \xrightarrow{f} (C', S') \xrightarrow{g} (C'', S'')$$

be two functors preserving fibrant or cofibrant objects and equivalences between them. Then, there exist natural isomorphisms

$$\mathbb{R}(g \circ f)^* \simeq \mathbb{R}f^* \circ \mathbb{R}g^* : \mathrm{Ho}((C'', S'')^\wedge) \longrightarrow \mathrm{Ho}((C, S)^\wedge),$$

$$\mathbb{L}(g \circ f)_! \simeq \mathbb{L}g_! \circ \mathbb{L}f_! : \mathrm{Ho}((C, S)^\wedge) \longrightarrow \mathrm{Ho}((C'', S'')^\wedge).$$

These isomorphisms are furthermore associative and unital in the arguments f and g.

Proof. The proof is the same as that of the usual property of composition for derived Quillen functors (see [Ho, Theorem 1.3.7]), and is left to the reader. $\square$

Examples of pairs of functors to which the previous proposition applies are given by pairs of right or left Quillen functors.

Proposition 4.8.2. *If* $f : (C, S) \longrightarrow (C, S)$ *is a* (*right or left*) *Quillen equivalence between pseudo-model categories, then the induced functors*

$$\mathbb{L}f_! : \mathrm{Ho}((C, S)^\wedge) \longrightarrow \mathrm{Ho}((C', S')^\wedge) \quad \mathrm{Ho}((C, S)^\wedge) \longleftarrow \mathrm{Ho}((C', S')^\wedge) : \mathbb{R}f^*$$

are equivalences, quasi-inverse of each others.

Proof. This is a straightforward application of Corollary 2.3.6. $\square$

Let (C, S) and (C', S') be pseudo-model categories and let us consider a functor $f : C \longrightarrow C'$ such that $f(S^{\mathrm{cf}}) \subset S'$. We will denote by $f_{\mathrm{cf}} : (C, S) \longrightarrow (C', S')$ the composition

$$f_{\mathrm{cf}} : \quad (C, S) \xrightarrow{RQ} (C, S)^{\mathrm{cf}} \xrightarrow{f} (C', S'),$$

where R (respectively, Q) denotes the fibrant (resp., cofibrant) replacement functor in (C, S). We deduce an adjunction on the model categories of pre-stacks

$$(f_{\mathrm{cf}})_! : (C, S)^\wedge \longrightarrow (C', S')^\wedge \quad (C, S)^\wedge \longleftarrow (C', S')^\wedge : f_{\mathrm{cf}}^*.$$

Note that the right derived functor $\mathbb{R}f_{\mathrm{cf}}^*$ is isomorphic to the functor $\mathbb{R}f^*$ defined above.

Proposition 4.8.3. *Let $(C, S; \tau)$ and $(C', S'; \tau')$ be pseudo-model sites and $f : C \longrightarrow C'$ a functor such that $f(S^{\mathrm{cf}}) \subset S'$. Then the following properties are equivalent*:

1. *The right derived functor* $\mathbb{R}f^*_{\mathrm{cf}} \simeq \mathbb{R}f^* : \mathrm{Ho}((C', S')^\wedge) \to \mathrm{Ho}((C, S)^\wedge)$ *sends the subcategory* $\mathrm{Ho}((C', S')^{\sim,\tau'})$ *into the subcategory* $\mathrm{Ho}((C, S)^{\sim,\tau})$.
2. *If* $F \in (C', S')^\wedge$ *has* τ'*-hyperdescent, then* $f^*F \in SPr(C)$ *has* τ*-hyperdescent.*
3. *For any pseudo-representable hypercover* $H_* \longrightarrow h_x$ *in* $(C, S)^\wedge$ (*see Definition* 4.4.1), *the morphism*

$$\mathbb{L}(f_{\mathrm{cf}})_!(H_*) \longrightarrow \mathbb{L}(f_{\mathrm{cf}})_!(h_x) \simeq h_{f_{\mathrm{cf}}(x)}$$

is a local equivalence in $(C', S')^\wedge$.
4. *The functor* $f^*_{\mathrm{cf}} : (C', S')^{\sim,\tau} \longrightarrow (C, S)^{\sim,\tau}$ *is right Quillen.*

Proof. The equivalence between (1)–(3) follows immediately from the fact that fibrant objects in $(C, S)^{\sim,\tau}$ (resp. in $(C', S')^{\sim,\tau}$) are exactly those fibrant objects in $(C, S)^\wedge$ (resp. in $(C', S')^\wedge$) which satisfy τ-hyperdescent (resp. τ'-hyperdescent) (see Corollary 4.6.3). Finally, (4) and (2) are equivalent by adjunction. □

Definition 4.8.4. Let $(C, S; \tau)$ and $(C', S'; \tau')$ be pseudo-model sites. A functor $f : C \to C'$ such that $f(S^{\mathrm{cf}}) \subseteq S'$, is said to be *continuous* or *a morphism of pseudo-model sites*, if it satisfies one of the equivalent conditions of Proposition 4.8.3.

Remark 4.8.5. By the comparison Theorem 4.7.1, a functor $f : (C, S; \tau) \to (C', S'; \tau')$ such that $f(S^{\mathrm{cf}}) \subseteq S'$, is continuous if and only if the induced functor $(L(C, S), \tau) \simeq (L(C^{\mathrm{cf}}, S^{\mathrm{cf}}), \tau) \to (L(C', S'), \tau')$ between the simplicially localized associated S-sites is continuous according to Definition 3.5.1.

It is immediate to check that if f is a continuous functor, then the functor

$$\mathbb{R}f^* : \mathrm{Ho}((C', S')^{\sim,\tau'}) \longrightarrow \mathrm{Ho}((C, S)^{\sim,\tau})$$

has as left adjoint

$$\mathbb{L}(f_!)^\sim \simeq \mathbb{L}(f_{cf\,!}) : \mathrm{Ho}((C, S)^{\sim,\tau}) \longrightarrow \mathrm{Ho}((C', S')^{\sim,\tau'}),$$

the functor defined by the formula

$$\mathbb{L}(f_!)^\sim(F) := a(\mathbb{L}f_!(F)),$$

for $F \in \mathrm{Ho}((C, S)^{\sim,\tau}) \subset \mathrm{Ho}((C, S)^\wedge)$, where $a : \mathrm{Ho}((C, S)^\wedge) \to \mathrm{Ho}((C, S)^{\sim,\tau})$ is the associated stack functor.

The basic properties of the associated stack functor a imply that the functoriality result of Proposition 4.8.1 still holds by replacing the model categories of pre-stacks with the model categories of stacks, if f and g are continuous.

Now we define the obvious notion of continuous equivalence between pseudo-model sites.

Definition 4.8.6. A continuous functor $f : (C, S; \tau) \to (C', S'; \tau')$ is said to be a *continuous equivalence* or an *equivalence of pseudo-model sites* if the induced right Quillen functor $f^*_{\mathrm{cf}} : (C', S')^{\sim,\tau'} \to (C, S)^{\sim,\tau}$ is a Quillen equivalence.

The following criterion will be useful in the next section.

Proposition 4.8.7. *Let $(C, S; \tau)$ and $(C', S'; \tau')$ be pseudo-model sites, $f : C \longrightarrow C'$ a functor such that $f(S^{\mathrm{cf}}) \subseteq S'$ and $f_{\mathrm{cf}} : (C, S) \longrightarrow (C', S')$ the induced functor. Let us denote by τ (resp. by τ') the induced Grothendieck topology on the homotopy category* $\mathrm{Ho}(C, S)$ (*resp.* $\mathrm{Ho}(C', S')$). *Suppose that*

1. *The induced morphism $Lf_{\mathrm{cf}} : L(C, S) \longrightarrow L(C', S')$ between the Dwyer-Kan localizations is an equivalence of S-categories.*
2. *The functor*

$$\mathrm{Ho}(f_{\mathrm{cf}}) : \mathrm{Ho}(C, S) \longrightarrow \mathrm{Ho}(C', S')$$

reflects covering sieves (i.e., a sieve R over $x \in \mathrm{Ho}(C, S)$ is τ-covering iff the sieve generated by $\mathrm{Ho}(f_{\mathrm{cf}})(R)$ is a τ'-covering sieve over $f_{\mathrm{cf}}(x)$.

Then f is a continuous equivalence.

Proof. This follows easily from the comparison statement Theorem 4.7.1 and from Theorem 2.3.1. □

4.9. A Giraud's theorem for model topoi

In this section we prove a Giraud's type theorem characterizing model topoi internally. Applied to t-complete model topoi, this will give an internal description of model categories that are Quillen equivalent to some model category of stacks over an S-site. We like to consider this result as an extension of Dugger characterization of combinatorial model categories ([Du2]), and as a model category analog of J. Lurie's theorem characterizing ∞-topoi (see [Lu, Theorem 2.4.1]). Using the strictification theorem of Hirschowitz and Simpson (stated in Section 4.2 of [To-Ve 1]) it also gives a proof of the Giraud's theorem for Segal topoi conjectured in [To-Ve 1, Conjecture 5.1.1]. The statement presented here is very close in spirit to the statement presented in [Re], with some minor differences in that our conditions are weaker than [Re], and closer to the original ones stated by Giraud (see [SGA4-I, Exp. IV, Theoreme 1.2]).

We start with some general definitions.

Definition 4.9.1. Let M be any $\mathbb{U}$-model category.

The model category has *disjoint homotopy coproducts* if for any $\mathbb{U}$-small family of objects $\{x_i\}_{i\in I}$, and any $i \neq j$ in I, the following square is homotopy cartesian:

$$\begin{array}{ccc} \emptyset & \longrightarrow & x_i \\ \downarrow & & \downarrow \\ x_j & \longrightarrow & \coprod^{\mathbb{L}}_{i\in I} x_i. \end{array}$$

2. *The homotopy colimits are stable under pullbacks in M* if for any morphism $y \longrightarrow z$ in M, such that z is fibrant, and any $\mathbb{U}$-small diagram $x_* : I \longrightarrow M/z$ of objects over z, the natural morphism

$$\operatorname{hocolim}_{i\in I}(x_i \times^h_z y) \longrightarrow \left(\operatorname{hocolim}_{i\in I} x_i\right) \times^h_z y$$

is an isomorphism in $\mathrm{Ho}(M)$.

3. A *Segal groupoid object in M* is a simplicial object

$$X_* : \Delta^{op} \longrightarrow M,$$

such that

- for any $n > 0$, the natural morphism

$$X_n \longrightarrow \underbrace{X_1 \times^h_{X_0} X_1 \times^h_{X_0} \dots \times^h_{X_0} X_1}_{n\ times}$$

  induced by the n morphisms $s_i : [1] \longrightarrow [n]$, defined as $s_i(0) = i$, $s_i(1) = i+1$, is an isomorphism in $\mathrm{Ho}(M)$.
- The morphism

$$d_0 \times d_1 : X_2 \longrightarrow X_1 \times^h_{d_0, X_0, d_0} X_1$$

  is an equivalence in $\mathrm{Ho}(M)$.

4. We say that *Segal equivalences relation are homotopy effective in M* if for any Segal groupoid object X_* in M with homotopy colimit

$$|X_*| := \operatorname{hocolim}_{n\in\Delta} X_n,$$

and any $n > 0$, the natural morphism

$$X_n \longrightarrow \underbrace{X_0 \times^h_{|X_*|} X_0 \times^h_{|X_*|} \dots \times^h_{|X_*|} X_0}_{n\ times}$$

induced by the n distinct morphisms $[0] \to [n]$, is an isomorphism in $\mathrm{Ho}(M)$.

We are now ready to state our version of Giraud's theorem for model topoi.

Theorem 4.9.2. *Let M be a $\mathbb{U}$-combinatorial model category* (*see Definition* A.2.1). *Then, M is a $\mathbb{U}$-model topos if and only if it satisfies the following conditions*:

1. *M has disjoint homotopy coproducts.*
2. *Homotopy colimits in M are stable under homotopy pullbacks.*
3. *Segal equivalence relations are homotopy effective in M.*

Proof. The fact that the conditions are satisfied in any model topos follows easily from the well known fact that they are satisfied in the model category *SSet*. The hard point is to prove they are sufficient conditions.

Let M be a $\mathbb{U}$-model category satisfying the conditions of the theorem.

We chose a regular cardinal λ as in the proof of [Du2, Proposition 3.2], and let $C := M_\lambda$ be a $\mathbb{U}$-small full sub-category of M consisting of a set of representatives of λ-small objects in M. By increasing λ if necessary, one can assume that the full sub-category C of M is $\mathbb{U}$-small, and is stable under fibered products in M and under the fibrant and cofibrant replacement functors (let us suppose these are fixed once for all). By this last condition we mean that for any morphism $x \to y$ in C, the functorial factorizations $x \to x' \to y$ are again in C. Let Γ_* and Γ^* be fibrant and cofibrant resolution functors on M [Hi, Chapter 16]. We can also assume that C is stable by Γ_* and Γ^* (i.e. that for any $x \in C$ and any $[n] \in \Delta$, $\Gamma_n(x)$ and $\Gamma^n(x)$ belong to C). We note that C is not strictly speaking a pseudo-model category but will behave pretty much the same way.

We consider the functor

$$\underline{h}^C : M \longrightarrow SPr(C),$$

sending an object $x \in M$ to the simplicial presheaf

$$\begin{array}{rcl} \underline{h}^C_x : C^{op} & \longrightarrow & SSet_{\mathbb{U}}, \\ y & \longmapsto & Hom(\Gamma^*(y), x). \end{array}$$

The functor $\underline{h}$ has a left adjoint

$$L : SPr(C) \longrightarrow M,$$

sending a $\mathbb{U}$-simplicial presheaf F to its geometric realization with respect to Γ. By the standard properties of mapping spaces, one sees that for any fibrant object $x \in M$ the simplicial presheaf $\underline{h}_x^C$ is fibrant in the model category of restricted diagrams $(C, W)^\wedge$. This, and the general properties of left Bousfield localizations imply that the pair $(\underline{h}^C, L)$ defines a Quillen adjunction

$$L : (C, W)^\wedge \longrightarrow M, \quad (C, W)^\wedge \longleftarrow M : \underline{h}^C.$$

Lemma 4.9.3. *The right derived functor*

$$\mathbb{R}\underline{h}^C : \mathrm{Ho}(M) \longrightarrow \mathrm{Ho}((C, W)^\wedge)$$

is fully faithful.

Proof. By the choice of C, any object $x \in M$ is a λ-filtered colimit $x \simeq \mathrm{colim}_{i \in I} x_i$ of objects $x_i \in C$. As all objects in C are λ-small, this implies that

$$\mathbb{R}\underline{h}_x^c \simeq \operatorname*{hocolim}_{i \in I} \mathbb{R}\underline{h}_{x_i}^C.$$

From this, one sees that to prove that $\mathbb{R}\underline{h}^C$ is fully faithful, it is enough to prove it is fully faithful when restricted to objects of C. This last case can be treated exactly as in the proof of our Yoneda Lemma 4.2.3. □

By the previous lemma and by Proposition 3.2 of [Du2], we can conclude that there is a $\mathbb{U}$-small set of morphisms S in $(C, W)^\wedge$ such that the above adjunction induces a Quillen equivalence

$$L : L_S(C, W)^\wedge \longrightarrow M, \quad L_S(C, W)^\wedge \longleftarrow M : \underline{h}^C.$$

By Corollary 3.8.5(2), it only remains to show that the left Bousfield localization of $(C, W)^\wedge$ along S is exact, or equivalently that the functor $\mathbb{L}L$ commutes with homotopy pull backs.

We start by the following particular case. Let $c \in C$ and h_c be the presheaf represented by c. One can see h_c as an object in $(C, W)^\wedge$ by considering it as a presheaf of discrete simplicial sets. Let $F \longrightarrow h_c$ and $G \longrightarrow h_c$ be two morphisms in $(C, W)^\wedge$.

Lemma 4.9.4. *The natural morphism*

$$\mathbb{L}L(F \times^h_{h_c} G) \longrightarrow \mathbb{L}L(F) \times^h_{\mathbb{L}L(h_c)} \mathbb{L}L(G)$$

is an isomorphism in $\mathrm{Ho}(M)$.

Proof. Up to an equivalence, we can write F as a homotopy colimit $\mathrm{hocolim}_{i \in I} h_{x_i}$ for some $x_i \in C$. As homotopy pull-backs commutes with homotopy colimits this shows that one can suppose F and G of the form h_a and h_b, for a and b two objects in C.

Now, as in Lemma 4.2.2, one checks that h_x and $\mathbb{R}\underline{h}_x^{\mathrm{c}}$ are naturally isomorphic in $\mathrm{Ho}((C,W)^{\wedge})$. For this, we easily deduce that the natural morphism

$$h_a \times^h_{h_c} h_b \longrightarrow h_{a\times^h_c b},$$

is an equivalence in $(C,W)^{\wedge}$ (here $h_{a\times^h_c b}$ can be seen as an object of C because of our stability assumptions). Therefore, to prove the lemma it is enough to check that for any $x \in C$ the natural morphism $h_x \longrightarrow \underline{h}^C(x)$ induces by adjunction a morphism $L(h_x) \longrightarrow x$ which is an equivalence in M. But, as h_x is always a cofibrant object in $(C,W)^{\wedge}$, one has

$$L(h_x) \simeq \mathbb{L}L(h_x) \simeq \mathbb{L}L(\underline{h}_x^{\mathrm{c}}) \simeq x$$

by Lemma 4.9.3. □

Let $\coprod_{i\in I} h_{c_i}$ be a coproduct with $c_i \in C$, and

$$F \longrightarrow \coprod_{i\in I} h_{c_i} \longleftarrow G$$

be two morphisms in $(C,W)^{\wedge}$.

Lemma 4.9.5. *The natural morphism*

$$\mathbb{L}L\left(F \times^h_{\coprod_{i\in I} h_{c_i}} G\right) \longrightarrow \mathbb{L}L(F) \times^h_{\mathbb{L}L\left(\coprod_{i\in I} h_{c_i}\right)} \mathbb{L}L(G)$$

is an isomorphism in $\mathrm{Ho}(M)$.

Proof. As for Lemma 4.9.4, one can reduce to the case where F and G are of the form h_a and h_b. Lemma 4.9.5 will then follows easily from our assumption (1) on M. □

We are now ready to treat the general case.

Lemma 4.9.6. *The functor $\mathbb{L}L$ preserves homotopy pull-backs.*

Proof. Let $F \longrightarrow H \longleftarrow G$ be two morphisms in $(C,W)^{\wedge}$. One can, as for lemma 4.9.4 suppose that F and G are of the form h_a and h_b. We can also suppose that H is fibrant in $(C,W)^{\wedge}$.

We let $\coprod_i h_{x_i} \longrightarrow H$ be an epimorphism of simplicial presheaves with $x_i \in C$, and we replace it by an equivalent fibration $p : X_0 \longrightarrow H$. We set X_* the nerve of p, which is the simplicial object of $(C, W)^\wedge$ given by

$$X_n := \underbrace{X_0 \times_H X_0 \times_H \ldots \times_H X_0}_{n\ times},$$

and for which faces and degeneracies are given by the various projections and generalized diagonals. As p is a fibration between fibrant objects one sees that X_* is a Segal groupoid object in $(C, W)^\wedge$. Furthermore, as p is homotopically surjective (as a morphism of simplicial presheaves), the natural morphism

$$|X_*| \longrightarrow H$$

is an equivalence in $(C, W)^\wedge$. Finally, as X_0 is equivalent to $\coprod_i h_{x_i}$, Lemma 4.9.5 implies that $\mathbb{L}L(X_*)$ is a Segal groupoid object in M, and one has $|\mathbb{L}L(X_*)| \simeq \mathbb{L}L(H)$ as L is left Quillen. Assumption (3) on M implies that

$$\mathbb{L}L(X_0 \times^h_H X_0) \simeq \mathbb{L}L(X_1) \simeq \mathbb{L}L(X_0) \times^h_{\mathbb{L}L(H)} \mathbb{L}L(X_0).$$

To finish the proof of Lemma 4.9.6 it is then enough to notice that since $X_0 \longrightarrow H$ is surjective up to homotopy, the morphisms $h_a, h_b \longrightarrow H$ can be lifted up to homotopy to morphisms to X_0 (because they correspond to elements in $H(a)$ and $H(b)$), and therefore

$$h_a \times^h_H h_b \simeq h_a \times^h_{X_0} (X_0 \times^h_H X_0) \times^h_H h_b.$$

One can then apply Lemma 4.9.5. □

Theorem 4.9.2 is proven. □

The following corollary is an internal classification of t-complete model topoi.

Corollary 4.9.7. *Let M be a $\mathbb{U}$-combinatorial model category. Then the following are equivalent.*

1. *The model category M satisfies the conditions of* Theorem 4.9.2 *and is furthermore t-complete.*
2. *There exists a $\mathbb{U}$-small S-site (T, τ) such that M is Quillen equivalent to $SPr_\tau(T)$.*

Proof. Conditions (1) and (2) follow from Theorem 4.9.2 combined with our Theorem 3.8.3. □

From the proof of Theorem 4.9.2 one also extracts the following consequence.

Corollary 4.9.8. *Let M be a $\mathbb{U}$-combinatorial model category. Then the following are equivalent*:

1. *The model category M satisfies the conditions of Theorem 4.9.2 and is furthermore t-complete.*

2. *There exists a* $\mathbb{U}$*-model category N, and a* $\mathbb{U}$*-small full subcategory of cofibrant object* $C \subset N^c$*, and a topology* τ *on* $\mathrm{Ho}(C) := (W \cap C)^{-1}C$*, such that M is Quillen equivalent to* $(C, W)^{\sim,\tau}$*. Furthermore, the natural functor* $\mathrm{Ho}(C) \longrightarrow \mathrm{Ho}(N)$ *is fully faithful and its image is stable under homotopy pull backs.*

This last corollary states that *M* is Quillen equivalent to the model category of stacks over something which is "almost" a pseudo-model site. However, the sub-category *C* produced during the proof of Theorem 4.9.2 is not a pseudo-model site as it is not stable by equivalences in *N*. On the other hand, one can show that the closure $\overline{C}$ of *C* by equivalences in *N* is a pseudo-model site, and that the natural morphism $LC \longrightarrow L\overline{C}$ is an equivalence of *S*-categories.

Corollary 4.9.9. *If M is a* $\mathbb{U}$*-model topos* (*resp. a t-complete* $\mathbb{U}$*-model topos*) *then so is* M/x *for any fibrant object* $x \in M$*.*

Proof. Indeed, if *M* is a $\mathbb{U}$-combinatorial model category satisfying the conditions of Theorem 4.9.2 then so does M/x for any fibrant object *x*. Furthermore, one can check that for any *S*-site (T, τ), and any object *F* the model category $SPr_\tau(T)/F$ is *t*-complete. This implies that if *M* is furthermore *t*-complete then so is M/x. □

Corollary 4.9.10. 1. *Any* $\mathbb{U}$*-model topos M is Quillen equivalent to a left proper model category for which avery object is cofibrant and which is furthermore internal* (*i.e. is a symmetric monoidal model category for the direct product moniodal structure*).

2. *For any* $\mathbb{U}$*-model topos M and any fibrant object* $x \in M$*, the category* $Ho(M/x)$ *is cartesian closed.*

Proof. It is enough to check this for $M = L_S SPr(T)$, for some $\mathbb{U}$-small *S*-category *T* and some $\mathbb{U}$-small set of morphisms *S* in *SPr*(*T*) such that $Id : SPr(T) \longrightarrow L_S SPr(T)$ preserves homotopy fiber products. We can also replace the projective model structure *SPr*(*T*) by the injective one $SPr_{inj}(T)$ (see Proposition 3.6.1), and therefore can suppose *M* of the form $L_S SPr_{inj}(T)$, again with $Id : SPr_{inj}(T) \longrightarrow L_S SPr_{inj}(T)$ preserving homotopy fiber products. We know that $SPr_{inj}(T)$ is an internal model category in which every object is cofibrant, and from this one easily deduces that the same is true for the exact localization $L_S SPr_{inj}(T)$.

Condition (2) follows from (1) and Corollary 4.9.9. □

5. Étale *K*-theory of commutative $\mathbb{S}$-algebras

In this section we apply the theory of stacks over pseudo-model sites developed in the previous section to the problem of defining a notion of étale *K*-theory of a commutative $\mathbb{S}$-algebra i.e. of a commutative monoid in Elmendorf–Kriz–Mandell-May's category of $\mathbb{S}$-modules (see [EKMM]). The idea is very simple. We only need two ingredients: the first is a notion of an *étale topology* on the model category $(\mathrm{Alg}_{\mathbb{S}})$ of commutative $\mathbb{S}$-algebras and the second is the corresponding model category of *étale stacks* on $(\mathrm{Alg}_{\mathbb{S}})$.

Then, in analogy with the classical situation (see [Ja1, Section 3]), *étale K-theory* will be just defined as a fibrant replacement of algebraic K-theory in the category of étale stacks over $(\mathrm{Alg}_{\mathbb{S}})$. The first ingredient is introduced in Section 5.2 as a natural generalization of the conditions defining étale coverings in Algebraic Geometry; the second ingredient is contained in the general theory developed in Section 4. We also study some basic properties of this étale K-theory and suggest some further lines of investigation.

A remark on the choice of our setting for *commutative ring spectra* is in order. Although we choosed to build everything in this Section starting from [EKMM]'s category $\mathscr{M}_{\mathbb{S}}$ of $\mathbb{S}$-modules, completely analogous constructions and results continue to hold if one replaces from the very beginning $\mathscr{M}_{\mathbb{S}}$ with any other model for spectra having a well behaved smash product. Therefore, the reader could replace $\mathscr{M}_{\mathbb{S}}$ with Hovey–Shipley–Smith's category $\mathbf{Sp}^{\Sigma}$ of symmetric spectra (see [HSS]) or with Lydakis' category **SF** of simplicial functors (see [Ly]), with no essential changes.

Moreover, one could also apply the constructions we give below for commutative $\mathbb{S}$-algebras, to the category of E_{∞}-algebras over any symmetric monoidal model category of the type considered by Markus Spitzweck in [Sp, Section 8, 9]. In particular, one can repeat with almost no changes what is in this Section starting from Spitzweck's generalization of $\mathbb{S}$-modules as presented in [Sp, Section 9].

The problem of defining an étale K-theory of ring spectra was suggested to us by Paul-Arne Ostvær and what we give below is a possible answer to his question. We were very delighted by the question since it looks as a particularly good test of applicability of our theory. For other applications of the theory developed in this paper to moduli spaces in algebraic topology we refer the reader to [To-Ve 3].

5.1. $\mathbb{S}$-modules, $\mathbb{S}$-algebras and their algebraic K-theory

The basic reference for what follows is [EKMM]. We fix two universes $\mathbb{U}$ and $\mathbb{V}$ with $\mathbb{U} \in \mathbb{V}$. These universes are, as everywhere else in this paper, to be understood in the sense of [SGA4-I, Exp. I, Appendice] and *not* in the sense of [EKMM, 1.1].

Definition 5.1.1.
- We will denote by $\mathscr{M}_{\mathbb{S}}$ the category of $\mathbb{S}$-modules in the sense of [EKMM, II, Definition 1.1] which belong to $\mathbb{U}$.
- $\mathrm{Alg}_{\mathbb{S}}$ will denote the category of commutative $\mathbb{S}$-algebras in $\mathbb{U}$, i.e. the category of commutative monoids in $\mathscr{M}_{\mathbb{S}}$. Its opposite category will be denoted by $\mathrm{Aff}_{\mathbb{S}}$. Following the standard usage in algebraic geometry, an object A in $\mathrm{Alg}_{\mathbb{S}}$, will be formally denoted by $\mathrm{Spec}A$ when considered as an object in $\mathrm{Aff}_{\mathbb{S}}$.
- If A is a commutative $\mathbb{S}$-algebra, $\mathscr{M}_A$ will denote the category of A-modules belonging to $\mathbb{U}$ and Alg_A the category of commutative A-algebras belonging to $\mathbb{U}$ (i.e. the comma category $A/\mathrm{Alg}_{\mathbb{S}}$ of objects in $\mathrm{Alg}_{\mathbb{S}}$ under A or equivalently the category of commutative monoids in $\mathscr{M}_A$).
- We denote by $\mathrm{Alg}_{\mathrm{conn},\,\mathbb{S}}$ the full subcategory of $\mathrm{Alg}_{\mathbb{S}}$ consisting of *connective algebras*; its opposite category will be denoted by $\mathrm{Aff}_{\mathrm{conn},\,\mathbb{S}}$. If A is a (connective) algebra, we denote by $\mathrm{Alg}_{\mathrm{conn},\,A}$ the full subcategory of Alg_A consisting of connective A-algebras; its opposite category will be denoted by $\mathrm{Aff}_{\mathrm{conn},\,A}$.

Recall that $\mathcal{M}_A$ is a topologically enriched, tensored and cotensored over the category (Top) of topological spaces in $\mathbb{U}$, left proper $\mathbb{U}$-cofibrantly generated $\mathbb{V}$-small model category where equivalences are morphisms inducing equivalences on the underlying spectra (i.e. equivalences are created by the forgetful functor $\mathcal{M}_A \to \mathcal{S}$, where $\mathcal{S}$ denotes the category of spectra [EKMM, I and VII, Theorem 4.6] belonging to $\mathbb{U}$) and cofibrations are retracts of relative cell A-modules [EKMM, III, Definition 2.1 (i), (ii); VII, Theorem 4.15]. Note that since the realization functor $|-| : SSet \to$ Top is monoidal, we can also view $\mathcal{M}_{\mathbb{S}}$ and $\mathcal{M}_A$ as tensored and cotensored over *SSet*.

Moreover, a crucial property of $\mathcal{M}_{\mathbb{S}}$ and $\mathcal{M}_A$, for any commutative $\mathbb{S}$-algebra A, is that they admit a refinement of the usual "up to homotopy" smash product of spectra giving them the structure of (topologically enriched, tensored and cotensored over the category (Top) of topological spaces or over *SSet*) symmetric monoidal model categories [EKMM, III, Theorem 7.1].

Finally, both $\mathrm{Alg}_{\mathbb{S}}$ and Alg_A for any commutative $\mathbb{S}$-algebra A are topologically or simplicially tensored and cotensored model categories [EKMM, VII, Corollary 4.10].

Proposition 5.1.2. *Let* $\imath : \mathrm{Alg}_{\mathrm{conn},\, \mathbb{S}} \hookrightarrow \mathrm{Alg}_{\mathbb{S}}$ *be the full subcategory of connective algebras and* $W_|$ *the set of equivalences in* $\mathrm{Alg}_{\mathrm{conn},\, \mathbb{S}}$. *Then* $(\mathrm{Aff}_{\mathrm{conn},\, \mathbb{S}} = (\mathrm{Alg}_{\mathrm{conn},\, \mathbb{S}})^{op}, W_|^{op}, \imath^{op})$ *is a* $\mathbb{V}$*-small pseudo-model category* (*see Definition* 4.1.1).

Proof. The only nontrivial property to check is stability of $(\mathrm{Alg}_{\mathrm{conn},\, \mathbb{S}})^{op}$ under homotopy pullbacks, i.e. stability of $\mathrm{Alg}_{\mathrm{conn},\, \mathbb{S}}$ under homotopy push-outs in $\mathrm{Alg}_{\mathbb{S}}$. Let $B \leftarrow A \to C$ be a diagram in $\mathrm{Alg}_{\mathrm{conn},\, \mathbb{S}}$; by Spitzweck [Sp, p. 41, after Lemma 9.14], there is an isomorphism $B \wedge_A^{\mathbb{L}} C \simeq B \coprod_A^h C$ in $\mathrm{Ho}(\mathcal{M}_{\mathrm{A}})$, where the left hand side is the derived smash product over A while the right hand side is the homotopy pushout in Alg_A. Therefore it is enough to know that for any connective A-modules M and N, one has $\pi_i(M \wedge_A^{\mathbb{L}} N) \equiv \mathrm{Tor}_i^A(M, N) = 0$ if $i < 0$; but this is exactly [EKMM, Chapter IV, Proposition 1.2 (i)]. $\square$

For any commutative $\mathbb{S}$-algebra A, the smash product $- \wedge_A -$ on $\mathcal{M}_A$ induces (by derivation) on the homotopy category $\mathrm{Ho}(\mathcal{M}_A)$ the structure of a closed symmetric monoidal category [EKMM, III, Theorem 7.1]. One can therefore define the notion of *strongly dualizable objects* in $\mathrm{Ho}(\mathcal{M}_A)$ (as in [EKMM, Section III.7, (7.8)]). The full subcategory of the category $\mathcal{M}_A^{\mathrm{c}}$ of cofibrant objects in $\mathcal{M}_A$, consisting of strongly dualizable objects will be denoted by $\mathcal{M}_A^{\mathrm{sd}}$, and will be endowed with the induced classes of cofibrations and equivalences coming from $\mathcal{M}_A$. It is not difficult to check that with this structure, $\mathcal{M}_A^{\mathrm{sd}}$ is then a Waldhausen category (see [EKMM, Section VI]). Furthermore, if $A \longrightarrow B$ is a morphism of commutative $\mathbb{S}$-algebras, then the base change functor

$$f^* := B \wedge_A (-) : \mathcal{M}_A^{\mathrm{sd}} \longrightarrow \mathcal{M}_B^{\mathrm{sd}},$$

being the restriction of a left Quillen functor, preserves equivalences and cofibrations. This makes the lax functor

$$\begin{array}{rccl} \mathscr{M}^{\mathrm{sd}}_{-}: & \mathrm{Aff}_{\mathbb{S}} & \longrightarrow & \mathrm{Cat}_{\mathbb{V}}, \\ & \mathrm{Spec}\, A & \mapsto & \mathscr{M}^{\mathrm{sd}}_{A}, \\ & (\mathrm{Spec}\, f : \mathrm{Spec}\, B \to \mathrm{Spec}\, A) & \mapsto & f^{*} \end{array}$$

into a lax presheaf of Waldhausen $\mathbb{V}$-small categories. Applying standard strictification techniques (e.g. [May1, Theorem 3.4]) and then taking the simplicial set (denoted by $|wS_{\bullet}\mathscr{M}^{\mathrm{sd}}_{A}|$ in [Wa]) whose Ω-spectrum is the Waldhausen K-theory space, we deduce a presheaf of $\mathbb{V}$-simplicial sets of K-theory

$$\begin{array}{rccl} K(-): & \mathrm{Aff}_{\mathbb{S}} & \longrightarrow & SSet_{\mathbb{V}}, \\ & \mathrm{Spec}\, A & \mapsto & K(\mathscr{M}^{\mathrm{sd}}_{A}). \end{array}$$

The restriction of the simplicial presheaf K to the full subcategory $\mathrm{Aff}^{\mathrm{conn}}_{\mathbb{S}}$ of *connective* affine objects will be denoted by

$$K_{|}(-) : \mathrm{Aff}^{\mathrm{conn}}_{\mathbb{S}} \longrightarrow SSet_{\mathbb{V}}.$$

Following Section 4.1, we denote by $\mathrm{Aff}^{\wedge}_{\mathbb{S}}$ (resp. by $\mathrm{Aff}^{\mathrm{conn}}_{\mathbb{S}})^{\wedge}$) the model category of pre-stacks over the $\mathbb{V}$-small pseudo-model categories $\mathrm{Aff}_{\mathbb{S}}$ (resp. $\mathrm{Aff}^{\mathrm{conn}}_{\mathbb{S}}$).

Definition 5.1.3. The presheaf K (respectively, the presheaf $K_{|}$) will be considered as an object in $\mathrm{Aff}^{\wedge}_{\mathbb{S}}$ (resp. in $(\mathrm{Aff}^{\mathrm{conn}}_{\mathbb{S}})^{\wedge}$) and will be called the *presheaf of algebraic K-theory over the symmetric monoidal model category $\mathscr{M}_{\mathbb{S}}$* (resp. the *restricted presheaf of algebraic K-theory over the category $\mathscr{M}^{\mathrm{conn}}_{\mathbb{S}}$* of connective $\mathbb{S}$-modules). For any $\mathrm{Spec}\, A \in \mathrm{Aff}_{\mathbb{S}}$, we will write

$$\mathbb{K}(A) := K(\mathrm{Spec}\, A).$$

Remark 5.1.4. 1. Note that we adopted here a slightly different definition of the algebraic K-theory space $\mathbb{K}(A)$ as compared to [EKMM, VI, Definition 3.2]. In fact our Waldhausen category $\mathscr{M}^{\mathrm{sd}}_{A}$ (of strongly dualizable objects) contains [EKMM] category $f\mathscr{C}_{A}$ of finite cell A-modules [EKMM, III, Definition 2.1] as a full subcategory; this follows from [EKMM, III, Theorem 7.9]. The Waldhausen structure on $f\mathscr{C}_{A}$ [EKMM, VI, Section 3] is however different from the one induced (via the just mentioned fully faithful embedding) by the Waldhausen structure we use on $\mathscr{M}^{\mathrm{sd}}_{A}$: the cofibrations in $f\mathscr{C}_{A}$ are fewer. However, the same arguments used in [EKMM, p. 113] after Proposition 3.5, shows that the two definitions give isomorphic K_i groups for $i > 0$ while not, in general, for $i = 0$. One should think of objects in $f\mathscr{C}_{A}$ as *free modules* while objects in $\mathscr{M}^{\mathrm{sd}}_{A}$ should be considered as *projective modules*.

2. Given any commutative $\mathbb{S}$-algebra A, instead of considering the simplicial set $\mathbb{K}(A) = |wS_{\bullet}\mathscr{M}^{\mathrm{sd}}_{A}|$ whose Ω-spectrum is the Waldhausen K-theory spectrum of the

Waldhausen category $\mathscr{M}_A^{\mathrm{sd}}$, we could as well have taken this spectrum itself and have defined a *spectra*-, or better an $\mathbb{S}$*-modules-valued presheaf* on $\mathrm{Aff}_{\mathbb{S}}$. Since $\mathbb{S}$-modules forms a nice simplicial model category, a careful inspection shows that all the constructions we made in the previous section still make sense if we replace from the very beginning the model category of simplicial presheaves (i.e. of contravariant functors from the source pseudo-model category to simplicial sets in $\mathbb{V}$) with the model category of $\mathscr{M}_{\mathbb{S}}$-valued presheaves (i.e. of contravariant functors from the source pseudo-model category to the simplicial model category of $\mathbb{S}$-modules). This leads naturally to a theory of *prestacks* or, given a topology on the source pseudo-model or simplicial category, to a theory of *stacks in* $\mathbb{S}$*-modules* (or in any other equivalent good category of spectra).

3. The objects K and $K_{|}$ are in fact underlying simplicial presheaves of presheaves of ring spectra, which encodes the ring structure on the K-theory spaces. We leave to the reader the details of this construction.

4. A similar construction as the one given above, also yields a K-theory presheaf on the category of E_∞-algebras in a general symmetric monoidal model category $\mathscr{M}$. It could be interesting to investigate further the output of this construction when $\mathscr{M}$ is one of the *motivic* categories considered in [Sp, 14.8].

Definition 5.1.5. Let τ (resp. τ') be a model pretopology on the model category $\mathrm{Aff}_{\mathbb{S}}$ (resp. on the pseudo-model category $\mathrm{Aff}_{\mathbb{S}}^{\mathrm{conn}}$), as in Definition 4.3.1, and let $\mathrm{Aff}_{\mathbb{S}}^{\sim,\tau}$ (resp. $(\mathrm{Aff}_{\mathbb{S}}^{\mathrm{conn}})^{\sim,\tau'}$) the associated model category of stacks (Theorem 4.6.1). Let $K \longrightarrow K_\tau$ (resp. $K_{|} \longrightarrow K_{|\tau'}$) be a fibrant replacement of K (resp. of $K_{|}$) in $\mathrm{Aff}_{\mathbb{S}}^{\sim,\tau}$ (resp. in $(\mathrm{Aff}_{\mathbb{S}}^{\mathrm{conn}})^{\sim,\tau'}$). The K_τ*-theory space* of a commutative $\mathbb{S}$-algebra A (resp. the restricted $K_{\tau'}$*-theory space* of a commutative connective $\mathbb{S}$-algebra A) is defined as $\mathbb{K}_\tau(A) := K_\tau(\mathrm{Spec}\, A)$ (resp. as $\mathbb{K}_{|\tau'}(A) := K_{|\tau'}(\mathrm{Spec}\, A)$). The natural morphism $K \longrightarrow K_\tau$ (resp. $K_{|} \longrightarrow K_{|\tau'}$) induces a natural augmentation (localization morphism) $\mathbb{K}(A) \longrightarrow \mathbb{K}_\tau(A)$ (resp. $\mathbb{K}_{|}(A) \longrightarrow \mathbb{K}_{|\tau'}(A)$).

Remark 5.1.6. Though we will not give all the details here, one can define also an algebraic K-theory and K_τ-theory space of *any stack* $X \in \mathrm{Aff}_{\mathbb{S}}^{\sim,\tau}$. The only new ingredient with respect to the above definitions is the notion of *1-Segal stack* Perf_X *of perfect modules over* X, that replaces $\mathscr{M}_A^{\mathrm{sd}}$ in the definition above. This notion is defined and studied in the forthcoming paper [To-Ve 6]. Of course, a similar construction is also available for the restricted *K-theory.*

5.2. *The étale topology on commutative* $\mathbb{S}$*-algebras*

In this section we define an analog of the étale topology in the category of commutative $\mathbb{S}$-algebras, by extending homotopically to these objects the notions of *formally étale* morphism and of morphism *of finite presentation*.

The notion of formally étale morphisms we will use has been previously considered by Rognes [Ro] and by McCarthy [MCM] and Minasian [Min].

We start with the following straightforward homotopical variation of the algebraic notion of finitely presented morphism between commutative rings (cf. [EGAI, Chapter 0, Proposition 6.3.11]).

Definition 5.2.1. A morphism $f : A \to B$ in $\mathrm{Ho}(\mathrm{Alg}_{\mathbb{S}})$ will be said to be *of finite presentation* if for any filtered direct diagram $C : J \to \mathrm{Alg}_A$, the natural map

$$\operatorname{hocolim}_{j\in J} \mathrm{Map}_{\mathrm{Alg}_A}(B, C_j) \longrightarrow \mathrm{Map}_{\mathrm{Alg}_A}\left(B, \operatorname{hocolim}_{j\in J} C_j\right)$$

is an equivalence of simplicial sets. Here $\mathrm{Map}_{\mathrm{Alg}_A}(-,-)$ denotes the mapping space in the model category Alg_A.

Remark 5.2.2. 1. It is immediate to check that the condition for $\mathrm{Map}_{\mathrm{Alg}_A}(-,-)$ of commuting (up to equivalences) with *hocolim* is invariant under equivalences. Hence the definition of finitely presented is well posed for a map in the homotopy category $\mathrm{Ho}(\mathrm{Alg}_{\mathbb{S}})$.

2. Since any commutative A-algebra can be written as a colimit of finite CW A-algebras, it is not difficult to show that $A \to B$ is of finite presentation if and only if B is a retract of a finite CW A-algebra. However, we will not use this characterization in the rest of this section.

We refer to [Ba] for the definition and basic properties of topological André–Quillen cohomology of commutative $\mathbb{S}$-algebras. Recall [Ba, Definition 4.1] that if $A \to B$ is a map of commutative $\mathbb{S}$-algebras, and M a B-module, the *topological André-Quillen cohomology* of B relative to A with coefficient in M is defined as

$$\mathrm{TAQ}^*(B|A, M) := \pi_{-*}F_B(\Omega_{B|A}, M) = Ext^*_B(\Omega_{B|A}, M),$$

where $\Omega_{B|A} := \mathbb{L}Q\mathbb{R}I(B \wedge^{\mathbb{L}}_A B)$, Q being the *module of indecomposables* functor [Ba, Section 3] and I the *augmentation ideal* functor [Ba, Section 2]. We call $\Omega_{B|A}$ the *topological cotangent complex* of B over A. In complete analogy to the (discrete) algebraic situation where a morphism of commutative rings is formally étale if the cotangent complex is homologically trivial (or equivalently has vanishing André–Quillen cohomology), we give the following (compare, on the algebro-geometric side, with [Ill, Chapter III, Proposition 3.1.1])

Definition 5.2.3. • A morphism $f : A \to B$ in $\mathrm{Ho}(\mathrm{Alg}_{\mathbb{S}})$ will be said to be *formally étale* if the associated topological cotangent complex $\Omega_{B|A}$ is weakly contractible.

- A morphism $f : A \to B$ $\mathrm{Ho}(\mathrm{Alg}_{\mathbb{S}})$ is *étale* if it is of finite presentation and formally étale. A morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ in $\mathrm{Ho}(\mathrm{Aff}_{\mathbb{S}})$ is *étale* if the map $A \to B$ in $\mathrm{Ho}(\mathrm{Alg}_{\mathbb{S}})$ inducing it, is étale.

Remark 5.2.4. 1. Note that if $A' \to B'$ and $A'' \to B''$ are morphisms in $\mathrm{Alg}_{\mathbb{S}}$, projecting to isomorphic maps in $\mathrm{Ho}(\mathrm{Alg}_{\mathbb{S}})$, then $\Omega_{B'|A'}$ and $\Omega_{B''|A''}$ are isomorphic in the homotopy category of $\mathbb{S}$-modules. Therefore, the condition given above of being formally étale is well-defined for a map in $\mathrm{Ho}(\mathrm{Alg}_{\mathbb{S}})$.

2. THH-*étale morphisms.* If A is a commutative $\mathbb{S}$-algebra, B a commutative A-algebra, we recall that Alg_A is tensored and cotensored over Top or equivalently over *SSet*; therefore it makes sense to consider the object $S^1 \otimes^{\mathbb{L}} B$ in $\mathrm{Ho}(\mathrm{Alg}_A)$, where the derived tensor product is performed in Alg_A. By a result of McClure, Schwänzl and Vogt (see [EKMM, IX, Theorem 3.3]), $S^1 \otimes^{\mathbb{L}} B$ is isomorphic to $\mathrm{THH}^A(B; B) \equiv \mathrm{THH}(B|A)$ in $\mathrm{Ho}(\mathrm{Alg}_A)$ and is therefore a model for *topological Hochschild homology* as defined e.g in [EKMM, IX.1]. Moreover, note that any choice of a point $* \to S^1$ gives to $S^1 \otimes^{\mathbb{L}} B$ a canonical structure of A-algebra.

A map $A \to B$ of commutative $\mathbb{S}$-algebras, will be called *formally* THH-*étale* if the canonical map $B \to S^1 \otimes^{\mathbb{L}} B$ is an isomorphism in $\mathrm{Ho}(\mathrm{Alg}_A)$; consequently, a map $A \to B$ of commutative $\mathbb{S}$-algebras, will be called THH-*étale* if it is formally THH-étale and of finite presentation. As shown by Minasian [Min] THH-étale morphisms are in particular étale.

3. It is easy to see that a morphism of commutative $\mathbb{S}$-algebras $A \to B$ is formally THH-étale if and only if B is a *co-discrete* object in the model category Alg_A i.e., if for any $C \in \mathrm{Alg}_A$ the mapping space $\mathrm{Map}_{\mathrm{Alg}_A}(B, C)$ is a discrete (i.e. 0-truncated) simplicial set. From this description, one can produce examples of étale morphisms of $\mathbb{S}$-algebras which are not THH-étale. The following example was communicated to us by Michael Mandell. Let $A = H\mathbb{F}_p = K(\mathbb{F}_p, 0)$ (H denotes the Eilenberg-Mac Lane $\mathbb{S}$-module functor, see [EKMM, IV, Section 2]), and perform the following construction. Start with $F_1(A)$, the free commutative A-algebra on a cell in degree -1. In $\pi_{-1}(F_1(A))$ there is a fundamental class but also lots of other linearly independent elements as for example the Frobenius F. We let B to be the A-algebra defined by the following homotopy co-cartesian square:

$$\begin{array}{ccc} F_1(A) & \xrightarrow{1-F} & F_1 A \\ \downarrow & & \downarrow \\ A & \longrightarrow & B. \end{array}$$

The morphism $1 - F$ being étale, we have that B is an étale A-algebra. However, one has $\pi_1(Map_{Alg_A}(B, A)) \simeq \mathbb{Z}/p \neq 0$, and therefore $A \longrightarrow B$ is not THH-étale (because $Map_{Alg_A}(B, A)$ is not 0-truncated).

Proposition 5.2.5. *If $C \leftarrow A \to B$ is a diagram in* $\mathrm{Ho}(\mathrm{Alg}_{\mathbb{S}})$ *and $A \to B$ is étale, then the homotopy co-base change map $C \to B \coprod_A^h C$ is again étale.*

Proof. The co-base change invariance of the finite presentation property is easy and left to the reader. The co-base change invariance of the formally étale property follows at once from [Sp, p. 41, after Lemma 9.14] and the "flat base change" formula for the cotangent complex [Ba, Proposition 4.6]

$$\Omega_{B\wedge^{\mathbb{L}}_A C|C} \simeq \Omega_{B|A} \wedge_A C.$$

As an immediate consequence we get the following corollary. □

Corollary 5.2.6. *Let A be a commutative $\mathbb{S}$-algebra. The subcategory $\mathrm{Aff}_A^{\acute{e}t}$ of Aff_A consisting of étale maps $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$, is a pseudo-model category.*

For any (discrete) commutative ring R, we denote by $HR = \mathrm{K}(R,0)$ the *Eilenberg-Mac Lane* commutative $\mathbb{S}$-algebra associated to R [EKMM, IV, Section 2].

Proposition 5.2.7. *A morphism of discrete commutative rings $R \to R'$ is étale iff $HR \to HR'$ is étale.*

Proof. By Pirashvili and Richater [Pi-Ri] and Basterra and McCarthy [Ba-MC], we can apply to topological André–Quillen homology and André–Quillen homology the two spectral sequences at the end of [Schw, Section 7.9] to conclude that the algebraic cotangent complex $\mathbb{L}_{R'/R}$ is acyclic iff the topological cotangent complex $\Omega_{HR'|HR}$ is weakly contractible; therefore the two formal etaleness do imply each other. Also the two finite presentation condition easily imply each other, since the functor π_0 is left adjoint and therefore preserves finitely presented objects. So we only have to observe that a finitely presented morphism of discrete commutative rings $R \longrightarrow R'$ is étale iff it has an acyclic algebraic cotangent complex [Ill, Chapter III, Proposition 3.1.1]. □

The following proposition compare the notions of étale morphisms of commutative rings and commutative $\mathbb{S}$-algebras in the connective case.

Proposition 5.2.8. *Let k be a commutative ring (in $\mathbb{U}$), and $Hk \longrightarrow B$ be an étale morphism of connective commutative $\mathbb{S}$-algebras. Then, the natural map $B \longrightarrow H(\pi_0(B))$ [EKMM, Proposition IV.3.1] is an equivalence of commutative $\mathbb{S}$-algebras. Therefore, up to equivalences, $Hk \longrightarrow B$ is of the form $Hk \longrightarrow Hk'$ where $k \to k'$ is an étale extension of discrete commutative rings.*

Proof. Consider the sequence of maps of commutative $\mathbb{S}$-algebras $Hk \longrightarrow B \longrightarrow H\pi_0(B)$; this gives a fundamental cofibration sequence [Ba, Proposition 4.3])

$$\Omega_{B|Hk} \wedge_B H\pi_0(B) \longrightarrow \Omega_{H\pi_0(B)|Hk} \longrightarrow \Omega_{H\pi_0(B)|B}.$$

Since $Hk \longrightarrow B$ is étale, by McCarthy and Minasian [MCM, Proposition 3.8(2)] also $Hk \longrightarrow H\pi_0(B)$ is étale; therefore the first two terms are contractible, hence $\Omega_{H\pi_0(B)|B} \simeq *$, too. Now, the map $B \longrightarrow H\pi_0(B)$ is a 1-equivalence (see also [Ba, Proof of Theorem 8.1]) and therefore, $\Omega_{H\pi_0(B)|B} \simeq *$ and [Ba, Lemma 8.2], tell us that $\pi_1 B \simeq 0$. Then, $B \longrightarrow H\pi_0(B)$ is also a 2-equivalence and the same argument shows then that $\pi_2 B \simeq 0$, etc. Therefore $\pi_i B \simeq 0$, for any $i \geqslant 1$ and we get the first statement. The second one follows from this and Proposition 5.2.7. □

Remark 5.2.9. Note that Proposition 5.2.8 is false if we remove the connectivity hypothesis. In fact, the $H\mathbb{F}_p$-algebra B described in Remark 5.2.4(3) is étale but has, by construction, non-vanishing homotopy groups in infinitely many negative degrees. Actually, even restricting to THH-etale characteristic zero will not be enough in order to avoid this kind of phenomenon (see e.g. [To-Ve 3, Rem. 2.19]).

Definition 5.2.10. For each $\mathrm{Spec} A \in \mathrm{Ho}(\mathrm{Aff}_{\mathbb{S}})$, let us define $\mathrm{Cov}_{ét}(\mathrm{Spec} A)$ as the set of *finite* families $\{f_i : \mathrm{Spec}\, B_i \longrightarrow \mathrm{Spec}\, A\}_{i\in I}$ of morphisms in $\mathrm{Ho}(\mathrm{Aff}_{\mathbb{S}})$, satisfying the following two conditions:

1. for any $i \in I$, the morphism $A \longrightarrow B_i$ is *étale*;
2. the family of base change functors

$$\{\mathbb{L}f_i^* : \mathrm{Ho}(\mathscr{M}_A) \longrightarrow \mathrm{Ho}(\mathscr{M}_{B_i})\}_{i\in I}$$

   conservative, i.e. a morphism in $\mathrm{Ho}(\mathscr{M}_A)$ is an isomorphism if and only if, for any $i \in I$, its image in $\mathrm{Ho}(\mathscr{M}_{B_i})$ is an isomorphism.

We leave to the reader the easy task of checking that this actually defines a model pre-topology (*ét*) (see Definition 4.3.1), called the *étale topology* on $\mathrm{Aff}_{\mathbb{S}}$. By restriction to the sub-pseudo-model category (see Proposition 5.1.2) $\mathrm{Aff}_{\mathrm{conn},\,\mathbb{S}}$ of connective objects, we also get a pseudo-model site $(\mathrm{Aff}_{\mathrm{conn},\,\mathbb{S}}, ét)$, called the *restricted étale site*.

If A is a commutative (resp. commutative and connective) $\mathbb{S}$-algebra, the pseudo-model category (see Corollary 5.2.6) $\mathrm{Aff}_{ét/A}$ (resp. $\mathrm{Aff}_{\mathrm{conn},\, ét/A}$), together with the "restriction" of the étale topology, will be called the *small étale site* (resp. the *restricted small étale site*) over A. More precisely, let us consider the obvious forgetful functors

$$F : \mathrm{Aff}_{ét/A} \longrightarrow \mathrm{Aff}_{\mathbb{S}},$$

$$F' : \mathrm{Aff}_{\mathrm{conn},\, ét/A} \longrightarrow \mathrm{Aff}_{\mathbb{S}}.$$

By definition of the pseudo-model structures on $\mathrm{Aff}_{ét/A}$ (resp. on $\mathrm{Aff}_{\mathrm{conn},\, ét/A}$), F (resp. F') preserves (actually, creates) equivalences. Therefore, we say that family of morphisms $\{\mathrm{Spec}(C_i) \rightarrow \mathrm{Spec}(B)\}$ in $\mathrm{Ho}(\mathrm{Aff}_{ét/A})$ (resp. in $\mathrm{Ho}(\mathrm{Aff}_{\mathrm{conn},\, ét/A})$) is an étale covering family of $(\mathrm{Spec} B \rightarrow \mathrm{Spec} A)$ in $\mathrm{Aff}_{ét/A}$ (resp. $\mathrm{Aff}_{\mathrm{conn},\, ét/A}$) iff its image via $\mathrm{Ho}(F)$ (resp. via $\mathrm{Ho}(F')$) is an étale covering family of $\mathrm{Spec}(B)$ in $\mathrm{Aff}_{\mathbb{S}}$ i.e. belongs to $\mathrm{Cov}_{ét}(\mathrm{Spec} A)$ (Definition 5.2.10).

We finish this paragraph by the following corollary that compare the small étale sites of a ring k and of its associated Eilenberg–Mac Lane $\mathbb{S}$-algebra Hk.

Corollary 5.2.11. *Let k be a discrete commutative ring,* $(\mathrm{aff}_{ét/k}, ét)$ *be the small étale affine site over* $\mathrm{Spec}(k)$ *consisting of affine étale schemes* $\mathrm{Spec}(k') \rightarrow \mathrm{Spec}(k)$, *and* $H : \mathrm{aff}_{\mathrm{ét/k}} \longrightarrow \mathrm{Aff}_{\mathrm{conn},\, ét/Hk}$ *be the Eilenberg-Mac Lane space functor*. *Then H induces* *a continuous equivalence of étale pseudo-model sites*

$$H : (\mathrm{aff}_{\acute{e}t/k}, \acute{e}t) \to (\mathrm{Aff}_{\mathrm{conn},\ \acute{e}t/HK}, \acute{e}t).$$

Proof. Propositions 5.2.8 and 5.2.7 imply that the conditions of Proposition 4.8.7 are satisfied. □

5.3. Étale K-theory of commutative $\mathbb{S}$-algebras

The following one is the main definition of this section.

Definition 5.3.1.
- For any $A \in \mathrm{Alg}_{\mathbb{S}}$, we define its *étale K-theory space* $\mathbb{K}_{\acute{e}t}(A)$ by applying Definition 5.1.5 to $\tau = (\acute{e}t)$.
- For any $A \in \mathrm{Alg}_{\mathbb{S}}^{\mathrm{conn}}$, we define its *restricted étale K-theory space* $\mathbb{K}_{\acute{e}t}(A)$ by applying Definition 5.1.5 to $\tau' = (\acute{e}t)$.

The following proposition shows that, as in the algebraic case (cf. [Ja1, Theorem 3.10]), also in our context, étale K-theory can be computed on the small étale sites.

Proposition 5.3.2. *Let A be a commutative* (*resp. commutative and connective*) $\mathbb{S}$-*algebra and* $(\mathrm{Aff}_{\acute{e}t/A})^{\sim,\ \acute{e}t}$ (resp. $(\mathrm{Aff}_{\mathrm{conn},\ \acute{e}t/A})^{\sim,\ \acute{e}t}$) the model category of stacks on the small étale site (resp. on the restricted small étale site) over A. For any presheaf F on $\mathrm{Aff}_{\mathbb{S}}$, we denote by F^{sm} (resp. $F_{|}^{sm}$) its restriction to $\mathrm{Aff}_{\acute{e}t/A}$ (resp. to $\mathrm{Aff}_{\mathrm{conn},\ \acute{e}t/A}$). Then the map $K^{sm} \to K_{\acute{e}t}^{sm}$ (resp. $K_{|}^{sm} \to K_{\acute{e}t}^{sm}$) induced via restriction by a fibrant replacement $K \to K_{\acute{e}t}$ (resp. $K_{|} \to K_{\acute{e}t}$) in $(\mathrm{Aff}_{\mathbb{S}})^{\sim,\ \acute{e}t}$ (resp. in $(\mathrm{Aff}_{\mathrm{conn},\mathbb{S}})^{\sim,\ \acute{e}t}$), is a fibrant replacement in $(\mathrm{Aff}_{\acute{e}t/A})^{\sim,\ \acute{e}t}$ (resp. in $(\mathrm{Aff}_{\mathrm{conn},\ \acute{e}t/A})^{\sim,\ \acute{e}t}$).

Proof. We prove the proposition in the non-connective case, the connective case is the similar.

Let us consider the natural functor

$$f : Aff_{\acute{e}t/A} \longrightarrow Aff_{\mathbb{S}},$$

from the small étale site of $\mathrm{Spec}\, A$ to the big étale site. It is clear that the associated restriction functor

$$f^* : Aff_{\mathbb{S}}^{\sim,\acute{e}t} \longrightarrow Aff_{\acute{e}t/A}^{\sim,\acute{e}t}$$

preserves equivalences (one can apply for example Lemma 4.5.4). Furthermore, if $\mathrm{Spec}\, B \longrightarrow \mathrm{Spec}\, A$ is a fibrant object in $\mathrm{Aff}_{\acute{e}t/A}$, then the pseudo-representable hypercovers (see Definition 4.4.1) of $\mathrm{Spec}\, B$ are the same in $\mathrm{Aff}_{\acute{e}t/A}$ and in $\mathrm{Aff}_{\mathbb{S}/A}$ (because each structure map of a pseudo-representable hypercover is étale). This implies by Corollary 4.6.3, that the functor f^* preserves fibrant objects. In particular, if $K \longrightarrow K_{\acute{e}t}$ is a fibrant replacement in $\mathrm{Aff}_{\mathbb{S}}^{\sim,\acute{e}t}$, so is its restriction to $\mathrm{Aff}_{\acute{e}t/A}^{\sim,\acute{e}t}$. □

As a consequence, we get the following comparison result to algebraic étale *K*-theory for fields; if *R* is a (discrete) commutative ring, we denote by $K_{\acute{e}t}(R)$ its étale *K*-theory space (e.g. [Ja1]).

Corollary 5.3.3. *For any discrete commutative ring k, we have an isomorphism* $\mathbb{K}_{\acute{e}t}(Hk) \simeq K_{\acute{e}t}(k)$ in $\mathrm{Ho}(SSet)$.

Proof. This follows from corollaries 5.2.11, 5.3.2 and from the comparison between algebraic *K*-theory of a commutative ring *R* and algebraic *K*-theory of the $\mathbb{S}$-algebra *HR* (see [EKMM, VI, Remark 6.1.5(1)]). □

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Appendix A Model categories and universes

In this appendix we have collected the definitions of $\mathbb{U}$-cofibrantly generated, $\mathbb{U}$-cellular and $\mathbb{U}$-combinatorial model categories for a universe $\mathbb{U}$, that have been used all along this work.

Throughout this appendix, we fix a universe $\mathbb{U}$.

A.1. $\mathbb{U}$-cofibrantly generated model categories

Recall that a category is a $\mathbb{U}$-category, or equivalently a locally $\mathbb{U}$-small category, if for any pair of objects (x, y) in C the set $Hom_{\mathrm{C}}(x, y)$ is a $\mathbb{U}$-small set.

Definition A.1.1. A $\mathbb{U}$-model category is a category M endowed with a model structure in the sense of [Ho, Definition 1.1.3] and satisfying the following two conditions:

1. The underlying category of M is a $\mathbb{U}$-category.
2. The underlying category of M has all kind of $\mathbb{U}$-small limits and colimits.

Let α be the cardinal of a $\mathbb{U}$-small set (we will simply say α *is a* $\mathbb{U}$*-small cardinal*). Recall from [Ho, Definition 2.1.3] that an object x in a $\mathbb{U}$-model category M, is α*-small*, if for any $\mathbb{U}$-small α-filtered ordinal λ, and any λ-sequence

$$y_0 \to y_1 \to \dots y_\beta \to y_{\beta+1} \to \dots$$

the natural map

$$\mathrm{colim}_{\beta<\lambda} Hom(x, y_\beta) \longrightarrow Hom(x, \mathrm{colim}_{\beta<\lambda} y_\beta)$$

is an isomorphism.

We will use (as we did in the main text) the following variation of the notion of *cofibrantly generated model category* of [Ho, Definition 2.1.17].

Definition A.1.2. Let M be a $\mathbb{U}$-model category. We say that M is $\mathbb{U}$*-cofibrantly generated* if there exist $\mathbb{U}$-small sets I and J of morphisms in M, and a $\mathbb{U}$-small cardinal α, such that the following three conditions are satisfied:

1. The domains and codomains of the maps of I and J are α-small.
2. The class of fibrations is J-inj.
3. The class of trivial fibrations is I-inj.

The main example of a $\mathbb{U}$-cofibrantly generated model category is the model categories $SSet_{\mathbb{U}}$ of $\mathbb{U}$-small simplicial sets.

The main "preservation" result is the following easy proposition (see [Hi, Section 13.8, 13.9, 13.10]).

Proposition A.1.3. *Let M be a $\mathbb{U}$-cofibrantly generated model category.*

1. *If C is a $\mathbb{U}$-small category, then the category M^C of C-diagrams in M is again a $\mathbb{U}$-cofibrantly generated model category in which equivalences and fibrations are defined objectwise.*
2. *Let us suppose that M is furthermore a $SSet_{\mathbb{U}}$-model category in the sense of* [Ho, Definition 4.2.18] (*in other words, M is a simplicial $\mathbb{U}$-cofibrantly generated model*

category), and let T be a $\mathbb{U}$-small S-category. Then, the category M^T of simplicial functors from T to M is again a $\mathbb{U}$-cofibrantly generated model category in which the equivalences and fibrations are defined objectwise. The model category M^T is furthermore a $SSet_{\mathbb{U}}$-model category in the sense of [Ho, Definition 4.2.18].

A standard construction we have been using very often in the main text is the following. We start by the model category $SSet_{\mathbb{U}}$ of $\mathbb{U}$-small simplicial sets. Now, if $\mathbb{V}$ is a universe with $\mathbb{U} \in \mathbb{V}$, then the category $SSet_{\mathbb{U}}$ is $\mathbb{V}$-small. Therefore, the category

$$SPr(SSet_{\mathbb{U}}) := SSet_{\mathbb{V}}^{SSet_{\mathbb{U}}^{op}}$$

of $\mathbb{V}$-small simplicial presheaves on $SSet_{\mathbb{U}}$, is a $\mathbb{V}$-cofibrantly generated model category.

This is the way we have considered, in the main text, model categories of diagrams over a base model category avoiding any set-theoretical problem.

A.2. $\mathbb{U}$-cellular and $\mathbb{U}$-combinatorial model categories

The following notion of *combinatorial model category* is due to Jeff Smith (see, for example, [Du2, Bek, Section 2, I, Section 1].

Definition A.2.1. 1. A category C is called $\mathbb{U}$-*locally presentable* (see [Du2]) if there exists a $\mathbb{U}$-small set of objects C_0 in C, which are all α-small for some cardinal α in $\mathbb{U}$ and such that any object in C is an α-filtered colimit of objects in C_0.

2. A $\mathbb{U}$-*combinatorial model category* is a $\mathbb{U}$-cofibrantly generated model category whose underlying category is $\mathbb{U}$-locally presentable.

The following localization theorem is due to J. Smith (unpublished). Recall that a model category is *left proper* if the equivalences are closed with respect to pushouts along cofibrations.

Theorem A.2.2. *Let M be a left proper, $\mathbb{U}$-combinatorial model category, and $S \subset M$ be a $\mathbb{U}$-small subcategory. Then the left Bousfield localization $L_S M$ of M with respect to S exists.*

Let us recall from [Hi, Section 12.7] the notion of *compactness*. We will say that an object x in a $\mathbb{U}$-cofibrantly generated model category M is *compact* is there exists a $\mathbb{U}$-small cardinal α such that x is α-compact in the sense of [Hi, Definition 13.5.1]. The following definition is our variation of the notion of *cellular model category* of [Hi].

Definition A.2.3. A $\mathbb{U}$-cellular model category M is a $\mathbb{U}$-cofibrantly generated model category with generating $\mathbb{U}$-small sets of cofibrations I and of trivial cofibrations J, such that the following two conditions are satisfied:

1. The domains and codomains of maps in I are compact.
2. Monomorphisms in M are effective.

The main localization theorem of [Hi] is the following.

Theorem A.2.4 (*Hirschhorn [Hi, Theorem 4.1.1]*). *Let M be a left proper, $\mathbb{U}$-cellular model category and $S \subset M$ be a $\mathbb{U}$-small subcategory. Then the left Bousfield localization $L_S M$ of M with respect to S exists.*

Finally, let us mention the following "preservation" result.

Proposition A.2.5. *If in Proposition A.1.3, M is $\mathbb{U}$-combinatorial (resp. $\mathbb{U}$-cellular), then so are M^C and M^T.*

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