SHARP AFFINE STABILITY ESTIMATES FOR HAMMER'S PROBLEM

P.DULIO-M.LONGINETTI-C.PERI-A.VENTURI

Abstract. We prove that the area distance between two convex bodies K and K' with the same parallel X-rays in a set of n mutually non parallel directions is bounded from above by the area of their intersection times a constant depending only on n. Equality holds if and only if K is a regular n-gon and K' is K rotated by π/n about its center, up to affine transformations. This and similar sharp affine invariant inequalities lead to stability estimates for Hammer's problem if the n directions are known up to an error or in case X-rays emanating from n collinear points are considered. For $n = 4$, the order of these estimates is compared with the cross ratio of the given directions and given points, respectively.

1. Introduction

The parallel X-ray of a planar convex body K in a direction θ gives the length of each chord of K parallel to θ. In 1963, P.C. Hammer asked [7]: *How many parallel X-ray pictures of a convex body must be taken in order to permit its exact reconstruction?* The following basic example (see [5]) shows that there are finite sets of directions, with arbitrary large cardinality, such that the corresponding X-rays do not distinguish a convex body among the others. Consider a regular q-gon Q centred at a fixed point p, and its rotation Q' by π/q about p. The convex hull of Q and Q' is a $2q$-gon and let θ be a direction parallel to one of its edges. It is easy to see that Q and Q' have the same parallel X-rays in the direction θ. It is important to note that Hammer's problem is one of an affine nature, since a nonsingular affine transformation preserves the ratios of lengths of parallel line segments. Thus, for uniqueness of the reconstruction one should avoid subsets of directions of the edges of an affinely regular polygon, i.e. the image of a regular polygon under some affine transformation.

R.J. Gardner and P. McMullen proved in [2] that *convex bodies are determined by X-rays taken in any set of directions that is not a subset of the directions of the edges of an affinely regular polygon.* Since the cross ratio of any four directions of the edges of a regular polygon is an algebraic number, any set of four directions with a transcendental cross ratio uniquely determines a convex body by means of the corresponding X-rays. See Section 2 for the formal definitions.

It might be objected that two congruent polygons, as well as their affine images, are not really that different at all. By answering this objection, for any set of directions of the edges of a regular $2n$-gon A. Volčič [17] constructed a family, with cardinality equal to that of the real numbers, of mutually non congruent convex bodies all with the same X-rays in the considered directions, see also [4, Theorem 1.2.13].

Motivated by genuine applications in the material sciences, Hammer's problem has been studied recently within discrete tomography [9], a new area of geometric tomography which focuses on determination of finite sets of the integer lattice by means of their discrete parallel X-rays. In this field, important and deep results have been obtained by R. Gardner and P. Gritzmann, which imply the following surprising result for continuous X-rays (see [3, Theorem 6.2]): convex bodies in the Euclidean plane are determined by their parallel X-rays in *any* set of seven mutually nonparallel lattice directions.

Hammer's problem can be seen as a particular case of the more general inverse problem of reconstructing a homogeneous planar body from a finite set of tomographic data. Usually, in the applications data contain errors, and uniqueness and stability estimates are crucial in order to show when Hammer's problem is well posed. A. Volčič proved in [16] that the reconstruction of a convex set K is well posed when the set of directions guarantees uniqueness. Roughly speaking, if we know the parallel X-rays of K in a finite set of directions, with the unique determination, and the data X-rays contain an error ε, then the corresponding reconstruction K_ε converges to K when ε tends to zero.

Denoting by $|K \triangle K_\varepsilon|$ the area of the symmetric difference of K and K_ε, the L_2-distance of the difference of the characteristic functions of K and K_ε measures how K_ε is close to the desired solution K. Such a distance is usually compared with the L_2-distance of the corresponding X-rays data and with the number n of data. The order of stability for this distance is known explicitly in the class of smooth (not necessarily convex) domains and it is of order $1/2$ with respect to ε and $1/n$ (see [13]).

When the parallel X-rays of K are known up to an error ε for any planar direction and under the a-priori assumption that the planar body is convex (not necessarily smooth), as considered in Hammer's problem, such order of stability estimate is improved to 1 with respect to ε (see [12]).

It is worth remarking that the uniqueness results obtained in [2] and in [3] are, unfortunately, unstable in the sense that a small perturbation of a finite set of directions providing uniqueness may cause the uniqueness property to be lost, so that the above results of well-posedness cannot be used. In view of this, it is of interest to investigate further stability estimates for non uniqueness situations. Observe that in the basic example considered above the convex q-gons Q and Q' are close enough for q sufficiently large. In particular the symmetric difference $Q \triangle Q'$ has area of order 2 with respect to q^{-1}, so that the distance $|Q \triangle Q'|^{1/2}$ is invariant under equi-affine transformation and it is of order 1.

The main goal of this paper is to show that any non uniqueness situation in Hammer's problem has the same upper bound for the area distance, i.e the "worst" case for the area distance of two sets K, K', with same parallel X-rays in the directions of the edges of an affinely regular polygon is essentially the basic example provided by the polygons Q and Q'. In particular, the following theorem provides an affine invariant stability estimate for the reconstruction of a convex body which is optimal in the constant and in the order for any of n distinct directions.

Theorem 1.1. *If K and K' are two planar convex bodies with the same X-rays in n different directions ($n \geq 3$), then*

$$|K \triangle K'| \leq |K \cap K'| \frac{1 - \cos(\pi/n)}{\cos(\pi/n)}\,. \tag{1}$$

Equality holds if and only if, up to an affine transformation, the directions are equally spaced, the convex bodies K and K' are two congruent regular n-gons, and K' is K rotated by π/n about its center.

In [11] inequality (1) was proved for $n = 3$, and in [6, Theorem 6.1] it was obtained for arbitrary n, under a strong additional assumption about the connected components of $K \triangle K'$: the case (A) in the proof of Theorem 4.1. Such assumption has been dropped here.
We also recall that the following stability result was obtained in [10] :

$$|K \triangle K'| \leq l^2(8n)^{-1} \tan\frac{\pi}{n}, \tag{2}$$

where l is the length of the boundary of $K \cap K'$. However, inequality (2) suffers from the disadvantage that it is not affine invariant. In [4, notes 1.3] R. Gardner asked to look for an affine invariant inequality for the distance between K and K'. Inequality (1) is affine invariant and solves a question raised by M. Longinetti in [11, Teorema 5] on the same estimate of $K \triangle K'$.

The proof of inequality (1) is given at the end of Section 4 and comes down from the stronger inequality (32), which is similar to (1) but with $2n$ replaced by the number of the edges of a suitable polygon inscribed in $K \cap K'$. It hinges on sharp inequalities for affinely invariant geometric functionals on convex polygons proved in Section 3. We believe that these inequalities are of independent interest from a purely geometrical point of view. In Section 5 we improve inequality (1) by replacing n with the smallest number $q(S)$ for an affinely regular $2q(S)$-gon to have edges with directions including the given set S of n directions (see Theorem 5.2). Then we consider the cross ratio ρ of four directions in S given up to a certain error and in Theorem 5.8 we obtain an explicit upper bound of the area distance between K and K' depending on such cross ratio. Finally, in Corollary 5.9 we explicit the order of this stability estimate with respect to the distance of ρ from $0, 1, \infty$. In Section 6 we reformulate the affine stability estimates to situations where the tomographic data are related to X-rays emanating from finite collinear points.

2. Definitions and preliminaries

For a finite set A we denote by $\#A$ the number of its elements. We also denote the interior of a set A by A^o, and the symmetric difference of A and B by $A \triangle B$. We recall that the *parallel X-ray* of a planar convex body K in a direction θ gives the length of each chord of K parallel to θ. Let K and K' be two planar compact convex bodies with the same parallel X-rays in the directions of a set S consisting of n different directions, where ($n \geq 3$). Let C be a connected component (i.e. a maximal connected subset) of $(K \triangle K')^o$. For instance, let $C \subset (K \setminus K')$ and let θ be a direction in S. Let $\theta C \subset (K' \setminus K)$ be a connected component with the same X-rays of C in the direction θ. We say that θC is a *connected component associated to* C. Similarly, for a connected component $C' \subset (K' \setminus K)$ we consider the connected component $\theta C'$ of $K \setminus K'$. Let

$$\mathcal{C} = \{\theta_{i_h} \cdots \theta_{i_1} C, \theta_{i_j} \in S\} \ ,$$

be the *system of components associated to* C. Let

$$W(\mathcal{C}) = \bigcup_{\theta_{i_j} \in S} \theta_{i_h} \cdots \theta_{i_1} C \ ,$$

be the *switching domain associated to* C. Let us note that $\mathcal{C}$ is a finite set of elements (see [4, Lemma 1.2.3 and Lemma 1.2.6]), and $W(\mathcal{C})$ is a measurable set.

In the following lemma, proved in [10, Proposition 2, Lemma of Theorem 4], we collect the main properties of $\mathcal{C}$. For a more extensive description see also [4, Section 1.2].

Proposition 2.1. *For any connected component* C *of* $(K \triangle K')^o$ *we have:*
a) $C \subset K \setminus (K \cap K')$ *or* $C \subset K' \setminus (K \cap K')$*;*
b) the boundary ∂C *is the union of two arcs, one on* ∂K*, the other on* $\partial K'$*, with the same endpoints called* terminals *of* C*;*
c) $\theta_i C$ *is a connected component of* $(K \triangle K')^o$ *with terminals on lines of direction* θ_i *passing through the terminals of* C*;*
d) $\theta_i \theta_i C = C$*;*
e) $\theta_i C \cap \theta_j C = \emptyset$ *for* $\theta_i \neq \theta_j \in S$*;*
f) $\#\mathcal{C} \geq 2n$*;*
g) all components in $\mathcal{C}$ *have the same area.*

A nondegenerate convex polygon P is said to be S*-regular* if for each vertex v of P and $\theta_i \in S$, the line through v parallel to θ_i meets a different vertex v' of P. Every affinely regular polygon is S-regular where S is any subset of the directions of its edges. In [4, Corollary 1.2.10] it is proved that if there is a S-regular polygon then S is a subset of the directions of the edges of an affinely regular polygon.

Moreover, the following result is proved in [4, Corollary 1.2.8].

Proposition 2.2. *The centroids of the connected components in* $\mathcal{C}$ *form the vertices of a* S*-regular polygon.*

As usual S^1 denotes the unite circle in $\mathbb{R}^2$. Let $u \in S^1$ be represented in complex form $u = e^{\theta i}$. For simplicity, we shall identify a direction $e^{\theta i}$ with θ. The *cross ratio* of four directions $e^{\theta_1 i}, e^{\theta_2 i}, e^{\theta_3 i}, e^{\theta_4 i} \in S^1$ is defined by

$$\rho = (\theta_1 \, \theta_2 \, \theta_3 \, \theta_4) = \frac{\sin(\theta_3 - \theta_1)}{\sin(\theta_3 - \theta_2)} : \frac{\sin(\theta_4 - \theta_1)}{\sin(\theta_4 - \theta_2)}$$

Note that rearrangements of four directions θ_1, θ_2, θ_3 and θ_4 change the cross ratio according to the rules:

$$
\begin{aligned}
&(\theta_1\,\theta_2\,\theta_3\,\theta_4) = \rho, && (\theta_1\,\theta_2\,\theta_4\,\theta_3) = \frac{1}{\rho},\\
&(\theta_1\,\theta_3\,\theta_2\,\theta_4) = 1-\rho, && (\theta_1\,\theta_3\,\theta_4\,\theta_2) = \frac{1}{1-\rho},\\
&(\theta_1\,\theta_4\,\theta_2\,\theta_3) = \frac{\rho-1}{\rho}, && (\theta_1\,\theta_4\,\theta_3\,\theta_2) = \frac{\rho}{\rho-1}.
\end{aligned}
\tag{3}
$$

We also need the following result which provides a function of the cross ratio that does not depend on possible rearrangements of four directions. For the proof see [15, pagg. 326-327] or [1, Lemma 2].

Lemma 2.3. *For $\rho \in \mathbb{R}\setminus\{0,1\}$, let j be the function defined by*

$$j(\rho) = \frac{(\rho^2-\rho+1)^3}{\rho^2(\rho-1)^2}.$$

Then

$$j(\rho) = j(\rho') \text{ if and only if } \rho' \in \left\{\rho, \frac{1}{\rho}, \frac{\rho-1}{\rho}, \frac{\rho}{\rho-1}, 1-\rho, \frac{1}{1-\rho}\right\}.$$

It can be easily seen that $j(\rho) \geq \frac{27}{4}$ for all ρ with equality if $\rho = -1, \frac{1}{2}, 2$.

For any set of four distinct planar directions $(\theta_1, \theta_2, \theta_3, \theta_4)$, let us consider the equivalent 4-ples of planar directions obtained from it by permutations and affine transformations, i.e.

$$[(\theta_1,\theta_2,\theta_3,\theta_4)] = \{(A(\theta_{\lambda(1)}), A(\theta_{\lambda(2)}), A(\theta_{\lambda(3)}), A(\theta_{\lambda(4)})),\ A \in AL,\ \lambda \in S_4\}. \tag{4}$$

Here AL denotes the group of affine transformations of the plane, and S_4 is the set of permutations on $\{1,2,3,4\}$.

If $\overline{\Theta^4}$ is the set of 4-ples of distinct planar directions, we denote by $\overline{\Theta^4}/\sim$ the related quotient space. Since the cross ratio is invariant under affine transformations, from the previous lemma the function $J = j\circ\rho$ is constant on each equivalent class $[(\theta_1,\theta_2,\theta_3,\theta_4)]$. Moreover, it is easy to prove the following result.

Lemma 2.4. *The function J is injective and continuous on $\overline{\Theta^4}/\sim$.*

The example quoted at the beginning of the introduction is relevant for the results we are going to present and we introduce the following definition to refer to it. Let us suppose that up to an affine transformation Q and Q' are two congruent regular q-gons, and Q' is Q rotated by π/q about its center. Then we say that Q and Q' are *affinely rotated regular q-gons.*

3. Sharp affine invariant inequalities for convex polygons

We now introduce some affine functionals on the class $\mathcal{P}_m$ of planar convex m-gons which will be used to prove inequality (1).

Given a convex polygon P with m vertices z_j, we define $T_j(P)$ to be the (possibly infinite) triangle outside P bounded by an edge L_j, with endpoints z_j, z_{j+1}, and the continuations of its two contiguous edges. For any point $o \in P$ we define $S_j(P, o)$ to be the triangle, possibly degenerate, with edge L_j and vertex o (see Figure 1).

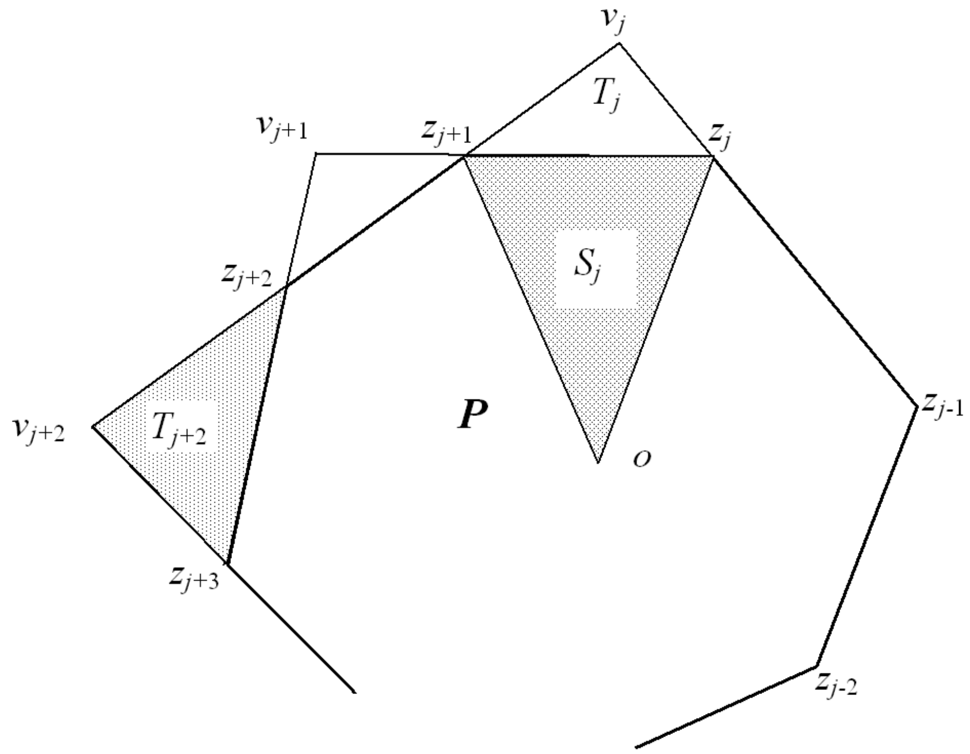

Figure 1. Triangles T_j of P.

For $P \in \mathcal{P}_m$, we define

$$G(P) = \min_{j=1,\dots,m} \frac{|T_j(P)|}{|P|}. \tag{5}$$

The initial source of the theorems of this section comes from the following proposition which was proved in [6, Theorem 4.1].

Proposition 3.1. *Let $P \in \mathcal{P}_m$, where $m > 4$. Then*

$$G(P) \leq \frac{2\sin^2(\pi/m)}{m\cos(2\pi/m)}. \tag{6}$$

Equality holds if and only if P is an affinely regular m-gon.

The next functional extends $G(P)$ to the case when just a subset of the triangles $T_j(P)$ is considered. For $P \in \mathcal{P}_m$ and $I \subseteq \mathbb{Z}_m$ let

$$G(P, I) = \frac{\#I \min_{j \in I} |T_j(P)|}{|P|}. \tag{7}$$

Let $o \in P$. If we replace $|P|$ by $\sum_{j \in I} |S_j(P, o)|$ we get

$$\widehat{G}(P, I, o) = \frac{\#I \min_{j \in I} |T_j(P)|}{\sum_{j \in I} |S_j(P, o)|}. \tag{8}$$

To obtain upper bounds for these functionals, we need the following geometric constructions: let P be a convex polygon with m vertices z_j, and let L_j be the edge with endpoints z_j and z_{j+1}. The polygon P' obtained by replacing the vertex z_{j+1} with a near point z'_{j+1} on L_j is called *a replacement by a local L_j-up cut* (see Figure 2). If instead, we replace the vertex z_j by a near point z'_j on L_j then we say that P' is *a replacement by a local L_j-down cut.*

Proposition 3.2. *Let $P \in \mathcal{P}_m$ and let P' be a replacement of P by a local L_j-up cut or a L_j-down cut, where $1 \le j \le m$. Then*

$$|P'| < |P| \quad \textit{and} \quad \min_{i \in I} |T_i(P')| \ge \min_{i \in I} |T_i(P)| \tag{9}$$

for every $I \subseteq \mathbb{Z}_m$ such that $j \notin I$.

Proof. For a L_j-up cutting we have

$$T_i(P') \supset T_i(P), \quad \text{for} \quad i = j+1, j+2,$$

$$T_i(P') = T_i(P), \quad \text{for} \quad i \neq j, j+1, j+2.$$

Similarly, for a L_j-down cutting we have

$$T_i(P') \supset T_i(P), \quad \text{for} \quad i = j-1, j-2,$$

$$T_i(P') = T_i(P), \quad \text{for} \quad i \neq j, j-1, j-2.$$

In both case we have

$$|P'| < |P|$$

and

$$\min_{i \in I} |T_i(P')| \ge \min_{i \in I} |T_i(P)|.$$

□

In this section we extend Proposition 3.1 to the functionals (7) and (8). These results will play an important role in proving Theorem 1.1.

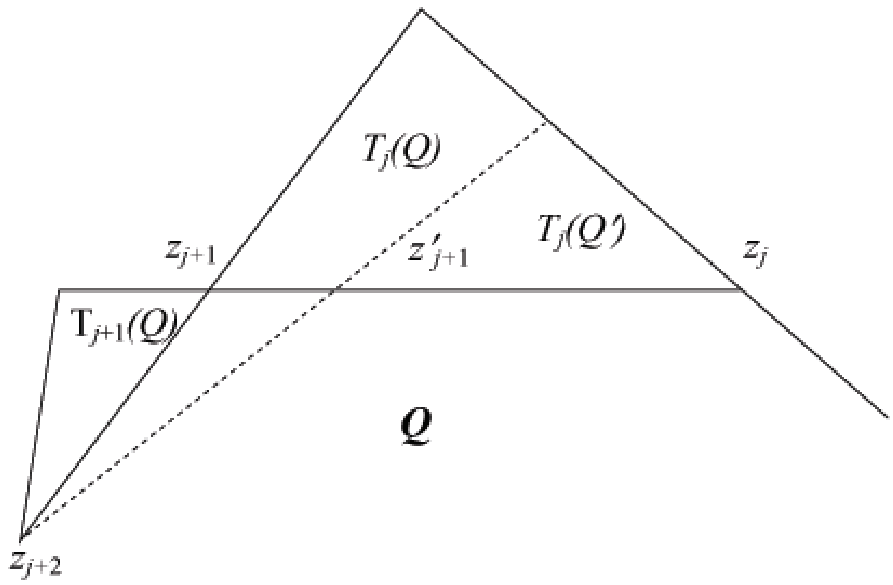

FIGURE 2. Replacement with local L_j-up cut.

Theorem 3.3. *Let h, m be integers, where $4 < h \leq m$. For any convex polygon $P \in \mathcal{P}_m$ and for any $I \subseteq \mathbb{Z}_m$, with $\#I \geq h$ we have:*

$$G(P, I) \leq \frac{2\sin^2(\pi/h)}{\cos(2\pi/h)}\,. \tag{10}$$

Equality holds if and only if $\#I = m = h$ and P is an affinely regular h-gon.

Proof. The proof is similar to that given in [10, Theorem 2] concerning the not affine invariant functional $\#I \min_{j\in I} |T_j(P)|/l^2(P)$, where $l(P)$ denotes the perimeter of P. This functional was introduced to obtain (2). Here we prefer to present an outline of the proof to show the principal techniques which will be used again in the sequel.

Let $P_{m,h}$ denote the class of convex polygons P with p edges, where $h \leq p \leq m$. For $P \in P_{m,h}$ we define the functional:

$$H(P) = \max\{G(P, I) : I \subseteq \mathbb{Z}_p,\ \#I \geq h\}. \tag{11}$$

The existence of the maximum of H on $P_{m,h}$ follows by a standard argument of semicontinuity. In fact, H is semicontinous on the subset Γ of $P_{m,h}$ consisting of polygons which have unit area and whose boundaries are contained in a fixed annulus (see [6, Lemma 2.1] for a similar argument). We recall the principal ingredients: let α_j be the exterior angle of P at the vertex z_j, and let l_j be the length of the edge L_j with endpoints z_j, z_{j+1}. If $T_j(P)$ is bounded then

$$|T_j(P)| = \frac{1}{2} l_j^2 (\cot\alpha_{j+1} + \cot\alpha_j)^{-1}. \tag{12}$$

Then we can prove that the lengths l_j, with $j \in I$, of any convergent maximizing sequence in Γ are uniformly bounded from below. This proves that this sequence converges to a polygon with at least h vertices, so that it belongs to $P_{m,h}$. Let $Q \in P_{m,h}$ be such that $H(Q)$ attains its maximum value on $P_{m,h}$. If Q has q edges, then there exists $I \subseteq \mathbb{Z}_q$, such that

$$H(Q) = \frac{\#I \min_{i \in I} |T_i(Q)|}{|Q|}. \tag{13}$$

Let us prove that $\#I = q$. In fact if $\#I < q$, there exists an edge L_j of Q with $j \notin I$. Let Q' be the replacement of Q by a local L_j-up cut (see Figure 2). By Proposition 3.2 we have

$$|Q'| < |Q| \quad \text{and} \quad \min_{i \in I} |T_i(Q')| \geq \min_{i \in I} |T_i(Q)| \tag{14}$$

so that

$$H(Q') > H(Q).$$

This contradicts the assumption that $H(Q)$ is maximal. Thus $\#I = q$ and

$$H(Q) = \frac{q \min_{i \in I} |T_i(Q)|}{|Q|}.$$

Since Q is also a maximizer for H on $\mathcal{P}_q$, from Proposition 3.1 we have

$$H(Q) = \frac{2 \sin^2(\pi/q)}{\cos(2\pi/q)}.$$

Since the function on the right hand side is decreasing with q, the functional H attains its maximum for $q = h = \#I$. This proves (10). Equality holds if and only if we have equality in (6), namely when P is an affinely regular polygon with h edges. $\square$

We now get an analogous result for the functional (8). To this end, we first prove some technical lemmas. Let h and p be integers such that $4 < h \leq m$, and $h \leq p \leq m$. For any polygon $P \in P_{m,h}$ with p vertices, we define

$$\widehat{H}(P) = \max \left\{ \frac{\#I \min_{j \in I} |T_j(P)|}{\sum_{j \in I} |S_j(P, o)|} \ : \ o \in P, \ I \subseteq \mathbb{Z}_p, \ \#I \geq h \right\}. \tag{15}$$

Lemma 3.4. *The functional $\widehat{H}$ attains its maximum on $P_{m,h}$ at a polygon P^*, for a subset I^* and for a point o^* such that*

$$o^* \quad \textit{is} \quad \textit{vertex of} \quad P^*, \tag{16}$$

$$|T_i(P^*)| \quad = \quad |T_j(P^*)| \quad \textit{for} \quad i, j \in I^*. \tag{17}$$

Proof. The existence of the maximum for $\widehat{H}$ is proved similarly to the existence of the maximum of the functional H on Γ. Since $\widehat{H}$ is continuous with respect to o , and o belongs to a fixed circle, we can use the same proof. Property (16) is an obvious consequence of the linearity of expression $\sum_{j \in I} |S_j(P, o)|$ with respect to o, which attains its minimum on vertices of P. Property (17) is the so called *equal-area property* . This property played a key role in characterising the maximizers of the functional (5) in [6, Proposition 1.1].

Equality (17) is obtained by contradiction. Assume that (17) does not hold, then there exists $k \in I^*$ such that

$$\min_{j \in I^*} |T_j(P^*)| < |T_k(P^*)|. \tag{18}$$

By a local L_k-cut (up or down) we replace P^* with a polygon P', sufficiently close to P^* such that

$$\min_{j \in I^*} |T_j(P^*)| \le \min_{j \in I^*} |T_j(P')|. \tag{19}$$

Note that equality can now occur, since the $\#I^* > 4$ and the replacement by a local L_k-cut changes just T_k and two other triangles in sequence, namely T_{k+1}, T_{k+2} for a local up-cut, T_{k-1}, T_{k-2} for a local down-cut, respectively. We consider two different cases:
i) $o^* \notin L_k$,
ii) $o^* = z_k$ or $o^* = z_{k+1}$.
In case i) the area of the union of the two triangles $S_k(P^*, o^*), S_{k-1}(P^*, o^*)$ decreases by a local L_k-down cut; similarly, the area of the union of the two triangles $S_k(P^*, o^*), S_{k+1}(P^*, o^*)$ decreases by a local L_k-up cut. Since the other triangles $S_j(P^*, o^*)$ do not change, we have:

$$\sum_{j \in I^*} |S_j(P^*, o^*)| > \sum_{j \in I^*} |S_j(P', o^*)|. \tag{20}$$

From the two previous inequalities we get

$$\widehat{H}(P^*) < \frac{\#I^* \min_{j \in I^*} |T_j(P')|}{\sum_{j \in I^*} |S_j(P', o^*)|} \le \widehat{H}(P'). \tag{21}$$

This contradicts the assumption that P^* is a maximizer.

ii) Assume, for instance, $o^* = z_{k+1}$. We distinguish two subcases: $k-1 \in I^*$, or $k-1 \notin I^*$. If $k-1 \in I^*$, then by a local L_k-down cut we can find a polygon P' which satisfies (19) and (20), so that (21) holds, which gives again a contradiction. If $k-1 \notin I^*$ then we consider the index i', which is the maximum between the indices in I^* not exceeding $k-1$. Let P' be the polygon obtained from P^* by erasing the vertices between $z_{i'+1}$ and z_k, i.e $P' = conv\{z_1, \ldots, z_{i'+1}, z_k, z_{k+1}, \ldots, z_p\}$. Since all the indices between $i'+1$ and $k-1$ do not belong to I^*, equality holds in (20). Moreover, $T_{i'}(P^*) \subset T_{i'}(P')$ and since $i' \in I^*$, inequality (19) holds. If in (19) the strict inequality holds we get (21) again, so that we have a contradiction. If in (19) the equality sign holds then inequality (18) holds with k replaced by i' and P' by P^*. Thus, P' is a maximizer for $\widehat{H}$ as well as P^*, with the same o^* and I^* and $o^* \notin L_{i'}$. Therefore, P' satisfies case i) with k replaced by i', and we get a contradiction. The proof is then completed for the case ii), under the assumption $o^* = z_{k+1}$. If $o^* = z_k$ the proof is similar. □

Lemma 3.5. *Let P^* be a maximizer for $\widehat{H}$ on the class $P_{m,h}$ with p edges, then $\#I^* = p$.*

Proof. We argue by contradiction by assuming $p > \#I^*$. We distinguish three different cases depending on the position of o^* with respect to the edges L_j, where $j \notin I^*$:
i) there exists at least an edge $L_k, k \notin I^*$ not having o^* as endpoint;
ii) $p = \#I^* + 1$ and o^* is an endpoint of the edge $L_k, k \notin I^*$;
iii) $p = \#I^* + 2$ and o^* is the vertex $z_{k+1} = L_k \cap L_{k+1}$ with $k, k+1 \notin I^*$.

In the cases ii) and iii) we have

$$\sum_{j\in I^*} |S_j(P^*, o^*)| = |P^*| > \sum_{j\in I^*} |S_j(P^*, z_{k-1})|,$$

so that we have a contradiction, since $\sum_{j\in I^*} |S_j(P^*, o)|$ attains its minimum value at o^*.

It remains to consider the case i). From (16) it follows that we can assume that the point o^* coincides with a vertex z_l. By a suitable affine transformation we can assume that $S_k(P^*, o^*)$ is an isosceles triangle with base L_k, (see Figure 3). Let σ be the measure of the angle of $S_k(P^*, o^*)$ opposite to the edge L_k. We can suppose that the exterior angles α_k, α_{k+1} at z_k and z_{k+1}, respectively, satisfy the following inequality

$$\min\{\alpha_k, \alpha_{k+1}\} > \sigma. \tag{22}$$

This can be obtained, for instance, by a suitable dilation centred at the middle point of L_k in the direction of the axis through o^*, with fixed points on the edge L_k. Such a dilatation increases α_k and α_{k+1} towards $\pi/2$ and reduces σ towards zero. The triangle $S_k(P^*, o^*)$ separates P^* into two disjoint convex polygons: P_1, P_2 (see Figure 3) such that

$$P^* = P_1 \cup S_k(P^*, o^*) \cup P_2.$$

Note that P_1 or P_2, but not both, can be empty, corresponding to the cases $o^* \equiv z_{k+2}$ or $o^* \equiv z_{k-1}$. We consider a rotation Φ by σ about o^*, such that $\Phi(z_k) \equiv z_{k+1}$. Notice that the polygon $P' = P_1 \cup \Phi(P_2)$, obtained by a "global L_k-cut " of $S_k(P^*, o^*)$ and by rotating P_2 into $\Phi(P_2)$, satisfies again (19). Moreover, the inequality (22) implies that the line through z_k, z_{k-1} is rotated by Φ into a line which supports $P' \cup T_{k+1}(P^*)$. This proves that P' is convex and

$$T_{k+1}(P') \supset T_{k+1}(P^*), \quad T_{k-1}(P') \supset \Phi(T_{k-1}(P^*)). \tag{23}$$

Note that $k \notin I^*$ and P' has one edge, L_k, less than P^*. So $T_k(P') = \emptyset$. The edges of P' should be relabelled, but this is not relevant for the proof. Moreover, since o^* equals the vertex z_l, we have

$$T_l(P') \supset \Phi(T_l(P^*)), \quad T_{l-1}(P') \supset T_{l-1}(P^*). \tag{24}$$

Therefore, the areas of the triangles T_l and T_{l-1} increase when we replace P^* by P', and the areas of the triangles $T_i(P^*)$ and $T_i(P')$, for $i \neq l, l-1, k-1, k+1$, remain the same, since they are congruent. By the equal area property (17) for $i, j \in I^*$, where $\#I^* > 4$, we have equality in (19).

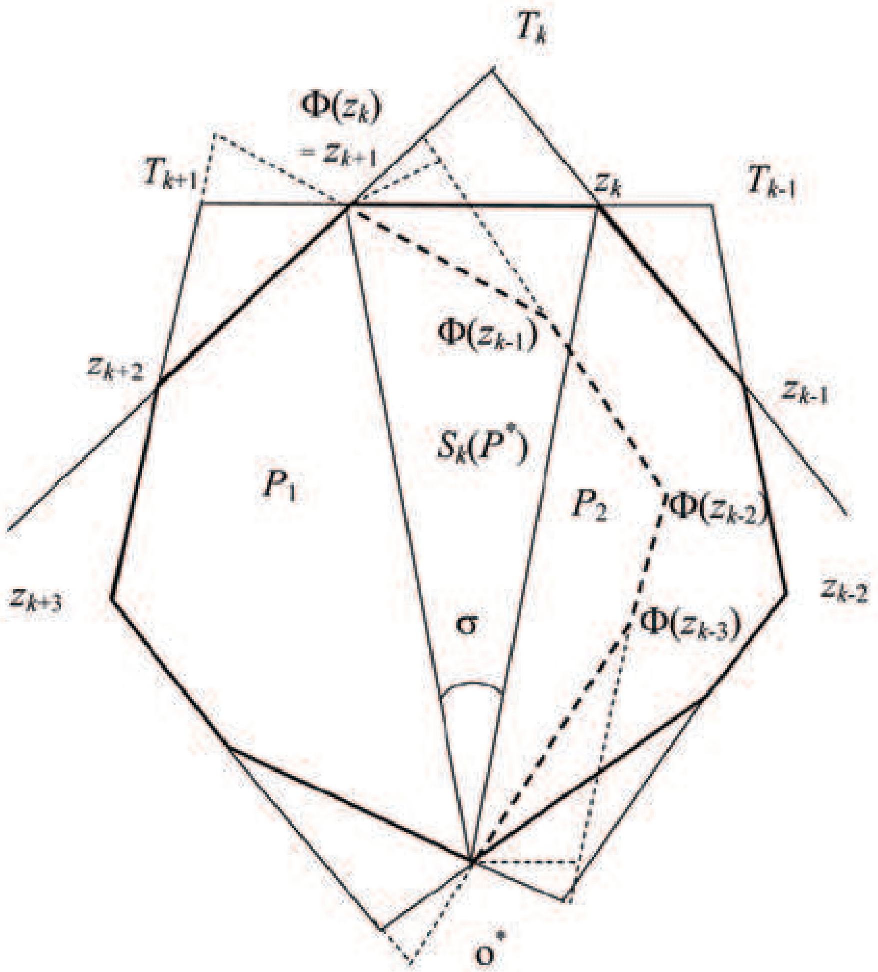

FIGURE 3. Replacement with a global L_k-cut.

Moreover, it is easy to see that

$$\sum_{j\in I^*} |S_j(P^*, o^*)| =$$

$$\sum_{j\in I^*,\ L_j \text{ side of } P_1} |S_j(P_1, o^*)| + \sum_{j\in I^*,\ L_j \text{ side of } P_2} |S_j(\Phi(P_2), o^*)| =$$

$$= \sum_{j\in I^*} |S_j(P', o^*)|.$$

Therefore $\widehat{H}(P') = \widehat{H}(P^*)$ and P' is a maximizer of $\widehat{H}$. If one of the indices $k+1, k-1, l, l-1$ belongs to I^* , from (23) and (24), it follows that P' does not satisfy the equal-area property

(17) and we get a contradiction. On the contrary, if these indices do not belong to I^*, then P' is a polygon with $p-1$ edges, which is a maximizer of $\widehat{H}$ with the same o^* and the same set of index I^*, so that case i) occurs. Then the same global cutting argument can be repeated to obtain a maximizer polygon which does not satisfy the equal-area property, against Lemma 3.4. □

We can now prove the analog of Theorem 3.3 for the functional $\widehat{G}(P, I, o)$.

Theorem 3.6. *For any convex polygon $P \in \mathcal{P}_m$, let h be an integer, where $4 < h \leq m$. For any $o \in P$ and $I \subseteq \mathbb{Z}_m$, with $\#I \geq h$ we have:*

$$\widehat{G}(P, I, o) \leq \frac{2\sin^2(\pi/h)}{\cos(2\pi/h)}. \tag{25}$$

Equality holds if and only if $\#I = h = m$ and P is an affinely regular h-gon.

Proof. Let $P^* \in P_{m,h}$ be a maximizer for the functional $\widehat{H}$. By Lemma 3.5 P^* has exactly $\#I^*$ edges, so that

$$\sum_{j \in I^*} |S_j(P^*, o^*)| = |P^*|.$$

and

$$G(P^*, I^*) = \widehat{G}(P^*, I^*, o^*).$$

Therefore, inequality (25) follows from inequality (10), and the equality case occurs in the same situation.

□

Theorem 3.7. *For any convex polygon $P \in \mathcal{P}_m$, let h be an integer, where $4 < h \leq m$. For $r = 1, \ldots, s$, let I_r be non-empty mutually disjoint subsets of $\mathbb{Z}_m$ such that $\#I_r \geq h$. Then*

$$\frac{\sum_{r=1}^{s} \#I_r \min_{j \in I_r} |T_j(P)|}{|P|} \leq \frac{2\sin^2(\pi/h)}{\cos(2\pi/h)}. \tag{26}$$

Equality holds if and only if $s = 1$, $\#I_1 = h = m$ and P is an affinely regular h-gon.

Proof. Let p be an integer such that $h \leq p \leq m$. We denote by $\mathcal{I}^p$ the class of finite collections of $\{I_r\}_{r=1,\ldots,s}$ of mutually disjoint subsets of $\mathbb{Z}_p$ such that $\#I_r \geq h$ and $\sum_{r=1}^{s} \#I_r \leq p$. For a polygon P with p edges we consider the following functional

$$E(P) = \max\{\frac{\sum_{r=1}^{s} \#I_r \min_{j \in I_r} |T_j(P)|}{|P|} \ : \ \{I_r\}_{r=1,\ldots,s} \in \mathcal{I}^p\}. \tag{27}$$

By using an argument similar to that used for the functional H in Theorem 3.3, one can see that E attains its maximum in the class $P_{m,h}$. Assume that P^* is a maximizers of E, with a corresponding family of indeces $\{I_r^*\}_{r=1,\ldots,s}$.

Let us fix a point $o \in P^*$. As in the previous theorem, for each edge L_i of P^* we consider the triangle $S_i(P^*, o)$. We have

$$|P^*| \geq \sum_{r=1}^{s} \sum_{i \in I_r^*} |S_i(P^*, o)|. \tag{28}$$

Moreover, let us define

$$\begin{aligned} a_r &= \# I_r^* \min_{j \in I_r^*} |T_j(P)|, \\ b_r &= \sum_{i \in I_r^*} |S_i(P^*, o)|. \end{aligned}$$

Then, from inequality (28), we get

$$E(P^*) \leq \frac{\sum_{r=1}^{s} a_r}{\sum_{r=1}^{s} b_r}.$$

Furthermore, from inequality (25) we get

$$\frac{a_r}{b_r} \leq \frac{2\sin^2(\pi/h)}{\cos(2\pi/h)}. \tag{29}$$

Since for any finite set of positive constant $\{a_r, b_r\}_{r=1,\ldots,s}$

$$\frac{\sum_{r=1}^{s} a_r}{\sum_{r=1}^{s} b_r} \leq \max_{r=1,\ldots,s} \frac{a_r}{b_r},$$

we obtain the desired inequality (26).

Equality holds in (26) if equality holds in (25), so that $s = 1$. □

In order to show the relations between the geometric inequalities of this section and inequality (1) we recall the following result proved in [6, Corollary 5.2].

Proposition 3.8. *Let K and K' be two planar convex bodies such that $K \triangle K'$ has $m > 4$ connected components $C_1, \cdots C_m$. Then*

$$\min_{i=1,\ldots,m} |C_i| \leq |K \cap K'| \frac{2\sin^2(\pi/m)}{m\cos(2\pi/m)}. \tag{30}$$

Equality holds if and only if $K \cap K'$ is an affinely regular m-gon, $m = 2q$ and K, K' are affinely rotated regular q-gons.

4. An affine invariant stability estimate

This section is devoted to obtain stability estimates for Hammer's problem in the case when X-rays of a planar convex body are exactly known in n distinct directions.

We point out that Proposition 3.8 provides an upper bound for the distance of two convex bodies when the number of the connected components of their symmetric difference is known. If we also assume that the two bodies have same parallel X-rays in n distinct directions, we can improve such a result by using the family of components associated to a given one. Let $S = \{\theta_1, \dots, \theta_n\}$ be a set of n distinct directions. We denote by $\{\mathcal{C}^r\}$ the family of the systems of associated components.

Theorem 4.1. *Let K and K' be two planar compact convex bodies with the same X-rays in n different directions ($n \geq 3$), and let*

$$h_{min} = \min_r \{\#\mathcal{C}^r\}. \tag{31}$$

Then

$$|K \triangle K'| \leq |K \cap K'| \frac{2sin^2(\pi/h_{min})}{\cos(2\pi/h_{min})}. \tag{32}$$

Equality holds if and only if, up to an affine transformation, the directions are equally spaced and K and K' are two congruent regular $(h_{min}/2)$-gons, and K' is K rotated by $2\pi/h_{min}$ about its center.

Proof. From Proposition 2.1, g) it follows that for any connected component C of $(K \triangle K')^o$

$$|W(\mathcal{C})| = \#\mathcal{C} \cdot |C|. \tag{33}$$

Let us first consider the simplest case when the following assumption holds:

(A) *There exists a connected component C such that $W(\mathcal{C}) = (K \triangle K')^o$.*

Then, $h_{min} = \#\mathcal{C}$ and $|K \triangle K'| = \#\mathcal{C} \cdot |C|$ so that the required inequality (32) follows from (30), where $m = \#\mathcal{C} = h_{min}$.

To avoid the restrictive assumption (A) on the set of connected components we shall use the functional E introduced in (27). Thus in the general case we argue as follows.

Let us suppose that there exists a finite number of switching domains $W(\mathcal{C}^1), \dots, W(\mathcal{C}^s)$ such that

$$(K \triangle K')^o = \cup_{r=1}^s W(\mathcal{C}^r). \tag{34}$$

Let P be the polygon whose vertices are the terminals of all connected components in $(K \triangle K')^o$. Let m be the number of the vertices of P. We label the edges of P by L_j for $j \in \mathbb{Z}_m$. We determine $I_1, \dots, I_s$ disjoint subsets of $\mathbb{Z}_m$, so that the endpoints of each edge L_j, where $j \in I_r$, are the terminals of a suitable connected component C_j of $\mathcal{C}^r$. Obviously $\#I_r = \#\mathcal{C}^r$. If $m > \sum \#I_r$, the remaining edges L_j, where $j \notin \cup I_r$, have endpoints equal to the terminals of an arc in $\partial K \cap \partial K'$. Therefore, from (34) and (33) for any choice of an index $j(r) \in I_r$ we have

$$|K \triangle K'| = \sum_{r=1}^s |W(\mathcal{C}^r)| = \sum_{r=1}^s \#I_r |C_{j(r)}|. \tag{35}$$

Let $j(r)$ be such that $|T_{j(r)}(P)| = \min_{j \in I_r} |T_j(P)|$. Since

$$C_j \subset T_j(P), \quad \forall\, j \in I_r, \quad r = 1, \dots s,$$

the area of $C_{j(r)}$ does not exceeds $\min_{j\in I_r}|T_j(P)|$ so that

$$|K \triangle K'| \leq \sum_{r=1}^{s} \#I_r \min_{j\in I_r} |T_j(P)|.$$

Since $P \subset K \cap K'$, it follows that

$$\frac{|K \triangle K'|}{|K \cap K'|} \leq \frac{\sum_{r=1}^{s} \#I_r \min_{j\in I_r} |T_j(P)|}{|P|}. \tag{36}$$

Since

$$h_{min} = \min_{r}\{\#I_r\}, \tag{37}$$

from Theorem 3.7 it follows that

$$\frac{\sum_{r=1}^{s} \#I_r \min_{j\in I_r} |T_j(P)|}{|P|} \leq \frac{2\sin^2(\pi/h_{min})}{\cos(2\pi/h_{min})}. \tag{38}$$

The two previous inequalities imply

$$\frac{|K \triangle K'|}{|K \cap K'|} \leq \frac{2\sin^2(\pi/h_{min})}{\cos(2\pi/h_{min})}. \tag{39}$$

By the assumption (34) the equality case follows from the equality case in Theorem 3.7. To complete the proof it remains to consider the case

$$(K \triangle K')^o = \cup_{r=1}^{\infty} W(\mathcal{C}^r).$$

Then inequality (32) follows from the previous case by standard limit arguments, and such inequalit is a strict inequality. In fact if we have a numerable family of $W(\mathcal{C}^r)$, then for every sufficiently small ε there exists a finite number $s \geq 2$ such that

$$\frac{|K \triangle K'|}{|K \cap K'|} - \varepsilon \leq \frac{|\cup_{r=1}^{s} W(\mathcal{C}^r)|}{|K \cap K'|}. \tag{40}$$

For any s, let $P(s)$ be the convex hull of the terminals of all connected components of $\cup_{r=1}^{s} W(\mathcal{C}^r)$. Note that such terminals are all vertices of $P(s)$. Then, by arguing as above, we obtain

$$\frac{|\cup_{r=1}^{s} W(\mathcal{C}^r)|}{|K \cap K'|} \leq E(P(s)), \tag{41}$$

where E is defined by (27). By construction each $P(s)$ has at least $s \cdot h_{min}$ edges, and the vertices of each polygon $P(s)$ are also vertices of the polygon $P(s+1)$. Therefore, the sequence of the polygons $P(s)$ is not contained in a neighbourhood of the set of affinely regular h_{min}-gons. Thus, from Theorem 3.7 we get

$$\sup_{s} E(P(s)) < \frac{2\sin^2(\pi/h_{min})}{\cos(2\pi/h_{min})},$$

and there exist a positive δ independent on s, as well on ε, such that

$$E(P(s)) < \frac{2\sin^2(\pi/h_{min})}{\cos(2\pi/h_{min})} - \delta.$$

From (40) and (41) we then obtain

$$\frac{|K \triangle K'|}{|K \cap K'|} < \varepsilon + \frac{2\sin^2 \pi/h_{min}}{\cos 2\pi/h_{min}} - \delta.$$

Since we can choose ε smaller than δ, equality cannot occur in (32). □

Proof of Theorem 1.1. If $K = K'$, then (1) is trivially true. Thus, let $K \neq K'$ and let $\{\mathcal{C}^r\}$ be the family of systems of associated components. Proposition 2.1 f) implies that $\#\mathcal{C}^r \geq 2n$. Thus, we have $h_{min} \geq 2n > 4$. From the monotonicity of the right term in (32) with respect to h_{min}, we get (1) by replacing h_{min} with $2n$, and the equality case as well as. □

Remarks:
a) By the identity

$$2|K \cap K'| + |K \triangle K'| = 2|K| = 2|K'|,$$

inequality (1) can be rewritten as follows

$$|K \triangle K'| \leq 2|K|\frac{1-\cos(\pi/n)}{1+\cos(\pi/n)}.$$

Let us note that the right hand side of the previous inequality is exactly known since the parallel X-rays provides $|K|$ by the Cavalieri's principle.

b) Inequality (2) follows from inequality (1) by using the standard isoperimetric inequality.

c) $h_{min} = 2n$ for instance when $q = 2n$ in the exceptional couple of regular q-gons Q, Q' of the introduction.

Since usually the number h_{min} is expected much bigger than $2n$, inequality (32) contains more information than (1). Some of these situations will be explicited in the next section which provides cases where h_{min} is large with respect to n as we want, by choosing suitable sets S of $n = 4$ directions (Theorem 5.8 and Corollary 5.9).

5. Sets of directions with error and estimates through the cross ratio

Let q be a positive integer and let

$$\Theta_q = \{e^{i\pi k/q}, k = 1, \dots, q\}$$

be the set of q equally spaced directions.
For a given set $S = \{\theta_1, \dots, \theta_n\}$ let

$$q(S) = \min\{q \in \mathbb{N} : \exists A \in AL,\ A(S) \subseteq \Theta_q\}. \tag{42}$$

Note that $q(S) \geq n$ and $q(S) = +\infty$ when there does not exist an integer q and an affine transformation $A \in AL$ such that $A(S) \subseteq \Theta_q$. From this point of view, the uniqueness result of Gardner and McMullen [2] claims that *the set S has the unique determination property if and only if $q(S) = +\infty$.*

The following proposition play a key role in order to characterize the set not having the unique determination property, and its proof essentially repeat the proof of [4, Corollary 1.2.10]. We present it in a little bit stronger formulation.

Lemma 5.1. *If there is a S-regular polygon of $2q$ vertices then up an affine transformation $S \subseteq \Theta_q$.*

Proof. We point out only some additional aspects with respect to the proof of [4, Corollary 1.2.10]. Let Q_0 be a S-regular polygon of 2q vertices. Let $M(Q_0)$ be the midpoint polygon whose vertices are the midpoints of the edges of Q_0.

Let us consider $Q_k = \sec(\pi/2q)M(Q_{k-1})$ for $k \in \mathbb{N}$. Clearly by induction each Q_k is also S-regular. In [4, Lemma 1.2.9] it is proved that the sequence $\{Q_{2k}\}$ converges in the Hausdorff metric to an affinely regular polygon R of $2q$ vertices. Of course the limit R is S-regular too. Since R has an even number of sides the only directions θ for which R is $\{\theta\}$-regular are the directions of the edges, which are equispaced of π/q. Hence there exists $A \in AL$ such that $A(S)$ is contained in the set Θ_q. □

The following theorem provides a sharp stability estimate when S does not satisfy the unique determination property.

Theorem 5.2. *Let S be a set of n distinct directions. If K and K' are two planar compact convex bodies, with the same parallel X-rays in the directions in S, then*

$$|K \triangle K'| \leq |K \cap K'| \frac{1 - \cos(\pi/q(S))}{\cos(\pi/q(S))}\,. \tag{43}$$

Equality holds if and only if K, K' are affinely rotated regular $q(S)$-gons.

Proof. Inequality (43) follows from (39) provided that the number h_{min}, defined by (31) satisfies the inequality:

$$h_{min} \geq 2q(S). \tag{44}$$

Let us consider the centroids of the elements of the system of components $\mathcal{C}$ such that $h_{min} = \#\mathcal{C}$. From Proposition 2.2 such centroids form the vertices of a S-regular polygon Q_0. Moreover, since it is S-regular, Q_0 has necessarily an even number of vertices, say $2q$. From previous lemma there exists $A \in AL$ such that $A(S)$ is contained in the set Θ_q. This implies $2q(S) \leq 2q = h_{min}$. Equality follows from the equality case in Theorem 4.1. □

Let us now consider the case when S is known up to an error δ. For any finite set S of n directions $\{\theta_1, \ldots, \theta_n\}$, and for any $\delta > 0$, we write $S_\delta = \prod_{i=1}^{n}[\theta_i - \delta, \theta_i + \delta]$. We also assume that δ is small enough so that $[\theta_i - \delta, \theta_i + \delta]$ and $[\theta_j - \delta, \theta_j + \delta]$ are mutually disjoint for $i \neq j$. We remark that for q big enough there exists a set $S' \subset \Theta_q$ of n directions θ'_i with $\theta'_i \in [\theta_i - \delta, \theta_i + \delta]$, so that $q(S') < +\infty$, even if $q(S) = +\infty$. This means that, even if the set S has the unique determination property, a small perturbation of S may cause the uniqueness to be lost.

Definition 5.3. *For any finite set S of n directions $\{\theta_1, \dots, \theta_n\}$, and $\delta > 0$ let*

$$q(S_\delta) = \min\{q(S'), S' \in S_\delta\}. \tag{45}$$

From (42) it follows

$$q(S_\delta) = \min\{q \in \mathbb{N} : \exists A \in AL,\ A[\theta_i - \delta, \theta_i + \delta] \cap \Theta_q \neq \emptyset \quad \forall i = 1, \dots, n\}.$$

Then, Theorem 5.2 can be extended as follows.

Theorem 5.4. *Let K and K' be two planar compact convex bodies, with the same parallel X-rays in a set S' of directions contained in S_δ. Then*

$$|K \triangle K'| \leq |K \cap K'| \frac{1 - \cos(\pi/q(S_\delta))}{\cos(\pi/q(S_\delta))}\ . \tag{46}$$

Proof. If $q(S') = +\infty$, then $K = K'$ and (46) is trivial. Otherwise, if $q(S') < +\infty$, by Definition 5.3 we have

$$q(S') \geq q(S_\delta). \tag{47}$$

Theorem 5.2 can be applied to the set of directions S' and inequality (46) follows from (47). □

The remainder of this section is devoted to estimate $q(S)$ by means of the cross ratios of any four directions in S. For simplicity, we confine ourselves to a set S of exactly four directions.

Given four distinct directions $\theta_1, \theta_2, \theta_3, \theta_4$ in the plane, let us consider the equivalent sets of 4 directions defined in (4). By Lemma 2.4 the function J is injective and continuous on $\overline{\Theta^4}/\sim$. In particular if we know the value of J with a small error δ' we can determine a corresponding 4-ple of distinct directions, with a small error δ, up to permutations and affine transformations. It turns out that the values of the function J play an important role to compute the values of $q(S_\delta)$. We define $\overline{\Theta^4_q}$ to be the set of distinct 4-ples in Θ_q.

Lemma 5.5. *For any finite set S of 4 directions $\{\theta_1, \dots, \theta_4\}$, and $\delta > 0$ we have*

$$q(S_\delta) = \min\{q \in \mathbb{N} : J(S_\delta) \cap J(\overline{\Theta^4_q}) \neq \emptyset\}. \tag{48}$$

Proof. It is enough to show that

$$\{q \in \mathbb{N} : J(S_\delta) \cap J(\overline{\Theta^4_q}) \neq \emptyset\} = \{q \in N : \exists A \in AL,\ A[\theta_i - \delta, \theta_i + \delta] \cap \Theta_q \neq \emptyset \quad \forall i = 1, \dots, 4\}.$$

Since $\cup_q \overline{\Theta^4_q}$ is dense in $\overline{\Theta^4}$, by the continuity of J there exists $q' \in \mathbb{N}$ such that $J(\overline{\Theta^4_{q'}}) \bigcap J(S_\delta) \neq \emptyset$, for all $\delta > 0$. Namely, for any such q', there exists a set of 4 directions of $S' \in S_\delta$ having the same cross ratio as that of a set $\overline{S}$ of 4 directions taken in $\overline{\Theta^4_{q'}}$. Therefore, there exists an affine map A such that $A(S') = \overline{S} \in \overline{\Theta^4_{q'}}$. This proves that

$$q' \in \{q \in \mathbb{N} : \exists A \in AL,\ A[\theta_i - \delta, \theta_i + \delta] \cap \Theta_q \neq \emptyset \quad \forall i = 1, \dots, 4\}.$$

By the same argument we get the opposite inclusion. □

Note that if $q(S) < +\infty$, then $q(S_\delta)$ is bounded from above by $q(S)$. Otherwise, the estimate(46) is more relevant once we know the order of divergence of $q(S_\delta)$ with respect to the infinitesimum δ. So, in view of the previous lemma, we have to study the value of the cross ratio on $\overline{\Theta^4_q}$. To this end we consider the following subset of four directions. Let W^4_q be the collection of ordered sets of 4 directions taken in $\overline{\Theta^4_q}$, i.e.

$$W^4_q = \{(w_{h_1}, w_{h_2}, w_{h_3}, w_{h_4}) : \ w_{h_j} = e^{h_j \frac{\pi}{q} i}, 0 \le h_1 < h_2 < h_3 < h_4 \le q-1\}.$$

Let $\tilde{W}^4_q$ be the subset of W^4_q, such that the first direction coincide with the direction of the x axis, and the angles between the first and third directions, and the second and fourth directions are both less than $\pi/2$.

Lemma 5.6. *For any q the values of J on $\overline{\Theta^4_q}/\sim$ are the same as those of J on $\tilde{W}^4_q/\sim$.*

Proof. It is enough to show that the values of J on $\tilde{W}^4_q/\sim$ equal the values of J on $W^4_q/\sim$. In fact we shall prove that the set $\tilde{W}^4_q/\sim$ equals the set $W^4_q/\sim$. Up to a rotation, we can suppose that $h_1 = 0$. Further, we can suppose that at least three of the given directions belong to the first quadrant. Otherwise, if $\frac{h_3\pi}{q} > \frac{\pi}{2}$, then $\frac{h_4\pi}{q} > \frac{h_3\pi}{q} > \frac{\pi}{2}$. Thus, by rotating the vector $e^{h_1\frac{\pi}{q}i}$ of an angle π, we get three vectors $-w_{h_1}, w_{h_3}, w_{h_4}$ all belonging to the second quadrant. Note that the cross ratio of four directions does not depend on the orientation of such directions, so that a rotation of $e^{h_1\frac{\pi}{q}i}$ of an angle π does not change the cross ratio. With a suitable rotation and axial symmetry, we then obtain four directions $(w_{h'_1}, w_{h'_2}, w_{h'_3}, w_{h'_4})$, three of them in the first quadrant. This implies that the angle between the first and third directions is smaller than $\pi/2$. If the angle between the second and fourth directions is greater than $\pi/2$, then the angle between $w_{h'_2}$ and $-w_{h'_4}$ is smaller than $\pi/2$, so that we can replace $w_{h'_4}$ by $-w_{h'_4}$, without changing the set of non oriented directions. Thus, by the rotation which map $-w_{h'_4}$ to the x-axis we obtain four directions belonging to $\tilde{W}^4_q/\sim$. Since rotations and symmetries map a set of 4 directions to a set of directions belonging to the same class of equivalence , we have $\tilde{W}^4_q/\sim = W^4_q/\sim$. □

Up to rearrangements, the cross ratio of $(w_{h_1}, w_{h_2}, w_{h_3}, w_{h_4}) \in \tilde{W}^4_q$ is given by:

$$\rho = \frac{\sin\frac{(h_3-h_1)\pi}{q}\sin\frac{(h_4-h_2)\pi}{q}}{\sin\frac{(h_4-h_3)\pi}{q}\sin\frac{(h_2-h_1)\pi}{q}}.$$

Let

$$k_1 = h_2 - h_1, \quad k_2 = h_3 - h_2, \quad k_3 = h_4 - h_3$$

so that

$$\rho = \frac{\sin\frac{(k_1+k_2)\pi}{q}\sin\frac{(k_3+k_2)\pi}{q}}{\sin\frac{k_1\pi}{q}\sin\frac{k_3\pi}{q}}. \tag{49}$$

This shows that the cross ratio is a function depending only on the angles between the directions and not on w_{h_j}.

By the previous lemma we can simply consider the value of (49) on

$$D_q = \{(k_1, k_2, k_3) \in \mathbb{N}^3 : 1 \leq k_1, k_2, k_3 \leq q-1 \text{ and } \ k_1 + k_2 \leq [q/2], k_3 + k_2 \leq [q/2]\},$$

where $[x]$ denotes the integer part of x.

Lemma 5.7.

$$1 < \min_{\tilde{W}_q^4} \rho \leq 2 \leq \max_{\tilde{W}_q^4} \rho. \tag{50}$$

$$\min_{\tilde{W}_q^4} \rho = \frac{\sin^2 \frac{[q/2]\pi}{q}}{\sin^2 \frac{([q/2]-1)\pi}{q}}, \quad \max_{\tilde{W}_q^4} \rho = \frac{\sin^2 \frac{[q/2]\pi}{q}}{\sin^2 \frac{\pi}{q}}. \tag{51}$$

Proof. First we observe that on D_q for $q \geq 4$ the function ρ is increasing with respect to k_2 and decreasing with respect to k_1 and k_3. In fact by elementary differentiation of (49) we have

$$\frac{\partial \rho}{\partial k_1} = \frac{\pi}{q}\rho(\tan^{-1}(k_1+k_2)\frac{\pi}{q} - \tan^{-1} k_1)\frac{\pi}{q},$$

which is negative in D_q. A similar result holds for the derivative with respect to k_3. Moreover

$$\frac{\partial \rho}{\partial k_2} = \frac{\pi}{q}\rho(\tan^{-1}(k_1+k_2)\frac{\pi}{q} + \tan^{-1}(k_3+k_2))\frac{\pi}{q},$$

which is positive since $k_1 + k_2 \leq [q/2], k_3 + k_2 \leq [q/2]$. So the minimum value of ρ on D_q is obtained for $k_2 = 1$ and $k_1 = k_3 = k-1$, where $k = [q/2]$. This implies the first equality in (51). The corresponding inequalities in (50) follow.

For the maximum we argue similarly, and by monotonicity property of ρ its maximum is obtained for $k_2 = k-1$ and $k_1 = k_3 = 1$.

□

Now we prove a stability result depending on the cross ratio of four directions.

Theorem 5.8. *If K and K' are two planar compact convex bodies with the same X-rays in a set S of four directions with a given cross ratio $r \geq 2$, then*

$$|K \triangle K'| \leq |K \cap K'| \ \left((1-r^{-1})^{-1/2} - 1\right). \tag{52}$$

Proof. If $q(S) = +\infty$ then $K = K'$. Otherwise, if $q(S) < +\infty$, from Lemma 5.6 there exist four directions, in $\tilde{W}^4_{q(S)}$ having cross ratio r. Hence from (51) we get

$$r \leq \max_{\tilde{W}^4_{q(S)}} \rho \leq \frac{1}{\sin^2 \frac{\pi}{q(S)}}.$$

This implies that

$$\cos \frac{\pi}{q(S)} \geq \sqrt{1 - r^{-1}}.$$

By using this estimate of $q(S)$ in Theorem 5.2 we get (52). □

The previous stability estimate extends to the case when the cross ratio r is given with error. Roughly speaking, the previous theorem means that if we choose a cross ratio r such that r is big enough then exact parallel X-rays along four directions with cross ratio approximately r determine a convex body K up to an area distance which goes to zero when r goes to $+\infty$. Moreover the order of this stability is $1/2$ with respect to $1/r$. Such a order of stability cannot to be compared with the order of stability with respect to $1/n$ in Theorem 1.1, but is a quantitative measure of the result of unique determination given in [2]. Moreover, since permutations of the directions allow us to reduce the cross ratio to be greater than 2, by (3) a similar stability estimate holds for other singular values, i.e in the following cases: i) ρ is close to zero, ii) ρ is close to 1, iii) ρ is close to ∞, which correspond to the singular points of j. More precisely

Corollary 5.9. *If K and K' are two planar convex bodies in a given compact set with the same X-rays in four different directions with cross ratio ρ, then*

$$\| \chi_K - \chi_{K'} \|_{L_2} \quad = \quad O(1/\sqrt[4]{j(\rho)}) \qquad for \quad \rho \to \alpha,$$

where $\alpha \in \{0, 1, \infty\}$.

Proof. From the stability estimate (52) for $\rho \geq 2$ we get

$$|K \triangle K'|/|K \cap K'| \leq \frac{2(\sqrt{2}-1)}{\rho}.$$

Similarly by (3)

$$\begin{aligned}
|K \triangle K'|/|K \cap K'| \quad &\leq \quad \frac{2(\sqrt{2}-1)}{1-\rho} \qquad for \quad \rho < -1, \\
&\leq \quad 2(\sqrt{2}-1)\frac{\rho-1}{\rho} \qquad for \quad 1 < \rho \leq 2, \\
&\leq \quad 2(\sqrt{2}-1)(1-\rho) \qquad for \quad \frac{1}{2} \leq \rho < 1, \\
&\leq \quad 2(\sqrt{2}-1)\rho \qquad for \quad 0 < \rho \leq \frac{1}{2}, \\
&\leq \quad 2(\sqrt{2}-1)\frac{\rho}{\rho-1} \qquad for \quad -1 \leq \rho < 0.
\end{aligned}$$

□

At the end we remark that in order to obtain (1) and all the related stability estimates we used only the fact that S consists of n distinct directions, without exploiting any other property of S. This allows us to obtain the same stability results for the reconstruction by means of X-rays in directions in a limited angle problem. This has to be compared with the tomographic problem of reconstructing, in a limited angle, a planar homogeneous set, possibly not convex, which is expected to be severely ill-posed (see [14, Chapter VI]).

6. Stability estimates for -1-chord functions

In the previous sections we considered parallel X-rays. We reformulate our previous results to X-rays emanating from collinear finite points. Let l_u be the line through the origin parallel to $u \in S^1$. The point X-ray of a planar convex body K at p gives the length of each chord of K lying on the line through p in the direction u, i.e. the length of $K \cap (l_u + p)$. The results of the previous sections are mainly related with the -1-chord functions of K. We recall only the necessary definitions, most introduced in [4, Chapter 5 and 6] where an overview of these arguments and a complete classification of all sets of points for which the point X-rays determine a convex body K can be found.

Let o be the origin. Let us observe that any line through o meets K in a (possibly degenerate) line segment. The radial function ρ_K with respect to the origin o is defined by

$$\rho_K(x) = \max\{c : cx \in K\}$$

for $x \neq 0$. Usually we work with the restriction of ρ_K to the unit vector u. If the ray $\{cu,\ c > 0\}$ does not intersect K then u does not belong to the domain of ρ_K. Let $i \neq 0$ and $o \notin K$. The *i-chord function* $\rho_{i,K}$ of K at o is defined by

$$\rho_{i,K}(u) = ||\rho_k(u)|^i - |\rho_K(-u)|^i|,$$

for a unit vector u in the domain of ρ_K. If u is not in the domain of ρ_K, we define $\rho_{i,K}(u) = 0$. Thus the i-chord function of K at o is defined for any unit vector u. The 1-chord function of K at o coincides with the point X-rays of K at o. There is an interesting duality between -1-chord functions and parallel X-rays, described in next proposition [4, Theorem 6.2.8].

Proposition 6.1. *Let K and K' be convex bodies not meeting a line l. Suppose that ϕ is a nonsingular projective transformation taking l to the line at infinity. If p is a finite point on l, then K and K' have equal -1-chord functions at p if and only if ϕK and $\phi K'$ are convex bodies with equal parallel X-rays in the direction corresponding to ϕp.*

In the sequel we assume that l is the x-axis, so that ϕ is of the form

$$\phi(x, y) = \left(\frac{a_1x + b_1y + c_1}{y}, \frac{a_2x + b_2y + c_2}{y}\right).$$

Then ϕ is nonsingular if and only if $a_1c_2 \neq a_2c_1$.

Let $i \in \mathbb{R}$. For any bounded measurable set E in the plane, let

$$\nu_i(E) = \int\int_E |y|^{i-2}dxdy.$$

It easy to see that under the previous projective transformation ϕ the Lebesgue measure of $\phi(E)$ is a multiple of $\nu_{-1}(E)$. The results for parallel X-rays in previous sections can be reformulated for -1-chord functions, via a suitable projective transformation. Thus from the previous proposition and Theorem 1.1, we have

Theorem 6.2. *Let K and K' be convex bodies not meeting a line l. If K and K' have equal -1-chord functions at distinct points $p_1, \dots, p_n \in l$, then*

$$\nu_{-1}(K \triangle K') \leq \nu_{-1}(K \cap K') \frac{1 - \cos(\pi/n)}{\cos(\pi/n)}. \tag{53}$$

Equality holds if and only if $p_1, \dots, p_n$ are equivalent under a projective transformation ϕ to a subset of directions of the edges of a regular n-gon, and ϕK , $\phi K'$ are rotated regular n-gons.

Let PL be the space of nonsingular projective transformation of the plane. Definition (42) can be reformulated for a set of collinear points $\tau = \{p_1, \dots, p_n\}$ as follows

$$q(\tau) = \min\{q \in \mathbb{N} : \exists \phi \in PL,\ \phi(\tau) \subseteq \Theta_q\}. \tag{54}$$

Therefore Theorem 5.2 is equivalent to

Theorem 6.3. *Let K and K' be convex bodies not meeting the line l. If K and K' have equal -1-chord functions at distinct points $p_1, \dots, p_n \in l$, then*

$$\nu_{-1}(K \triangle K') \leq \nu_{-1}(K \cap K') \frac{1 - \cos(\pi/q(\tau))}{\cos(\pi/q(\tau))}. \tag{55}$$

Finally to Theorem 5.8 corresponds

Theorem 6.4. *Let K and K' be convex bodies with equal -1-chord functions at distinct collinear four points $p_1, \dots, p_4$ on a line l not intersecting K and K'. If their cross ratio $r \geq 2$ then*

$$\nu_{-1}(K \triangle K') \leq \nu_{-1}(K \cap K') \ \left((1 - r^{-1})^{-1/2} - 1\right). \tag{56}$$

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