Analytical results for 2-D non-rectilinear waveguides based on a Green's function

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SUMMARY

We consider the problem of wave propagation for a 2-D rectilinear optical waveguide which presents some perturbation. We construct a mathematical framework to study such a problem and prove the existence of a solution for the case of small imperfections. Our results are based on the knowledge of a Green's function for the rectilinear case. Copyright © 2008 John Wiley & Sons, Ltd.

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1. INTRODUCTION

An optical waveguide is a dielectric structure that guides and confines an optical signal along a desired path. Probably, the best known example is the optical fibre, where the light signal is confined in a cylindrical structure. Optical waveguides are largely used in long distance communications, integrated optics and many other applications.

In a rectilinear optical waveguide (Figure 1), the central region (the *core*) is surrounded by a layer with a lower index of refraction called *cladding*. A protective *jacket* covers the cladding. The difference between the indices of refraction of core and cladding makes it possible to guide an optical signal and to confine its energy in proximity of the core.

In recent years, the growing interest in optical integrated circuits stimulated the study of waveguides with different geometries. In fact, electromagnetic wave propagation along perturbed waveguides is still continuing to be widely investigated because of its importance in the design of optical devices such as couplers, tapers, gratings, bending imperfections of structures and so on (see, for instance, [1, 2]).

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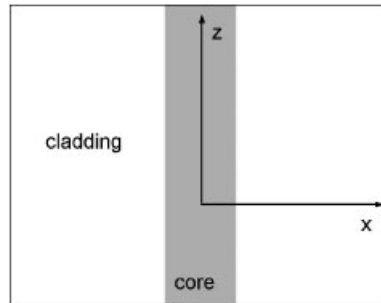†E-mail: ciraolo@math.unifi.it

Figure 1. The geometry of a rectilinear waveguide. The index of refraction depends only on the variable x, which is transversal to the axis of the waveguide.

In this paper we propose an analytical approach to the study of non-rectilinear waveguides. In particular, we will assume that the waveguide is a small perturbation of a rectilinear one and, in such a case, we prove a theorem that guarantees the existence of a solution.

There are two relevant ways of modelling wave propagation in optical waveguides. In *closed waveguides* one considers a tubular neighbourhood of the core and imposes Dirichlet, Neumann or Robin conditions on its boundary (see [3] and references therein). The use of these boundary conditions is efficient but somewhat artificial, since it creates spurious waves reflected by the interface jacket cladding. In this paper we will study *open waveguides*, i.e. we will assume that the cladding (or the jacket) extends to infinity. This choice provides a more accurate model to study the energy radiated outside the core (see [4, 5]).

Thinking of an optical signal as a superposition of waves of different frequencies (the *modes*), it is observed that in a rectilinear waveguide most of the energy provided by the source propagates as a finite number of such waves (the *guided modes*). The guided modes are mostly confined in the core; they decay exponentially transversally to the waveguide's axis and propagate along that axis without any significant loss of energy. The rest of the energy (the *radiating energy*) is made of *radiation* and *evanescent* modes, according to their different behaviour along the waveguide's axis (see Section 3 for further details). The electromagnetic field can be represented as a discrete sum of guided modes and a continuous sum of radiation and evanescent modes.

As already mentioned, in this paper we shall present an analytical approach to the study of time harmonic wave propagation in perturbed 2-D optical waveguides (see Figure 2(a)). As a model equation, we will use the following *Helmholtz equation* (or *reduced wave equation*):

$$\Delta u(x,z) + k^2 n(x,z)^2 u(x,z) = f(x,z) \tag{1}$$

with $(x,z) \in \mathbb{R}^2$, where $n(x,z)$ is the index of refraction of the waveguide, k is the wavenumber and f is a function representing a source. The axis of the waveguide is assumed to be the z-axis, while x denotes the transversal coordinate.

Our work is strictly connected to the results in [6], where the authors derived a resolution formula for (1), obtained as a superposition of guided, radiation and evanescent modes, in the case in which the function n is of the form

$$n := n_0(x) = \begin{cases} n_{\mathrm{co}}(x), & |x| \leqslant h \\ n_{\mathrm{cl}}, & |x| > h \end{cases} \tag{2}$$
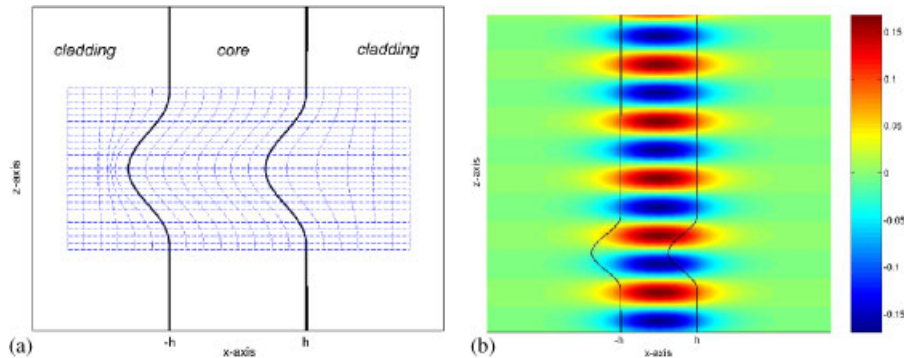
Figure 2. The perturbed waveguide and the real part of $w^{(0)}$: (a) the perturbed waveguide. The dashed lines show the effect of Γ on the plane. In particular, they show how a rectangular grid in the (s,t)-plane is mapped in the (x,z)-plane and (b) real part of $w^{(0)}$. Here, $w^{(0)}$ is a pure guided mode supported by the waveguide in the rectilinear configuration.

where n_{co} is a bounded function decreasing along the positive direction and $2h$ is the width of the core. Such a choice of n corresponds to an index of refraction depending only on the transversal coordinate and, thus, (1) describes the electromagnetic wave propagation in a rectilinear open waveguide. By using the approach proposed in [7], the results in [6] have been generalized in [8] to the case in which the index of refraction is not necessarily decreasing along the positive direction. The use of a rigorous transform theory guarantees that the superposition of guided, radiation and evanescent modes is complete. Such results are recalled in Section 3. The problem of studying the uniqueness of the obtained solution and its outgoing nature has been studied in [9].

In this paper we shall study small perturbations of rectilinear waveguides and present a mathematical framework that allows us to study the problem of wave propagation in perturbed waveguides. In particular, we shall assume that it is possible to find a diffeomorphism of $\mathbb{R}^2$ such that the non-rectilinear waveguide is mapped in a rectilinear one. Thanks to our knowledge of a Green's function for the rectilinear case, we are able to prove the existence of a solution for small perturbations of 2-D rectilinear waveguides by using the contraction mapping theorem.

In order to use such a theorem, we shall prove that an inverse of the operator obtained by linearizing the problem is continuous (see Theorem 5.5). Such a problem has been solved by using weighted Sobolev spaces, which are commonly used when dealing with the Helmholtz equation (see, for instance, [10]).

In a forthcoming work, the results obtained in this paper will be used to show several interesting numerical results for the applications.

In Section 2 we describe our mathematical framework for studying non-rectilinear waveguides. Since our results are based on the knowledge of a Green's function for rectilinear waveguides, in Section 3 we recall the main results obtained in [6]. Section 4 will be devoted to some technical lemmas needed in Section 5. The existence of a solution for the problem of perturbed waveguides will be proven in Theorem 5.5. Crucial to our construction are the estimates contained in Section 5, in particular, the ones in Theorem 5.4. In Section 6, we show how to apply our results to compute the first-order approximation of the solution of the perturbed problem. Appendix A contains results on the global regularity for solutions of the Helmholtz equation in $\mathbb{R}^N$, $N \geqslant 2$, that we need in Theorem 5.5.

2. NON-RECTILINEAR WAVEGUIDES: FRAMEWORK DESCRIPTION

When a rectilinear waveguide has some imperfection or the waveguide slightly bends from the rectilinear position, we cannot assume that its index of refraction n depends only on the transversal coordinate x (see Figure 2(a)). From the mathematical point of view, in this case, we shall study the Helmholtz equation:

$$\Delta u + k^2 n_\varepsilon(x,z)^2 u = f \quad \text{in } \mathbb{R}^2 \tag{3}$$

where $n_\varepsilon(x,z)$ is a perturbation of the function $n_0(x)$ defined in (2), representing a 'perfect' rectilinear configuration.

We denote by L_0 and L_ε the Helmholtz operators corresponding to $n_0(x)$ and $n_\varepsilon(x,z)$, respectively:

$$L_0 = \Delta + k^2 n_0(x)^2, \quad L_\varepsilon = \Delta + k^2 n_\varepsilon(x,z)^2 \tag{4}$$

In [6], the authors found a resolution formula for

$$L_0 u = f$$

i.e. they were able to express explicitly (in terms of a Green's function) an inverse of the operator L_0 and then a solution of (1). By making an abuse of notation, the inverse of L_0 found in [6] will be denoted here by L_0^{-1}. The issue of prescribing the appropriate boundary conditions for the problem will not be discussed here and we defer to [9] for a detailed study.

Now, we want to use L_0^{-1} to write higher-order approximations of solutions of (3), i.e. of

$$L_\varepsilon u = f \tag{5}$$

The existence of a solution of (5) will be proven in Theorem 5.5 by using a standard fixed point argument: since (5) is equivalent to

$$L_0 u = f + (L_0 - L_\varepsilon)u$$

we have

$$u = L_0^{-1} f + \varepsilon L_0^{-1} \left(\frac{L_0 - L_\varepsilon}{\varepsilon} \right) u$$

Our goal is to find suitable function spaces on which L_0^{-1} and $(L_0 - L_\varepsilon)/\varepsilon$ are continuous; then, by choosing ε sufficiently small, the existence of a solution will follow by the contraction mapping theorem.

It is clear that this procedure can be extended to more general elliptic operators; in Section 5 we will provide the details.

3. A GREEN'S FUNCTION FOR RECTILINEAR WAVEGUIDES

In this section we recall the expression of the Green's formula obtained by Magnanini and Santosa in [6] and generalized in [8] to a non-symmetric index of refraction.

We look for solutions of the homogeneous equation associated to (1) in the form

$$u(x,z) = v(x,\lambda)e^{ik\beta z}$$

$v(x,\lambda)$ satisfies the associated eigenvalue problem for v:

$$v'' + [\lambda - q(x)]v = 0 \quad \text{in } \mathbb{R} \tag{6}$$

with

$$n_* = \max_{\mathbb{R}} n, \quad \lambda = k^2(n_*^2 - \beta^2), \quad q(x) = k^2[n_*^2 - n(x)^2] \tag{7}$$

The solutions of (6) can be expressed in the following form:

$$v_j(x,\lambda) = \begin{cases} \phi_j(h,\lambda)\cos Q(x-h) + \dfrac{\phi_j'(h,\lambda)}{Q}\sin Q(x-h) & \text{if } x > h \\ \phi_j(x,\lambda) & \text{if } |x| \leqslant h \\ \phi_j(-h,\lambda)\cos Q(x+h) + \dfrac{\phi_j'(-h,\lambda)}{Q}\sin Q(x+h) & \text{if } x < -h \end{cases} \tag{8}$$

for $j = s,a$, with $Q = \sqrt{\lambda - d^2}$, $d^2 = k^2(n_*^2 - n_{\text{cl}}^2)$ and where the ϕ_j's are solutions of (6) in the interval $(-h,h)$ and satisfy the following conditions:

$$\begin{aligned} \phi_s(0,\lambda) &= 1, \quad \phi_s'(0,\lambda) = 0 \\ \phi_a(0,\lambda) &= 0, \quad \phi_a'(0,\lambda) = \sqrt{\lambda} \end{aligned} \tag{9}$$

The indices $j = s,a$ correspond to symmetric and antisymmetric solutions, respectively.

Remark 1 (*Classification of solutions*)
The eigenvalue problem (6) leads to three different types of solutions of (1) of the form $u_\beta(x,z) = v(x,\lambda)e^{ik\beta z}$.

- *Guided modes*: $0 < \lambda < d^2$. There exist a finite number of eigenvalues λ_m^j, $m = 1,\ldots,M_j$, satisfying the equations

$$\sqrt{d^2 - \lambda}\,\phi_j(h,\lambda) + \phi_j'(h,\lambda) = 0, \quad j \in \{s,a\}$$

and corresponding eigenfunctions $v_j(x,\lambda_m^j)$ which satisfy (6). In this case, $v_j(x,\lambda_m^j)$ decays exponentially for $|x| > h$:

$$v_j(x,\lambda_m^j) = \begin{cases} \phi_j(h,\lambda_m^j)e^{-\sqrt{d^2 - \lambda_m^j}(x-h)}, & x > h \\ \phi_j(x,\lambda_m^j), & |x| \leqslant h \\ \phi_j(-h,\lambda_m^j)e^{\sqrt{d^2 - \lambda_m^j}(x+h)}, & x < -h \end{cases}$$

In the z direction, u_β is bounded and oscillatory, because β is real.

- *Radiation modes*: $d^2 < \lambda < k^2 n_*^2$. In this case, u_β is bounded and oscillatory in both the x and z directions.
- *Evanescent modes*: $\lambda > k^2 n_*^2$. The functions v_j are bounded and oscillatory. In this case β becomes imaginary and hence u_β decays exponentially in one direction along the z-axis and increases exponentially in the other one.

By using the theory of Titchmarsh on eigenfunction expansion (see [17]), we can express a Green's function for (1) as superposition of guided, radiation and evanescent modes:

$$G(x, z; \xi, \zeta) = \sum_{j \in \{s, a\}} \int_0^{+\infty} \frac{e^{i|z - \zeta| \sqrt{k^2 n_*^2 - \lambda}}}{2i \sqrt{k^2 n_*^2 - \lambda}} v_j(x, \lambda) v_j(\xi, \lambda) \, d\rho_j(\lambda) \tag{10}$$

with

$$\langle d\rho_j, \eta \rangle = \sum_{m=1}^{M_j} r_m^j \eta(\lambda_m^j) + \frac{1}{2\pi} \int_{d^2}^{+\infty} \frac{\sqrt{\lambda - d^2}}{(\lambda - d^2) \phi_j(h, \lambda)^2 + \phi_j'(h, \lambda)^2} \eta(\lambda) \, d\lambda$$

for all $\eta \in C_0^\infty(\mathbb{R})$, where

$$r_m^j = \left[\int_{-\infty}^{+\infty} v_j(x, \lambda_m^j)^2 \, dx \right]^{-1} = \frac{\sqrt{d^2 - \lambda_m^j}}{\sqrt{d^2 - \lambda_m^j} \int_{-h}^h \phi_j(x, \lambda_m^j)^2 \, dx + \phi_j(h, \lambda_m^j)^2}$$

and where $v_j(x, \lambda)$ are defined by (8) (see [8] for further details).

We notice that (10) can be split up into three summands:

$$G = G^g + G^r + G^e$$

where

$$G^g(x, z; \xi, \zeta) = \sum_{j \in \{s, a\}} \sum_{m=1}^{M_j} \frac{e^{i|z - \zeta| \sqrt{k^2 n_*^2 - \lambda_m^j}}}{2i \sqrt{k^2 n_*^2 - \lambda_m^j}} v_j(x, \lambda_m^j) v_j(\xi, \lambda_m^j) r_m^j \tag{11a}$$

$$G^r(x, z; \xi, \zeta) = \frac{1}{2\pi} \sum_{j \in \{s, a\}} \int_{d^2}^{k^2 n_*^2} \frac{e^{i|z - \zeta| \sqrt{k^2 n_*^2 - \lambda}}}{2i \sqrt{k^2 n_*^2 - \lambda}} v_j(x, \lambda) v_j(\xi, \lambda) \sigma_j(\lambda) \, d\lambda \tag{11b}$$

$$G^e(x, z; \xi, \zeta) = -\frac{1}{2\pi} \sum_{j \in \{s, a\}} \int_{k^2 n_*^2}^{+\infty} \frac{e^{-|z - \zeta| \sqrt{\lambda - k^2 n_*^2}}}{2 \sqrt{\lambda - k^2 n_*^2}} v_j(x, \lambda) v_j(\xi, \lambda) \sigma_j(\lambda) \, d\lambda \tag{11c}$$

with

$$\sigma_j(\lambda) = \frac{\sqrt{\lambda - d^2}}{(\lambda - d^2) \phi_j(h, \lambda)^2 + \phi_j'(h, \lambda)^2} \tag{12}$$

G^g represents the guided part of the Green's function, which describes the guided modes, i.e. the modes propagating mainly inside the core; G^r and G^e are the parts of the Green's function

corresponding to the radiation and evanescent modes, respectively. The radiation and evanescent components altogether form the radiating part G^{rad} of G:

$$G^{\mathrm{rad}} = G^{\mathrm{r}} + G^{\mathrm{e}} = \frac{1}{2\pi} \sum_{j \in \{s,a\}} \int_{d^2}^{+\infty} \frac{\mathrm{e}^{\mathrm{i}|z-\zeta|\sqrt{k^2 n_*^2 - \lambda}}}{2\mathrm{i}\sqrt{k^2 n_*^2 - \lambda}} v_j(x,\lambda) v_j(\xi,\lambda) \sigma_j(\lambda) \, \mathrm{d}\lambda \tag{13}$$

4. ASYMPTOTIC LEMMAS

This section contains some lemmas that will be useful in the remainder of this paper. We note that, by assuming that n is a bounded function, Theorem 2.2.1 in [11] guarantees that $\phi_j(x,\lambda)$ and $\phi_j'(x,\lambda)$, $j \in \{s,a\}$ are absolutely continuous functions of the x-variable.

Lemma 4.1
Let $\lambda_0 = \min(\lambda_1^s, \lambda_1^a)$, where λ_1^s and λ_1^a are defined in Remark 1. Let ϕ_j, $j \in \{s,a\}$, be defined by (8). Then, the following estimates hold for $x \in [-h, h]$ and $\lambda \geqslant \lambda_0$:

$$|\phi_j(x,\lambda)| \leqslant \Phi_*, \quad |\phi_j'(x,\lambda)| \leqslant \Phi_* \sqrt{\lambda} \tag{14}$$

where

$$\Phi_* := \exp \left\{ \frac{1}{2\sqrt{\lambda_0}} \int_{-h}^{h} |q(t)| \, \mathrm{d}t \right\} \tag{15}$$

Proof
This lemma is a particular case of Theorem 2.5.3 in [11] and hence its proof is omitted. □

In the next two lemmas we study the asymptotic behaviour of the function $\sigma_j(\lambda)$ as $\lambda \to +\infty$ and $\lambda \to d^2$, respectively.

Lemma 4.2
Let $\sigma_j(\lambda)$, $j \in \{s,a\}$, be the quantities defined in (12). The following asymptotic expansions hold as $\lambda \to \infty$:

$$\sigma_s(\lambda) = \frac{1}{\sqrt{\lambda - d^2}} + \mathcal{O}\left(\frac{1}{\lambda}\right), \quad \sigma_a(\lambda) = \frac{1}{\sqrt{\lambda}} + \mathcal{O}\left(\frac{1}{\lambda}\right) \tag{16}$$

Proof
By multiplying

$$\phi_j''(x,\lambda) + [\lambda - q(x)]\phi_j(x,\lambda) = 0, \quad x \in [-h, h]$$

by $\phi_j'(x,\lambda)$ and integrating in x over $(0,h)$, we find

$$\phi_j'(h,\lambda)^2 - \phi_j'(0,\lambda)^2 + (\lambda - d^2)[\phi_j(h,\lambda)^2 - \phi_j(0,\lambda)^2] = 2 \int_0^h [q(x) - d^2]\phi_j(x,\lambda)\phi_j'(x,\lambda) \, \mathrm{d}x$$

Thus, by using (9), we obtain the following inequalities:

$$|\phi_s'(h,\lambda)^2 + (\lambda - d^2)\phi_s(h,\lambda)^2 - (\lambda - d^2)| \leqslant 2k^2(n_*^2 + n_{\mathrm{cl}}^2) \int_0^h |\phi_s(x,\lambda)\phi_s'(x,\lambda)| \, \mathrm{d}x$$

and

$$|\phi'_a(h,\lambda)^2 + (\lambda - d^2)\phi_a(h,\lambda)^2 - \lambda| \leqslant 2k^2(n_*^2 + n_{\text{cl}}^2)\int_0^h |\phi_a(x,\lambda)\phi'_a(x,\lambda)|\,\mathrm{d}x$$

The asymptotic formulas (16) follow from the two inequalities above, (12) and the bounds (14) for $\phi_j(x,\lambda)$ and $\phi'_j(x,\lambda)$. □

Lemma 4.3
Let $\sigma_j(\lambda)$, $j \in \{s,a\}$, be the quantities defined in (12). The following formulas hold for $\lambda \to d^2$:

$$\sigma_j(\lambda) = \begin{cases} \dfrac{\sqrt{\lambda - d^2}}{\phi'_j(h,d^2)^2} + \mathscr{O}(\lambda - d^2) & \text{if } \phi'_j(h,d^2) \neq 0 \\[2ex] \dfrac{1}{\phi_j(h,d^2)^2\sqrt{\lambda - d^2}} + \mathscr{O}(\sqrt{\lambda - d^2}) & \text{otherwise} \end{cases} \tag{17}$$

Proof
We recall that, if $q \in L^1_{\text{loc}}(\mathbb{R})$, for $x \in [-h,h]$, $\phi_s(x,\lambda)$ and $\phi_a(x,\lambda)$ are analytic in λ and $\sqrt{\lambda}$, respectively (see [12]). Thus, in a neighbourhood of $\lambda = d^2$, we express

$$\phi(h,\lambda) = \sum_{m=0}^{+\infty}(\lambda - d^2)^m a_m, \quad \phi'(h,\lambda) = \sum_{m=0}^{+\infty}(\lambda - d^2)^m b_m$$

$$\phi(h,\lambda)^2 = \sum_{m=0}^{+\infty}(\lambda - d^2)^m \alpha_m, \quad \phi'(h,\lambda)^2 = \sum_{m=0}^{+\infty}(\lambda - d^2)^m \beta_m$$

where we omitted the dependence on j to avoid very heavy notations.

We note that $\alpha_0 = a_0^2$, $\alpha_1 = 2a_0 a_1$ and the same for b_0 and b_1. From (12) we have

$$\sigma_j(\lambda)^{-1} = \sqrt{\lambda - d^2}\,\phi_j(h,\lambda)^2 + \frac{1}{\sqrt{\lambda - d^2}}\phi'_j(h,\lambda)^2$$

$$= \frac{\beta_0}{\sqrt{\lambda - d^2}} + \sum_{m=0}^{+\infty}(\lambda - d^2)^{m+1/2}(\alpha_m + \beta_{m+1}) \tag{18}$$

If $\beta_0 \neq 0$, since $\beta_0 = b_0^2$, (17) follows. If $\beta_0 = 0$ we have that the leading term in (18) is $\alpha_0 + \beta_1$. We note that $\alpha_0 \neq 0$; otherwise $\phi(x,d^2) \equiv 0$ for all $x \in \mathbb{R}$. We know that $\beta_1 = 2b_0 b_1 = 0$, because $b_0 = 0$. Then $\alpha_0 + \beta_1 = \alpha_0$ and (17) follows. □

5. EXISTENCE OF A SOLUTION

Let $\mu : \mathbb{R}^2 \to \mathbb{R}$ be a positive function. We will denote by $L^2(\mu)$ the weighted space consisting of all the complex-valued measurable functions $u(x,z)$, $(x,z) \in \mathbb{R}^2$, such that

$$\mu^{1/2}u \in L^2(\mathbb{R}^2)$$

equipped with the natural norm

$$\|u\|^2_{L^2(\mu)} = \int_{\mathbb{R}^2} |u(x,z)|^2 \mu(x,z) \, dx \, dz$$

In a similar way we define the weighted Sobolev spaces $H^1(\mu)$ and $H^2(\mu)$. The norms in $H^1(\mu)$ and $H^2(\mu)$ are given, respectively, by

$$\|u\|^2_{H^1(\mu)} = \int_{\mathbb{R}^2} |u(x,z)|^2 \mu(x,z) \, dx \, dz + \int_{\mathbb{R}^2} |\nabla u(x,z)|^2 \mu(x,z) \, dx \, dz$$

and

$$\|u\|^2_{H^2(\mu)} = \int_{\mathbb{R}^2} |u(x,z)|^2 \mu(x,z) \, dx \, dz + \int_{\mathbb{R}^2} |\nabla u(x,z)|^2 \mu(x,z) \, dx \, dz + \int_{\mathbb{R}^2} |\nabla^2 u(x,z)|^2 \mu(x,z) \, dx \, dz$$

Here, ∇u and $\nabla^2 u$ denote the gradient and Hessian matrix of u, respectively.

In this section we shall prove an existence theorem for the solutions of (5). We will make use of results on global regularity of the solution of (1); such results will be proven in Appendix A.

The proofs in this section and in Appendix A hold true whenever the weight μ has the following properties: $\mu \in C^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ is positive, bounded and such that

$$|\nabla \mu| \leqslant C_1 \mu, \quad |\nabla^2 \mu| \leqslant C_2 \mu \quad \text{in } \mathbb{R}^2 \tag{19}$$

where C_1 and C_2 are positive constants.

In this section, for the sake of simplicity, we will assume that μ is given by

$$\mu(x,z) = \mu_1(x)\mu_2(z) \tag{20}$$

with $\mu_j \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, $j = 1, 2$. Analogous results hold for every μ satisfying (19) and such that

$$\mu(x,z) \leqslant \mu_1(x)\mu_2(z) \tag{21}$$

For instance, it is easy to verify that the more commonly used weight function $\mu(x,z) = (1+x^2+z^2)^{-a}$, $a>1$, satisfies (19) and (21).

Before starting with the estimates on u, we prove a preliminary result on the boundness of the guided and radiated parts of the Green's function.

Lemma 5.1
Let G^g and G^r be the functions defined in (11a) and (11b), respectively. Then

$$|G^g(x,z;\xi,\zeta)| \leqslant \Phi^2_* \sum_{j \in \{s,a\}} \sum_{m=1}^{M_j} \frac{r_m^j}{2\sqrt{k^2 n_*^2 - \lambda_m^j}} \tag{22a}$$

and

$$|G^r(x,z;\xi,\zeta)| \leqslant \max \left\{ \frac{1}{2}, \frac{\Phi_*}{4\sqrt{\pi}} \sum_{j \in \{s,a\}} \Upsilon_j, \frac{\Phi^2_*}{2\pi} \sum_{j \in \{s,a\}} \Upsilon_j^2 \right\} \tag{22b}$$

Here,

$$\Upsilon_j = \left(\int_{d^2}^{k^2 n_*^2} \frac{\sigma_j(\lambda)}{2\sqrt{k^2 n_*^2 - \lambda}} \, \mathrm{d}\lambda \right)^{1/2}, \quad j \in \{s, a\}$$

where, as in Lemma 4.1, Φ_* is given by (15).

Proof

Since G^{g} is a finite sum, from Remark 1 and Lemma 4.1, it is easy to deduce (22a).

In the study of G^{r} we have to distinguish three different cases, according to whether (x, z) and (ξ, ζ) belong to the core or not. Furthermore, we observe that $\Upsilon_j < +\infty$ as follows form Lemma 4.3.

Case 1: $x, \xi \in [-h, h]$. From Lemma 4.1 we have that $v_j(x, \lambda)$ are bounded by Φ_*. From (11b) we have

$$|G^{\mathrm{r}}| \leqslant \frac{1}{2\pi} \sum_{j \in \{s, a\}} \Phi_*^2 \Upsilon_j^2$$

Case 2: $|x|, |\xi| \geqslant h$. We can use the explicit formula for v_j (see (8)) and obtain by the Hölder inequality:

$$|v_j(x, \lambda)| \leqslant \sqrt{\phi_j(h, \lambda)^2 + Q^{-2} \phi_j'(h, \lambda)^2} = [Q \, \sigma_j(\lambda)]^{-1/2}$$

Therefore, we have

$$|G^{\mathrm{r}}(x, z; \xi, \zeta)| \leqslant \frac{1}{2\pi} \sum_{j \in \{s, a\}} \int_{d^2}^{k^2 n_*^2} \frac{\mathrm{d}\lambda}{2\sqrt{\lambda - d^2} \sqrt{k^2 n_*^2 - \lambda}} = \frac{1}{2}$$

and hence (22b) follows.

Case 3: $|x| \leqslant h$ and $|\xi| \geqslant h$. We estimate $|v_j(x, \lambda)|$ by Φ_* and $v_j(\xi, \lambda)$ by $[Q \sigma_j(\lambda)]^{-1/2}$ and express

$$|G^{\mathrm{r}}(x, z; \xi, \zeta)| \leqslant \frac{1}{2\pi} \Phi_* \sum_{j \in \{s, a\}} \int_{d^2}^{k^2 n_*^2} \frac{1}{2\sqrt{k^2 n_*^2 - \lambda}} \left[\frac{\sigma_j(\lambda)}{Q} \right]^{1/2} \mathrm{d}\lambda$$

$$\leqslant \frac{1}{4\pi} \Phi_* \sum_{j \in \{s, a\}} \Upsilon_j \left(\int_{d^2}^{k^2 n_*^2} \frac{\mathrm{d}\lambda}{\sqrt{\lambda - d^2} \sqrt{k^2 n_*^2 - \lambda}} \right)^{1/2}$$

Again (22b) follows.   □

In the next lemma we prove estimates that will be useful in Theorem 5.4.

Lemma 5.2

Let $\mu_1 \in L^1(\mathbb{R})$ and $\mu_2 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. For $j, l \in \{s, a\}$ and $\lambda, \eta \geqslant k^2 n_*^2$, set

$$p_{j,l}(\lambda, \eta) = \int_{-\infty}^{+\infty} v_j(x, \lambda) v_l(x, \eta) \mu_1(x) \, \mathrm{d}x \tag{23}$$

and

$$q(\lambda, \eta) = \int_{-\infty}^{+\infty} (e_{\lambda,\eta} \star \mu_2)(z)\mu_2(z)\,\mathrm{d}z \tag{24}$$

where $e_{\lambda,\eta}(z) = \mathrm{e}^{-|z|(\sqrt{\lambda - k^2 n_*^2} + \sqrt{\eta - k^2 n_*^2})}$.

Then $p_{j,l}(\lambda, \eta) = 0$ for $j \neq l$,

$$p_{j,j}(\lambda, \eta)^2 \sigma_j(\lambda)\sigma_j(\eta) \leqslant 4\|\mu_1\|_1^2 \left(\Phi_*^2 \sigma_j(\lambda) + \frac{1}{\sqrt{\lambda - d^2}} \right) \left(\Phi_*^2 \sigma_j(\eta) + \frac{1}{\sqrt{\eta - d^2}} \right) \tag{25}$$

and

$$|q(\lambda, \eta)| \leqslant \min\left(\|\mu_2\|_1^2, \frac{\|\mu_2\|_2^2}{\sqrt[4]{\lambda - k^2 n_*^2}\sqrt[4]{\eta - k^2 n_*^2}} \right) \tag{26}$$

Proof
Since v_s and v_a are, respectively, even and odd functions of x, $p_{j,l}(\lambda, \eta) = 0$ for $j \neq l$.

By the Hölder inequality, $p_{j,j}(\lambda, \eta)^2 \leqslant p_{j,j}(\lambda, \lambda)p_{j,j}(\eta, \eta)$; also, by formula (8), we obtain

$$|p_{j,j}(\lambda, \lambda)| \leqslant 2\int_0^h |\phi_j(x,\lambda)|^2 |\mu_1(x)|\,\mathrm{d}x + 2\int_h^{+\infty} |v_j(x,\lambda)|^2 |\mu_1(x)|\,\mathrm{d}x$$

$$\leqslant 2\Phi_*^2 \int_0^h |\mu_1(x)|\,\mathrm{d}x + \frac{2}{\sqrt{\lambda - d^2}\sigma_j(\lambda)} \int_h^{+\infty} |\mu_1(x)|\,\mathrm{d}x$$

and hence (25).

Now we have to estimate $q(\lambda, \eta)$. Firstly, we observe that

$$|q(\lambda, \eta)| \leqslant \int_{-\infty}^{+\infty} \mu_2(z)\,\mathrm{d}z \int_{-\infty}^{+\infty} \mu_2(\zeta)\,\mathrm{d}\zeta = \|\mu_2\|_1^2$$

which proves part of (26). Furthermore, from Young's inequality (see Theorem 4.2 in [13]) and the arithmetic–geometric mean inequality, we have

$$|q(\lambda, \eta)| \leqslant \|e_{\lambda,\eta}\|_1 \|\mu_2\|_2^2 = \frac{2\|\mu_2\|_2^2}{\sqrt{\lambda - k^2 n_*^2} + \sqrt{\eta - k^2 n_*^2}}$$

$$\leqslant \frac{\|\mu_2\|_2^2}{\sqrt[4]{\lambda - k^2 n_*^2}\sqrt[4]{\eta - k^2 n_*^2}}$$

which completes the proof. □

Theorem 5.3
Let G be the Green's function (10) and let $\mu \in C^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ be a bounded positive function such that (19) and (20) hold. Then

$$\|G\|_{L^2(\mu \times \mu)} < +\infty \tag{27}$$

Proof

We write $G = G^g + G^r + G^e$, as in (11a)–(11c), and use Minkowski inequality:

$$\|G\|_{L^2(\mu \times \mu)} \leqslant \|G^g\|_{L^2(\mu \times \mu)} + \|G^r\|_{L^2(\mu \times \mu)} + \|G^e\|_{L^2(\mu \times \mu)} \tag{28}$$

From Lemma 5.1 and (19) it follows that

$$\|G^g\|_{L^2(\mu \times \mu)}, \|G^r\|_{L^2(\mu \times \mu)} < +\infty \tag{29}$$

It remains to prove that $\|G^e\|_{L^2(\mu \times \mu)} < +\infty$. From (20) we have

$$\|G^e\|^2_{L^2(\mu \times \mu)} = \int_{\mathbb{R}^2 \times \mathbb{R}^2} |G^e(x, z; \xi, \zeta)|^2 \mu_1(x) \mu_2(z) \mu_1(\xi) \mu_2(\zeta) \, \mathrm{d}x \, \mathrm{d}z \, \mathrm{d}\xi \, \mathrm{d}\zeta$$

$$= \int_{\mathbb{R}^2 \times \mathbb{R}^2} G^e(x, z; \xi, \zeta) \overline{G^e(x, z; \xi, \zeta)} \mu_1(x) \mu_2(z) \mu_1(\xi) \mu_2(\zeta) \, \mathrm{d}x \, \mathrm{d}z \, \mathrm{d}\xi \, \mathrm{d}\zeta$$

hence, thanks to Lemma 5.2, the definition (11c) of G^e and Fubini's theorem, we obtain

$$\|G^e\|^2_{L^2(\mu \times \mu)} \leqslant \frac{1}{16\pi^2} \sum_{j,l \in \{s,a\}} \int_{k^2 n_*^2}^{+\infty} \int_{k^2 n_*^2}^{+\infty} p_{j,l}(\lambda, \eta)^2 \sigma_j(\lambda) \sigma_l(\eta) \frac{q(\lambda, \eta) \, \mathrm{d}\lambda \, \mathrm{d}\eta}{\sqrt{\lambda - k^2 n_*^2} \sqrt{\eta - k^2 n_*^2}}$$

The conclusion follows from Lemmas 5.2, 4.2 and 4.3.   □

Corollary 5.4

Let u be the solution of (1) given by

$$u(x, z) = \int_{\mathbb{R}^2} G(x, z; \xi, \zeta) f(\xi, \zeta) \, \mathrm{d}\xi \, \mathrm{d}\zeta, \quad (x, z) \in \mathbb{R}^2 \tag{30}$$

with G as in (10) and let $f \in L^2(\mu^{-1})$. Then

$$\|u\|_{H^2(\mu)} \leqslant C \|f\|_{L^2(\mu^{-1})} \tag{31}$$

where

$$C^2 = \frac{5}{2} + 2C_2 + \left[\frac{3}{2} + 4C_2 + 8C_2^2 + (1 + 4C_2)k^2 n_*^2 + 2k^4 n_*^4\right] \|G\|^2_{L^2(\mu \times \mu)} \tag{32}$$

Proof

From (30) and by using the Hölder inequality, it follows that

$$\|u\|^2_{L^2(\mu)} = \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} G(x, z; \xi, \zeta) f(\xi, \zeta) \, \mathrm{d}\xi \, \mathrm{d}\zeta \right|^2 \mu(x, z) \, \mathrm{d}x \, \mathrm{d}z$$

$$\leqslant \|f\|^2_{L^2(\mu^{-1})} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |G(x, z; \xi, \zeta)|^2 \mu(\xi, \zeta) \mu(x, z) \, \mathrm{d}\xi \, \mathrm{d}\zeta \, \mathrm{d}x \, \mathrm{d}z$$

$$= \|f\|^2_{L^2(\mu^{-1})} \|G\|^2_{L^2(\mu \times \mu)}$$

and then we obtain (31) and (32) from Lemmas A.1 and A.3   □

Remark 2

In the next theorem we shall prove the existence of a solution of $L_\varepsilon u = f$. It will be useful to assume in general that L_ε is of the form

$$L_\varepsilon = \sum_{i,j=1}^{2} a_{ij}^\varepsilon \partial_{ij} + \sum_{i=1}^{2} b_i^\varepsilon \partial_i + c^\varepsilon \tag{33}$$

This choice of L_ε is motivated by our project to treat non-rectilinear waveguides. Our idea is that of transforming a non-rectilinear waveguide into a rectilinear one by a change of variables $\Gamma : \mathbb{R}^2 \to \mathbb{R}^2$.

For this reason, we suppose that Γ is a C^2 invertible function:

$$\Gamma(s,t) = (x(s,t), z(s,t))$$

By setting $w(s,t) = u(x,z)$, a solution u of (1) is converted into a solution w of

$$|\nabla s|^2 w_{ss} + |\nabla t|^2 w_{tt} + 2\nabla s \cdot \nabla t\, w_{st} + \Delta s \cdot w_s + \Delta t \cdot w_t + c(s,t)^2 w = F(s,t) \tag{34}$$

where $c(s,t) = kn(x(s,t), z(s,t))$ and $F(s,t) = f(x(s,t), z(s,t))$.

If our waveguide is a slight perturbation of a rectilinear one, we may choose Γ as a perturbation of the identity map

$$\Gamma(s,t) = (s + \varepsilon \varphi(s,t), t + \varepsilon \psi(s,t))$$

and obtain $L_\varepsilon w = F$ from (34), where L_ε is given by (33), with

$$a_{ij}^\varepsilon = \delta_{ij} + \varepsilon \tilde{a}_{ij}^\varepsilon, \ \ i,j = 1,2, \quad b_i = \varepsilon \tilde{b}_i^\varepsilon, \ \ i = 1,2, \quad c^\varepsilon = k^2 n_0(x)^2 + \varepsilon \tilde{c}^\varepsilon \tag{35}$$

we also may assume that

$$\left[\sum_{i,j=1}^{2} (\tilde{a}_{ij}^\varepsilon)^2 \right]^{1/2}, \quad \left[\sum_{i=1}^{2} (\tilde{b}_i^\varepsilon)^2 \right]^{1/2}, \quad |\tilde{c}^\varepsilon| \leqslant K\mu \quad \text{in } \mathbb{R}^2 \tag{36}$$

for some constant K independent of ε.

Theorem 5.5

Let L_ε be as in Remark 2 and let $f \in L^2(\mu^{-1})$. Then there exists a positive number ε_0 such that, for every $\varepsilon \in (0, \varepsilon_0)$, equation $L_\varepsilon u = f$ admits a (weak) solution $u^\varepsilon \in H^2(\mu)$.

Proof

We express

$$L_\varepsilon = L_0 + \varepsilon \tilde{L}_\varepsilon \tag{37}$$

clearly, the coefficients of $\tilde{L}_\varepsilon$ are $\tilde{a}_{ij}^\varepsilon$, $\tilde{b}_i^\varepsilon$ and $\tilde{c}^\varepsilon$ defined in (35). We can express (37) as

$$u + \varepsilon L_0^{-1} \tilde{L}_\varepsilon u = L_0^{-1} f$$

$L_0^{-1} f$ is nothing but the solution to (1) defined in (30).

We shall prove that $L_0^{-1}\tilde{L}_\varepsilon$ maps $H^2(\mu)$ continuously into itself. In fact, for $u \in H^2(\mu)$, we easily have

$$\|\tilde{L}_\varepsilon u\|_{L^2(\mu^{-1})}^2 \leqslant \int_{\mathbb{R}^2} \left[\sum_{i,j=1}^2 (\tilde{a}_{ij}^\varepsilon)^2 \sum_{i,j=1}^2 |u_{ij}|^2 + \sum_{i=1}^2 (\tilde{b}_i^\varepsilon)^2 \sum_{i=1}^2 |u_i|^2 + (\tilde{c}^\varepsilon)^2 |u|^2 \right] \mu^{-1}\, \mathrm{d}x\, \mathrm{d}z$$

$$\leqslant K^2 \|u\|_{H^2(\mu)}^2$$

Moreover, Corollary 5.4 implies that

$$\|L_0^{-1} f\|_{H^2(\mu)} \leqslant C \|f\|_{L^2(\mu^{-1})}$$

and hence

$$\|L_0^{-1}\tilde{L}_\varepsilon u\|_{H^2(\mu)} \leqslant CK \|u\|_{H^2(\mu)}$$

Therefore, we choose $\varepsilon_0 = (CK)^{-1}$ so that, for $\varepsilon \in (0, \varepsilon_0)$, the operator $\varepsilon L_0^{-1}\tilde{L}_\varepsilon$ is a contraction and hence our conclusion follows from Picard's fixed point theorem. □

6. NUMERICAL RESULTS

In this section we show how to apply our results to compute the first-order approximation of the solution of the perturbed problem. The example presented here has only an illustrative scope; a more extensive and rigorous description of the computational issues can be found in [8, 14], where we apply our results to real-life optical devices. We notice that, even if not fully comparable, our numerical results are consistent with the ones in [2], where the authors numerically study a problem similar to ours.

In this section we study a perturbed slab waveguide as the one shown in Figure 2(a). In the case of a rectilinear slab waveguide, it is possible to express the Green's formula explicitly and numerically evaluate it (see [6]).

Having in mind the approach proposed in Remark 2, we change the variables by using a C^2-function $\Gamma : \mathbb{R}^2 \to \mathbb{R}^2$ of the following form:

$$\Gamma(s, t) = (s, t + \varepsilon S(s)T(t))$$

where $S, T \in C_c^2(\mathbb{R})$; a good choice of S and T is represented in Figure 3. In Figure 2(a) we also show how Γ transforms the plane, by plotting in the (x, z)-plane the image of a rectangular grid in the (s, t)-plane.

By expanding L_ε and w by their Neumann series, we find that $w^{(0)}$ and $w^{(1)}$ (the zeroth- and first-order approximations of w, respectively) satisfy

$$\Delta w^{(0)} + k^2 n(s)^2 w^{(0)} = F(s, t) \tag{38a}$$

and

$$\Delta w^{(1)} + k^2 n(s)^2 w^{(1)} = -2S'(s)T(t)w_{ss}^{(0)} - 2S(s)T'(t)w_{st}^{(0)} - [S''(s)T(t) + S(s)T''(t)]w_t^{(0)} \tag{38b}$$
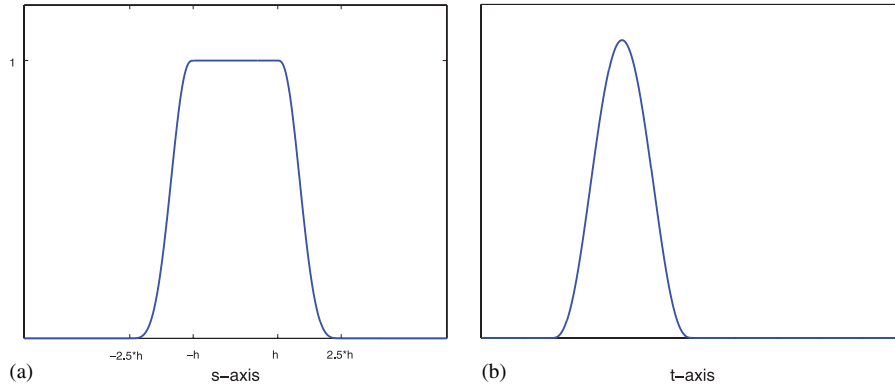
respectively.

Figure 3. Our choice of the functions S and T. Such a choice corresponds to a perturbed waveguide as in Figure 2(a): (a) the function S and (b) the function T.
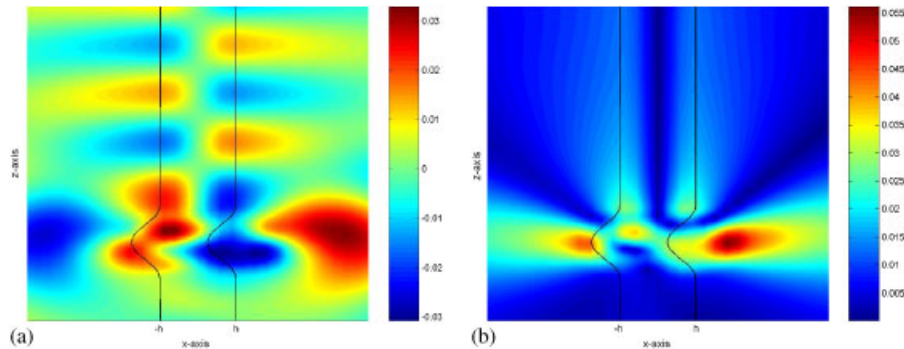

Figure 4. The real part and modulus of $w^{(1)}$ (the first-order approximation of w): (a) real part of $w^{(1)}$ and (b) modulus of $w^{(1)}$.

In our simulations, we assume that $w^{(0)}$ is a pure guided mode and calculate $w^{(1)}$ by using (38) and the Green's function (10). In other words, we take a special choice of f and see what happens to the propagation of a pure guided mode in the presence of an imperfection of the waveguide.

In Figures 2(b), 4 and 5, we set $k=5.0$, $h=0.2$, $n_{co}=2$, $n_{cl}=1$. With such parameters, the waveguide supports two guided modes, corresponding to the following values of the parameter λ: $\lambda_1^s = 23.7$ and $\lambda_1^a = 73.5$.

As already mentioned, we are assuming that $w^{(0)}$ is a pure guided mode. Here, $w^{(0)}$ is forward propagating and corresponds to λ_1^s:

$$w^{(0)}(s,t) = v_s(s, \lambda_1^s) e^{it\sqrt{k^2 n_*^2 - \lambda_1^s}}$$

the real part of $w^{(0)}$ is shown in Figure 2(b).

Figure 4(a) and (b) shows the real part and the absolute value of $w^{(1)}$, respectively. Here we do not write the numerical details of our computation and refer to [8, 14] for a more detailed description.
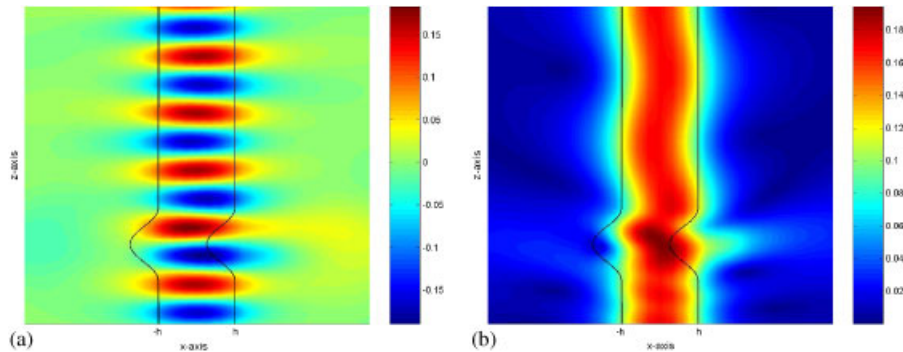
Figure 5. The real part and modulus of $w^{(0)} + \varepsilon w^{(1)}$. The pictures clearly show the effect of a perturbation of the waveguide: due to the presence of an imperfection, the waveguide does not support the pure guided mode $w^{(0)}$ and the other supported guided mode and the radiating energy appear: (a) real part of $w^{(0)} + \varepsilon w^{(1)}$ and (b) modulus of $w^{(0)} + \varepsilon w^{(1)}$.

In Figure 5(a) and (b) we show the real part and the absolute value of $w^{(0)} + \varepsilon w^{(1)}$, respectively. Here, we choose $\varepsilon = 1$ to emphasize the effect of the perturbation on the wave propagation. As is clear from Theorem 5.5, our existence result holds for $\varepsilon \in [0, \varepsilon_0]$, where $\varepsilon_0 = (CK)^{-1}$ (which will be presumably less than 1). The computation of ε_0 and the convergence of the Neumann series related to w have not been considered here; again, we refer to [8, 14] for a detailed study of such issues.

7. CONCLUSIONS

In this paper, we studied the electromagnetic wave propagation for non-rectilinear waveguides, assuming that the waveguide is a small perturbation of a rectilinear one. Thanks to the knowledge of a Green's function for the rectilinear configuration, we provided a mathematical framework by which the existence of a solution for the scalar 2-D Helmholtz equation in the perturbed case is proven. Our work is based on careful estimates in suitable weighted Sobolev spaces which allow us to use a standard fix-point argument.

For the case of a slab waveguide (piecewise constant indices of refraction), numerical examples were also presented. We showed that our approach provides a method for evaluating how imperfections of the waveguide affect the wave propagation of a pure guided mode.

In a forthcoming paper, we will address the computational issues arising from the design of optical devices.

APPENDIX A: REGULARITY RESULTS

In this Appendix we study the global regularity of weak solutions of the Helmholtz equation. Since our results hold in $\mathbb{R}^N$, $N \geqslant 2$, it will be useful to denote a point in $\mathbb{R}^N$ by x, i.e. $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$.

The results in this section can be found in the literature in a more general context for $N \geqslant 3$ (see [15]). Here, under stronger assumptions on n and μ, we provide an *ad hoc* treatment that holds for $N \geqslant 2$.

We will suppose $f \in L^2(\mu^{-1})$. Since μ is bounded, it is clear that $f \in L^2(\mu)$ too.

Lemma A.1
Let $u \in H^1_{\text{loc}}(\mathbb{R}^N)$ be a weak solution of

$$\Delta u + k^2 n(x)^2 u = f, \quad x \in \mathbb{R}^N \tag{A1}$$

with $n \in L^\infty(\mathbb{R}^N)$. Let μ satisfy the assumptions in (19). Then

$$\int_{\mathbb{R}^N} |\nabla u|^2 \mu \, dx \leqslant \frac{1}{2} \int_{\mathbb{R}^N} |f|^2 \mu \, dx + \left(2C_2 + k^2 n_*^2 + \frac{1}{2}\right) \int_{\mathbb{R}^N} |u|^2 \mu \, dx \tag{A2}$$

where $n_* = \|n\|_{L^\infty(\mathbb{R}^N)}$.

Proof
Let $\eta \in C_0^\infty(\mathbb{R}^N)$ be such that

$$\eta(0) = 1, \quad 0 \leqslant \eta \leqslant 1, \quad |\nabla \eta| \leqslant 1, \quad |\nabla^2 \eta| \leqslant 1 \tag{A3}$$

and consider the function defined by

$$\mu_m(x) = \mu(x) \eta\left(\frac{x}{m}\right) \tag{A4}$$

Then $\mu_m(x)$ increases with m and converges to $\mu(x)$ as $m \to +\infty$; furthermore,

$$\begin{aligned}
|\nabla \mu_m(x)| &\leqslant |\nabla \mu(x)| + \frac{1}{m} \mu(x) \leqslant \left(C_1 + \frac{1}{m}\right) \mu(x) \\
|\nabla^2 \mu_m(x)| &\leqslant |\nabla^2 \mu(x)| + \frac{2}{m} |\nabla \mu(x)| + \frac{1}{m^2} \mu(x) \leqslant \left(C_2 + \frac{2C_1}{m} + \frac{1}{m^2}\right) \mu(x)
\end{aligned} \tag{A5}$$

for every $x \in \mathbb{R}^N$.

Since u is a weak solution of (A1), we have that

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla \phi \, dx - k^2 \int_{\mathbb{R}^N} n(x)^2 u \phi \, dx = -\int_{\mathbb{R}^N} f \phi \, dx$$

for every $\phi \in H^1_{\text{loc}}(\mathbb{R}^N)$. We choose $\phi = \bar{u} \mu_m$ and obtain by Theorem 6.16 in [13]:

$$\int_{\mathbb{R}^N} |\nabla u|^2 \mu_m \, dx = -\int_{\mathbb{R}^N} \bar{u} \nabla u \cdot \nabla \mu_m \, dx + k^2 \int_{\mathbb{R}^N} n(x)^2 |u|^2 \mu_m \, dx - \int_{\mathbb{R}^N} f \bar{u} \mu_m \, dx \tag{A6}$$

Integration by parts gives

$$\text{Re} \int_{\mathbb{R}^N} \bar{u} \nabla u \cdot \nabla \mu_m \, dx = -\frac{1}{2} \int_{\mathbb{R}^N} |u|^2 \Delta \mu_m \, dx$$

hence, by considering the real part of (A6), we obtain

$$\int_{\mathbb{R}^N} |\nabla u|^2 \mu_m \, dx \leqslant 2\left(C_2 + \frac{2C_1}{m} + \frac{1}{m^2}\right)\int_{\mathbb{R}^N} |u|^2 \mu \, dx + k^2 n_*^2 \int_{\mathbb{R}^N} |u|^2 \mu_m \, dx$$

$$+ \left(\int_{\mathbb{R}^N} |f|^2 \mu_m \, dx\right)^{1/2}\left(\int_{\mathbb{R}^N} |u|^2 \mu_m \, dx\right)^{1/2}$$

here we have used (A5) and the Hölder inequality.

The Young inequality and the fact that $\mu_m \leqslant \mu$ then yield

$$\int_{\mathbb{R}^N} |\nabla u|^2 \mu_m \, dx \leqslant \left[2\left(C_2 + \frac{2C_1}{m} + \frac{1}{m^2}\right) + k^2 n_*^2 + \frac{1}{2}\right]\int_{\mathbb{R}^N} |u|^2 \mu \, dx + \frac{1}{2}\int_{\mathbb{R}^N} |f|^2 \mu \, dx$$

The conclusion then follows by the monotone convergence theorem. □

Lemma A.2
The following identity holds for every $u \in H_{loc}^2(\mathbb{R}^N)$ and every $\phi \in C_0^2(\mathbb{R}^N)$:

$$\int_{\mathbb{R}^N} |\Delta u|^2 \phi \, dx + \int_{\mathbb{R}^N} |\nabla u|^2 \Delta\phi \, dx = \int_{\mathbb{R}^N} |\nabla^2 u|^2 \phi \, dx + \text{Re}\int_{\mathbb{R}^N}(\nabla^2\phi\nabla u, \nabla u)\, dx \quad (A7)$$

Proof
It is obvious that, without loss of generality, we can assume that $u \in C^3(\mathbb{R}^N)$; a standard approximation argument will then lead to the conclusion.

For $u \in C^3(\mathbb{R}^N)$, (A7) follows by integrating over $\mathbb{R}^N$ the differential identity

$$\phi\sum_{i,j=1}^N u_{ii}\bar{u}_{jj} - \phi\sum_{i,j=1}^N u_{ij}\bar{u}_{ij} + \sum_{i,j=1}^N u_i\bar{u}_i\phi_{jj} - \text{Re}\sum_{i,j=1}^N u_i\bar{u}_j\phi_{ij}$$

$$= \text{Re}\left\{\sum_{i,j=1}^N [(\phi\bar{u}_j u_{ii})_j + (u_i\bar{u}_i\phi_j)_j - (\phi\bar{u}_j u_{ij})_i - (u_j\bar{u}_i\phi_j)_i]\right\}$$

and by the divergence theorem. □

Lemma A.3
Let $u \in H^1(\mu)$ be a weak solution of (A1). Then

$$\int_{\mathbb{R}^N} |\nabla^2 u|^2 \mu \, dx \leqslant 2\int_{\mathbb{R}^N} |f|^2 \mu \, dx + 2k^4 n_*^4 \int_{\mathbb{R}^N} |u|^2 \mu \, dx + 4C_2\int_{\mathbb{R}^N} |\nabla u|^2 \mu \, dx \quad (A8)$$

where C_2 is the constant in (A5).

Proof
From well-known interior regularity results on elliptic equations (see Theorem 8.8 in [16]), we have that if $u \in H_{loc}^1(\mathbb{R}^N)$ is a weak solution of (A1), then $u \in H_{loc}^2(\mathbb{R}^N)$. Then we can apply Lemma A.2 to u by choosing $\phi = \mu_m$:

$$\int_{\mathbb{R}^N} |\Delta u|^2 \mu_m \, dx + \int_{\mathbb{R}^N} |\nabla u|^2 \Delta\mu_m \, dx = \int_{\mathbb{R}^N} |\nabla^2 u|^2 \mu_m \, dx + \text{Re}\int_{\mathbb{R}^N}(\nabla^2\mu_m\nabla u, \nabla u)\, dx$$

From (A1), (A5) and the above formula, we have

$$\int_{\mathbb{R}^N} |\nabla^2 u|^2 \mu_m \, dx = \int_{\mathbb{R}^N} |f - k^2 n(x)^2 u|^2 \mu_m \, dx$$

$$+ \int_{\mathbb{R}^N} |\nabla u|^2 \Delta \mu_m \, dx - 2 \operatorname{Re} \int_{\mathbb{R}^N} (\nabla^2 \mu_m \nabla u, \nabla u) \, dx$$

$$\leqslant 2 \int_{\mathbb{R}^N} |f|^2 \mu_m \, dx + 2 k^4 n_*^4 \int_{\mathbb{R}^N} |u|^2 \mu_m \, dx$$

$$+ \left(C_2 + \frac{2C_1}{m} + \frac{1}{m^2} \right) \int_{\mathbb{R}^N} |\nabla u|^2 \mu \, dx + 2 \int_{\mathbb{R}^N} |\nabla^2 \mu_m| |\nabla u|^2 \, dx$$

$$\leqslant 2 \int_{\mathbb{R}^N} |f|^2 \mu_m \, dx + 2 k^4 n_*^4 \int_{\mathbb{R}^N} |u|^2 \mu_m \, dx$$

$$+ 3 \left(C_2 + \frac{2C_1}{m} + \frac{1}{m^2} \right) \int_{\mathbb{R}^N} |\nabla u|^2 \mu \, dx$$

Since $\mu_m \leqslant \mu$, the proof is completed by taking the limit as $m \to \infty$. $\qquad\square$

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REFERENCES

1. Ammari H, Iakovleva E, Kang H. Reconstruction of a small inclusion in a two-dimensional open waveguide. *SIAM Journal on Applied Mathematics* 2005; **65**(6):2107–2127.
2. Hesthaven JS, Dinesen PG, Lynov JP. Spectral collocation time-domain modeling of diffractive optical elements. *Journal of Computational Physics* 1999; **155**:287–306.
3. Oliner AA. Historical perspectives on microwave field theory. *IEEE Transactions on Microwave Theory and Techniques* 1984; **32**(9):1022–1045.
4. Snyder AW, Love D. *Optical Waveguide Theory*. Chapman & Hall: London, 1974.
5. Marcuse D. *Light Transmission Optics*. Van Nostrand Reinhold Company: New York, 1982.
6. Magnanini R, Santosa F. Wave propagation in a 2-D optical waveguide. *SIAM Journal on Applied Mathematics* 2001; **61**:1237–1252.
7. Alexandrov O, Ciraolo G. Wave propagation in a 3-D optical waveguide. *Mathematical Models and Methods in Applied Sciences* (*M3AS*) 2000; **14**(6):819–852.
8. Ciraolo G. Non-rectilinear waveguides: analytical and numerical results based on the Green's function. *Ph.D. Thesis*. (Available from: http://www.math.unifi.it/~ciraolo/.)
9. Ciraolo G, Magnanini R. A radiation condition for uniqueness in a wave propagation problem for 2-D open waveguides. arXiv:0710.1950.
10. Leis R. *Initial Boundary Value Problems in Mathematical Physics*. Wiley: New York, 1986.
11. Zettl A. *Sturm-Liouville Theory*. Mathematical Surveys and Monographs, vol. 121. American Mathematical Society: Providence, RI, 2005.
12. Coddington EA, Levinson N. *Theory of Ordinary Differential Equations*. McGraw-Hill: New York, 1955.
13. Lieb EH, Loss M. *Analysis*. American Mathematical Society: Providence, RI, 1997.

14. Ciraolo G. A method of variation of boundaries for waveguide grating couplers. Preprint. http://www.math.unifi. it/~ciraolo/.
15. Agmon S. Spectral properties of Schrödinger operators and scattering theory. *Annali della Scuola Normale Superiore di Pisa—Classe di Scienze Série 4* 1975; **2**:151–218.
16. Gilbarg D, Trudinger NS. *Elliptic Partial Differential Equations of Second Order*. Springer: Berlin, 1983.
17. Titchmarsh EC. *Eigenfunction Expansions Associated with Second-order Differential Equations*. Clarendon Press: Oxford, 1946.