POLYGONAL HEAT CONDUCTORS WITH A STATIONARY HOT SPOT *

By

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Abstract. We consider a convex polygonal heat conductor whose inscribed circle touches every side of the conductor. Initially, the conductor has constant temperature and, at every time, the temperature of its boundary is kept at zero. The hot spot is the point at which temperature attains its maximum at each given time. It is proved that, if the hot spot is stationary, then the conductor must satisfy two geometric conditions. In particular, we prove that these geometric conditions yield some symmetries provided the conductor is either pentagonal or hexagonal.

1 Introduction

A hot spot in a heat conductor is a point at which temperature attains its maximum at each given time. Let Ω be a bounded convex domain in the Euclidean space $\mathbb{R}^N$, $N \geq 2$, and consider a heat conductor Ω having initial constant temperature and zero boundary temperature at every time. The physical situation can be modeled as the following initial-boundary value problem for the heat equation:

$$u_t = \Delta u \quad \text{in} \quad \Omega \times (0, \infty), \tag{1.1}$$

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty), \tag{1.2}$$

$$u = 1 \quad \text{on} \quad \Omega \times \{0\}, \tag{1.3}$$

where $u = u(x, t)$ denotes the normalized temperature at a point $x \in \Omega$ at a time $t > 0$.

Since Ω is convex, a result of [BL] shows that $\log u(x, t)$ is concave in x; this fact together with the analyticity of u in the spatial variable x, implies that for each time $t > 0$, there exists a unique point $x(t) \in \Omega$ satisfying

$$\{x \in \Omega : \nabla u(x, t) = 0\} = \{x(t)\}, \tag{1.4}$$

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where ∇ denotes the spatial gradient. The point $x(t)$ is the unique hot spot for each time $t > 0$. Put $\mathcal{M} = \{x \in \Omega : d(x) = \max_{z\in\Omega} d(z)\}$, where $d(z)$ is the distance of z to $\partial\Omega$, defined by

$$d(z) = \text{ dist}(z, \partial\Omega) \ (= \inf\{|z - y| : y \in \partial\Omega\}) \quad \text{for } z \in \overline{\Omega}. \tag{1.5}$$

Then we have

$$\text{dist}(x(t), \mathcal{M}) \to 0 \quad \text{as } t \to 0^+, \tag{1.6}$$

since the function $-4t \log[1-u(x,t)]$ attains its maximum at $x = x(t)$ for each $t > 0$ and a result of Varadhan [V] shows that

$$-4t \log[1 - u(x,t)] \to d(x)^2 \quad \text{as } t \to 0^+ \text{ uniformly on } \overline{\Omega}. \tag{1.7}$$

In conclusion, the hot spot $x(t)$ starts from $\mathcal{M}$. Also, as $t \to \infty,\ x(t)$ tends to the point at which the positive first eigenfunction of $-\Delta$ with homogeneous Dirichlet boundary condition attains its maximum (see [MS 3], Introduction, for details).

From now on, without loss of generality, we assume that Ω contains the origin 0.

A conjecture of Klamkin [Kl] stated that if the origin is a stationary hot spot, that is, if $x(t) \equiv 0$, then Ω must be centro-symmetric with respect to 0. This was disproved by Gulliver–Willms [GW] and Kawohl [Ka]. A typical counterexample is an equilateral triangle in the plane. After that, Chamberland–Siegel [CS] posed the following conjecture.

Conjecture 1.1 (Chamberland–Siegel)**.** *If 0 is a stationary hot spot in a bounded convex domain Ω, then Ω is invariant under the action of an essential subgroup G of orthogonal transformations.*

A subgroup G of orthogonal transformations is said to be **essential** if for every $x \neq 0$, there exists an element $g \in G$ such that $gx \neq x$. As observed in [CS], it is quite easy to prove that, if Ω is invariant under the action of an essential subgroup G of orthogonal transformations, then the origin must be a stationary hot spot. Indeed, if Ω enjoys that invariance, then by the unique solvability of the initial-Dirichlet problem (1.1)–(1.3), the solution u itself is invariant under the action of G. Namely, we have $u(x,t) \equiv u(gx,t)\ (x \in \Omega,\ t > 0,\ g \in G)$. Taking the gradient of both sides of the last identity, together with the assumption that G is essential, implies that $\nabla u(0,t) = 0\ (t > 0)$; and then it follows from (1.4) that the origin is a stationary hot spot.

A proof of Conjecture 1.1 appears to be a much harder task. So far, the only progress in this direction is the following theorem, that was proved by the authors in [MS 3] as a consequence of a more general result.

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^2$.*

(1) *If Ω is a triangle and 0 is a stationary hot spot, then Ω must be an equilateral triangle centered at 0.*
(2) *If Ω is a convex quadrangle and 0 is a stationary hot spot, then Ω must be a parallelogram centered at 0.*
(3) *If Ω is a non-convex quadrangle, then there is no stationary critical point of u in Ω. In particular, there is no stationary hot spot.*

In (1) of Theorem 1.2, G is the cyclic group generated by the rotation of the angle $(2\pi)/3$; and in (2), $G = \{I, -I\}$, where I is the identity mapping. The proof is based on two ingredients: the balance law around stationary critical points of the heat flow (see [MS 1]) and the asymptotic behavior as $t \to 0^+$ of solutions of the heat equation, due to Varadhan [V].

In the present paper, we treat the case of certain pentagons and hexagons.

Theorem 1.3. *Let Ω be a convex polygon in $\mathbb{R}^2$ and suppose that its inscribed circle touches every side of Ω.*

(1) *If Ω is a pentagon and 0 is a stationary hot spot, then Ω must be a regular pentagon centered at 0.*
(2) *If Ω is a hexagon and 0 is a stationary hot spot, then Ω is invariant under the action of the rotation of one of angles $\pi/3,\ (2\pi)/3,\ \pi$.*

This theorem is a consequence of the following general statement.

Theorem 1.4. *Let Ω be a convex polygon in $\mathbb{R}^2$ with m sides, $m \geq 5$, and let $B_R(0)$ be the open disk with radius $R > 0$ centered at 0.*

Suppose that 0 is a stationary hot spot and the circle $\partial B_R(0)$ touches every side of Ω at the points $p_1, \dots, p_m \in \partial\Omega \cap \partial B_R(0)$. Let $q_1, \dots, q_k$ be the k $(1 \leq k \leq m)$ nearest vertices of Ω to 0.

Then

$$\sum_{i=1}^{m} p_i = 0 \tag{1.8}$$

and

$$\sum_{j=1}^{k} q_j = 0. \tag{1.9}$$

We observe that in the special case in which the vertices $q_1, \dots, q_k$ are consecutive, equation (1.9) easily implies that $k = m$ and Ω must be a regular polygon.

While (1.8) was already obtained in [MS 3], (1.9) is new; it is derived by coupling a suitable extension argument to a careful analysis of the short-time behavior of $u(x,t)$ near the vertices of Ω.

The present paper is organized as follows. Sections 2 and 3 are devoted to the proof of Theorem 1.4. In Section 2, we introduce the function $v = 1-u$ and give sub- and supersolutions $v^-,\ v^+$ for the initial-boundary value problem solved by v. Then, by folding back v with respect to each side of Ω, we extend v to a solution of the heat equation in a domain larger than Ω and by using the balance law around a stationary critical point, we obtain (1.8) and the main identity (2.13). In Section 3, with the aid of $v^-,\ v^+$, we exploit a more detailed initial behavior of v and eventually obtain (1.9). Finally, in Section 4, we prove Theorem 1.3.

2 Barriers for an extension of the solution

In this section, we extend the solution of (1.1)–(1.3) to a larger domain, in order to prove (1.8) and prepare the proof of (1.9).

Let Ω be a convex m-gon in $\mathbb{R}^2$ with $m \geq 5$. Suppose that the circle $\partial B_R(0)$ touches every side of Ω, say $\partial\Omega \cap \partial B_R(0) = \{p_1,\dots,p_m\}$. Let $q_1,\dots,q_k$ be the k $(1 \leq k \leq m)$ nearest vertices of Ω to the origin; we can set $R^* = |q_1| = |q_2| = \cdots = |q_k|$, and hence $R^* > R$.

Denote by $\nu_1,\dots,\nu_m$ the interior normal unit vectors to $\partial\Omega$ at the points $p_1,\dots,p_m$, respectively. Note that

$$p_i = -R\nu_i \quad (i = 1,\dots,m). \tag{2.1}$$

For notational convenience, we deal with the function $v = 1-u$ instead of u and consider the cold spot of v instead of the hot spot of u; so v satisfies

$$v_t = \Delta v \quad \text{in} \quad \Omega \times (0,\infty), \tag{2.2}$$

$$v = 1 \quad \text{on} \quad \partial\Omega \times (0,\infty), \tag{2.3}$$

$$v = 0 \quad \text{on} \quad \Omega \times \{0\}. \tag{2.4}$$

We now introduce a subsolution $v^- = v^-(x,t)$ and a supersolution $v^+ = v^+(x,t)$ for problem (2.2)–(2.4).

Define

$$f(\xi) = \frac{1}{\sqrt{\pi}} \int_\xi^\infty e^{-\frac{1}{4}\eta^2} d\eta \qquad \text{for all } \xi \in \mathbb{R}; \tag{2.5}$$

and note that

$$\int_0^\infty \xi f(\xi)\, d\xi = 1. \tag{2.6}$$

The function $w = w(s,t)$ given by

$$w(s,t) = f\left(t^{-1/2}s\right) \quad \text{for } (s,t) \in \mathbb{R} \times (0,\infty) \tag{2.7}$$

satisfies the one-dimensional heat equation $w_t = w_{ss}$ in $\mathbb{R} \times (0,\infty)$. Hence, we easily see that the functions

$$v^-(x,t) = \max_{1 \le i \le m} f\left(t^{-1/2}(x-p_i)\cdot\nu_i\right), \tag{2.8}$$

$$v^+(x,t) = \sum_{i=1}^{m} f\left(t^{-1/2}(x-p_i)\cdot\nu_i\right). \tag{2.9}$$

are, respectively, a sub- and a supersolution for problem (2.2)–(2.4). By the comparison principle, it follows that

$$v^- \le v \le v^+ \quad \text{in } \Omega \times (0,\infty). \tag{2.10}$$

The following result is used in Section 3.

Lemma 2.1. *For any compact set K contained in Ω, there exist two positive constants $A > 0, B > 0$ satisfying*

$$0 < v(x,t) \le Ae^{-B/t} \quad \textit{for all } (x,t) \in K \times (0,\infty).$$

Proof. This follows directly from (2.10) and the convexity of Ω. □

Note that Lemma 2.1 holds for general (not necessarily convex) domains $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ because of Varadhan's result (1.7).

By following the procedure employed in [MS 3], we extend v to a solution $v^* = v^*(x,t)$ of the heat equation in a larger domain domain $\Omega^* \times (0,\infty) \supset \Omega \times (0,\infty)$. Here Ω^* is obtained by putting together Ω and all its reflections with respect to each of its sides and by eliminating possible overlaps; v^* equals $1 - u^*$, where u^* is obtained by odd reflections of u with respect to each side of Ω. It is clear that $B_{R^*}(0) \subset \Omega^*$ (see Fig. 1 (a)).

Since 0 is a stationary cold spot of v, we infer that it is a stationary critical point of v^*.

Therefore we can use the balance law obtained in [MS 1], Theorem 2 (see also [MS 2], Corollary 2.2, for another proof) to infer that

$$\int_{B_{R^*}(0)} xv^*(x,t)\, dx = 0 \quad \text{for any } t > 0. \tag{2.11}$$

Letting $t \to 0^+$ yields

$$2\int_{B_{R^*}(0)\setminus\Omega} x\, dx = 0, \tag{2.12}$$

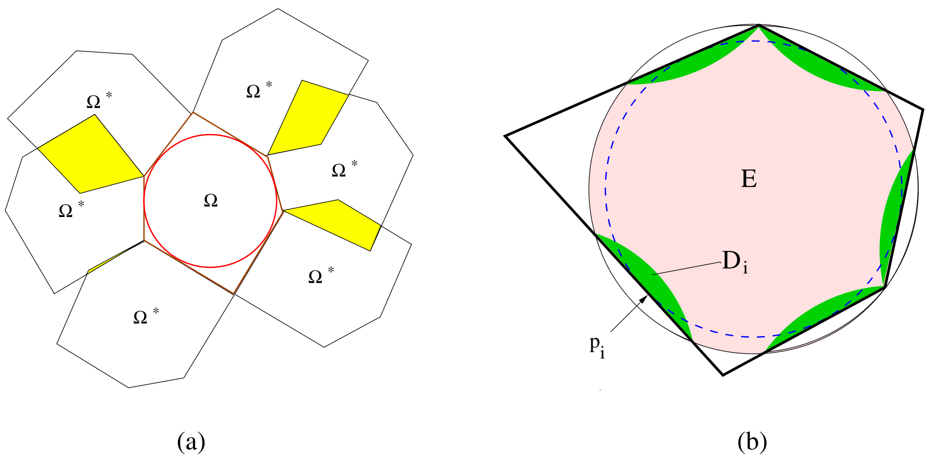

Figure 1: (a) The construction of the set Ω^* and (b) the sets E, D_i and $D = \bigcup D_i$.

since v^* tends to 0 inside Ω and to 2 outside; (2.12) easily implies (1.8).

Denote by D the region obtained as the union of the reflections of each connected component of $B_{R^*}(0) \setminus \overline{\Omega}$ with respect to each relevant side of Ω; let $D_j, 1 \le j \le m$, be the connected components of D; and put $E = (B_{R^*}(0) \cap \Omega) \setminus \overline{D}$. Note that both D and E are contained in Ω. For $x \in D_j, 1 \le j \le m$, denote by x^* the reflection of x with respect to the side of Ω containing $\overline{D}_j \cap \partial\Omega$. Then $v^*(x,t) \equiv 2 - v^*(x^*,t)$ because of (2.3) (see Fig. 1 (b)). Since

$$\int_{B_{R^*}(0)} x v^*(x,t)\, dx = \int_E x v(x,t)\, dx + \int_D x v(x,t)\, dx + \int_{B_{R^*}(0) \setminus (D \cup E)} x v(x,t)\, dx$$
$$= \int_E x v(x,t)\, dx + \int_D x v(x,t)\, dx + \int_D x^* [2 - v(x,t)]\, dx,$$

it follows from (2.11) and (2.12) that for any $t > 0$,

$$\int_E x v(x,t)\, dx + \int_D (x - x^*)\, v(x,t)\, dx = 0. \tag{2.13}$$

In the next section, in order to prove (1.9), we calculate

$$\lim_{t \to 0^+} \frac{1}{t} \left\{ \int_E x v(x,t)\, dx + \int_D (x - x^*)\, v(x,t)\, dx \right\}. \tag{2.14}$$

in a different way.

3 Proof of Theorem 1.4: asymptotic lemmas

When $k < m$, let $s_1, \dots, s_\ell \quad (\ell = 2m - 2k)$ be all the points such that

$$\partial\Omega \cap \partial B_{R^*}(0) = \{q_1, \dots, q_k, s_1, \dots, s_\ell\}. \tag{3.1}$$

Since each p_i is the midpoint of a pair of points in $\partial\Omega \cap \partial B_{R^*}(0)$, we have from (1.8)

$$2\sum_{j=1}^{k} q_j + \sum_{j=1}^{\ell} s_j = 2\sum_{i=1}^{m} p_i = 0. \tag{3.2}$$

Notice that when $k = m$, the definition of $q_1, \dots, q_m$ implies that all the angles of the m-gon Ω are equal, so that Ω must be a regular polygon. Thus (1.9) holds when $k = m$. In the sequel,we assume that $k < m$.

Since the circle $\partial B_R(0)$ touches every side of Ω, all the angles between the circle $\partial B_{R^*}(0)$ and the sides of Ω at q_j or at s_j are equal. Denote by $\alpha \in (0, \pi/2)$ the measure of these angles.

In view of Lemma 2.1, it is enough to replace the sets in the integrals in (2.14) with small neighborhoods of the points q_j, s_j, and small neighborhoods of $\partial\Omega$ in D_j. Choose a number $\delta_0 > 0$ so small that, for any $x \in \{q_1, \dots, q_k, s_1, \dots, s_\ell\}$,

$$\overline{B_{\delta_0}(x)} \cap (\{p_1, \dots, p_m, s_1, \dots, s_\ell\} \cup \{\text{ vertices of } \Omega\}) = \{x\}.$$

Lemma 3.1. *For $\varepsilon > 0$ and $1 \le j \le \ell$, set*

$$E^\varepsilon(s_j) = \{x \in E : 0 < (x - s_j) \cdot \nu_i < \varepsilon\} \cap B_{\delta_0}(s_j),$$

where ν_i is the interior unit normal vector to the side of Ω containing the point s_j (see Fig. 2).

Then, if ε is sufficiently small, we have

$$\lim_{t \to 0^+} \frac{1}{t} \int_{E^\varepsilon(s_j)} x\, v(x,t)\, dx = 2\cot\alpha\; s_j \quad \textit{for } 1 \le j \le \ell. \tag{3.3}$$

Proof. Since Ω is convex and s_j is not a vertex of Ω, (2.8), (2.9), and (2.10) imply that there exist positive constants A_j and B_j such that

$$\left|v(x,t) - f\big(t^{-1/2}(x - s_j) \cdot \nu_i\big)\right| \le A_j e^{-\frac{B_j}{t}} \quad \text{for all } \ x \in \Omega \cap B_{\delta_0}(s_j), t > 0. \tag{3.4}$$

Here we have used the fact that $(x - s_j) \cdot \nu_i = (x - p_i) \cdot \nu_i$.

Set $e_i = (p_i - s_j)/|p_i - s_j|$; if $\varepsilon > 0$ is sufficiently small, we can write

$$E^\varepsilon(s_j) = \{x = s_j + z_1\, \nu_i + z_2\, e_i : 0 < z_1 < \varepsilon,\ \varphi_-(z_1) < z_2 < \varphi_+(z_1)\},$$

where $\varphi_-(z_1) < 0 < \varphi_+(z_1)$ for $z_1 \in (0, \varepsilon)$ and the functions φ_- and φ_+ represent $\partial E^\varepsilon(s_j) \cap \partial B_{R^*}(0)$ and $\partial E^\varepsilon(s_j) \cap \partial D$, respectively. Note that $\varphi'_-(0) = -\cot\alpha$ and $\varphi'_+(0) = \cot\alpha$.

In view of (3.4), we calculate

$$\begin{aligned}\frac{1}{t}\int_{E^\varepsilon(s_j)} f\left(t^{-\frac{1}{2}}(x-s_j)\cdot\nu_i\right)\,dx =& \frac{1}{t}\int_0^\varepsilon\left[f\left(t^{-\frac{1}{2}}z_1\right)\int_{\varphi_-(z_1)}^{\varphi_+(z_1)} dz_2\right]dz_1 \\ =& \int_0^{t^{-\frac{1}{2}}\varepsilon} \frac{\varphi_+(t^{\frac{1}{2}}\xi)-\varphi_-(t^{\frac{1}{2}}\xi)}{t^{\frac{1}{2}}\xi}\,\xi f(\xi)d\xi.\end{aligned}$$

Since

$$\lim_{t\to 0^+}\frac{\varphi_+(t^{\frac{1}{2}}\xi)-\varphi_-(t^{\frac{1}{2}}\xi)}{t^{\frac{1}{2}}\xi} = \varphi'_+(0)-\varphi'_-(0) = 2\cot\alpha \quad \text{for } \xi > 0,$$

by Lebesgue's dominated convergence theorem we get

$$\lim_{t\to 0^+}\frac{1}{t}\int_{E^\varepsilon(s_j)} f\left(t^{-\frac{1}{2}}(x-s_j)\cdot\nu_i\right)\,dx = 2\cot\alpha\int_0^\infty \xi f(\xi)\,d\xi. \tag{3.5}$$

In a similar way, we obtain

$$\lim_{t\to 0^+}\frac{1}{t}\int_{E^\varepsilon(s_j)} (x-s_j)\,f\left(t^{-\frac{1}{2}}(x-s_j)\cdot\nu_i\right)\,dx = 0, \tag{3.6}$$

since

$$\lim_{t\to 0^+}\frac{1}{t}\int_0^\varepsilon\left[f\left(t^{-\frac{1}{2}}z_1\right)\int_{\varphi_-(z_1)}^{\varphi_+(z_1)} z_i\,dz_2\right]dz_1 = 0 \quad \text{for } i = 1, 2.$$

With the aid of (3.4), (3.5), (3.6), and (2.6), we then get (3.3). □

Lemma 3.2. *For $\varepsilon > 0$ and $1 \le j \le k$, set*

$$E^\varepsilon(q_j) = \{x \in E : 0 < (x-q_j)\cdot\nu_i < \varepsilon \text{ or } 0 < (x-q_j)\cdot\nu_{i+1} < \varepsilon\} \cap B_{\delta_0}(q_j),$$

where ν_i and ν_{i+1} are the interior unit normal vectors to the two sides of Ω containing the vertex q_j (see Fig. 2).

Then, if ε is sufficiently small, we have

$$4\cot 2\alpha \le \limsup_{t\to 0^+}\frac{1}{t}\int_{E^\varepsilon(q_j)} v(x,t)\,dx \le 8\cot 2\alpha \tag{3.7}$$

and

$$\lim_{t\to 0^+}\frac{1}{t}\int_{E^\varepsilon(q_j)} (x-q_j)\,v(x,t)\,dx = 0, \tag{3.8}$$

for $1 \le j \le k$.

Proof. Let β be the angle of Ω at the vertex q_j; observe that $\beta + 2\,\alpha = \pi$. Since β is the largest angle in Ω, we have $\pi(1-2/m) < \beta < \pi$, $\alpha < \pi/m$ and hence

$$\beta - 2\,\alpha > 0,$$

for every $m \geq 4$.

Let γ be the bisectrix of the angle of Ω at q_j; γ divides $E^\varepsilon(q_j)$ into two parts, $E^\varepsilon_i(q_j)$ and $E^\varepsilon_{i+1}(q_j)$, corresponding to ν_i and ν_{i+1}, respectively.

Since q_j is a vertex of Ω, (2.8), (2.9), and (2.10) imply that there exist positive constants A_j and B_j such that

$$0 < f\big(t^{-1/2}(x-q_j)\cdot\nu_i\big) \leq v(x,t) \leq 2\,f\big(t^{-1/2}(x-q_j)\cdot\nu_i\big) + A_j e^{-B_j/t} \quad \text{for all } \ x \in E^\varepsilon_i(q_j),\ t>0. \tag{3.9}$$

Here we have used the fact that $(x-q_j)\cdot\nu_i = (x-p_i)\cdot\nu_i$.

Set $e_i = (p_i - q_j)/|p_i - q_j|$. If ε is sufficiently small, we can write

$$E^\varepsilon_i(q_j) = \{x = q_j + z_1\,\nu_i + z_2\,e_i : 0 < z_1 < \varepsilon,\ z_1 \tan\alpha < z_2 < \varphi(z_1)\}.$$

Note that $\varphi'(0) = \cot\alpha$ and $\varphi'(z_1) > 0$ for $z_1 > 0$.

We now write

$$\begin{aligned}\frac{1}{t}\int_{E^\varepsilon_i(q_j)} f\left(t^{-\frac12}(x-q_j)\cdot\nu_i\right)\,dx &= \frac{1}{t}\int_0^\varepsilon \left[f\left(t^{-\frac12}z_1\right)\int_{z_1\tan\alpha}^{\varphi(z_1)} dz_2\right]dz_1 \\ &= \frac{1}{t}\int_0^\varepsilon f\left(t^{-\frac12}z_1\right)[\varphi(z_1) - z_1\tan\alpha]\,dz_1 \\ &= \int_0^{t^{-\frac12}\varepsilon} \xi f(\xi)\,\frac{\varphi(t^{\frac12}\xi) - t^{\frac12}\xi\tan\alpha}{t^{\frac12}\xi}\,d\xi.\end{aligned}$$

Thus, since

$$\lim_{t\to0^+}\varphi(t^{\frac12}\xi)/(t^{\frac12}\xi) = \varphi'(0) = \cot\alpha \qquad \text{for } \ \xi > 0,$$

by Lebesgue's dominated convergence theorem we get

$$\lim_{t\to0^+}\frac{1}{t}\int_{E^\varepsilon_i(q_j)} f\left(t^{-\frac12}(x-q_j)\cdot\nu_i\right)\,dx = 2\cot 2\alpha\int_0^\infty \xi f(\xi)\,d\xi. \tag{3.10}$$

By a similar calculation, we have

$$\lim_{t\to0^+}\frac{1}{t}\int_{E^\varepsilon_i(q_j)} |x-q_j|\,f\left(t^{-\frac12}(x-q_j)\cdot\nu_i\right)\,dx = 0, \tag{3.11}$$

since

$$\lim_{t\to0^+}\frac{1}{t}\int_0^\varepsilon\left[f\left(t^{-\frac12}z_1\right)\int_{z_1\tan\alpha}^{\varphi(z_1)} z_i\,dz_2\right]dz_1 = 0 \qquad \text{for } \ i=1,2.$$

From (3.9), (3.10), and (2.6), it follows that

$$2\cot 2\alpha \le \limsup_{t\to 0^+} \frac{1}{t}\int_{E_i^\varepsilon(q_j)} v(x,t)\,dx \le 4\cot 2\alpha. \tag{3.12}$$

Also, since

$$\left|\frac{1}{t}\int_{E_i^\varepsilon(q_j)} (x-q_j)\,v(x,t)\,dx\right| \le \frac{1}{t}\int_{E_i^\varepsilon(q_j)} |x-q_j|\,v(x,t)\,dx,$$

we have from (3.9) and (3.11)

$$\lim_{t\to 0^+} \frac{1}{t}\int_{E_i^\varepsilon(q_j)} (x-q_j)\,v(x,t)\,dx = 0.$$

By the same arguments, we obtain the last two formulas with $E_i^\varepsilon(q_j)$ replaced by $E_{i+1}^\varepsilon(q_j)$; and hence (3.7) and (3.8) follow at once. □

Lemma 3.3. *For any $j, s \in \{1,\dots,k\}$,*

$$\lim_{t\to 0^+} \frac{1}{t}\left[\int_{E^\varepsilon(q_j)} v(x,t)\,dx - \int_{E^\varepsilon(q_s)} v(x,t)\,dx\right] = 0. \tag{3.13}$$

Proof. Since the angles of Ω at two distinct vertices q_j and q_s are equal, by a translation and an orthogonal transformation we can superpose one angle on the other. Thus, there exists an orthogonal matrix T such that the function

$$w(x,t) = v(x,t) - v(q_s + T(x-q_j), t)$$

satisfies

$$w_t = \Delta w \quad \text{in} \quad \left(\Omega \cap \overline{B_{\delta_0}(q_j)}\right) \times (0,\infty), \tag{3.14}$$

$$w = 0 \quad \text{on} \quad \left(\partial\Omega \cap \overline{B_{\delta_0}(q_j)}\right) \times (0,\infty), \tag{3.15}$$

$$w = 0 \quad \text{on} \quad \left(\Omega \cap \overline{B_{\delta_0}(q_j)}\right) \times \{0\}. \tag{3.16}$$

Since $\overline{\Omega} \cap \partial B_{\delta_0}(q_j)$ does not contain any vertices of Ω, it follows from (2.8), (2.9), and (2.10) that there exist positive constants $G > 0, H > 0$ satisfying

$$|w(x,t)| \le G e^{-\frac{H}{t}} \quad \text{for all } (x,t) \in \left(\overline{\Omega} \cap \partial B_{\delta_0}(q_j)\right) \times (0,\infty). \tag{3.17}$$

Observe that

$$(\partial_t - \Delta)\left(G e^{-\frac{H}{t}}\right) = G H t^{-2} e^{-\frac{H}{t}} > 0 \ \text{ for } (x,t) \in \mathbb{R}^2 \times (0,\infty). \tag{3.18}$$

Therefore, in view of (3.14)–(3.18), we obtain by the comparison principle

$$|w(x,t)| \le Ge^{-\frac{H}{t}} \text{ for all } (x,t) \in (\Omega \cap B_{\delta_0}(q_j)) \times (0,\infty). \tag{3.19}$$

Since for $t > 0$,

$$\begin{aligned}\frac{1}{t}\left| \int_{E^\varepsilon(q_j)} v(y,t)\,dy - \int_{E^\varepsilon(q_s)} v(y,t)\,dy \right| \\ &= \frac{1}{t}\left| \int_{E^\varepsilon(q_j)} v(y,t)\,dy - \int_{E^\varepsilon(q_j)} v(q_s + T(x-q_j),t)\,dx \right| \\ &\le \frac{1}{t}\int_{\Omega\cap B_{\delta_0}(q_j)} |w(x,t)|\,dx,\end{aligned}$$

(3.19) implies (3.13). □

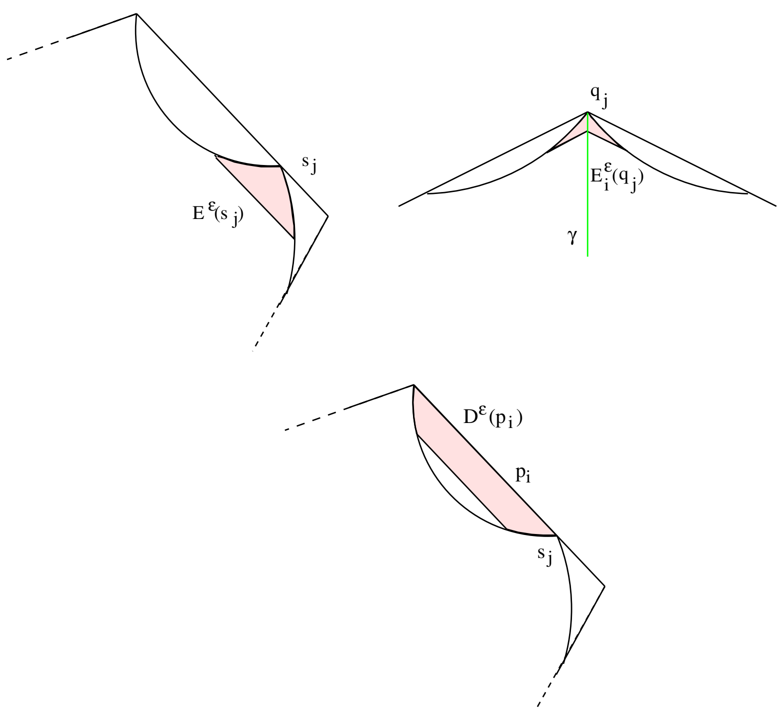

Figure 2: The sets $E^\varepsilon(s_j)$, $E^\varepsilon(q_j)$, and $D^\varepsilon(p_i)$.

Lemma 3.4. *If $\varepsilon > 0$ is sufficiently small, then there exist a positive sequence $\{t_n\}_{n\in\mathbb{N}}$ with $t_n \to 0$ as $n \to \infty$ and a number $\lambda \in [4\cot 2\alpha, 8\cot 2\alpha]$ such that for* *any* $j \in \{1, \dots, k\}$,

$$\lim_{n \to \infty} \frac{1}{t_n} \int_{E^\varepsilon(q_j)} v(x, t_n)\, dx = \lambda. \tag{3.20}$$

Proof. It is clear that (3.7) guarantees the existence of a positive sequence $\{t_n\}_{n \in \mathbb{N}}$ with $t_n \to 0$ as $n \to \infty$ and a number $\lambda \in [4 \cot 2\alpha, 8 \cot 2\alpha]$ such that (3.20) holds for $j = 1$. Thus it follows from Lemma 3.3 that (3.20) holds for any $j \in \{1, \dots, k\}$. $\square$

Lemma 3.5. *Let*

$$\rho = \sqrt{(R^*)^2 - R^2} > 0 \tag{3.21}$$

and, for $\varepsilon > 0$ *and* $1 \le i \le m$*, set*

$$D^\varepsilon(p_i) = \{\, x \in D_i : 0 < (x - p_i) \cdot \nu_i < \varepsilon \,\},$$

where ν_i *is the interior unit normal vector to the side of* Ω *containing* p_i *(see Fig. 2).*

Then if ε *is sufficiently small, we have for* $1 \le i \le m$,

$$\lim_{t \to 0^+} \frac{1}{t} \int_{D^\varepsilon(p_i)} (x - x^*)\, v(x, t)\, dx = 4\rho\nu_i = -\frac{4\rho}{R}\, p_i. \tag{3.22}$$

Proof. We consider three cases: (a) the set $\partial D_i \cap \{q_1, \dots, q_k\}$ is empty; (b) the set $\partial D_i \cap \{q_1, \dots, q_k\}$ has exactly one point; (c) the set $\partial D_i \cap \{q_1, \dots, q_k\}$ has exactly two points. The treatment of case (c) is completely similar to that of case (b), so, its proof is omitted.

(a) Since $\overline{D_i}$ does not contain any vertex of Ω, (2.8), (2.9), and (2.10) imply that there exist positive constants A_i and B_i such that

$$\left| v(x, t) - f\big(t^{-1/2}(x - p_i) \cdot \nu_i\big) \right| \le A_i\, e^{-B_i/t} \quad \text{for all } x \in D_i, t > 0. \tag{3.23}$$

Let e_i be a unit vector orthogonal to ν_i. If ε is sufficiently small, we can parametrize $D^\varepsilon(p_i)$ as

$$D^\varepsilon(p_i) = \{x = p_i + z_1\, \nu_i + z_2\, e_i : 0 < z_1 < \varepsilon,\ \varphi_-(z_1) < z_2 < \varphi_+(z_1)\}, \tag{3.24}$$

where now $\varphi_-(0) = -\rho,\ \varphi_+(0) = \rho$, and $\varphi_-'(0) = \cot \alpha,\ \varphi_+'(0) = -\cot \alpha$. Note that x^*, the reflection of $x \in D^\varepsilon(p_i)$, is given by

$$x^* = p_i - z_1\, \nu_i + z_2\, e_i.$$

We compute

$$\frac{1}{t}\int_{D^\varepsilon(p_i)}(x-x^*)\, f\left(t^{-\frac{1}{2}}(x-p_i)\cdot\nu_i\right)dx = \frac{1}{t}\int_0^\varepsilon\left[2z_1\nu_i f\left(t^{-\frac{1}{2}}z_1\right)\int_{\varphi_-(z_1)}^{\varphi_+(z_1)}dz_2\right]dz_1$$
$$= 2\nu_i\int_0^{t^{-\frac{1}{2}}\varepsilon}[\varphi_+(t^{\frac{1}{2}}\xi)-\varphi_-(t^{\frac{1}{2}}\xi)]\,\xi f(\xi)\,d\xi,$$

so by Lebesgue's dominated convergence theorem,

$$\lim_{t\to 0^+}\frac{1}{t}\int_{D^\varepsilon(p_i)}(x-x^*)\, f\left(t^{-\frac{1}{2}}(x-p_i)\cdot\nu_i\right)dx = 4\rho\nu_i\int_0^\infty \xi f(\xi)\,d\xi.$$

With the aid of (3.23) and (2.6), we obtain (3.22).

(b) As in case (a), we consider the parametrization $x = p_i + z_1\,\nu_i + z_2\,e_i$ of a point in the set $D^\varepsilon(p_i)$ given in (3.24); additionally, we assume that $p_i - \rho e_i$ is the point of $\partial D_i \cap \{q_1,\dots,q_k\}$.

Take a small number $\delta \in (0, \varphi_-(\varepsilon)+\rho)$ and set

$$D_+^\varepsilon(p_i) = \{x : 0 < z_1 < \varepsilon,\ \max(\varphi_-(z_1), \delta-\rho) < z_2 < \varphi_+(z_1)\},$$
$$D_-^\varepsilon(p_i) = \{x : 0 < z_1 < \varepsilon,\ \min(\varphi_-(z_1), \delta-\rho) < z_2 \le \delta-\rho\}. \tag{3.25}$$

Then $D^\varepsilon(p_i) = D_+^\varepsilon(p_i)\cup D_-^\varepsilon(p_i)$.

Since $\overline{D_+^\varepsilon(p_i)}$ does not contain any vertex of Ω, from (2.8), (2.9) and (2.10) it follows that for some positive constants A_i^+ and B_i^+

$$\left|v(x,t) - f\big(t^{-1/2}(x-p_i)\cdot\nu_i\big)\right| \le A_i^+ e^{-B_i^+/t} \quad \text{for all } x\in D_+^\varepsilon(p_i),\ t>0. \tag{3.26}$$

Since the point $p_i - \rho\, e_i$ is a vertex of Ω, we have from (2.9) and (2.10) that for some positive constants A_i^- and B_i^-,

$$0 < v(x,t) \le 2\, f\big(t^{-1/2}(x-p_i)\cdot\nu_i\big) + A_i^-\, e^{-B_i^-/t} \quad \text{for all } x\in D_-^\varepsilon(p_i),\ t>0. \tag{3.27}$$

We now compute

$$\frac{1}{t}\int_{D_+^\varepsilon(p_i)}(x-x^*)\, f\left(t^{-\frac{1}{2}}(x-p_i)\cdot\nu_i\right)dx$$
$$= \frac{2\nu_i}{t}\int_0^\varepsilon z_1 f\left(t^{-\frac{1}{2}}z_1\right)[\varphi_+(z_1) - \max(\varphi_-(z_1),\delta-\rho)]\,dz_1$$
$$= 2\nu_i\int_0^{t^{-\frac{1}{2}}\varepsilon}[\varphi_+(t^{\frac{1}{2}}\xi) - \max(\varphi_-(t^{\frac{1}{2}}\xi),\delta-\rho)]\,\xi f(\xi)\,d\xi;$$

hence, by Lebesgue's dominated convergence theorem and (2.6),

$$\lim_{t\to 0^+}\frac{1}{t}\int_{D_+^\varepsilon(p_i)}(x-x^*)\, f\left(t^{-\frac{1}{2}}(x-p_i)\cdot\nu_i\right)dx = 2(2\rho-\delta)\nu_i.$$

As before, we conclude that

$$\lim_{t\to 0^+} \frac{1}{t} \int_{D_+^\varepsilon(p_i)} (x - x^*)\ v(x,t)\ dx = 2(2\rho - \delta)\nu_i. \tag{3.28}$$

On the other hand, we have

$$\begin{aligned}\frac{1}{t} \int_{D_-^\varepsilon(p_i)} &|x - x^*|\ f\left(t^{-\frac{1}{2}}(x - p_i)\cdot \nu_i\right) dx \\ &= \frac{2}{t} \int_0^\varepsilon z_1 f\left(t^{-\frac{1}{2}} z_1\right) [\delta - \rho - \min(\varphi_-(z_1), \delta - \rho)]\ dz_1 \\ &= 2 \int_0^{t^{-\frac{1}{2}}\varepsilon} [\delta - \rho - \min(\varphi_-(t^{\frac{1}{2}}\xi), \delta - \rho)]\ \xi f(\xi)\ d\xi,\end{aligned}$$

so by Lebesgue's dominated convergence theorem,

$$\lim_{t\to 0^+} \frac{1}{t} \int_{D_-^\varepsilon(p_i)} |x - x^*|\ f\left(t^{-\frac{1}{2}}(x - p_i)\cdot \nu_i\right) dx = 2\delta.$$

Therefore, (3.27) implies that

$$\limsup_{t\to 0^+} \left| \frac{1}{t} \int_{D_-^\varepsilon(p_i)} (x - x^*)\ v(x,t)\ dx \right| \le 4\delta$$

and thus, by (3.28), we have

$$\limsup_{t\to 0^+} \left| \frac{1}{t} \int_{D^\varepsilon(p_i)} (x - x^*)\ v(x,t)\ dx - 4\rho\nu_i \right| \le 6\delta.$$

Since $\delta > 0$ is chosen arbitrarily small, we again obtain (3.22). □

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. In view of Lemma 2.1, it suffices to consider the integrals in (2.14) over the unions of the sets $E^\varepsilon(s_j) \cup E^\varepsilon(q_j)$ and $D^\varepsilon(p_i)$, respectively, for $\varepsilon > 0$ sufficiently small. Lemma 2.1 thus guarantees that

$$\begin{aligned}\lim_{n\to\infty} &\frac{1}{t_n} \left[\int_E x v(x,t_n)\ dx + \int_D (x - x^*)\ v(x,t_n)\ dx \right] \\ &= \lim_{n\to\infty} \frac{1}{t_n} \sum_{j=1}^{\ell} \int_{E^\varepsilon(s_j)} x v(x,t_n)\ dx + \lim_{n\to\infty} \frac{1}{t_n} \sum_{j=1}^{k} \int_{E^\varepsilon(q_j)} x v(x,t_n)\ dx \\ &\quad + \lim_{n\to\infty} \frac{1}{t_n} \sum_{i=1}^{m} \int_{D^\varepsilon(p_i)} (x - x^*) v(x,t_n)\ dx.\end{aligned}$$

Lemmas 3.1, 3.2, 3.4, and 3.5 yield that

$$\lim_{n\to\infty} \frac{1}{t_n} \left[\int_E x v(x,t_n)\, dx + \int_D (x - x^*)\, v(x,t_n)\, dx \right] \\ = 2\cot\alpha \sum_{j=1}^{\ell} s_j + \lambda \sum_{j=1}^{k} q_j - \frac{4\rho}{R} \sum_{i=1}^{m} p_i.$$

Therefore, (2.13) implies

$$2\cot\alpha \sum_{j=1}^{\ell} s_j + \lambda \sum_{j=1}^{k} q_j - \frac{4\rho}{R} \sum_{i=1}^{m} p_i = 0,$$

so using (1.8) and (3.2), we get

$$(\lambda - 4\cot\alpha) \sum_{j=1}^{k} q_j = 0.$$

Therefore, since $\lambda \in [4\cot 2\alpha, 8\cot 2\alpha]$, we obtain (1.9). This completes the proof of Theorem 1.4. □

4 The proof of Theorem 1.3

Let $C_p = \partial B_R(0)$ and $C_q = \partial B_{R^*}(0)$ be the circles containing the points $p_1, \dots, p_m$ and $q_1, \dots, q_k$ respectively. As already observed, since $\partial\Omega$ is circumscribed to C_p, the angles of Ω at the vertices $q_1, \dots, q_k$ are all equal. Also, notice that (1.9) directly implies that $k \geq 2$.

(1) We distinguish four cases (see Fig. 3). (i) Let $k = 2$; then q_1 and q_2 are opposite. Label by p_1, p_2, p_3 and p_4 the points in $\partial\Omega \cap C_p$ lying on the sides of Ω issuing from q_1 and q_2. They must be the vertices of a rectangle centered at 0; hence $\sum_{i=1}^{4} p_i = 0$ and, by (1.8), $p_5 = 0$ — a contradiction.

(ii) If $k = 3$, q_1, q_2 and q_3 are the vertices of an equilateral triangle, which we call $\mathcal{T}$; and Ω and $\mathcal{T}$ have at least one side in common. Then C_p must be the inscribed circle of $\mathcal{T}$ and any side of Ω issuing from any vertex of Ω lying outside C_q cannot intersect C_p, since it must lie outside $\mathcal{T}$ — a contradiction.

(iii) Let $k = 4$. Since (1.9) holds, the q_j's must be pairwise opposite and also be the vertices of a rectangle, for they all lie on C_q. Such a rectangle and Ω must have at least three sides in common (tangent to C_p); this fact implies that the q_j's are the vertices of a square. Hence, two sides of Ω issuing from the vertex of Ω lying outside C_q cannot intersect C_p — a contradiction.

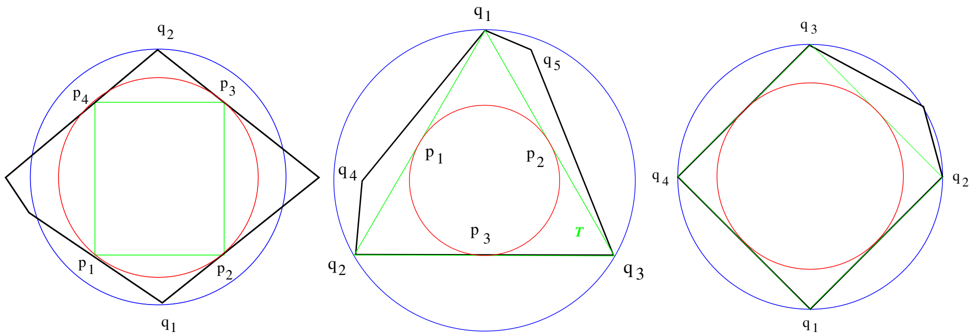

Figure 3: Proof of Theorem 1.3, item (1), cases (i), (ii) and (iii).

(iv) If $k = 5$, all the vertices of Ω lie on C_q, and hence all the angles of Ω must all be equal to one other. The fact that $\partial\Omega$ is circumscribed to C_p also implies that all the sides of Ω have equal length, that is Ω must be regular.

(2) We distinguish five cases (see Fig. 4). (i) If $k = 2$, then q_1 and q_2 are opposite. As in the proof of (1), $k = 2$, we let p_1, p_2, p_3 and p_4 be the points in $\partial\Omega \cap C_p$ lying on the sides of Ω issuing from q_1 and q_2. We have that $\sum_{i=1}^4 p_i = 0$, and it follows by (1.8) that $p_5 + p_6 = 0$. Therefore, all the points p_i's are pairwise opposite and so are vertices of Ω; hence, Ω is centrally symmetric. In other words, Ω is invariant under a rotation of an angle π.

(ii) If $k = 3$, q_1, q_2 and q_3 are the vertices of an equilateral triangle, which we call $\mathcal{T}$. If Ω and $\mathcal{T}$ have a side in common, then we get a contradiction, by the same argument used in the proof of (1), $k = 3$. If Ω and $\mathcal{T}$ have no side in common, then the vertices of Ω lying outside C_q must also be the vertices of an equilateral triangle. In fact, since $\partial\Omega$ is circumscribed to C_p, such vertices lie on the three half-lines through the origin and the points $q_1 + q_2$, $q_2 + q_3$, and $q_3 + q_1$, respectively, and have the same distance from the origin. Therefore, Ω is invariant under a rotation of an angle $2\pi/3$.

(iii) Let $k = 4$. Since (1.9) holds, the q_j's must be pairwise opposite and also be the vertices of a rectangle $\mathcal{R}$, for they all lie on C_q. Now Ω and $\mathcal{R}$ have at least one side in common. Call such a side σ_1; σ_1 must be a shorter side of $\mathcal{R}$, since otherwise C_p would be contained in $\mathcal{R}$, and hence at least one side of Ω would not intersect C_p. Thus, the side σ_2 of $\mathcal{R}$ opposite to σ_1 must also be a side of Ω, and the midpoints p_1 and p_2 of σ_1 and σ_2 are such that $p_1 + p_2 = 0$. By (1.8), we have $\sum_{i=3}^6 p_i = 0$. Therefore, the p_i's are pairwise opposite and, as in the case $k = 2$, Ω is invariant under a rotation of angle π.

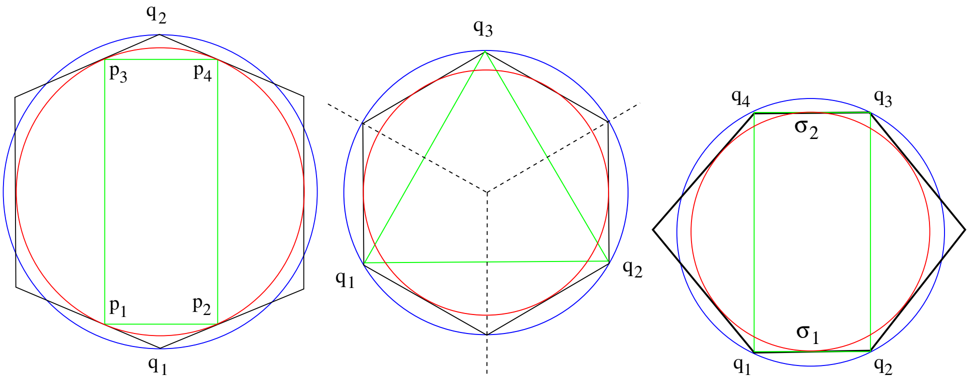

Figure 4: Proof of Theorem 1.3, item (2), cases (i), (ii) and (iii).

(iv) The case $k = 5$ cannot occur. We can assume that the segments joining q_1 to q_2, q_2 to q_3, q_3 to q_4, and q_4 to q_5 are sides of Ω. Since the angles of Ω at the points q_j's are all equal, we can suppose that $q_j = R^*(\cos(j-1)\theta, \sin(j-1)\theta),\ j = 1, \dots, 5$ for some positive angle θ. Then (1.9) implies that $\theta = 2\pi/5$, that is the q_j's are the vertices of a regular pentagon that contains C_p. Therefore, the sides of Ω issuing from the vertex of Ω outside C_q cannot intersect C_p because they lie outside the pentagon — a contradiction.

(v) If $k = 6$, all the vertices of Ω lie on C_q and hence the angles of Ω must all be equal. The fact that $\partial\Omega$ is circumscribed to C_p also implies that all the sides of Ω have equal length, that is, Ω must be regular.

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