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Gravity on a fuzzy sphere

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Abstract

We propose an action for gravity on a fuzzy sphere, based on a matrix model. We find striking similarities with an analogous model of two dimensional gravity on a noncommutative plane, i.e. the solution space of both models is spanned by pure $U(2)$ gauge transformations acting on the background solution of the matrix model, and there exist deformations of the classical diffeomorphisms which preserve the two-dimensional noncommutative gravity actions.

1 Introduction

A lot of knowledge has been successfully elaborated for noncommutative gauge theories on a quantum hyper-plane, but the ultimate goal remains to find consistent theories which unify curved manifolds with noncommutative algebras [1]-[12]. Usually theories of gravity are realized through the metric tensor which can be thought as a field deforming the space-time from flatness to a curved space. However one can write down gravity theories whose basic fields deform the space-time starting from a given curved manifold, for example in two dimensions a sphere. This procedure will be applied in the present paper to the noncommutative case.

Recently, in [8] it has been suggested a model for two-dimensional gravity on a noncommutative plane, made of gravity fields and an auxiliary scalar field. However it is possible to choose another noncommutative manifold as background, i.e. the fuzzy sphere. The purpose of this article is to postulate the action for noncommutative gravity on a fuzzy sphere and make comparison with the results contained in [8], concerning the flat case. We are aware of another work which has another proposal on this subject [12], but we believe that our point of view is fresh and more suited to the comparison with the flat case. We start from a very simple matrix model action, based on $U(2)$ invariance, which contains the fuzzy sphere solution as a background, and then construct the gravity fields as fluctuations from this background. In our model the scalar field and gravity fields are unified in three undifferentiated fluctuations. Only in the commutative limit one can disentangle the gravity degrees of freedom from the scalar one. We find that the solutions of this model are made by pure $U(2)$ gauge transformations acting on the background solution of the matrix model, and we explicitly study some example. This property has a perfect analogy with the model on the plane confirming that the two models of noncommutative gravity are strictly related. Moreover we find that although the classical diffeomorphism group is broken, it is possible to define deformed diffeomorphisms which preserve the fuzzy sphere action, in complete analogy with the quantum plane [8]. Again two-dimensional theories are good examples on which to test new original ideas on noncommutative gravity.

2 The model

The fuzzy sphere [13]-[22] is a noncommutative manifold represented by the following algebra:

$$[\hat{x}_i, \hat{x}_j] = i\rho\epsilon_{ijk}\hat{x}_k \qquad \hat{x}^i = \rho L^i, \tag{2.1}$$

where L^i is the usual angular momentum operator. The radius of the sphere, obtained by the following condition

$$\hat{x}_i\hat{x}_i = R^2 = \rho^2 L_i L_i = \rho^2 \frac{N(N+2)}{4} \tag{2.2}$$

is kept fixed in the commutative limit $N \to \infty$, therefore

$$\rho \sim \frac{1}{N}. \tag{2.3}$$

We postulate that the action of gravity on a fuzzy sphere is given by the following action :

$$S = \frac{1}{g^2} Tr[\frac{i}{3}\epsilon^{ijk} X_i X_j X_k + \frac{\rho}{2} X_i X_i], \tag{2.4}$$

where X_i is not only an $(N+1) \times (N+1)$ hermitian matrix, but it has also values in $U(2)$:

$$X_i = X_i^A t_A = X_i^0 + X_i^a \tau_a \tag{2.5}$$

where τ_a are the Pauli matrices. Having chosen the generators of the $U(2)$ algebra hermitian , all the component matrices X_i^0, X_i^a are therefore $(N+1) \times (N+1)$ hermitian matrices.

Let us note the analogy of this action with the proposed action for gravity on a noncommutative two-dimensional plane [8]

$$S_p = Tr[\phi\epsilon^{ij}([X_i, X_j] - i\theta_{ij}^{-1})]. \tag{2.6}$$

In this case there are also three independent hermitian operators as in our case, but we note that a pure scalar operator ϕ is disentangled from the true gravity degrees of freedom, the two matrices $X_i (i = 1, 2)$. In our case (2.4) we can think that the scalar operator ϕ and the gravity degrees of freedom are unified in our operators $X_i (i = 1, 2, 3)$ and that it is possible to disentangle the scalar field from the gravity fields only in the commutative limit $\rho \to 0$ (2.3).

Let us describe in details how to obtain such a picture. Firstly we need to develop the matrix X_i as a background plus fluctuations (see also [14]-[17]):

$$X_i = \hat{x}_i + \rho R \hat{a}_i. \tag{2.7}$$

In front of the $U(2)$ noncommutative connection $\hat{a}_i$ there is a factor ρ, which means that in the $N \to \infty$ limit the difference between X_i and the background $\hat{x}_i$ must be negligible. The action S (2.4) is invariant under the $U(2)$ unitary transformation

$$X_i \to U^{-1} X_i U \tag{2.8}$$

which extends the local Lorentz invariance of the gravity theory to a global symmetry of the matrix model.

By developing U in terms of an infinitesimal transformation

$$U \sim 1 + i\hat{\lambda} \tag{2.9}$$

the fluctuations around the fixed background transform as

$$\hat{a}_i \to \hat{a}_i - \frac{i}{R}[\hat{L}_i, \hat{\lambda}] + i[\hat{\lambda}, a_i]. \tag{2.10}$$

To reformulate the action in terms of the fluctuations $\hat{a}_i$ we have to define a star product on the fuzzy sphere, analogous to the Moyal star product for the plane.

Recall that a matrix on the fuzzy sphere can be developed in terms of the noncommutative analogue of the spherical harmonics $\hat{Y}_{lm}$:

$$\hat{Y}_{lm} = R^{-l} \sum_a f^{(lm)}_{a_1, a_2, \ldots a_l} \; \hat{x}_{a_1} \ldots \hat{x}_{a_l} \tag{2.11}$$

while the classical spherical harmonics are defined with $\hat{x}_i$ substituted with the commutative coordinates x_i.

A general hermitian matrix

$$\hat{a} = \sum_{l=0}^{N} \sum_{m=-l}^{l} a_{lm} \; \hat{Y}_{lm} \qquad a^*_{lm} = a_{l-m} \tag{2.12}$$

corresponds therefore to an ordinary function on the commutative sphere as:

$$a(\Omega) = \frac{1}{N+1} \sum_{l=0}^{N} \sum_{m=-l}^{l} Tr(\hat{Y}^{\dagger}_{lm} \hat{a}) Y_{lm}(\Omega) \tag{2.13}$$

and the ordinary product of matrices is mapped to the star product on the commutative sphere:

$$\hat{a}\hat{b} \rightarrow a * b$$

$$a(\Omega) * b(\Omega) = \frac{1}{N+1} \sum_{l=0}^{N} \sum_{m=-l}^{l} Tr(\hat{Y}_{lm}^{\dagger} \hat{a}\hat{b}) Y_{lm}(\Omega). \tag{2.14}$$

Derivative operators can be constructed using the adjoint action of $\hat{L}_i$ and tend to the classical Lie derivative L_i in the $N \rightarrow \infty$ limit:

$$Ad(\hat{L}_i) \rightarrow L_i = \frac{1}{i} \epsilon_{ijk} x_j \partial_k. \tag{2.15}$$

L_i can be expanded in terms of the Killing vectors of the sphere

$$L_i = -i K_i^a \partial_a. \tag{2.16}$$

In terms of K_i^a we can form the metric tensor $g_{ab} = K_a^i K_b^i$. The explicit form of these Killing vectors is

$$\begin{aligned} K_1^\theta &= -sin\phi \qquad K_1^\phi = -cotg\theta cos\phi \\ K_2^\theta &= cos\phi \qquad K_1^\phi = -cotg\theta sin\phi \\ K_3^\theta &= 0 \qquad K_3^\phi = 1. \end{aligned} \tag{2.17}$$

Trace over matrices can be mapped to the integration over functions:

$$\frac{1}{N+1} Tr(\hat{a}) \rightarrow \int \frac{d\Omega}{4\pi} \, a(\Omega). \tag{2.18}$$

Having introduced the star product (2.14), we can map the action $S(\lambda)$ to the following field theory action as in [14]-[17]

$$\begin{aligned} S &= \frac{i}{4g_f^2} \epsilon_{ijk} Tr \int d\Omega ((L_i a_j) a_k + \frac{R}{3} [a_i, a_j] a_k - \frac{i}{2} \epsilon_{ijl} a_l a_k)_* \\ &+ \frac{\pi}{12 g_f^2} \frac{N(N+2)}{R^2} \end{aligned} \tag{2.19}$$

where the residual trace is for the $U(2)$ degrees of freedom. The classical limit is realized as

$$R = \text{fixed} \qquad g_f^2 = \frac{4\pi g^2}{(N+1)\rho^4 R^2} = \text{fixed} \qquad N \to \infty. \tag{2.20}$$

In the commutative limit, the star product becomes the commutative product, and the scalar field ϕ is separable from the gravity fields as in the following formula

$$Ra_i(\Omega) = K_i^a A_a(\Omega) + \frac{x_i}{R}\phi(\Omega) \tag{2.21}$$

where $A_a(\Omega)$ are the gravity fields on the sphere.

Defining the gauge covariant field strength $\hat{F}_{ij}$ as

$$\begin{aligned}\hat{F}_{ij} &= \frac{1}{\rho^2 R^2}([X_i, X_j] - i\rho\epsilon_{ijk}X_k) \\ &= [\frac{\hat{L}_i}{R}, \hat{a}_j] - [\frac{\hat{L}_j}{R}, \hat{a}_i] + [\hat{a}_i, \hat{a}_j] - \frac{i}{R}\epsilon_{ijk}\hat{a}_k\end{aligned} \tag{2.22}$$

it can be expanded, only in the commutative limit in terms of the components fields (ϕ, A_a) (2.21) as follows:

$$F_{ij}(\Omega) = \frac{1}{R^2}K_i^a K_j^b F_{ab} + \frac{i}{R^2}\epsilon_{ijk}x_k\phi + \frac{1}{R^2}x_j K_i^a D_a\phi - \frac{1}{R^2}x_i K_j^a D_a\phi \tag{2.23}$$

where $F_{ab} = -i(\partial_a A_b - \partial_b A_a) + [A_a, A_b]$ and $D_a = -\partial_a + [A_a, ...]$ for the $U(2)$ case.

The classical limit of the action S is determined to be:

$$\begin{aligned}S &= \frac{1}{2g_f^2 R^2} Tr \int d\Omega (i\epsilon_{ijk}K_i^a K_j^b F_{ab}\frac{x_k}{R}\phi - \phi^2) = \\ &= \frac{1}{2g_f^2 R^2} Tr \int d\Omega (\frac{i\epsilon^{ab}}{\sqrt{g}}F_{ab}\phi - \phi^2)\end{aligned} \tag{2.24}$$

where ϵ^{ab} is defined as $\epsilon^{\theta\phi} = 1$.

The corresponding equations of motion are the null condition of the field strength

$$\hat{F}_{ij} \propto [X_i, X_j] - i\rho\epsilon_{ijk}X_k = 0. \tag{2.25}$$

In the classical limit these correspond to

$$
\begin{aligned}
& i\epsilon^{ab}F_{ab} - 2\sqrt{g}\phi = 0 \\
& D_a\phi = 0.
\end{aligned} \tag{2.26}
$$

Let us discuss them in components. Firstly we identify the gravity fields A_a as:

$$
A_a = e_a^1\tau_1 + e_a^2\tau_2 + \omega_a^3\tau_3 + \omega_a^0 \tag{2.27}
$$

where $e_a^i (i = 1, 2)$ are the two components of the vierbein, ω_a^3 is the usual abelian two-dimensional spin connection and ω_a^0 is an extra scalar spin connection which appears for consistency in the noncommutative case.

To compute the classical equations of motion (2.26) in components we need:

$$
F_{ab} = F_{ab}^A\tau_A = F_{ab}^1\tau_1 + F_{ab}^2\tau_2 + F_{ab}^3\tau_3 + F_{ab}^0 \tag{2.28}
$$

where

$$
\begin{aligned}
F_{ab}^\alpha &= -i(\partial_a e_b^\alpha - \partial_b e_a^\alpha) + i\epsilon^{\alpha\beta}(\omega_a^3 e_b^\beta - \omega_b^3 e_a^\beta) \qquad \alpha = 1, 2 \\
F_{ab}^3 &= \partial_a\omega_b^3 - \partial_b\omega_a^3 + i\epsilon^{\alpha\beta}e_a^\alpha e_b^\beta \\
F_{ab}^0 &= \partial_a\omega_b^0 - \partial_b\omega_a^0
\end{aligned} \tag{2.29}
$$

Analogously the scalar field ϕ has an expansion

$$
\phi = \phi^A\tau_A = \phi^0 + \phi^a\tau_a \qquad a = 1, 2, 3. \tag{2.30}
$$

Finally we obtain :

$$
\begin{aligned}
& i\epsilon^{ab}F_{ab}^A - 2\sqrt{g}\phi^A = 0 \\
& D_a\phi = 0.
\end{aligned} \tag{2.31}
$$

3 Classical solutions of the model

Let us discuss the classical solutions of the model. We will find in this respect a striking similarity with the corresponding model on the plane. Recalling that the model (2.6) has as equations of motions the following ones:

$$
\begin{aligned}
&[X_i, \phi] = 0 \\
&[X_i, X_j] = i\theta_{ij}^{-1}.
\end{aligned} \tag{3.1}
$$

A particular solution to this equation, which we call background solution is given by the choice

$$
\begin{aligned}
X_i &= \hat{p}_i \qquad \hat{p}_i = -i\theta_{ij}^{-1}\hat{x}_j \\
\phi_0 &= \phi_0^0 + \phi_0^a \tau_a = \phi_0^A \tau^A \qquad [\hat{p}_i, \phi_0^A] = 0
\end{aligned} \tag{3.2}
$$

with the scalar components all constants ($\partial_i \phi_0^A = 0$).

The general solution of this model can be found by applying a pure $U(2)$ gauge transformation on the background solution (3.2) as follows

$$
\begin{aligned}
\phi &= U^{-1}\phi_0 U \\
X_i &= U^{-1}\hat{p}_i U
\end{aligned} \tag{3.3}
$$

where U is an unitary operator, with values in the group $U(2)$.

The corresponding equations of motions for the model on the fuzzy sphere are again solved by a pure gauge transformation acting on the background of the matrix model

$$
\begin{aligned}
&\hat{F}_{ij} \propto [X_i, X_j] - i\rho\epsilon_{ijk}X_k = 0 \\
&X_i = U^{-1}\hat{x}_i U.
\end{aligned} \tag{3.4}
$$

As an exercise, let us compute an example of such solutions, and identify it with a classical solution of the corresponding gravity theory on a sphere. Let us pose

$$
X_i = \hat{x}_i + (U^{-1}\hat{x}_i U - \hat{x}_i) = \hat{x}_i + A_i. \tag{3.5}
$$

The field A_i must be a fluctuation of order ρ, as discussed at the beginning. Let us compute the solution generated by the following unitary operator for the simplest case $N = 2$, where $\hat{x}_i$ is proportional to the Pauli matrices:

$$U = e^{2i\alpha n_i^1 \hat{x}_i} P + e^{2i\beta n_i^2 \hat{x}_i}(1-P) = U_L P + U_R(1-P) \tag{3.6}$$

where α, β are pure constants, n_i^1, n_i^2 are two generic unit vectors, $\hat{x}_i$ are proportional to the Pauli matrices and the group factor is a projector

$$P = \frac{1 + n_i \tau_i}{2} \tag{3.7}$$

with n_i another unit vector.

A simple algebra shows that

$$\begin{aligned} A_i &= [-2sin(\rho\alpha)cos(\rho\alpha)\epsilon_{ijk} n_j^1 \hat{x}_k + 2sin^2(\rho\alpha)(n_i^1(n_j^1 \hat{x}_j) - \hat{x}_i)]P + \\ &+ [-2sin(\rho\beta)cos(\rho\beta)\epsilon_{ijk} n_j^2 \hat{x}_k + 2sin^2(\rho\beta)(n_i^1(n_j^2 \hat{x}_j) - \hat{x}_i)](1-P). \end{aligned} \tag{3.8}$$

It is possible to check directly that A_i satisfies the equations of motion

$$[\hat{x}_i, A_j] - [\hat{x}_j, A_i] + [A_i, A_j] = i\rho\epsilon_{ijk} A_k. \tag{3.9}$$

By developing the matrix A_i for $N \to \infty$ ($\rho \to 0$), it is easy to show that $A_i \sim O(\rho)$ as in (2.7) , confirming that it is possible to generate, through unitary transformations, fluctuations which tend to a nontrivial classical limit.

In the commutative limit the fluctuation must be truncated to order $O(\rho)$:

$$A_i = -2\rho\alpha\epsilon_{ijk} n_j^1 x_k P - 2\rho\beta\epsilon_{ijk} n_j^2 x_k (1-P) \tag{3.10}$$

and it satisfies the classical equation

$$L_i A_j - L_j A_i = i\epsilon_{ijk} A_k \qquad L_i = -i\epsilon_{ijk} x_j \frac{\partial}{\partial x_k}. \tag{3.11}$$

In fact the action of the first member gives rise to

$$\begin{aligned} L_i A_j - L_j A_i &= -2i\rho\alpha(n_i^1 x_j - n_j^1 x_i)P - 2i\rho\beta(n_i^2 x_j - n_j^2 x_i)(1-P) = \\ &= i\epsilon_{ijk} A_k. \end{aligned} \tag{3.12}$$

This truncated fluctuation (3.10) satisfies the gauge condition

$$x_i A_i = 0 \tag{3.13}$$

and therefore it will not contribute to the scalar field, but only to the gravity fields. By developing

$$A_i = \frac{x_i}{R}\phi + K_i^a A_a \tag{3.14}$$

we can extract the gravity fields as

$$A_a = g_{ab} K_i^b A_i = \sqrt{g}\epsilon_{ab}[(-2\rho\alpha n_i^1 K_i^b)P + (-2\rho\beta n_i^2 K_i^b)(1-P)]. \tag{3.15}$$

The classical equations of motion

$$\begin{aligned} F_{ab} &= \partial_a A_b - \partial_b A_a + [A_a, A_b] = 0 \\ \phi &= 0 \end{aligned} \tag{3.16}$$

are solved since $[A_a, A_b] = 0$, thanks to $[P, P] = [P, 1-P] = 0$ and since

$$\partial_\theta A_\phi = \partial_\phi A_\theta = sin\theta[2\rho\alpha(n_1^1 cos\phi + n_2^1 sin\phi) + 2\rho\beta(n_1^2 cos\phi + n_2^2 sin\phi)]. \tag{3.17}$$

4 Deformed diffeomorphisms on the fuzzy sphere

Let us recall the steps that are necessary to prove the existence of deformed diffeomorphisms on the noncommutative plane. In the commutative limit, the action for gravity on a plane is invariant not only under local Lorentz invariance, but also under infinitesimal diffeomorphisms (Lie derivatives) along an arbitrary vector field $v = v^\mu \partial_\mu$

$$\begin{aligned} \mathcal{L}_v A_i &= (di_v + i_v d)A_i \qquad A_i = X_i - \hat{p}_i \\ \mathcal{L}_v \phi &= i_v d\phi \end{aligned} \tag{4.1}$$

where i_v is the inner product on differential forms. Using the Leibnitz rule for the inner product is is possible to prove that

$$\mathcal{L}_v A = \delta_{i_v A} A + i_v F$$
$$\mathcal{L}_v \phi = \delta_{i_v A} \phi + i_v D\phi \tag{4.2}$$

the infinitesimal diffeomorphisms contains a gauge transformation with parameter $\lambda = i_v A$, and a residual part which is vanishing on shell (i.e. on the equations of motion).

In the noncommutative case the situation is quite different. It is easy to generalize the local Lorentz invariance, but it seems quite difficult to define the corresponding infinitesimal diffeomorphisms. However in the paper [8] this question has been solved for the noncommutative plane. We are going to prove that the same property holds also for our action (2.4) on the fuzzy sphere, but we recall the solution on a noncommutative plane as a suggestion.

Let us generalize the inner product in the noncommutative case; starting from a p-form ω^p we define

$$\omega^p = \frac{1}{p!}\omega_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots . \wedge dx^{\mu_p}$$
$$i^*_v \omega^p = \frac{1}{2(p-1)!}[v^\rho * \omega^p_{\rho\mu_1 \ldots \mu_{p-1}} + \omega^p_{\rho\mu_1 \ldots \mu_{p-1}} * v^\rho] dx^{\mu_1} \wedge \ldots . \wedge dx^{\mu_p}. \tag{4.3}$$

In this case the Leibnitz rule is not valid anymore for i^*_v. We then define the deformed diffeomorphisms on the noncommutative plane as

$$\Delta^*_v = i^*_v D + \delta_{i^*_v A}$$
$$\Delta^*_v A = i^*_v F + \delta_{i^*_v A} A$$
$$\Delta^*_v \phi = i^*_v D\phi + \delta_{i^*_v A}. \tag{4.4}$$

Since it is known that the action is invariant under local Lorentz transformations, in order to prove its invariance under the transformation (4.4) it is enough to prove it for the residual transformation:

$$\delta'_v A = i^*_v F$$
$$\delta'_v \phi = i^*_v D\phi. \tag{4.5}$$

Let us reformulate it in terms of the matrix model variables $(\phi, X_i (i = 1, 2))$:

$$S = Tr(\phi\epsilon^{ij}F_{ij}) \qquad F_{ij} = [X_i, X_j] - i\theta_{ij}^{-1}$$
$$\delta_v^{'}X_i = v^{\alpha}F_{i\alpha} + F_{i\alpha}v^{\alpha}$$
$$\delta_v^{'}\phi = v^{\alpha}[X_{\alpha}, \phi] + [X_{\alpha}, \phi]v^{\alpha}. \tag{4.6}$$

Then the variation of the action is given by

$$\begin{aligned} \delta^{'}S &= Tr(\delta\phi\epsilon^{ij}F_{ij} + 2\epsilon^{ij}\delta X_i[X_j, \phi]) = \\ &= Tr((\epsilon^{ij}v^{\alpha} + \epsilon^{j\alpha}v^i + \epsilon^{\alpha i}v^j)([X_{\alpha}, \phi]F_{ij} + F_{ij}[X_{\alpha}, \phi])). \end{aligned} \tag{4.7}$$

The tensor under parenthesis is zero, due to the cyclic dependence from three indices which can take only two values. Therefore it is proved that (4.5) is indeed a deformed diffeomorphism on the noncommutative plane.

Now comes the interesting part since we have found for our postulated action (2.4) an analogous property, i.e. it is possible to generalize the diffeomorphism group of classical gravity on the sphere to a nontrivial deformed diffeomorphism group on a fuzzy sphere.

We start directly from the matrix variable $X_i (i = 1, 2, 3)$ of our model

$$S = \frac{1}{g^2}Tr(\frac{i}{3}\epsilon^{ijk}X_iX_jX_k + \frac{\rho}{2}X_iX_i). \tag{4.8}$$

We note that the variation of S is vanishing on shell

$$\delta S \propto \frac{i}{g^2}Tr[\epsilon^{ijk}(\partial X_i\hat{F}_{jk})]. \tag{4.9}$$

We define the deformed diffeomorphism as generated by

$$\delta_v^{'}X_i = v^{\alpha}\hat{F}_{i\alpha} + \hat{F}_{i\alpha}v^{\alpha}$$
$$\hat{F}_{ij} \propto [X_i, X_j] - i\rho\epsilon_{ijk}X_k. \tag{4.10}$$

Let us note the nontrivial dependence on ρ of the deformed rules which tends to zero in the commutative limit. Therefore we obtain for the generic variation (4.9) the following result

$$\delta_v^{'}S = \frac{i}{g^2}Tr[v^{\alpha}\epsilon^{ijk}(\hat{F}_{i\alpha}\hat{F}_{jk} + \hat{F}_{jk}\hat{F}_{i\alpha})]. \tag{4.11}$$

This particular variation can be rewritten as

$$
\delta'_v S = \frac{i}{g^2} Tr[(v^{\alpha}\epsilon^{ijk} - v^i\epsilon^{jk\alpha} - v^j\epsilon^{ki\alpha} - v^k\epsilon^{ij\alpha})
(\hat{F}_{i\alpha}\hat{F}_{jk} + \hat{F}_{jk}\hat{F}_{i\alpha})] \tag{4.12}
$$

and again the variation is proportional to a tensor which is zero because of the cyclic dependence from more indices than the number of values. It is this property which finally permits to extend the diffeomorphism group for all the known two-dimensional noncommutative manifolds.

We note that to reach this result the choice of the action is crucial and that this property doesn't hold for the general matrix model action of the fuzzy sphere, defined in [16]-[17].

5 Conclusion

In this article we have introduced in two dimensions an alternative description of noncommutative gravity starting from the fuzzy sphere background, instead of the noncommutative plane. We have found several properties in common between the two models which may suggest their equivalence.

Let us recapitulate them; the classical solutions of the two models are made by pure $U(2)$ unitary transformations acting on the background solutions, and the diffeomorphism group can be deformed in both cases to a nontrivial noncommutative invariance, being the first examples of general covariance in a matrix model.

We can think the solution space of noncommutative gravity as an extension of the simplest noncommutative algebras (quantum plane and fuzzy sphere) to a generic two-dimensional noncommutative manifold. Indeed the fuzzy sphere solution can be found by deforming the quantum plane with non trivial gravity fields [8].

A possible generalization of the present research would be to extend these results to the case of four dimensions [23]-[24], in which however the issue of a deformed general covariance is a harder problem.

Returning to two dimensions, we would like to point out that our postulate action is probably unique if we demand the presence of deformed diffeomorphisms, and previous proposals on this subject [12] are surely different because of the lack of this property. Moreover in our model the scalar field and the gravity fields are unified in three undifferentiated fluctuations, resulting in a more symmetric description. Only in the commutative limit one recovers the distinction between the scalar field and the gravity fields. The fuzzy sphere case is also interesting because it gives a finite description of noncommutative gravity (with a function space truncated to a finite Hilbert space). Therefore it would be interesting to study the quantum properties of this model, as a noncommutative counterpart of two dimensional quantum gravity which has been deeply studied.

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