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ABSTRACT. We prove an existence and uniqueness theorem for the abstract version of a semilinear integrodifferential inverse problem. We apply such result to the Kermack-McKendrick model with diffusion. Our main tools are: the analytic semigroups theory, optimal regularity results and fixed point arguments.

1 INTRODUCTION

We study an identification problem for a semilinear integrodifferential system arising in the theory of the spread of infections. More precisely, we investigate the Kermack-McKendrick model with diffusion. For more details about this model see [10].
In this section we state our problem then, we formulate it in an abstract setting and finally we point out the novelty of our results.

We will use the spaces $\mathcal{B}([0,T];D_A(\theta,\infty))$ of bounded functions with values in the interpolation spaces $D_A(\theta,\infty)$. For precise definitions see the beginning of Section 2.

Let Ω be an open bounded set in $\mathbf{R}^3$ and let T be a positive constant. If u and v denote the susceptible and the infective populations densities, respectively their evolution equations are

$$u_t(t,x) = K_1\Delta u(t,x) - a_1 u(t,x)$$

$$-b_1 u(t,x)\int_0^t h(t-s)v(s,x)\,ds, \qquad (t,x)\in(0,T]\times\Omega, \tag{1.1}$$

$$v_t(t,x) = K_2 \Delta v(t,x) - a_2 v(t,x)$$

$$+b_2 u(t,x) \int_0^t h(t-s) v(s,x)\, ds, \qquad (t,x) \in (0,T] \times \Omega, \tag{1.2}$$

with the related initial–boundary conditions

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in \Omega, \tag{1.3}$$

$$D_\nu u(t,x) = D_\nu v(t,x) = 0, \qquad (t,x) \in [0,T] \times \partial\Omega, \tag{1.4}$$

where D_ν denotes the external normal derivative, a_i and b_i, $i = 1, 2$ are called the constant rates, $K_i > 0$ are the diffusion coefficients, $u_0, v_0 : \Omega \to \mathbf{R}$ are given functions. The kernel $h(t)$ is considered unknown. The addition of the terms $K_1 \Delta u$ and $K_2 \Delta v$ to the original Kermack-McKendrick model allows the migration of both infective and susceptible populations, and the integral term in (1.1) and (1.2) represents a transfer mechanism of infection.

Because of the interpretation of the model we assume $a_i \leq 0$ and $b_i > 0$ for $i = 1, 2$.

The fact that the kernel h is not directly measurable leads us to consider the inverse problem, i.e. u, v and h have to be determined simultaneously. To obtain our purpose we need additional information on u, which can be analytically represented in integral form as follows

$$\int_\Omega \phi(x) u(t,x)\, dx = g(t), \tag{1.5}$$

where $\phi : \Omega \to \mathbf{R}$ and $g : [0,T] \to \mathbf{R}$ are given functions whose regularity will be specified in the sequel. More precisely, condition (1.5) represents additional measurements on the susceptible population u in some parts of Ω, so ϕ is a suitable compact support function that depends on the type of device used for the measure, while g stands for the results of the measurements on u.

We are in position to formulate our **Inverse Problem** (IP).
Determine three continuous functions: $u : [0,T] \times \Omega \to \mathbf{R}$, $v : [0,T] \times \Omega \to \mathbf{R}$ *and* $h : [0,T] \to \mathbf{R}$ *satisfying system (1.1)-(1.5).*

We now formulate our problem in a more general setting relating it to a Banach algebra X.

Let X be a Banach algebra and assume that $A : D(A) \subset X \to X$ and $B : D(B) \subset X \to X$ are two linear and closed operators with domains $D(A)$, and $D(B)$, respectively. We denote by $D_A(\theta, \infty)$ the interpolation spaces, for their definition see Section 2, with the inclusions $D(A) \subset D_A(\theta, \infty) \subset X$. Analogously we define $D_B(\theta, \infty)$.

Consider now three functions $u : [0,T] \to D_A(\theta, \infty)$, $v : [0,T] \to D_B(\theta, \infty)$ and $h : [0,T] \to \mathbf{R}$ satisfying the system

$$u'(t) = Au(t) - a_1 u(t) - b_1 u(t) \int_0^t h(t-s) v(s)\, ds, \tag{1.6}$$

$$v'(t) = Bv(t) - a_2 v(t) + b_2 u(t) \int_0^t h(t-s)v(s)\,ds, \tag{1.7}$$

$$u(0) = u_0, \quad v(0) = v_0, \tag{1.8}$$

$$\Phi[u(t)] = g(t), \quad t \in [0,T], \tag{1.9}$$

where Φ is a given bounded linear functional on $D_A(\theta,\infty)$, $g : [0,T] \to \mathbf{R}$, $u_0 : \overline{\Omega} \to \mathbf{R}$ and $v_0 : \overline{\Omega} \to \mathbf{R}$ are given functions and $a_i \le 0$ and $b_i > 0$ for $i = 1, 2$.

We can formulate our **Abstract Inverse Problem** (AIP).
Determine three functions $u : [0,T] \to D_A(\theta,\infty)$, $v : [0,T] \to D_B(\theta,\infty)$ *and* $h : [0,T] \to \mathbf{R}$ *satisfying (1.6)–(1.9).*

Optimal regularity results and the analytic semigroup theory are fundamental tools in the study of direct and inverse parabolic problems. Our strategy is to formulate the abstract version of the inverse problem in terms of a system of equivalent fixed point equations. Several optimal regularity results are at our disposal for such equivalent formulation.

Consider the following optimal regularity results for the Cauchy Problem (CP):

$$\begin{cases} u'(t) = Au(t) + f(t), & t \in [0,T], \\ u(0) = u_0. \end{cases}$$

Let $A : D(A) \subset X \to X$ be the infinitesimal generator of the analytic semigroup e^{tA} and $\beta \in (0,1)$. In [9] [12] we can find the proofs of the following results:

THEOREM 1.1. **(Strict solution in Hölder spaces $C^\beta([0,T];X)$)** *For any* $f \in C^\beta([0,T];X)$, $u_0 \in D(A)$ *with* $Au_0 + f(0) \in D_A(\beta,\infty)$ *the Cauchy problem (CP) admits a unique solution* $u \in C^{1+\beta}([0,T];X) \cap C^\beta([0,T];D(A))$.

THEOREM 1.2. **(Strict solution in spaces $\mathcal{B}([0,T];D_A(\beta,\infty))$)** *For any* $f \in C([0,T];X) \cap \mathcal{B}([0,T];D_A(\beta,\infty))$, $u_0 \in D_A(\beta+1,\infty)$ *the Cauchy problem (CP) admits a unique solution* $u \in C^1([0,T];X) \cap C([0,T];$
$D(A)) \cap \mathcal{B}([0,T];D_A(\beta+1,\infty))$.

What we prove in this paper is an existence and uniqueness theorem for the (AIP) and we apply it to the (IP) in the case X is the space of continuous functions $C(\overline{\Omega})$. Using the generation results in [6] we could also prove a similar result if we set $X = C^1(\overline{\Omega})$ if A and B are second order differential operators.

On the optimal regularity Theorem 1.1 are based many results that we can find in the recent literature also for the Phase-Field models, for the theory of materials with memory and for the population dynamic models see [1] [4] [5]. In [7], [2] and in [3] the authors study the above model requiring a different set of conditions on the data without investigating the $\mathcal{B}([0,T];D_A(\theta,\infty))$ regularity for some $\theta \in \mathbf{R}^+$.

The novelty of this paper is that we apply Theorem 1.2, (and not Theorem 1.1) which is particularly suitable for the semilinear model introduced above.

We recall that in the paper [8], for the first time, the analytic semigroup theory and Theorem 1.1 have been used to face integrodifferential parabolic inverse problems for the heat equation with memory.

The plan of the paper is the following.

In section 2 we define the function spaces and we introduce some preliminary material related to the analytic semigroups theory.
In section 3 we state our main result for the abstract problem (1.6)-(1.9) (see Theorem 3.1) and we give an application to a concrete case (see Theorem 3.2).
In section 4 we prove, that under suitable conditions the integrodifferential system (1.6)-(1.9) is equivalent to a fixed point system of three Volterra integral equations of the second kind (see Theorem 4.1).
Finally in section 5 we prove Theorem 3.1.

2 PRELIMINARY MATERIAL

The results that we are going to recall in this section hold in the case X is a Banach space only. For our purpose, since we apply our results to the (IP) we consider X to be a Banach algebra with norm $\|\cdot\|$. Let $T > 0$, we denote by $C([0,T];X)$ the usual space of continuous functions with values in X, while we denote by $\mathcal{B}([0,T];X)$ the space of bounded functions with values in X. Equipped both with the sup–norm:

$$\|u\|_{C([0,T];X)} = \|u\|_{\mathcal{B}([0,T];X)} := \sup_{0\le t\le T} \|u(t)\| \tag{2.1}$$

they become Banach spaces. For $\beta \in (0,1)$ we define

$$C^\beta([0,T];X) = \{u \in C([0,T];X) : |u|_{C^\beta([0,T];X)} = \sup_{0\le s<t\le T} \frac{\|u(t)-u(s)\|}{(t-s)^\beta} < \infty\} \tag{2.2}$$

and we endow it with the norm

$$\|u\|_{C^\beta([0,T];X)} = \|u\|_{C([0,T];X)} + |u|_{C^\beta([0,T];X)}. \tag{2.3}$$

Let Ω a bounded open set in $\mathbf{R}^n$, $n \in \mathbf{N}$, for $\beta \in (0,1)$ we define

$$C^\beta(\Omega) = \{u \in C(\Omega) : |u|_{C^\beta(\Omega)} = \sup_{x,y\in\Omega,\ x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\beta} < \infty\} \tag{2.4}$$

and we endow it with the norm

$$\|u\|_{C^\beta(\Omega)} = \|u\|_{C(\Omega)} + |u|_{C^\beta(\Omega)}, \tag{2.5}$$

where

$$\|u\|_{C(\Omega)} := \sup_{x\in\Omega} |u(x)|. \tag{2.6}$$

By $\mathcal{L}(X)$ we denote the space of all bounded linear operators from X into itself equipped with the sup–norm, while $\mathcal{L}(X;\mathbf{R})$ is the space of all bounded linear functionals on X considered with the natural norm.

DEFINITION 2.1. Let $A : D(A) \subset X \to X$, be a linear operator, possibly with $\overline{D(A)} \neq X$. A is said to be sectorial if it satisfies the following assumptions:

(H1) there exists $\theta \in (\pi/2, \pi)$ such that any $\lambda \in \mathbf{C}\backslash\{0\}$ with $|\arg\lambda| \le \theta$ and $\lambda = 0$ belong to the resolvent set of A;

(H2) there exists $M > 0$ such that $\|\lambda(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \le M$ for any $\lambda \in \mathbf{C}\backslash\{0\}$ with $|\arg\lambda| \le \theta$.

The fact that the resolvent set of A is not void implies that A is closed, so that $D(A)$ endowed with the graph norm becomes a Banach space.
According to assumptions *H1, H2*, it is possible to define the semigroup $\{e^{tA}\}_{t\geq 0}$, of bounded linear operators in $\mathcal{L}(X)$, so that $t \to e^{tA}$ is an analytic function from $(0,\infty)$ to $\mathcal{L}(X)$ satisfying for $k \in \mathbb{N}$ the relations

$$\frac{d^k}{dt^k}e^{tA} = A^k e^{tA}, \quad t > 0, \tag{2.7}$$

and $Ae^{tA}x = e^{tA}Ax$, for all $x \in D(A)$ and $t \geq 0$. Moreover there exist positive constants M_k, for $k \in \mathbb{N}_0$ such that

$$||t^k A^k e^{tA}||_{\mathcal{L}(X)} \leq M_k, \qquad t > 0. \tag{2.8}$$

For more details see for example [9] [11]. Let us define the family of interpolation spaces ([11] or [13]) $D_A(\beta,\infty)$, $\beta \in (0,1)$, between $D(A)$ and X by

$$D_A(\beta,\infty) = \left\{x \in X : \ |x|_{D_A(\beta,\infty)} := \sup_{t>0} t^{1-\beta}||Ae^{tA}x|| < \infty\right\} \tag{2.9}$$

with the norm (2.11) for $j = 0$ only. We also set

$$\mathcal{D}_A(1+\beta,\infty) = \{x \in \mathcal{D}(A) : Ax \in \mathcal{D}_A(\beta,\infty)\}, \tag{2.10}$$

$D_A(1+\beta,\infty)$ turn out to be Banach spaces when equipped with the norms

$$||x||_{D_A(1+\beta,\infty)} = \sum_{j=0}^{1} ||A^j x|| + |Ax|_{D_A(\beta,\infty)}. \tag{2.11}$$

We reconsider Theorem 1.1 with the related fundamental estimates, related to the Cauchy problem

$$u'(t) = Au(t) + f(t), \qquad t \in [0,T], \tag{2.12}$$

$$u(0) = u_0. \tag{2.13}$$

THEOREM 2.2. *Let $A : D(A) \subset X \to X$ be the generator of the analytic semigroups e^{tA}. Then for any $f \in C([0,T];X) \cap \mathcal{B}([0,T];D_A(\beta,\infty))$, $u_0 \in D_A(\beta+1,\infty)$ problem (2.12)–(2.13) admits a unique solution $u \in C^1([0,T];X) \cap C([0,T];D(A)) \cap \mathcal{B}([0,T];D_A(\beta+1,\infty))$ represented by the formula*

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} f(s)\, ds := e^{tA}u_0 + (e^{tA} * f)(t), \tag{2.14}$$

and u', $Au \in C([0,T];X) \cap \mathcal{B}([0,T];D_A(\beta,\infty))$ and $Au \in C^\beta([0,T];X)$. Moreover, the following estimate holds

$$||e^{tA} * f||_{C([0,T];X)\cap\mathcal{B}([0,T];D_A(\beta,\infty))} \leq TM_0||f||_{C([0,T];X)\cap\mathcal{B}([0,T];D_A(\beta,\infty))}, \tag{2.15}$$

$$||e^{tA}u_0||_{C([0,T];X)\cap\mathcal{B}([0,T];D_A(\beta,\infty))} \leq C||u_0||_{D_A(\beta+1,\infty)}. \tag{2.16}$$

Proof. One can argue as in Corollary 4.3.9 iii) of [9]. To prove estimate (2.15) we note that

$$\|e^{tA} * f\|_{C([0,T];X)} \leq TM_0 \|f\|_{C([0,T];X)}, \tag{2.17}$$

$$\|e^{tA} * f\|_{\mathcal{B}([0,T];D_A(\beta,\infty))} \leq TM_0 \|f\|_{\mathcal{B}([0,T];D_A(\beta,\infty))}. \tag{2.18}$$

Adding (2.17) and (2.18) we get (2.15). As in Corollary 4.3.9 iii) [9] we get estimate (2.16). ■

THEOREM 2.3. *Let $h \in C([0,T];\mathbf{R})$ and $u \in C([0,T];X) \cap \mathcal{B}([0,T];D_A(\beta,\infty))$. Define the convolution operator*

$$h * u(t) := \int_0^t h(t-s)u(s)ds. \tag{2.19}$$

Then $*$ maps $C([0,T];\mathbf{R}) \times [C([0,T];X) \cap \mathcal{B}([0,T];D_A(\beta,\infty))]$ into $C([0,T];X)$ $\cap \mathcal{B}([0,T];D_A(\beta,\infty))$ and the following estimate holds:

$$\|h * u\|_{C([0,T];X)\cap\mathcal{B}([0,T];D_A(\beta,\infty))} \leq T\|h\|_{C([0,T];\mathbf{R})}\|u\|_{C([0,T];X)\cap\mathcal{B}([0,T];D_A(\beta,\infty))}. \tag{2.20}$$

Proof. Consider the estimates

$$\|h * u\|_{C([0,T];X)} \leq T\|h\|_{C([0,T];\mathbf{R})}\|u\|_{C([0,T];X)} \tag{2.21}$$

and

$$|h * u(t)|_{D_A(\beta,\infty)} = \sup_{\tau>0} \tau^{1-\beta}\|Ae^{\tau A}\int_0^t h(t-s)v(s)ds\|$$

$$\leq \sup_{\tau>0} \int_0^t h(t-s)\tau^{1-\beta}\|Ae^{\tau A}u(s)\|ds \leq T\|h\|_{C([0,T];\mathbf{R})}\|u\|_{\mathcal{B}([0,T];D_A(\beta,\infty))} \tag{2.22}$$

so that

$$\|h * u\|_{\mathcal{B}([0,T];D_A(\beta,\infty)} \leq T\|h\|_{C([0,T];\mathbf{R})}\|u\|_{\mathcal{B}([0,T];D_A(\beta,\infty))}. \tag{2.23}$$

From (2.21) and (2.23) we get the statement. ■

3 THE MAIN RESULTS

In this section we state our main abstract result and then we apply it to the case of the continuous functions space.

Suppose X be a Banach algebra, $\beta \in (0,1)$. Let us introduce the following conditions

Regularity Conditions

R_1 $u_0 \in D(A)$, $v_0 \in D(B)$;

R_2 $Au_0 - a_1u_0 \in D_A(\beta+1,\infty)$, $Bv_0 - a_2v_0 \in D_B(\beta+1,\infty)$;

R_3 $g \in C^2([0,T])$;

R_4 $\Phi \in \mathcal{L}(D_A(\beta+1,\infty);\mathbf{R})$;

R_5 $\Phi[Au] = \Psi[u]$, with $\Psi \in \mathcal{L}(D_A(\beta,\infty);\mathbf{R})$.

Compatibility Conditions
C_1 $\Phi[u_0] = g(0)$;
C_2 $\Phi[Au_0 - a_1 u_0] = g'(0)$;
C_3 $\Phi[v_0 u_0] \neq 0$ *and* $b_1 \neq 0$.

Our main abstract result is:

THEOREM 3.1. *Let A and B be sectorial operators (see Definition 2.1). Under assumptions R_1-R_5 and C_1-C_3 there exists $T^* \in (0,T]$ such that for any $\tau \in (0,T^*)$ problem (1.6)-(1.9) has a unique solution* $(u,v,h) \in [C^1([0,\tau];X) \cap C([0,\tau];D(A)) \cap \mathcal{B}([0,\tau];D_A(\beta+1,\infty)] \times [C^1([0,\tau];X) \cap C([0,\tau];D(B)) \cap \mathcal{B}([0,\tau];D_B(\beta+1,\infty)] \times C([0,\tau];\mathbf{R})$.

Proof. It is in Section 5. ∎

We are now in the position to apply Theorem 3.1 when we choose as reference space:

$$X = C(\bar{\Omega}). \tag{3.1}$$

We suppose that $D(A) = D(B)$ with the Neumann homogeneous boundary conditions, where

$$D(A) = \{u \in C(\bar{\Omega}) : Au \in C(\bar{\Omega}), \quad D_\nu u = 0 \ \ on \ \partial\Omega\}, \tag{3.2}$$

and

$$A := K_1 \Delta, \quad B := K_2 \Delta. \tag{3.3}$$

Moreover, in [9] it is proved that the operators defined in (3.3) whose domains are defined in (3.2) are sectorial in $C(\bar{\Omega})$. Then we recall the following characterizations concerning the interpolation spaces related to A (see [9]), for $\beta \neq 1/2$:

$$D_A(\beta,\infty) = C^{2\beta}(\overline{\Omega}), \quad if \ \ \beta \in (0,1/2), \tag{3.4}$$

$$D_A(\beta,\infty) = C^{2\beta}_\nu(\overline{\Omega}), \quad if \ \ \beta \in (1/2,1), \tag{3.5}$$

where

$$C^{2\beta}_\nu(\overline{\Omega}) = \{u \in C^{2\beta}(\overline{\Omega}) : D_\nu u = 0 \ \ on \ \ \partial\Omega \}. \tag{3.6}$$

Finally, we can define the set of admissible data consisting of all those functions u_0, v_0, ϕ, g, satisfying the following assumptions, for $\beta \in (0,1)/\{1/2\}$:
Regularity Conditions
$\mathcal{R}_1$ $u_0, v_0 \in D(A)$;
$\mathcal{R}_2$ $K_1\Delta u_0 - a_1 u_0,\ K_2\Delta v_0 - a_2 v_0 \in C^{2(\beta+1)}(\overline{\Omega})$, $\quad if \ \ \beta \in (0,1/2)$;
$\mathcal{R}_2'$ $K_1\Delta u_0 - a_1 u_0,\ K_2\Delta v_0 - a_2 v_0 \in C^{2(\beta+1)}_\nu(\overline{\Omega})$, $\quad if \ \ \beta \in (1/2,1)$;
$\mathcal{R}_3$ $g \in C^2([0,T])$;
$\mathcal{R}_4$ $\Phi \in \mathcal{L}(C^{2(\beta+1)}(\overline{\Omega});\mathbf{R})$, $\quad if \ \ \beta \in (0,1/2)$;
$\mathcal{R}_4'$ $\Phi \in \mathcal{L}(C^{2(\beta+1)}_\nu(\overline{\Omega});\mathbf{R})$, $\quad if \ \ \beta \in (1/2,1)$;
$\mathcal{R}_5$ $\int_\Omega \phi(x) K_1 \Delta u(t,x)\,dx = \Psi[u(t,\cdot)]$ with $D_\nu \phi = 0$ on $\partial\Omega$;
$\mathcal{R}_6$ $\Psi \in \mathcal{L}(C^{2\beta}(\overline{\Omega});\mathbf{R})$ $\quad if \ \ \beta \in (0,1/2)$;
$\mathcal{R}_6'$ $\Psi \in \mathcal{L}(C^{2\beta}_\nu(\overline{\Omega});\mathbf{R})$ $\quad if \ \ \beta \in (1/2,1)$;

where

$$\Psi[u(t,\cdot)] = \int_\Omega \psi(x)u(t,x)\,dx, \quad \psi(x) := K_1\Delta\phi(x) \tag{3.7}$$

Compatibility Conditions

$\mathcal{C}_1$ $\int_\Omega u_0(x)\,dx = g(0)$;

$\mathcal{C}_2$ $\int_\Omega [K_1\Delta u_0(x)\,dx - a_1 u_0(x)]\,dx = g'(0)$;

$\mathcal{C}_3$ $\int_\Omega v_0(x)u_0(x)\,dx \neq 0$ *and* $b_1 \neq 0$;

THEOREM 3.2. *Let A and B be sectorial operators (see Definition 2.1). Under assumptions $\mathcal{R}_1$-$\mathcal{R}_5$ and $\mathcal{C}_1$-$\mathcal{C}_3$ there exists $T^* \in (0,T]$ such that for any $\tau \in (0,T^*)$ problem (1.6)-(1.9) has a unique solution $(u,v,h) \in [C^1([0,\tau];C(\bar\Omega)) \cap \mathcal{B}([0,\tau];C^{2(\beta+1)}(\bar\Omega)] \times [C^1([0,\tau];C(\bar\Omega)) \cap \mathcal{B}([0,\tau];C^{2(\beta+1)}(\bar\Omega)] \times C([0,\tau];\mathbf{R})$ when $\beta \in (0,1/2)$ and $(u,v,h) \in [C^1([0,\tau];C(\bar\Omega)) \cap \mathcal{B}([0,\tau];C_\nu^{2(\beta+1)}(\bar\Omega)] \times [C^1([0,\tau];C(\bar\Omega)) \cap \mathcal{B}([0,\tau];C_\nu^{2(\beta+1)}(\bar\Omega)] \times C([0,\tau];\mathbf{R})$ when $\beta \in (1/2,1)$. Moreover $u,v \in C([0,\tau];D(A))$.*

Proof. It is an application of the Theorem 3.1, while condition (3.7) follows from the Green formulae for the Laplace operator and the homogeneous Neumann boundary conditions. ∎

4 AN EQUIVALENT FIXED POINT SYSTEM

In this section we formulate the abstract inverse problem (1.6)-(1.9) in terms of an equivalent non–linear fixed point system. The main result of this section is the equivalence Theorem 4.1, which can be considered the heart of the paper since the inverse problem (1.6)-(1.9), without additional conditions on the data is, in general, not well posed. The equivalence theorem gives a set of hypotheses on the data such that the inverse problem becomes well posed, and starting from the equivalence fixed point system we can obtain existence and uniqueness results for system (1.6)-(1.9). A theorem of continuous dependence on the data can also be proved starting from Theorem 4.1. We omit the statement and the proof for the sake of brevity. Here we prove:

THEOREM 4.1. *Let A and B be sectorial operators. Let us assume the data g, u_0, v_0, Φ and Ψ satisfy the regularity conditions R_1-R_5 and the compatibility conditions C_1-C_3. Suppose $\beta \in (0,1)$ and let $(u,v,h) \in [C^1([0,T];X) \cap C([0,T];D(A)) \cap \mathcal{B}([0,T];D_A(\beta+1,\infty))] \times [C^1([0,T];X) \cap C([0,T];D(B)) \cap \mathcal{B}([0,T];D_B(\beta+1,\infty))] \times C([0,T];\mathbf{R})$ be a solution of the problem (1.6)-(1.9). Then the triplet (w,z,h), where $w = u'$, $z = v'$, belongs to*

$$[C([0,T];X) \cap \mathcal{B}([0,T];D_A(\beta,\infty))] \times [C([0,T];X) \cap \mathcal{B}([0,T];D_B(\beta,\infty))] \times C([0,T];\mathbf{R})$$

and solves problem (4.15)(defined in the sequel). Conversely, if $(w,z,h) \in [C([0,T];X) \cap \mathcal{B}([0,T];D_A(\beta,\infty))] \times [C([0,T];X) \cap \mathcal{B}([0,T];D_B(\beta,\infty))] \times C([0,T];\mathbf{R})$ is a solution of the problem (4.15), then the triplet (u,v,h), where $u = u_0 + 1 * w$, $v = v_0 + 1 * z$, belongs to $[C^1([0,T];X) \cap C([0,T];D(A)) \cap \mathcal{B}([0,T];D_A(\beta+1,\infty))] \times [C^1([0,T];X) \cap C([0,T];D(B)) \cap \mathcal{B}([0,T];D_B(\beta+1,\infty))] \times C([0,T];\mathbf{R})$ and solves problem (1.6)-(1.9).

Proof. Let $(u,v,h) \in [C^1([0,T];X) \cap C([0,T];D(A)) \cap \mathcal{B}([0,T];D_A(\beta+1,\infty))] \times [C^1([0,T];X) \cap C([0,T];D(B)) \cap \mathcal{B}([0,T];D_B(\beta+1,\infty))] \times C([0,T];\mathbf{R})$ be a solution of the problem (1.6)–(1.9). With (u,v,h) we associate the triplet (w,v,h) where the first two components are defined by

$$w(t) = u'(t), \qquad u(t) = u_0 + 1 * w(t), \tag{4.1}$$

$$z(t) = v'(t), \qquad v(t) = u_0 + 1 * z(t). \tag{4.2}$$

Differentiating with respect to the time (1.6) and (1.7) we get

$$\begin{cases} u''(t) = Au'(t) - a_1 u'(t) - b_1[u'(t)[h * v(t)] \\ +u(t)h(t)v_0 - u(t)[h * v'(t)]], \\ v''(t) = Bv'(t) - a_2 v'(t) + b_2[u'(t)[h * v(t)] \\ +u(t)h(t)v_0 + u(t)[h * v'(t)]], \\ u'(0) = Au_0 - a_1 u_0, \qquad v'(0) = Bv_0 - a_2 v_0, \\ \Phi[u'(t)] = g'(t), \quad t \in [0,T], \end{cases} \tag{4.3}$$

and thanks to positions (4.1) and (4.2) we obtain

$$\begin{cases} w'(t) = Aw(t) - a_1 w(t) - b_1[w(t)[h * [u_0 + 1 * z(t)]] \\ +[u_0 + 1 * w(t)][h(t)v_0 + h * z(t)]], \\ z'(t) = Bz(t) - a_2 z(t) + b_2[z(t)[h * [u_0 + 1 * z(s)]] \\ +[u_0 + 1 * w(t)][h(t)v_0 + h * z(t)]], \\ w(0) = Au_0 - a_1 u_0, \qquad z(0) = Bv_0 - a_2 v_0, \\ \Phi[w(t)] = g'(t), \quad t \in [0,T], \end{cases} \tag{4.4}$$

thanks to the regularity conditions R_1-R_3 and compatibility condition C_1 and C_2 we have that the (AIP) is equivalent to (4.4). Apply the functional Φ to both hands sides of the first equation of system (4.4) we get

$$\begin{cases} w'(t) = Aw(t) - a_1 w(t) - b_1[w(t)[h * [u_0 + 1 * z(t)]] \\ +[u_0 + 1 * w(t)][h(t)v_0 + h * z(t)]], \\ z'(t) = Bz(t) - a_2 z(t) + b_2[z(t)[h * [u_0 + 1 * z(s)]] \\ +[u_0 + 1 * w(t)][h(t)v_0 + h * z(t)]], \\ w(0) = Au_0 - a_1 u_0, \qquad z(0) = Bv_0 - a_2 v_0, \\ g''(t) = \Phi[Aw](t) - a_1 g'(t) - b_1 h(t)\Phi[u_0 v_0] \\ -b_1 \Phi\Big([w(t)[h * [v_0 + 1 * z(t)]] \\ +u_0[h * z(t)] + [1 * w(t)][h(t)v_0 + h * z(t)]\Big), \end{cases} \tag{4.5}$$

where we have taken into account the regularity conditions R_4. Set

$$\chi^{-1} := b_1 \Phi[u_0 v_0] \neq 0 \tag{4.6}$$

and supposed C_3 holds. Thanks to the fundamental condition R_5, that is $\Phi[Aw] =$

$\Psi[w]$, we obtain the system

$$\begin{cases} w'(t) = Aw(t) - a_1 w(t) - b_1[w(t)[h * [u_0 + 1 * z(t)] \\ +[u_0 + 1 * w(t)][h(t)v_0 + h * z(t)]], \\ z'(t) = Bz(t) - a_2 z(t) + b_2[v'(t)[h * [u_0 + 1 * z(s)]] \\ +[u_0 + 1 * w(t)][h(t)v_0 + h * z(t)], \\ w(0) = Au_0 - a_1 u_0, \quad z(0) = Bv_0 - a_2 v_0, \\ h(t) = \chi\Big\{\Psi[w](t) - a_1 g'(t) - g''(t) - b_1 \Phi\Big([w(t)[h * [v_0 + 1 * z(t)] \\ +u_0[h * z(t)] + [1 * w(t)][h(t)v_0 + h * z(t)]]\Big)\Big\}. \end{cases} \tag{4.7}$$

Using the constant variation formula (see (2.14)), we represent in integral form the first two equations in (4.7) and we get the equivalent system

$$\begin{cases} w(t) = e^{tA}[Au_0 - a_1 u_0] + e^{tA} * \Big[- a_1 w(t) - b_1[w(t)[h * [v_0 + 1 * z(t)]] \\ +[u_0 + 1 * w(t)][h(t)v_0 + h * z(t)]]\Big], \\ z(t) = e^{tB}[Bv_0 - a_2 v_0] + e^{tB} * \Big[- a_2 z(t) + b_2[z(t)[h * [u_0 + 1 * z(t)]] \\ +[u_0 + 1 * w(t)][h(t)v_0 + h * z(t)]]\Big], \\ h(t) = \chi\Big\{\Psi[w](t) - a_1 g'(t) - g''(t) - b_1 \Phi\Big(w(t)[h * [v_0 + 1 * z(t)] \\ +u_0[h * z(t)] + [1 * w(t)][h(t)v_0 + h * z(t)]]\Big)\Big\}, \end{cases} \tag{4.8}$$

replacing the first equation into the term $\Psi[w]$ in the last equation in system (4.8), which is the only term which is not contractive, and define the nonlinear operators

$$\Gamma_1(w, z, h) := e^{tA} * \Big[- a_1 w(t) - b_1[w(t)[h * [v_0 + 1 * z(t)]] \\ +[u_0 + 1 * w(t)][h(t)v_0 + h * z(t)]]\Big], \tag{4.9}$$

$$\Gamma_2(w, z, h) := e^{tB} * \Big[- a_2 z(t) + b_2[z(t)[h * [u_0 + 1 * z(t)]] \\ +[u_0 + 1 * w(t)][h(t)v_0 + h * z(t)]]\Big], \tag{4.10}$$

we get

$$\begin{cases} w(t) = e^{tA}[Au_0 - a_1 u_0] + \Gamma_1(w, z, h), \\ z(t) = e^{tB}[Bv_0 - a_2 v_0] + \Gamma_2(w, z, h), \\ h(t) = \chi\Big\{\Psi\left[e^{tA}[Au_0 - a_1 u_0]\right] - a_1 g'(t) - g''(t) \\ +\Psi\left[\Gamma_1(w, z, h)\right](t) - b_1 \Phi\Big([w(t)[h * [v_0 + 1 * z(t)]] \\ +u_0[h * z(t)] + [1 * w(t)][h(t)v_0 + h * z(t)]]\Big)\Big\}, \end{cases} \tag{4.11}$$

with the positions

$$w_0(t) := e^{tA}[Au_0 - a_1 u_0], \qquad z_0(t) := e^{tB}[Bv_0 - a_2 v_0], \tag{4.12}$$

$$h_0(t) := \chi\Big\{\Psi[e^{tA}[Au_0 - a_1 u_0]] - a_1 g'(t) - g''(t)\Big\}, \tag{4.13}$$

$$\Gamma_3(w,z,h) := \chi\Big\{\Psi[\Gamma_1(w,z,h)](t) - b_1\Phi\Big([w(t)[h*[v_0+1*z(t)]]$$
$$+u_0[h*z(t)] + [1*w(t)][h(t)v_0 + h*z(t)]\Big)\Big\}, \tag{4.14}$$

we finally get the system (for $t \in [0,T]$)):

$$\begin{cases} w(t) = w_0(t) + \Gamma_1(w,z,h)(t), \\ z(t) = z_0(t) + \Gamma_2(w,z,h)(t), \\ h(t) = h_0(t) + \Gamma_3(w,z,h)(t), \end{cases} \tag{4.15}$$

and this completes the proof. ∎

5 PROOF OF THEOREM 3.1

In order to apply the Contraction Principle to system (4.15) we begin by defining the complete metric space

$$Y_m(\beta) := \{(w,z,h) \in Y \; : \|(w,z,h)\|_Y \le 2m\}, \quad m \in \mathbf{R}_+, \quad \beta \in (0,1) \tag{5.1}$$

where

$$Y := [C([0,T];X) \cap \mathcal{B}([0,T];D_A(\beta,\infty))]$$
$$\times[C([0,T];X) \cap \mathcal{B}([0,T];D_B(\beta,\infty))] \times C([0,T];\mathbf{R}). \tag{5.2}$$

Y is a Banach space when endowed with the norm

$$\|(w,z,h)\|_Y = \|w\|_{C([0,T];X)\cap\mathcal{B}([0,T];D_A(\beta,\infty))}$$
$$+\|z\|_{C([0,T];X)\cap\mathcal{B}([0,T];D_B(\beta,\infty))} + \|h\|_{C([0,T];\mathbf{R})}. \tag{5.3}$$

Moreover, we choose m to satisfy the inequality

$$\|w_0\|_{C([0,T];X)\cap\mathcal{B}([0,T];D_A(\beta,\infty))} + \|z_0\|_{C([0,T];X)\cap\mathcal{B}([0,T];D_B(\beta,\infty))}$$
$$+\|h_0\|_{C([0,T];\mathbf{R})} \le m, \tag{5.4}$$

where w_0, z_0, h_0 are defined in (4.12), (4.13). By virtue of Theorems 2.2 and 2.3 and the regularity conditions the vector function (w_0, z_0, h_0) belongs to Y. We consider the nonlinear vector operator

$$\Gamma(w,z,h) := (w_0 + \Gamma_1(w,z;h), z_0 + \Gamma_2(w,z,h), h_0 + \Gamma_3(w,z,h)), \tag{5.5}$$

$$\Gamma: \quad Y_m(\beta) \to Y_m(\beta). \tag{5.5'}$$

Then we proceed to estimate the nonlinear operators Γ_i $(i = 1,2,3)$ defined in (4.9), (4.10) and (4.14), respectively. In the following we denote by $C_i(m,T)$, $i = 1,...,6$ positive constants continuously depending on the arguments pointed out. We first consider the estimates

$$\|\Gamma_1(w,z,h)\|_{C([0,T];X)\cap\mathcal{B}([0,T];D_A(\beta,\infty))}$$
$$\le a_1\|e^{tA} * w\|_{C([0,T];X)\cap\mathcal{B}([0,T];D_A(\beta,\infty))}$$
$$+b_1\|e^{tA} * [w[h*[v_0+1*z]]]\|_{C([0,T];X)\cap\mathcal{B}([0,T];D_A(\beta,\infty))}$$

$$+b_1\|e^{tA}*\big([u_0+1*w][hv_0+h*z]\big)\|_{C([0,T];X)\cap\mathcal{B}([0,T];D_A(\beta,\infty))}$$
$$\leq TC_1(m,T), \tag{5.6}$$
$$\|\Gamma_2(w,z,h)\|_{C([0,T];X)\cap\mathcal{B}([0,T];D_B(\beta,\infty))}$$
$$\leq a_2\|e^{tB}*z\|_{C([0,T];X)\cap\mathcal{B}([0,T];D_B(\beta,\infty))}$$
$$+b_2\|e^{tB}*[z[h*[u_0+1*z]]]\|_{C([0,T];X)\cap\mathcal{B}([0,T];D_B(\beta,\infty))}$$
$$\|e^{tB}*\big([u_0+1*w(t)][h(t)v_0+h*z(t)]\big)\|_{C([0,T];X)\cap\mathcal{B}([0,T];D_B(\beta,\infty))}$$
$$\leq TC_2(m,T) \tag{5.7}$$

and finally

$$\|\Gamma_3(w,z,h)\|_{C([0,T];\mathbf{R})}\leq$$
$$\chi\max\left\{\|\Psi\|_{\mathcal{L}(D_A(\beta,\infty);\mathbf{R})};\|\Phi\|_{\mathcal{L}(D_A(\beta+1,\infty);\mathbf{R})}\right\}$$
$$\times\Big\{\|\Gamma_1(w,z,h)\|_{C([0,T];X)\cap\mathcal{B}([0,T];D_A(\beta,\infty))}$$
$$+b_1\|w(t)[h*[v_0+1*z(t)]]\|_{C([0,T];X)\cap\mathcal{B}([0,T];D_A(\beta,\infty))}$$
$$+b_1\|u_0[h*z(t)]\|_{C([0,T];X)\cap\mathcal{B}([0,T];D_A(\beta,\infty))}$$
$$+b_1\|[1*w(t)][h(t)v_0+h*z(t)]\|_{C([0,T];X)\cap\mathcal{B}([0,T];D_A(\beta,\infty))}\Big\}$$
$$\leq TC_3(m,T). \tag{5.8}$$

Adding (5.6)-(5.8) we have

$$\|\Gamma(w,z,h)\|_{Y_m}\leq T\sum_{j=1}^{3}C_j(m,T)$$
$$:=T\delta_1(m,T), \tag{5.9}$$

for a suitable T^* such that

$$T^*\delta_1(m,T^*)\leq m \tag{5.10}$$

operator $\Gamma(w,z,h)$ maps Y_m into itself. Analogously, by Theorems 2.2 and 2.3, we can prove the estimates

$$\|\Gamma_1(w_2,z_2,h_2)-\Gamma_1(w_1,z_1,h_1)\|_{C([0,T];X)\cap\mathcal{B}([0,T];D_A(\beta,\infty))}\leq TC_4(m,T)$$
$$\times\Big[\|w_2-w_1\|_{C([0,T];X)\cap\mathcal{B}([0,T];D_A(\beta,\infty))}$$
$$+\|z_2-z_1\|_{C([0,T];X)\cap\mathcal{B}([0,T];D_B(\beta,\infty))}+\|h_2-h_1\|_{C([0,T];\mathbf{R})}\Big], \tag{5.11}$$

$$\|\Gamma_2(w_2,z_2,h_2)-\Gamma_2(w_1,z_1,h_1)\|_{C([0,T];X)\cap\mathcal{B}([0,T];D_B(\beta,\infty))}\leq TC_5(m,T)\}$$
$$\times\Big[\|w_2-w_1\|_{C([0,T];X)\cap\mathcal{B}([0,T];D_A(\beta,\infty))}$$
$$+\|z_2-z_1\|_{C([0,T];X)\cap\mathcal{B}([0,T];D_B(\beta,\infty))}+\|h_2-h_1\|_{C([0,T];\mathbf{R})}\Big], \tag{5.12}$$

and therefore

$$\|\Gamma_3(w_2,z_2,h_2)-\Gamma_3(w_1,z_1,h_1)\|_{C([0,T];\mathbf{R})}\le TC_6(m,T)$$
$$\times\Big[\|w_2-w_1\|_{C([0,T];X)\cap B([0,T];D_A(\beta,\infty))}$$
$$+\|z_2-z_1\|_{C([0,T];X)\cap B([0,T];D_B(\beta,\infty))}+\|h_2-h_1\|_{C([0,T];\mathbf{R})}\Big]. \qquad (5.13)$$

Lastly adding (5.11)-(5.13) we get

$$\|\Gamma(w_2,z_2,h_2)-\Gamma(w_1,z_1,h_1)\|_{Y_m}\le T\delta_2(m,T)$$
$$\times\Big[\|w_2-w_1\|_{C([0,T];X)\cap B([0,T];D_A(\beta,\infty))}$$
$$+\|z_2-z_1\|_{C([0,T];X)\cap B([0,T];D_B(\beta,\infty))}+\|h_2-h_1\|_{C([0,T];\mathbf{R})}\Big] \qquad (5.14)$$

where we have set

$$\delta_2(m,T):=\sum_{j=4}^{6}C_j(m,T). \qquad (5.15)$$

If we now choose T^+ such that Γ is a contraction operator in Y_m

$$(T^+)\delta_2(m,T^+)<1, \qquad (5.16)$$

for

$$T_0:=\min\{T^*,T^+\} \qquad (5.17)$$

we have that Γ, has a unique fixed point in Y_m. Hence we get the statement thanks to the equivalence Theorem 4.1 and the existence of a unique fixed point of Γ . ∎

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