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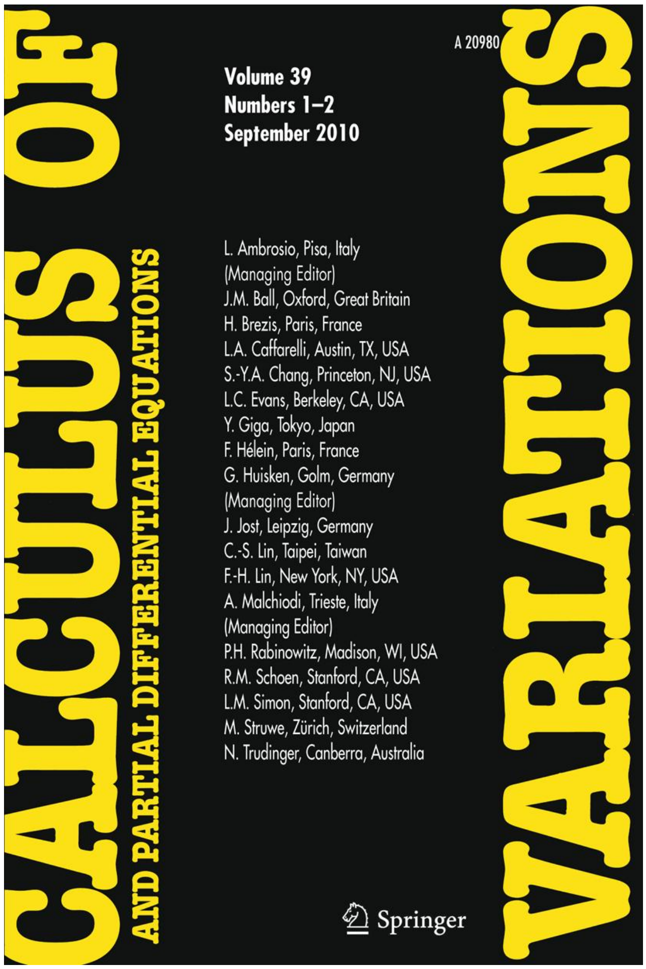

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Calculus of Variations

Critical points of solutions of degenerate elliptic equations in the plane

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Abstract We study the minimizer u of a convex functional in the plane which is not Gâteaux-differentiable. Namely, we show that the set of critical points of any C^1-smooth minimizer can not have isolated points. Also, by means of some appropriate approximating scheme and viscosity solutions, we determine an Euler–Lagrange equation that u must satisfy. By applying the same approximating scheme, we can pair u with a function v which may be regarded as the stream function of u in a suitable generalized sense.

Mathematics Subject Classification (2000) 35B05 · 35B38 · 35J20 · 35J60

1 Introduction

1.1 Motivations: a case study

This paper will mainly focus on the properties of certain convex coercive *non-differentiable* functionals and their extremals. We are partly motivated by the investigations that the second author and Talenti pursued in a series of papers [22–24] about *complex-valued* solutions of the classical *eikonal equation* in the plane.

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One of the main characters acting in those papers is the functional

$$\mathcal{J}(u) = \int_{\Omega} f(|\nabla u|)\, dx, \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ (therein, with $N = 2$), $|\nabla u|$ denotes the modulus of the gradient of a scalar function u defined in Ω and

$$f(\rho) = \frac{1}{2}\left[\rho\sqrt{1+\rho^2} + \log\left(\rho + \sqrt{1+\rho^2}\right)\right] \quad \text{for } \rho \geq 0. \tag{2}$$

Notice that f is strictly convex and grows quadratically at infinity, thus the existence and uniqueness of a function minimizing $\mathcal{J}$ subject to a Dirichlet boundary condition is not hard to prove. However, since $f'(0) > 0$, $\mathcal{J}$ is not always differentiable—non-differentiability occurring when the Lebesgue measure of the set $\{x \in \Omega : \nabla u(x) = 0\}$ is positive—and a standard Euler–Lagrange equation may not be available for $\mathcal{J}$, though a differential inclusion

$$0 \in \partial\mathcal{J}(u),$$

by means of the subdifferential $\partial\mathcal{J}$, still characterizes a minimizing u (see [23] for details). A formal Euler–Lagrange equation would read

$$\operatorname{div}\left\{\mathrm{f}'(|\nabla \mathrm{u}|)\,\frac{\nabla \mathrm{u}}{|\nabla \mathrm{u}|}\right\} = 0 \tag{3}$$

with $f'(\rho) = \sqrt{1+\rho^2}$—a clearly singular equation exactly at the *critical points* of u. Nevertheless, away from its critical points, a smooth minimizer u certainly satisfies the quasilinear *elliptic degenerate* differential equation

$$\operatorname{tr}[\mathcal{A}(\nabla u)\,\nabla^2 u] = 0 \tag{4}$$

(degeneration occurring, of course, at critical points), where the $N \times N$ matrix $A(p)$ has coefficients

$$\mathcal{A}_{ij}(p) = [\alpha(|p|) - 1]\, p_i p_j + |p|^2\delta_{ij}, \quad i, j = 1, \ldots, N$$

Here, δ_{ij} is the usual Kronecker's delta, while

$$\alpha(\rho) = \begin{cases} \dfrac{\rho f''(\rho)}{f'(\rho)}, & \rho > 0, \\ 0, & \rho = 0. \end{cases} \tag{5}$$

For $N = 2$ and f given by (2), Eq. 4 may be formally re-written as

$$-(|\nabla u|^4 + u_y^2)\, u_{xx} + 2\, u_x\, u_y\, u_{xy} - (|\nabla u|^4 + u_x^2)\, u_{yy} = 0. \tag{6}$$

Functional (1) and Eqs. 3, 4 show some interesting features.

Even if, as already mentioned, minimizers of $\mathcal{J}$ do not satisfy (3) in general, it is proved in [23] that they are *viscosity solutions* of (6). In [23], it is also shown that *classical* solutions of (6) exist which *can not* be minimizers of $\mathcal{J}$—thus proving that the Dirichlet problem for viscosity solutions of (6) is not uniquely solvable.

Another interesting feature concerns the set of critical points of solutions of (6)—a decisive information for a good understanding of the properties of (1), (3) and (4). For $N = 2$, sample solutions of (6) have their gradients which vanish on a set of *positive* Lebesgue measure [22]; also, it has been shown [22] that classical solutions of (6) cannot have isolated (non-degenerate) critical points, that is their gradients either never vanish or annihilate on a continuum. Notice that the latter property is known to occur for smooth solutions of the ∞-Laplace equation (see [5]), but not for elliptic equations (see [15]), even in some degenerate case (e.g. the p-Laplace equation; see [1]).

This phenomenon may be heuristically explained by observing that $f(\rho)$ grows *only linearly* near $\rho = 0$, forcing the gradient of a minimizer to be "smaller than usual" wherever it is possible. Also, a simple inspection informs us that the operator in (3) behaves like the 1-Laplace operator near critical points and the ordinary Laplace operator for large values of $|\nabla u|$. This set of remarks make us claim that Eqs. 3 and 4 are "more degenerate" than the p-Laplace equation for $1 < p < \infty$ but "less degenerate" than the 1-Laplace equation, and for this reason they deserve attention.

Let us finally observe that functionals and equations with a structure similar to the one described in this subsection have been considered in the study of torsional creep problems in *elasto-plastic* materials [18,21,25,26].

The aforementioned reasons motivate our interest on a more detailed analysis of such functionals and equations.

1.2 Main results

We shall consider a *strictly convex* functional of type (1). From now on, unless differently specified, Ω will be a *bounded* domain in $\mathbb{R}^N$, while f is assumed to abide to the requirements below:

$$f \text{ is strictly convex;} \tag{7a}$$

$$f \in C^1([0, +\infty)) \cap C^{2,\lambda}_{loc}((0, +\infty)), \quad 0 < \lambda < 1; \tag{7b}$$

$$f'(0) > 0. \tag{7c}$$

In order to avoid technicalities, unnecessary to the aims of our investigation, we require that the couple (Ω, ψ), where $\psi : \partial\Omega \to \mathbb{R}$ is a given continuous function, satisfies a *bounded slope condition* (referred to by BSC from now on; see Section 3 for details).

Under these assumptions, a classical result makes sure that the variational problem

$$\min\{\mathcal{J}(w) : w \in \text{Lip}(\overline{\Omega}), \quad w = \psi \text{ on } \partial\Omega\}, \tag{8}$$

$\text{Lip}(\overline{\Omega})$ being the space of Lipschitz continuous functions in $\overline{\Omega}$, admits a unique solution (see e.g. [14]).

In Theorem 3.7, we specify sufficient conditions on f that guarantee that each solution u of (8) is a viscosity solution of (4) subject to $u = \psi$ on $\partial\Omega$. The proof of Theorem 3.7 follows the outline of the one given in [23] for the special case (2): we uniformly approximate $\mathcal{J}$ by a sequence of strictly convex differentiable functionals

$$\mathcal{J}_n(u) = \int_\Omega f_n(|\nabla u|)\, dx \tag{9}$$

whose minimizers u_n are proven to be viscosity solutions of some relevant differential equations with coefficients that converge uniformly to those of (4). Differently from [22], the uniform convergence of the u_n's, needed to use the standard stability result of [17, Sect. 6], is easily obtained by means of the BSC.

The main result of this paper concerns the set of critical points of a solution of (8).

Theorem 1.1 *Let u be a C^1 solution of* (8)*, where f satisfies* (7).
Then u can not have isolated critical points.

We point out that on account of a recent result of De Silva and Savin, the solution of (8) is always C^1-regular (see [8]).

This result considerably improves the one obtained in [22] for solutions of class C^2 and settles a conjecture raised by G. Talenti (Personal communication). Its proof proceeds by contradiction and relies on two remarks:

(i) if a solution u of (8) has an isolated critical point at $z_0 \in \Omega$, then it is a weak solution of (3) in a neighborhood $\mathcal{U}$ of z_0;

(ii) even if u is assumed to be only C^1 in $\mathcal{U}$, yet one can define an *index* $I(z_0)$ for the vector field ∇u in z_0—a *winding number* defined on loops avoiding z_0.

Remark (i) then implies that u is a classical solution of (4) in $\mathcal{U} \setminus \{z_0\}$ and also that there exists a (distributional) *stream function* v for u in $\mathcal{U} \setminus \{z_0\}$, that is a function v such that

$$\partial_x v = -f'(|\nabla u|)\,\frac{\partial_y u}{|\nabla u|}, \quad \partial_y v = f'(|\nabla u|)\,\frac{\partial_x u}{|\nabla u|} \tag{10}$$

in $\mathcal{U} \setminus \{z_0\}$, in a distributional sense (stream functions will play a crucial rôle in the following sections. The reader may refer to [7] as a propaedeutic reading for what concerns Sect. 3, while [3] show how stream functions have been sometimes used to infer critical-point set properties). The modulus $|\nabla v|$ of the gradient of v is proven to extend continuously to z_0 and, since both ∇u and ∇v must have the same index, we infer that $\mathrm{I}(z_0) = 0$, which entails a contradiction versus the hypothesis of z_0 being isolated. It is clear that a possible generalization of Theorem 1.1 to general dimension should rely on different arguments.

The crucial role played by the stream function v in the proof of Theorem 1.1 motivates a better understanding of system (10) or its inverse

$$\partial_x u = g'(|\nabla v|)\,\frac{\partial_y v}{|\nabla v|}, \quad \partial_y u = -g'(|\nabla v|)\,\frac{\partial_x v}{|\nabla v|}, \tag{11}$$

which can be also viewed as sorts of Cauchy–Riemann systems for u and v. Here g is the *Fenchel conjugate of f* defined by

$$g(r) = \sup\{\rho r - f(\rho) : \rho \geq 0\}, \quad r \in [0, \infty). \tag{12}$$

In other words, we want to investigate on the possibility of defining a generalized *stream function* v associated to a solution u of (8). The main difficulty with this task is that, since by Theorem 1.1 u may not have isolated critical points, system (10) is in general severely singular.

In this paper, we do not completely succeed in our task, but we present a few results which may help to understand the problem. In fact, we collect the related results in Sect. 3. There we develop a method for approximating (1) by means of Gâteaux-differentiable functionals, whose Lagrangeans f_n give rise to systems

$$\partial_x v_n = -f_n'(|\nabla u_n|)\,\frac{\partial_y u_n}{|\nabla u_n|}, \quad \partial_y v_n = f_n'(|\nabla u_n|)\,\frac{\partial_x u_n}{|\nabla u_n|} \tag{13}$$

which can be uniquely solved by suitably normalized stream functions v_n; each one of the latter is in turn a critical point of the corresponding functional

$$\mathcal{K}_n(v) = \int_\Omega g_n(|\nabla v|)\,dx,$$

where $g_n' = (f_n')^{-1}$ (g_n is indeed the Fenchel conjugate of f_n defined according to (12)). It is evident that u_n and v_n also satisfy

$$\partial_x u_n = g_n'(|\nabla v_n|)\,\frac{\partial_y v_n}{|\nabla v_n|},\quad \partial_y u_n = -g_n'(|\nabla v_n|)\,\frac{\partial_x v_n}{|\nabla v_n|}. \tag{14}$$

In Theorem 3.5, under appropriate assumptions on the approximating sequence $(f_n)_{n\in\mathbb{N}}$, we show that the sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ contain subsequences which converge respectively to functions u and v satisfying (11) almost everywhere.

We are not able to prove that u and v also satisfy (10); however, by the same argument used in the proof of Theorem 3.7, we show that v is a viscosity solution v of

$$\mathrm{tr}[\mathcal{B}(\nabla v)\,\nabla^2 v] = 0, \tag{15}$$

where $\mathcal{B}(p)$ is a matrix whose coefficients are the uniform limits of

$$-\left[1-\alpha_n(g_n'(|p|))\right] p_i p_j - |p|^2\alpha_n(g_n'(|p|))\delta_{ij} \quad i, j = 1, \ldots, 2,$$

with

$$\alpha_n(\rho) = \begin{cases} \dfrac{\rho f_n''(\rho)}{f_n'(\rho)}, & \rho > 0, \\ \lim\limits_{\rho\to 0^+} \dfrac{\rho f_n''(\rho)}{f_n'(\rho)}, & \rho = 0. \end{cases} \tag{16}$$

This is the content of Theorem 3.8.

It is worth mentioning that the analytic form of $\mathcal{B}$ may depend upon the particular approximating sequence $(f_n)_{n\in\mathbb{N}}$ adopted, that is, different approximations lead to different limit equations (see Remark 3.10 for details). One of these choices leads to the following interesting equation for v:

$$-\left[1-\alpha(g'(|\nabla v|))\right]\Delta_\infty v - |\nabla v|^2\alpha(g'(|\nabla v|))\,\Delta\, v = 0,$$

where g is given by (12). Notice that, for values of $|\nabla v|$ less than or equal to $f'(0)$, v must be ∞-harmonic.

2 Critical points of minimizers

This section will be devoted to the proof of our main result (Theorem 1.1), which will be consequence of Lemma 2.1 and Theorem 2.5, which may be of independent interest.

Lemma 2.1 *Let u be a solution of* (8) *with f satisfying* (7)*. If $u \in C^1(\Omega)$ and the set $\{z \in \Omega : |\nabla u(z)| = 0\}$ has zero Lebesgue measure, then u is a weak solution of* (3) *in Ω.*

In particular, if $z_0 \in \Omega$ is an isolated critical point for u, then there exists a neighborhood of z_0 in which u is a weak solution of (3).

Proof It is easy to see that, for any test function φ whose support is contained in Ω, the derivative of $\mathcal{J}$ in the direction given by φ may be written as

$$\partial\mathcal{J}(u)(\varphi) = \int\limits_{\{|\nabla u|\neq 0\}} f'(|\nabla u|)\left\langle \frac{\nabla u}{|\nabla u|}, \nabla\varphi\right\rangle dxdy + f'(0)\int\limits_{\{|\nabla u|=0\}} |\nabla\varphi|\, dxdy.$$

By assumption, the second addendum vanishes, while the first one amounts to

$$\int_{\Omega} f'(|\nabla u|) \left\langle \frac{\nabla u}{|\nabla u|}, \nabla\varphi \right\rangle dxdy.$$

If u is a solution of (8), then $\partial\mathcal{J}(u)(\varphi) = 0$ for every test function compactly supported in Ω and hence u is a weak solution of (3) in Ω. □

We will next proceed to compute the index $\mathrm{I}(z_0)$ of an isolated critical point of a solution $u \in C^1(\Omega)$ of (8). We recall that $\mathrm{I}(z_0)$ is defined by the formula

$$\mathrm{I}(z_0) = \frac{1}{2\pi} \int_{+\gamma} \frac{u_x du_y - u_y du_x}{|\nabla u|^2}, \tag{17}$$

where $+\gamma$ is any loop which wraps z_0 counterclockwise and no other critical point.

Also recall the geometric meaning of the previous definition: the index of a critical point of a C^1-regular function is defined as the topological index of the vector field $\nabla u/|\nabla u|$ at the same point and that the latter corresponds to the topological degree of the field $\nabla u/|\nabla u|$ itself, considered as a map of the unit circle in itself.

Given any $r > 0$ such that $B = \overline{B(z_0, r)} \subset \Omega$ we set our first goal to proving that the differential form

$$\omega = \frac{f'(|\nabla u|)}{|\nabla u|} (-u_y dx + u_x dy), \tag{18}$$

which is continuous and bounded in $B' = B \setminus \{z_0\}$, may be integrated to obtain a so-called *stream function* (see [6]) v which is continuous in B.

Notice that (3) may be cast into the form

$$d\omega = 0, \quad \text{in } \Omega, \tag{19}$$

in the sense of currents/distributions, where d has to be interpreted as the boundary operator (see [28]).

Lemma 2.2 *Let $u : \Omega \longrightarrow \mathbb{R}$ be a $C^1(\Omega)$ distributional solution to (3), and let z_0 be an isolated critical point for u. Then the following claims hold.*

(i) *the period of the 1-form ω given in (18) around z_0 is null;*
(ii) *there exists a function $v \in C^1(B') \cap C^0(B)$ such that $dv = \omega$;*
(iii) *moreover, $|\nabla v| \in C^0(B)$.*

Proof (i) Let $B_1 = B(z_0, r/2)$. We must prove that $\int_{\partial B_1} \omega = 0$. Indeed, we know that, for any $\varphi \in C_0^\infty(B)$ it holds:

$$\int_B \omega \wedge d\varphi = 0.$$

(This is really what (19) means.) Let $(\eta_\varepsilon)_{\varepsilon>0}$ be a family of radially symmetric regularizing kernels, and define $\varphi_\varepsilon = \eta_\varepsilon * \chi_{B_1}$, where as usual χ_{B_1} is the characteristic function of the set B_1. Then $d\chi_{B_1} = \mathcal{H}^1 \llcorner \partial B_1(-\nu_1 dx - \nu_2 dy)$ ($\nu = (\nu_1, \nu_2)$ is the outer unit normal to the domain B_1, while $\llcorner$ denotes the operator of restriction of measures to subsets) and $\mathrm{supp}(d\varphi_\varepsilon)$ is contained in a tubular ε-neighborhood of ∂B_1, hence taking ε_0 small enough, $z_0 \notin \operatorname{supp}(d\varphi_\varepsilon)$ and $\operatorname{supp}(d\varphi_\varepsilon) \subset B$ for all $\varepsilon \le \varepsilon_0$ and it holds (since $\operatorname{supp}(d\varphi_\varepsilon) \subset \operatorname{supp}(d\varphi_{\varepsilon_0})$)

$$0 = \int_B \omega \wedge d\varphi_\varepsilon = \int_{\operatorname{supp}(d\varphi_{\varepsilon_0})} \omega \wedge d\varphi_\varepsilon.$$

The last integral tends to

$$\int_{\operatorname{supp}(d\varphi_{\varepsilon_0})} \omega \wedge d\chi_{B_1} = \int_{\partial B_1} \omega.$$

The alleged convergence is worth an explanation: we know (e.g. see [4, Thm. 2.2]) that both $d\varphi_\varepsilon$ tends to $d\chi_B$ and the total variation $|d\varphi_\varepsilon|$ tends to $|d\chi_B|$ in the sense of measures. This is enough to prove convergence in the stronger topology dual to the space of continuous and bounded functions on $\operatorname{supp}(d\varphi_{\varepsilon_0})$ (see [12, Prop. 2, p. 38]), to which ω belongs.

(ii) Thus we can integrate ω, to obtain a function $v \in C^1(B')$. We claim that v can be extended continuously to B. In fact, since $dv = \omega$, then (10) holds and hence $\nabla v \in L^\infty(B')$; therefore v is (Lipschitz and a fortiori) uniformly continuous on B' and then we can extend it continuously to the border of B', in particular to z_0. This allows us to *mend* the domain of definition of v from the topological point of view.

(iii) From the definition of ω, it turns out that $|\nabla v| = f'(|\nabla u|)$; since $u \in C^1(\Omega)$ and f' is continuous, $|\nabla v|$ is continuous on B too. □

Now we prove that the index of z_0 as a critical point of the function u is zero. We recall the following proposition (see e.g. [2, Lemma 3.1]).

Theorem A *Let w be a real-valued C^1 function in an open set Ω in the complex plane. Let $z_0 \in \Omega$ be an isolated critical point of w.*

Then, one of the following cases occurs.

- (i) *There exists a neighborhood $\mathcal{U}$ of z_0 such that $\{z \in \mathcal{U} : w(z) = w(z_0)\}$ is exactly z_0, and we have* $\mathrm{I}(z_0) = 1$.
- (ii) *There exists a positive integer L and a neighborhood $\mathcal{V}$ of z_0 such that the level set $\{z \in \mathcal{V} : w(z) = w(z_0)\}$ consists of L simple curves. If $L \ge 2$, each pair of such curves crosses at z_0 only. We have* $\mathrm{I}(z_0) = 1 - L$.

The previous theorem let us prove the following crucial statement.

Theorem 2.3 *Let u be a C^1 solution of (3) in the sense of distributions. Assume that z_0 is an isolated critical point for u.*

Then the index of z_0 as a critical point is null: $\mathrm{I}(z_0) = 0$.

Proof In the following, we will need to recall that the level lines of u correspond to the lines of steepest descent for v.

We apply Theorem A to the function u, to infer the geometry of its level set $\{u = u(z_0)\}$. We can exclude the case $\{u = u(z_0)\} = \{z_0\}$, for otherwise z_0 would be a local extremum for u and hence for some $\varepsilon > 0$) either $\{u = u(z_0) + \varepsilon\}$ or $\{u = u(z_0) - \varepsilon\}$ would be a closed curve γ winding around z_0 thus implying that

$$\int_{+\gamma} \omega \neq 0.$$

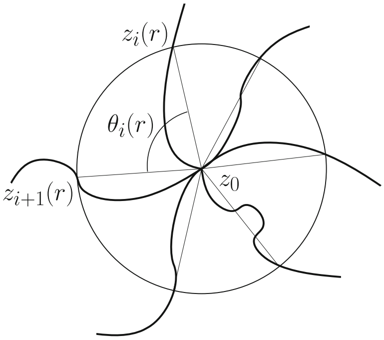

Fig. 1 The level set $\{u = u(z_0)\}$

But this would contradict Lemma 2.2, (i).

Let $2L$ be the number of branches of $\{u = u(z_0)\}$. (*Caveat*: in the present terminology, a branch is any arc in which any of the curves whose existence is stated by Theorem A is split into by z_0. Hence, when the curves are L, the branches are exactly $2L$.) Denote them by $\gamma_1 \dots \gamma_{2L}$; here the subscripts are assigned in the counterclockwise order of occurrence, starting from an arbitrary branch.

We preliminarily observe that each γ_i is rectifiable. Indeed, let $z, z' \in \gamma_i \cap B'$; since γ_i is a curve of steepest descent for v and $|\nabla v(z_0)| \le |\nabla v|$ on B, we can write that

$$\max_B |v - v(z')| \ge v(z) - v(z') = \int_0^l |\nabla v(\gamma_i(s))|\, ds \ge l\ |\nabla v(z_0)|,$$

where l is the length of the arc on γ_i joining z' to z. Thus, we discover that l remains bounded as $z' \to z_0$, since $|\nabla v(z_0)| > 0$.

When r is small enough, each γ_i crosses the circle $\{z : |z - z_0| = r\}$ in one point $z_i(r)$, for $i = 1 \dots 2L$ (see Fig. 1). Hence, setting $\theta_i(r)$ as the angle between the two directions $z_i(r) - z_0$ and $z_{i+1}(r) - z_0$ for $i = 1 \dots 2L$ and $z_{2L+1}(r) = z_1(r)$, it holds that

$$\sum_{i=1}^{2L} \theta_i(r) = 2\pi. \tag{20}$$

Now consider two consecutive branches, say γ_i, γ_{i+1}; we may assume that v is increasing along γ_i and decreasing along γ_{i+1}, away from z_0. (It is easy to see that the case in which v is increasing—or decreasing—along two consecutive branches is not consistent with the present case.) Since the two branches are rectifiable, then we can infer that

$$v(z_i) = v(z_0) + |\nabla v(z_0)|\ r + o(r),$$
$$v(z_{i+1}) = v(z_0) - |\nabla v(z_0)|\ r + o(r),$$

as r approaches 0, where we have set for short $z_i = z_i(r),\ z_{i+1} = z_{i+1}(r)$.

At the same time, there exists $\xi \in [z_i, z_{i+1}]$ (here $[z_i, z_{i+1}]$ is the line segment joining z_i to z_{i+1}) such that the following inequality holds

$$v(z_i) - v(z_{i+1}) \le |\nabla v(\xi)| \cdot |z_i - z_{i+1}|.$$

Thus, we conclude

$$|\nabla v(\xi)| \geq \frac{v(z_i) - v(z_{i+1})}{|z_i - z_{i+1}|} \geq \frac{2 \cdot |\nabla v(z_0)| + o(1)}{\sqrt{2\,[1 - \cos\,\theta_i]}}.$$

Let Θ be any limit point of $\theta_i = \theta_i(r)$, as r tends to 0. Then we obtain (eventually by taking subsequences) that

$$|\nabla v(z_0)| = \liminf_{r \to 0} |\nabla v(\xi)| \geq \frac{2}{\sqrt{2(1 - \cos\,\Theta)}} |\nabla v(z_0)| \geq |\nabla v(z_0)|.$$

Therefore $\Theta = \pi$.

Thus we have proved that two branches of $\{u = u(z_0)\}$ cannot exist such that the angle formed by the limit tangent versor to the branches is different from π. But then (20) informs that the level set $\{u = u(z_0)\}$ is made of no more than two branches, i.e. one curve. Hence $L = 1$ and then, as stated by Theorem A the index of z_0 as a critical point is zero. □

Lemma 2.4 *Let w be any C^2 function and let $t > 0$ be any regular value of $|\nabla w|$.*
Then

$$\int\limits_{+\gamma_t} \frac{w_x dw_y - w_y dw_x}{|\nabla w|^2} = \int\limits_{+\gamma_t} \frac{\det(\nabla^2 w)}{|\nabla^2 w \nabla w|}\, ds,$$

where $\gamma_t = \{z : w(z) = t\}$ and s denotes the arc-length.

Proof We have

$$\int\limits_{+\gamma_t} \frac{w_x dw_y - w_y dw_x}{|\nabla w|^2} = \int\limits_{+\gamma_t} \frac{(w_x w_{xy} - w_y w_{xx})dx + (w_x w_{yy} - w_y w_{xy})dy}{|\nabla w|^2}.$$

Since t is a regular value, γ_t is made of regular curves and

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \frac{ds}{|\nabla^2 w \nabla w|} \begin{pmatrix} -w_x w_{xy} - w_y w_{yy} \\ w_x w_{xx} + w_y w_{xy} \end{pmatrix}$$

on γ_t. The conclusion follows at once after simple algebraic manipulations. □

Theorem 2.5 *Let u be a C^1 solution in the sense of distributions of* (3). *Assume f satisfies* (7).
Then u can not have isolated critical points.

Proof Let Ω' denote the open subset of Ω in which $|\nabla u| > 0$. On any open subset A of Ω' whose closure is contained in Ω', $|\nabla u|$ is bounded away from zero; thus we can apply [14, Thm. 10.18] to infer that u has Hölder continuous second derivatives in any such A and hence in Ω' (notice that, under our assumptions (7) on f, it is a standard computation to prove that $u \in W^{2,2}(\Omega')$). Also, on any such A, Sard's lemma in the version of [20] may be applied to $|\nabla u|^2$ in A; in particular, we have that

$$\int\limits_A \frac{\det(\nabla^2 u)}{|\nabla u|} dx dy = \int\limits_m^M \left(\int\limits_{\gamma_t} \frac{\det(\nabla^2 u)}{|\nabla^2 u \nabla u|} \right) dt \tag{21}$$

by the coarea formula (see [11, Thm. 1, Sect. 3.4.2]), where $m = \min_{\overline{A}} |\nabla u|$, $M = \max_{\overline{A}} |\nabla u|$.

Now z_0 is a strict minimum point for $|\nabla u|$ such that the connected component $\tilde{A}$ of $\{z \in \Omega' : |\nabla u(z)| < \varepsilon_0\}$ containing z_0 is bounded by a simple closed curve. We now choose $A_\varepsilon = \{z \in \tilde{A} : \varepsilon < |\nabla u(z)| < \varepsilon_0\}$ for $0 < \varepsilon < \varepsilon_0$ and apply (21) and Lemma 2.4; we get

$$\int_{A_\varepsilon} \frac{\det(\nabla^2 u)}{|\nabla u|}\, dxdy = \int_\varepsilon^{\varepsilon_0} \left(\int_{\gamma_t} \frac{u_x du_y - u_y du_x}{|\nabla u|^2} \right) dt.$$

Formula (17) and Theorem 2.3 yield that the integrand on the right-hand side of the latter is null for almost every $t \in (\varepsilon, \varepsilon_0)$, hence

$$\int_{A_\varepsilon} \frac{\det(\nabla^2 u)}{|\nabla u|} dxdy = 0.$$

Now, $\det(\nabla^2 u) \leq 0$ in A_ε, because u satisfies an elliptic equation; thus $\det(\nabla^2 u) \equiv 0$ in A_ε and hence in A_0, since $\varepsilon \in (0, \varepsilon_0)$ is arbitrary. An application of Bernstein's inequality (e.g. see [13, Problem 12.3] or [29]) yields that

$$|\nabla^2 u|^2 \leq -\det(\nabla^2 u)\, \left[\alpha(|\nabla u|) + \alpha(|\nabla u|)^{-1}\right]$$

and hence that u is affine in A_0 and, by continuity, constant in a whole neighborhood of z_0 (recall that $|\nabla u(z_0)| = 0$). This is a contradiction. □

Proof of Theorem 1.1 By Lemma 2.1 u satisfies (3) in the sense of distributions. Therefore Theorem 2.5 applies. □

Remark 2.6 It is easy to see that a result similar to that of Theorem 1.1 can be proved also for C^1 solutions of (10): if (u, v) is a C^1 pair satisfying (10), then u cannot have isolated critical points. Indeed, in this case Theorem 2.3 holds for v taken as the second component of the solution pair.

Remark 2.7 A slight modification of the example in [22, Sec. 2.2] proves that there exist solutions of (8) whose set of critical points has zero Lebesgue measure. Indeed the function u defined by

$$u(x, y) = \begin{cases} 1 - \sqrt{\dfrac{1 - x^2 - y^2}{2} + \dfrac{1}{2}\sqrt{(1 - x^2 - y^2)^2 + 4y^2}} & x \geq 0, \\ -1 + \sqrt{\dfrac{1 - x^2 - y^2}{2} + \dfrac{1}{2}\sqrt{(1 - x^2 - y^2)^2 + 4y^2}} & x < 0, \end{cases}$$

is a distributional solution of class $C^{1,1}$ of (3) in every ball $B(0, R)$, $0 < R < 1$, and hence solves (8) with $\psi = u_{|\partial B(0,R)}$.

3 Stream functions and viscosity approximations

The reader should refer to [17] for a definition and relevant developments concerning viscosity solutions ([19] for an account more appropriate to novices). Lemma 3.2 resumes the only non-standard property of viscosity solutions needed in the following.

Here, we shall analyse the relationships between functional (1) and Eq. 4; we will also set up a framework that, in certain instances, leads to the construction of a stream function associated to the unique solution of (8).

We shall assume that f fulfills (7) and that the ratio $\dfrac{\rho f''(\rho)}{f'(\rho)}$ has finite limits as $\rho \to 0^+$ and $\rho \to +\infty$. It is easy to show that on account of (7c), the first limit is zero. Thus, we adopt the definition (5) of the function $\alpha : [0, +\infty) \to [0, +\infty)$ and we assume

$$\alpha_\infty := \lim_{\rho \to +\infty} \frac{\rho f''(\rho)}{f'(\rho)} > 0. \tag{22}$$

Observe incidentally that (22) and (7b) imply that there exist constants $q_1, q_2 \in (1, +\infty)$, $q_1 < q_2$ and $c_1, c_2 > 0$ such that, for $\rho \geq 0$

$$\begin{aligned} c_1 \rho^{q_1} - c_2 \leq f(\rho) \leq c_1 \rho^{q_2} + c_2, \\ c_1 \rho^{q_1 - 1} - c_2 \leq f'(\rho) \leq c_1 \rho^{q_2 - 1} + c_2. \end{aligned}$$

As starters, we recall a classical result, providing a short proof of it tailored on our purposes. We will say that a couple (Ω, ψ), where $\psi : \partial\Omega \to \mathbb{R}$ is a continuous function, satisfies a *bounded slope condition* (BSC for short) with constant $Q > 0$ if, for every $x_0 \in \partial\Omega$ there exist two *affine functions* L^+ and L^- such that

$$\begin{aligned} & L^- \leq \psi \leq L^+ \text{ in } \partial\Omega, \\ & L^-(x_0) = \psi(x_0) = L^+(x_0), \\ & \sup_{\substack{x, y \in \Omega \\ x \neq y}} \frac{|L^\pm(x) - L^\pm(y)|}{|x - y|} \leq Q. \end{aligned} \tag{23}$$

Proposition 3.1 *Let* (Ω, ψ) *satisfy a BSC with constant* Q. *Assume* $(f_n)_{n \in \mathbb{N}}$ *is a sequence of strictly convex functions converging uniformly to* f *on* $[0, +\infty)$. *Let* u *(resp.* u_n*) be the unique solution of* (8) *for* $\mathcal{J}$ *(resp. for* $\mathcal{J}_n$ *in* (9)*).*

Then

(a) u_n *is a minimizing sequence for* $\mathcal{J}$ *and* $\mathcal{J}_n(u_n) \to \mathcal{J}(u)$;
(b) *the sequence* $(u_n)_{n \in \mathbb{N}}$ *tends to* u *in the* sup norm *topology, and in the weak* topology of* $W^{1,\infty}(\Omega)$.

Proof Since $\mathcal{J}_n \to \mathcal{J}$ uniformly (a) is standard. To prove (b) apply [14, Thm 1.2] to get the bound

$$|\nabla u_n| \leq Q \quad \text{on } \overline{\Omega}. \tag{24}$$

Then [9, Chap. 2, B.1] and an application of the Ascoli-Arzelà's theorem yields the desired conclusion. □

In order to prove our next proposition, we need a preliminary lemma. We state it in quite a general form, since it will also be used further along, while considering the case of stream functions. The proof of Lemma 3.2 follows the lines of that of [16, Prop. 2.3] (which may also be considered as an appropriate source for a more advanced discussion of the current topic)

Lemma 3.2 *Let* $M : \mathbb{R}^N \to \mathbb{R}^N$ *be a monotone operator such that the mapping* $p \mapsto \dfrac{|p|^3}{|M(p)|} \nabla_p M(p)$ *is continuous on* $\mathbb{R}^N$.

Then every weak solution of

$$\mathrm{div}\,(\mathrm{M}(\nabla \mathrm{u})) = 0 \tag{25}$$

which is continuous and lies in $W^{1,q}(\Omega)$ *for some* $q > 1$ *is also a viscosity solution of*

$$-\frac{|\nabla u|^3}{|M(\nabla u)|}\operatorname{tr}\left\{\nabla_p M(\nabla u)\nabla^2 u\right\} = 0, \tag{26}$$

where tr *is the usual trace operator on matrices.*

Proof As usual for the viscosity setting, the proof splits into two steps: first prove that u is a subsolution, then that u is a supersolution. The two steps are nearly identical, thus we only go through the first one.

Assume by contradiction, that there exist $\hat{x} \in \Omega$ and $\varphi \in C^2(\Omega)$ satisfying

$$\begin{aligned} &u(\hat{x}) = \varphi(\hat{x}), \qquad \varphi(x) > u(x), x \in \Omega \setminus \{\hat{x}\}, \\ &-\frac{|\nabla\varphi(\hat{x})|^3}{|M(\nabla\varphi(\hat{x}))|}\operatorname{tr}\left\{\nabla_p M(\nabla\varphi(\hat{x}))\nabla^2\varphi(\hat{x})\right\} > 0. \end{aligned} \tag{27}$$

By our assumptions on M and φ, the last inequality implies that $|\nabla\varphi(\hat{x})| > 0$. Hence, by continuity, we can find numbers $\theta, r, t > 0$ with $\theta, r > t$ such that

$$\begin{aligned} &\overline{B_r} \subset \Omega, \qquad \min_{B_t}(u - \varphi) \geq -\theta, \qquad \max_{\partial B_r}(u-\varphi) \leq -3\theta, \\ &-\operatorname{tr}\left\{\nabla_p M(\nabla\varphi(\hat{x}))\nabla^2\varphi(\hat{x})\right\} > 2\theta, \qquad -\operatorname{tr}\left\{\nabla_p M(\nabla\varphi)\nabla^2\varphi\right\} > \theta \quad \text{on } B_r \end{aligned}$$

(here B_r denotes the ball of radius r centered at $\hat{x}$).

Given a non-negative $\psi \in W_0^{1,q}(\Omega)$, we integrate by parts on B_r the last inequality and obtain

$$\begin{aligned} &\int_{B_r} \langle M(\nabla\varphi(x)), \nabla\psi(x)\rangle \\ &\quad = \int_{B_r} -\operatorname{tr}\left(\nabla M(\nabla\varphi(x))\nabla^2\varphi(x)\right)\cdot\psi(x) \geq \int_{B_r} \theta\cdot\psi(x). \end{aligned} \tag{28}$$

Now we choose

$$\psi(x) = \begin{cases} (u(x) - \varphi(x) + 2\theta)^+ & \text{for any } x \in B_r, \\ 0 & \text{elsewhere;} \end{cases}$$

then $\psi \in W_0^{1,q}(\Omega)$ and

$$\nabla\psi(x) = \begin{cases} \nabla u(x) - \nabla\varphi(x) & \text{for any } x \in B_r \cap \{u - \varphi > 2\theta\}, \\ 0 & \text{elsewhere} \end{cases}$$

(see e.g. [14, (3.8) p. 86]). Thus, by (28), we have

$$\int_{B_r\cap\{u-\varphi>2\theta\}} \langle M(\nabla\varphi), \nabla u - \nabla\varphi\rangle \geq \int_{B_r\cap\{u-\varphi>2\theta\}} \theta(u - \varphi + 2\theta) \geq \int_{B_t} \theta^2,$$

while, since u is a weak solution of (25), it holds that

$$\int_{B_r\cap\{u-\varphi>2\theta\}} \langle M(\nabla u), \nabla u - \nabla\varphi\rangle = 0.$$

By subtracting the latter equation to the earlier one we get a contradiction:

$$0 \geq \int_{B_r \cap \{u-\varphi > 2\theta\}} -\langle M(\nabla u) - M(\nabla \varphi), \nabla u - \nabla \varphi \rangle \geq \int_{B_t} \theta^2,$$

(here the first inequality follows from the monotonicity of M). □

We now introduce some further assumptions on the approximating sequence $(f_n)_{n\in\mathbb{N}}$ considered in Proposition 3.1 and prove a couple of preliminary results.

Theorem 3.3 *Assume that the f_n's satisfy (7a)–(7b) and*

$$f_n'(0) = 0, \ n \in \mathbb{N}, \tag{29}$$

and let u_n be the solution of problem (8) for $\mathcal{J}_n$ in (9).

Then u_n is a viscosity solution in Ω of

$$-[\alpha_n(|\nabla u|) - 1]\, \Delta_\infty u - |\nabla u|^2\, \Delta u = 0, \tag{30}$$

where α_n is given by (16).

Proof It is easy to see that—owing to (29)—the Euler–Lagrange equation for the functional (9) is

$$\text{div}\,(A_\mathrm{n}(\nabla \mathrm{u})) = 0, \tag{31}$$

where $A_n : \mathbb{R}^N \to \mathbb{R}^N$ is the monotone operator defined by

$$A_n(p) = \begin{cases} f_n'(|p|)\dfrac{p}{|p|} & p \neq 0, \\ 0 & p = 0. \end{cases} \tag{32}$$

(See also Lemma 2.1.) Then apply Lemma 3.2, with $M = A_n$. □

When $N = 2$ (and Ω is simply connected), we can always define a stream function for each u_n.

Theorem 3.4 *Let $\Omega \subset \mathbb{R}^2$ be simply connected and let the assumptions of Theorem 3.3 be in force.*

Then the following assertions hold:

(i) *for every $n \in \mathbb{N}$ there exists a unique Lipschitz continuous solution in the sense of distributions v_n of the system*

$$\partial_x v_n = -f_n'(|\nabla u_n|)\frac{\partial_y u_n}{|\nabla u_n|}, \qquad \partial_y v_n = f_n'(|\nabla u_n|)\frac{\partial_x u_n}{|\nabla u_n|}, \tag{33}$$

such that

$$\int_\Omega v_n = 0; \tag{34}$$

(ii) *if $\dfrac{\rho f_n''(\rho)}{f_n'(\rho)}$ converges to a positive constant as $\rho \to 0^+$, then v_n is a viscosity solution of*

$$-\left[1 - \alpha_n(g_n'(|\nabla v|))\right] \Delta_\infty v - |\nabla v|^2 \alpha_n(g_n'(|\nabla v|))\, \Delta v = 0, \tag{35}$$

in Ω, where α_n is given by (16) and g_n' is the inverse function of f_n'.

Proof (i) As a minimizer of the differentiable functional $\mathcal{J}_n$, u_n is a weak solution of the corresponding Euler–Lagrange equation. The latter statement corresponds to saying that for any $\varphi \in W_0^{1,q}(\Omega)$ for $q \geq 2$, it holds that

$$\int_\Omega \langle A_n\left(\nabla u_n(x)\right), \nabla\varphi(x)\rangle dx = 0,$$

where A_n is defined in (32). The previous equation may be interpreted as the following differential form

$$\omega_n = \begin{cases} \dfrac{f_n'(|\nabla u_n|)}{|\nabla u_n|}(-\partial_y u_n\, dx + \partial_x u_n\, dy) & |\nabla u_n| \neq 0, \\ 0 & \text{elsewhere.} \end{cases}$$

being closed, as a form belonging to $L^q(\Omega)$, for $q \geq 2$.

We required the domain Ω to be simply connected. Thus (see [27, Lemma 3.2.1]) we can integrate ω_n to obtain a function $v_n \in W^{1,q}(\Omega)$ such that (33) holds. The function v_n is not completely defined by the condition (33): we are left with the choice of a constant to add. We choose such constant so that (34) holds.

We observe that since (24) holds and $|\nabla v_n| = f_n'(|\nabla u_n|)$, we can conclude

$$|\nabla v_n| \leq f_n'(Q) \quad \text{in } \Omega. \tag{36}$$

(ii) Since (i) holds, v_n is a weak solution of

$$\text{div}\,(B_\text{n}(\nabla \text{v})) = 0$$

where $B_n(p)$ is the monotone operator

$$B_n(p) = \begin{cases} g_n'(|p|)\dfrac{p}{|p|} & p \neq 0 \\ 0 & p = 0, \end{cases}$$

which happens to be the inverse of A_n.

Then, owing to the stated assumptions on α_n, the proof follows the lines of that of Lemma 3.2, by observing that, due to the fact that g_n' is the inverse of f_n', then

$$\frac{r g_n''(r)}{g_n'(r)} = \frac{1}{\alpha_n(g_n'(r))}.$$

□

Now, we want to take the limit in (33).

Theorem 3.5 *Let* (23) *be in force and assume that* $(f_n)_{n\in\mathbb{N}}$ *is a sequence of strictly convex functions converging uniformly to* f *on* $[0, +\infty)$.

Let u *(resp.* u_n*) be the unique solution of* (23)–(8) *for* $\mathcal{J}$ *(resp. for* $\mathcal{J}_n$ *in* (9)*) and let* v_n *be defined as in Theorem 3.4. Also, assume that the gradients* ∇u_n *converge to* ∇u *almost everywhere in* Ω.

Then $(v_n)_{n\in\mathbb{N}}$ *contains a subsequence which converges uniformly on* $\overline{\Omega}$ *to a function* $v \in W^{1,\infty}(\Omega)$ *and the pair* (u, v) *satisfies the system* (11) *almost everywhere in* Ω.

Proof Consider the functionals

$$\mathcal{F}(u, v) = \int_\Omega \{f(|\nabla u|) + g(|\nabla v|) - (u_x\, v_y - u_y\, v_x)\}\, dxdy,$$

$$\mathcal{F}_n(u, v) = \int_\Omega \{f_n(|\nabla u|) + g_n(|\nabla v|) - (u_x\, v_y - u_y\, v_x)\}\, dxdy.$$

From (33) it is clear that $\mathcal{F}_n(u_n, v_n) = 0$.

Now, observe that, by (36) and the uniform convergence of the g_n's, the gradients of the v_n's are uniformly bounded and, since (34) holds for every $n \in \mathbb{N}$, the v_n's satisfy on $\overline{\Omega}$ the assumptions of Ascoli-Arzelà's theorem. Thus, $(v_n)_{n\in\mathbb{N}}$ contains a subsequence (that we will still denote by $(v_n)_{n\in\mathbb{N}}$) which converges uniformly on $\overline{\Omega}$ to a function $v \in W^{1,\infty}(\Omega)$ and, by the boundedness of $(v_n)_{n\in\mathbb{N}}$ in $W^{1,\infty}(\Omega)$, we can always assume that $(v_n)_{n\in\mathbb{N}}$ weakly converges to v in any $W^{1,p}(\Omega),\ p > 1$. The latter property implies that

$$\lim_{n\to\infty} \int_\Omega (\partial_x u_n\, \partial_y v_n - \partial_y u_n\, \partial_x v_n)\}\, dxdy = \int_\Omega (u_x\, v_y - u_y\, v_x)\}\, dxdy$$

—since the gradients ∇u_n are bounded and are assumed to converge a.e. to ∇u—and also that

$$\liminf_{n\to\infty} \int_\Omega g_n(|\nabla v_n|)\, dxdy \geq \int_\Omega g(|\nabla v|)\, dxdy,$$

by the uniform convergence of g_n to g and the convexity of g. Therefore, we can infer that

$$\mathcal{F}(u, v) \leq \liminf_{n\to\infty} \mathcal{F}_n(u_n, v_n) = 0.$$

On the other hand, by the very definition (12) of g, the following inequalities hold almost everywhere in Ω:

$$f(|\nabla u|) + g(|\nabla v|) - (u_x\, v_y - u_y\, v_x) \geq |\nabla u|\, |\nabla v| - (u_x\, v_y - u_y\, v_x) \geq 0;$$

thus, $\mathcal{F}(u, v) = 0$ and hence

$$f(|\nabla u|) + g(|\nabla v|) - (u_x\, v_y - u_y\, v_x) = |\nabla u|\, |\nabla v| - (u_x\, v_y - u_y\, v_x) = 0$$

almost everywhere in Ω. These couple of equalities then yield (11) at once. □

Remark 3.6 The a.e. convergence of the gradients ∇u_n assumed in Theorem 3.5 can be obtained at least in two fashions.

The former consists in applying either [10, Thm. 1] or [9, Thm. 2, p. 21] (by possibly restricting our assumptions on f).

The latter consists in an adaptation of the arguments used in [23]: since u_n is a solution of (30), it also satisfies the Bernstein's inequality

$$|\nabla^2 u_n|^2 \leq -\det(\nabla^2 u_n)\, \left[\alpha_n(|\nabla u_n|) + \alpha_n(|\nabla u_n|)^{-1}\right],$$

which, by the properties of the coefficients α_n, implies the bound

$$\int_\Omega \alpha(|\nabla u_n|)\, |\nabla^2 u_n|^2\, dxdy \leq C_K,$$

where K is any compact subset of Ω and C_K is a constant depending on K.

This last inequality provides the expected compactness of the sequence $(u_n)_{n\in\mathbb{N}}$.

We conclude this section by showing that the functions u and v determined in Theorem 3.5 are solutions of second-order degenerate elliptic equations.

Theorem 3.7 *Assume the BSC* (23) *is in force and let f satisfy* (7) *and* (22).
Then the solution u of problem (8) *is a viscosity solution of*

$$-\left[\alpha(|\nabla u|)-1\right]\Delta_\infty u-|\nabla u|^2\,\Delta u=0\quad \text{in }\Omega, \tag{37}$$

where α is given by (5).

Proof We can always approximate f by a sequence of Lagrangeans f_n such that:

(a) the f_n's converge to f uniformly on $[0,+\infty)$;
(b) the functions $\alpha_n(|p|)p_i p_j$, with α_n given by (16), converge to $\alpha(|p|)p_i p_j$, $i,j=1\dots N$, uniformly on any compact subset of $\mathbb{R}^N$ (see also Remark 3.10).

By Proposition 3.1, the solutions u_n of the minimum problem (8) rephrased in terms of $\mathcal{J}_n$ converge to u uniformly in $\overline{\Omega}$. The conclusion then follows from [17, Lemma 6.1]. □

Theorem 3.8 *Let f satisfy* (7) *and* (22) *and suppose f is approximated, uniformly on* $[0,\infty)$, *by a sequence of functions f_n which obey* (7a), (7b) *and* (29).
Assume that the functions α_n satisfy the assumptions of Theorem 3.4 and that $\alpha_n\circ g_n'$ converge to a function β uniformly on the compact subsets of $[0,+\infty)$.
Then the sequence $(v_n)_{n\in\mathbb{N}}$ contains a subsequence that converges uniformly in Ω to a function a viscosity solution v of

$$-\left[1-\beta(|\nabla v|)\right]\Delta_\infty v-|\nabla v|^2\,\beta(|\nabla v|)\,\Delta v=0\quad \text{in }\Omega. \tag{38}$$

Proof By Theorem 3.4, each v_n satisfies (35); (36) and the properties of the f_n's imply that the sequence $(v_n)_{n\in\mathbb{N}}$ satisfies the assumptions of Ascoli-Arzelà's theorem. Therefore the conclusion follows again by applying [17, Lemma 6.1] to any uniformly converging subsequence of $(v_n)_{n\in\mathbb{N}}$. □

Corollary 3.9 *If the coefficients α_n also converge to α uniformly on every compact subset of* $[0,+\infty)$, *then the function v of Theorem 3.8 is a viscosity solution of*

$$-\left[1-\alpha(g'(|\nabla v|))\right]\Delta_\infty v-|\nabla v|^2\alpha(g'(|\nabla v|))\,\Delta v=0\quad \text{in }\Omega, \tag{39}$$

where $g(r)=\sup\{r\rho-f(\rho):\rho\geq 0\}$.

Proof It is enough to compute the function β. Since $g_n(r)$ always converge to $g'(r)$, we obtain that $\beta(r)=\alpha(g'(r))$ at once. □

Remark 3.10 Sequences of $(f_n)_{n\in\mathbb{N}}$ that satisfy the assumptions mentioned in the statement of Theorem 3.8 can be constructed in various fashions. A convenient way is to modify f only in a neighborhood of $\rho=0$.

Here we give two examples; we set $f_n'(\rho)=\sigma_n(\rho)f'(\rho)$, with $\sigma_n(\rho)\equiv 1$ for $\rho\geq 1$, while

(a) $\sigma_n^a(\rho)=1-(1-\rho^{\frac{1}{n}})^n$ for $\rho\in[0,1]$;
(b) $\sigma_n^b(\rho)=1-(1-\rho^s)^n$ for $\rho\in[0,1]$

where $s > 0$.

With the obvious notations it easy to see that the α_n^a's converge uniformly to α on $[0, +\infty)$, so that Corollary 3.9 applies. In particular, $\alpha \circ g' \equiv 0$ on $[0, f'(0)]$ so that, for $|\nabla v| \leq f'(0)$, (39) reads thus: $\Delta_\infty v = 0$.

The α_n^b's instead converge uniformly in the compact subsets of $(0, \infty)$ but not on those of $[0, \infty)$. Straightforward computations show that the function v of Theorem 3.8 satisfies (38), where

$$\beta(r) = \begin{cases} -s\,\dfrac{f'(0) - r}{r} \log[1 - r/f'(0)] & 0 \leq r \leq f'(0), \\ \alpha(g'(r)) & r > f'(0). \end{cases}$$

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