JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **227,** 359–381 (1998)
ARTICLE NO. AY986098

Flow of a Bingham Fluid in Contact with a Newtonian Fluid*

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Submitted by M. C. Nucci

Received November 20, 1997

We study the problem modeling the flow of a Bingham fluid in contact with a newtonian fluid, playing the role of lubricant. This is a free boundary problem coupled by means of diffraction conditions with a boundary value problem of parabolic type. We examine the steady state solutions, the evolutive case with particular regard for the asymptotic behaviour of the solution, and a regularized model related to the appearance of a new rigid zone. © 1998 Academic Press

Key Words: free boundary problem; Bingham fluid; stationary solutions; asymptotic behaviour of the solution.

INTRODUCTION

The lubrication of one fluid by another is a particularly important branch of two-fluid dynamics: if one fluid has a large viscosity, it may be lubricated by a less viscous fluid. A review of the most interesting models and techniques in chemical engineering and in medicine can be found in [1], [10], and [12].

In view of applications in lubricate transport, it can be interesting to study the flow of a Bingham fluid in contact with a newtonian fluid. In fact the Bingham fluid behaves like a viscous fluid if the shear stress exceeds a yield value, and like a rigid body otherwise. During the motion the possible formation of rigid zones inside the medium may cause a deceleration of the medium.

Then, in order to avoid the consequent possible stopping of the flow, a newtonian fluid with small viscosity playing the role of lubricant can be placed between the Bingham fluid and the boundary.

* Work partially supported by MURST National Project "Problemi non lineari. . . ."
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0022-247X/98 $25.00

We recall that the one-dimensional flow of Bingham fluids has been studied both in planar and in cylindrical symmetry [2–4]: there the problem is formulated as a free boundary problem, where the surfaces dividing fluid and rigid zones are the free boundaries.

Equations of motion are obtained from mass and momentum conservation, coupled with a constitutive law for the shear stress, expressed as a linear function of the shear rate. The equation of fluid motion yields a parabolic equation in the fluid zone, and the equation of motion of the solid zone reduces to a free boundary condition (see [2]).

In particular, in [3] and [4] the behaviour of the solution has been investigated, taking into account the possible formation of rigid zones inside the fluid.

Here we propose a model for the flow of two immiscible fluids between two parallel plates at a distance $2(L + l)$: the Bingham fluid flows between two symmetric slabs, both occupied by the newtonian fluid.

Let y be the coordinate along the motion direction and let x be the coordinate in the direction perpendicular to the plates.

We suppose that the Bingham fluid occupies the zone $|x| < L$, and the newtonian fluid occupies the two symmetric zones, $L < |x| < L + l$.

Under the hypothesis of laminar flow, the velocity field at each point of the representative xy plane is given by $\mathbf{v} = (0, v(x,t))$.

Let us denote by

$$\eta = \begin{cases} \eta_1, & L < |x| < L + l, \\ \eta_2, & |x| < L, \end{cases} \tag{0.1}$$

$$\rho = \begin{cases} \rho_1, & L < |x| < L + l, \\ \rho_2, & |x| < L, \end{cases} \tag{0.2}$$

respectively, the viscosity and the density of the two fluids.

In the zone $|x| < L$, occupied by the Bingham fluid, the only nonzero element σ of the stress tensor is given by the constitutive law

$$\sigma = \begin{cases} \eta_2 v_x + \tau_0 \dfrac{v_x}{|v_x|}, & v_x \neq 0, \\ [-\tau_0, \tau_0], & v_x = 0, \end{cases} \tag{0.3}$$

where $\tau_0 \geq 0$ represents the yield stress of the Bingham fluid.

In the zone $L < |x| < L + l$, occupied by the newtonian fluid, the stress is defined by

$$\sigma = \eta_1 v_x. \tag{0.4}$$

The equation of motion for the fluids reduces then to

$$\rho v_t = \sigma_x + f, \qquad |x| \le L + l, t > 0, \tag{0.5}$$

where f represents the term $-\partial p/\partial x$, depending only on t, with p being the pressure.

Note that Eq. (0.5) holds in the whole region occupied by both the fluids. However, the stress, which is continuous at each point of the slab, is not differentiable at the interfaces $x = \pm L$ between the two fluids.

Then we cannot provide a global formulation of the problem as in [9], but we have to split the problem into two coupled ones.

In the region occupied by the newtonian fluid, the governing equation (0.5) together with (0.4) gives

$$\rho_1 v_t - \eta_1 v_{xx} = f, \qquad \text{in } L < |x| < L + l. \tag{0.6}$$

In the region occupied by the Bingham fluid, we define a free boundary problem. Inserting (0.3) into (0.5) one obtains in the fluid zone the equation

$$\rho_2 v_t - \eta_2 v_{xx} = f, \qquad \text{in } \{-L < x < s^-(t)\} \cup \{s^+(t) < x < L\} \tag{0.7}$$

where we denote by $x = s^{\pm}(t)$ the equations of the rigid boundaries, where the stress assumes the yield value $\pm \tau_0$.

Free boundary conditions are given by the condition of no deformation of the free boundary, i.e., $v_x(s^{\pm}(t), t) = 0$, corresponding to imposing $\sigma = \pm \tau_0$, and by the equation of motion for the rigid core.

The problem is completed assuming boundary conditions at $x = \pm(L + l)$, e.g., no-slip conditions, and initial conditions.

As done in previous works [3, 4], we have to point out the existence of a unilateral constraint, related to the presence of rigid zones inside the Bingham fluid. In fact, in the fluid zone occupied by the Bingham fluid, the shear rate v_x cannot change its sign, because of the constitutive law (0.3). Then we have to impose that when v_x vanishes somewhere in the Bingham zone, then at these points it must be $\sigma = \pm \tau_0$.

A detailed description of the problem will be given in next section. The following two sections contain the study of how the steady state solutions depend on the parameters characterizing the two fluids.

The study of the evolutive case is the object of Section 4, with particular regard for the asymptotic behaviour of the solution. Finally, the possible onset of further rigid zones is considered in the regularized model presented in Section 5.

1. STATEMENT OF THE PROBLEM

Let $\bar{\tau}, \bar{\rho}, \bar{\eta}$ be fixed reference values.

Define the transformation

$$\tilde{x} = \frac{x}{L + l}, \qquad \tilde{t} = \frac{\bar{\eta}}{\bar{\rho}(L + l)^2} t.$$

Denoting by

$$\tilde{\sigma} = \frac{\sigma}{\bar{\tau}}, \qquad \tilde{v} = \frac{\bar{\eta}}{\bar{\tau}(L + l)} v, \qquad \tilde{\tau}_0 = \frac{\tau_0}{\bar{\tau}},$$

$$\tilde{f} = \frac{f}{\bar{\tau}}(L + l), \qquad \tilde{\eta}_i = \frac{\eta_i}{\bar{\eta}}, \qquad \tilde{\rho}_i = \frac{\rho_i}{\bar{\rho}}, \qquad \lambda = \frac{l}{L},$$

the nondimensional variables and constants involved in the problem, we can rewrite equations (0.3)–(0.5). Dropping the tildes we obtain

$$\rho v_t = \sigma_x + f, \qquad \text{in } \left\{|x| < \frac{1}{1 + \lambda}\right\} \cup \left\{\frac{1}{1 + \lambda} < |x| \le 1\right\}, t > 0, \quad (1.1)$$

with σ defined by

$$\sigma = \begin{cases} \eta_1 v_x, & \dfrac{1}{1 + \lambda} \le |x| \le 1, \\ \eta_2 v_x + \tau_0 \dfrac{v_x}{|v_x|}, & |x| < \dfrac{1}{1 + \lambda}, v_x \neq 0, \\ [-\tau_0, \tau_0], & |x| < \dfrac{1}{1 + \lambda}, v_x = 0, \end{cases} \quad (1.2)$$

and ρ, η as in (0.1) and (0.2).

In the region $|x| < 1/(1 + \lambda)$, the fluid zone of the Bingham fluid (where $v_x \neq 0$) is separated from the rigid zone (where $v_x = 0$) by the free boundaries $x = s^{\pm}(t)$.

Inserting (1.2) into (1.1), and writing

$$D_{1T} = \left\{\frac{1}{1 + \lambda} < |x| < 1, 0 < t < T\right\},$$

$$D_{2T} = \left\{-\frac{1}{1 + \lambda} < x < s^-(t), 0 < t < T\right\}$$
$$\cup \left\{s^+(t) < x < \frac{1}{1 + \lambda}, 0 < t < T\right\},$$

$$v = \begin{cases} v_1, & \text{in } D_{1T}, \\ v_2, & \text{in } D_{2T}, \end{cases}$$

we have

$$\rho_i v_{it} - \eta_i v_{ixx} = f, \qquad \text{in } D_{iT}, \, i = 1,2, \tag{1.3}$$

$$v_1(\pm 1, t) = 0, \qquad t > 0, \tag{1.4}$$

$$v_1\left(\pm\frac{1}{1+\lambda}, t\right) = v_2\left(\pm\frac{1}{1+\lambda}, t\right), \qquad t > 0, \tag{1.5}$$

$$\left[\sigma\left(\pm\frac{1}{1+\lambda}, t\right)\right] = 0, \qquad t > 0, \tag{1.6}$$

$$v_{2x}\big(s^{\pm}(t), t\big) = 0, \qquad t > 0, \tag{1.7}$$

$$\big(\rho_2 v_{2t}\big(s^{\pm}(t), t\big) - f\big)s^{\pm}(t) = -\tau_0, \qquad t > 0, \tag{1.8}$$

$$v_i(x,0) = v_{i0}(x), \qquad i = 1,2, \qquad 0 < x < 1, \qquad s^{\pm}(0) = s_0^{\pm}, \tag{1.9}$$

with $|s_0^{\pm}| < 1/(1+\lambda)$ and $[f]$ being the jump of f. Equation (1.4) is a no-slip boundary condition; (1.5) and (1.6) express the continuity of velocity and stress at the surfaces $x = \pm 1/(1+\lambda)$ separating the two fluids. Equation (1.7) is the condition of no deformation of the free boundaries $x = s^{\pm}(t)$ bounding the rigid core in the Bingham fluid, due to the assumption that on the free boundary the strain rate is zero. The further condition (1.8) on the free boundary can be derived directly from the equation of motion of the solid zone (see [13]).

For the sake of simplicity, we will suppose that f is a positive constant. The methods used here can be extended, with minor changes, to the general case $f = f(t)$ (see [2]).

We recall that we look for nonnegative solutions of our problem such that $v_x \neq 0$ inside the fluid region of the Bingham zone.

We remark that in the theory of flow involving two fluids is proved the nonuniqueness of the solution as discussed in [10] and [12]. However, restricting ourselves to symmetric solution w.r.t. the axis $x = 0$ guarantees that the solution of problem (1.3)–(1.9) is unique. This will be the argument of Sections 3 and 4.

2. STEADY STATE SOLUTIONS

For the sake of simplicity, we will study the stationary problem only in the half-plane $x > 0$, remarking that the same method can be applied for $x < 0$. Prescribing the value $v_0 > 0$ of the velocity of the Bingham fluid on $x = 0$, we will look for all the solutions analyzing how they depend on the parameters f, τ_0, and $\sigma(1)$.

Integrating Eq. (0.6),

$$\sigma(x) = -f(x-1) + \sigma_1, \qquad \sigma_1 = \sigma(1), 0 < x < 1. \tag{2.1}$$

Figure 1 shows that in the plane σ, x Eq. (2.1) gives, for any f and $\sigma(1)$, the value of the stress at each point of the domain.

From (1.2) and (2.1), we obtain v_∞, the stationary solution of (1.1) integrating

$$v'_\infty(x) = \begin{cases} \dfrac{\sigma(x)}{\eta_1}, & \dfrac{1}{1+\lambda} < x < 1, \\ \dfrac{\sigma(x) - \tau_0}{\eta_2} & 0 < x < \dfrac{1}{1+\lambda}, \sigma > \tau_0, \\ \dfrac{\sigma(x) + \tau_0}{\eta_2}, & 0 < x < \dfrac{1}{1+\lambda}, \sigma < -\tau_0, \\ 0, & 0 < x < \dfrac{1}{1+\lambda}, \sigma \in [-\tau_0, \tau_0]. \end{cases} \tag{2.2}$$

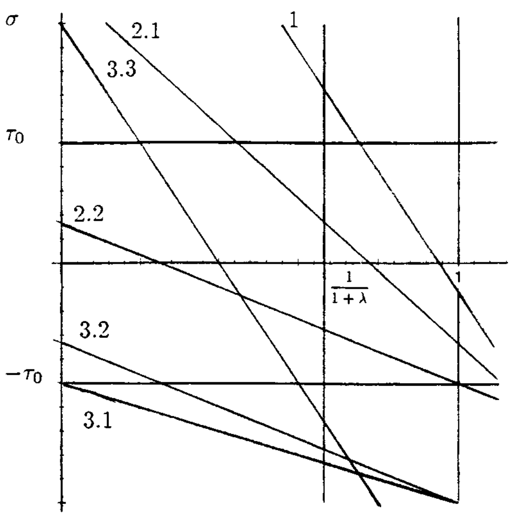

FIG. 1. Stationary stress.

Imposing $v_\infty(0) = v_0$, $v_\infty(1) = 0$ (no-slip conditions), we have that the following identity must hold:

$$v_0 + \int_0^1 v'_\infty(x)\,dx = 0. \tag{2.3}$$

As we can observe in Fig. 1, different kinds of behaviour of the stationary solution are allowed, with the possible presence of one or two boundaries dividing the rigid zone from the fluid one in the Bingham fluid.

Let us denote by

$$x_1 = \min\left\{\frac{1}{1+\lambda}, \max\left\{0, \sup_x \{\sigma(x) > \tau_0\}\right\}\right\},$$

$$x_2 = \min\left\{\frac{1}{1+\lambda}, \max\left\{0, \inf_x \{\sigma(x) < -\tau_0\}\right\}\right\},$$

with $0 \le x_1 \le x_2 \le 1/(1+\lambda)$, the abscissae of the boundaries separating rigid and fluid zones in the region occupied by the Bingham fluid.

Integrating (2.3) in the zone occupied by the newtonian fluid and recalling (2.1) and (2.2),

$$v_0 + \int_0^{x_1} \frac{-f(x-1) + \sigma_1 - \tau_0}{\eta_2}\,dx + \int_{x_2}^{1/(1+\lambda)} \frac{-f(x-1) + \sigma_1 + \tau_0}{\eta_2}\,dx$$

$$= -\frac{1}{\eta_1}\frac{\lambda}{1+\lambda}\left(\sigma_1 + \frac{f}{2}\frac{\lambda}{1+\lambda}\right). \tag{2.4}$$

Imposing $v_\infty(1/(1+\lambda)) \ge 0$, we obtain the condition

$$\sigma_1 \le -\frac{f}{2}\frac{\lambda}{1+\lambda}. \tag{2.5}$$

In order to evaluate the left-hand side of (2.4) we have to distinguish many cases according to the value of $\sigma(0)$ and $\sigma(1/(1+\lambda))$, taking into account the possible presence of either one or two boundaries dividing the rigid zone from the fluid one in the region occupied by the Bingham fluid. For each case, we obtain a relationship between σ_1, f, τ_0, supposing the constants η_1, η_2, λ fixed.

We refer to the lines represented in Fig. 1 in order to describe all possible stationary solutions, whose profiles are given in Fig. 3, at the end of the present section.

Case 1. $\sigma(1/(1+\lambda)) > \tau_0$. The whole region is fluid, because the applied stress is greater than the threshold value everywhere. Writing

$\eta = \eta_2/\eta_1$, Eq. (2.4) becomes

$$\frac{2}{1+\lambda}(1+\eta\lambda)\sigma_1 - \frac{2}{1+\lambda}\tau_0 + \left[1-(1-\eta)\frac{\lambda^2}{(1+\lambda)^2}\right]f + 2\eta_2 v_0 = 0. \tag{2.6}$$

Moreover σ_1, τ_0, f are subject to constraint (2.5) and, from the hypothesis $\sigma(1/(1+\lambda)) > \tau_0$,

$$\sigma_1 > -\alpha f + \tau_0, \qquad \alpha = \frac{\lambda}{1+\lambda}. \tag{2.7}$$

Case 2. $|\sigma(1/(1+\lambda))| < \tau_0$. Close to the newtonian fluid there is a layer in which the Bingham fluid is rigid. We have to distinguish two possibilities:

Case 2.1. $\sigma(0) > \tau_0$. The zone close to $x = 0$ is fluid and is divided from the rigid zone by a boundary located at $x_1 = (\sigma_1 - \tau_0)/f$.

The constants σ_1, f, τ_0 satisfy

$$\sigma_1^2 + (1+\eta\alpha^2)f^2 + 2(1+\eta\alpha)\sigma_1 f - 2\sigma_1\tau_0 - 2\tau_0 f + 2\eta_2 v_0 f + \tau_0^2 = 0. \tag{2.8}$$

Moreover because of the constraints for $\sigma(0)$ and $\sigma(1/(1+\lambda))$, besides (2.5) the following inequalities hold:

$$-\alpha f - \tau_0 < \sigma_1 < -\alpha f + \tau_0, \tag{2.9}$$

$$\sigma_1 > -f + \tau_0. \tag{2.10}$$

Case 2.2. $|\sigma(0)| < \tau_0$. The whole region occupied by the Bingham fluid is rigid and the conditions are

$$2\alpha\sigma_1 + \alpha^2 f + 2\eta_1 v_0 = 0, \tag{2.11}$$

together with (2.5), (2.10), and

$$-f - \tau_0 < \sigma_1 < -f + \tau_0. \tag{2.12}$$

Case 3. $\sigma(1/(1+\lambda)) < -\tau_0$. The zone of Bingham fluid close to the newtonian one is fluid. In this case either one or two boundaries can exist.

We have to impose

$$\sigma_1 < -\alpha f - \tau_0. \tag{2.13}$$

Case 3.1. $\sigma(0) < -\tau_0$. Again the two regions are fluid. Because of the condition $\sigma(0) < -\tau_0$, v is decreasing everywhere and condition (2.4) gives the equation

$$\frac{2}{1+\lambda}(1+\eta\lambda)\sigma_1 + \frac{2}{1+\lambda}\tau_0 + \left[1 - (1-\eta)\frac{\lambda^2}{(1+\lambda)^2}\right]f + 2\eta_2 v_0 = 0 \tag{2.14}$$

together with (2.5), (2.13), and

$$\sigma_1 < -\tau_0 - f. \tag{2.15}$$

Case 3.2. $|\sigma(0)| < \tau_0$. Close to $x = 0$ there is a rigid layer divided from a fluid one by the boundary $x_2 = (\sigma_1 + \tau_0)/f + 1$.

Equation (2.4) becomes

$$(\sigma_1 + \tau_0)^2 + 2\alpha(1-\eta)\sigma_1 f + 2\alpha\tau_0 f + (1-\eta)\alpha^2 f^2 - 2\eta_2 v_0 f = 0 \tag{2.16}$$

and the σ_1, f, τ_0 have to satisfy (2.5), (2.13), and

$$-f - \tau_0 < \sigma_1 < -f + \tau_0. \tag{2.17}$$

Case 3.3. $\sigma(0) > \tau_0$. In this case there are two boundaries dividing a rigid zone from two fluid external ones. The equations of the boundaries are

$$x_1 = \frac{\sigma_1 - \tau_0}{f} + 1, \qquad x_2 = \frac{\sigma_1 + \tau_0}{f} + 1.$$

The equation for σ_1, f, τ_0 is

$$-4\sigma_1\tau_0 + \frac{2}{1+\lambda}(1+\eta\lambda)f\sigma_1 + 2\left(\frac{1}{1+\lambda} - 2\right)f\tau_0 - \left[(1-\eta)\frac{\lambda^2}{(1+\lambda)^2} - 1\right]f^2 + 2\eta_2 v_0 f = 0, \tag{2.18}$$

together with conditions (2.5), (2.13), and

$$\sigma_1 > -f + \tau_0. \tag{2.19}$$

The study of the equations obtained from (2.4) in different cases can be carried on fixing the values of one of the parameters. The representation in the plane τ_0, σ_1, e.g. (see Fig. 2), has been obtained setting $f = 1$.

Remark 2.1. The stationary flow of two newtonian fluids can be investigated with same methods setting $\tau_0 = 0$ (see [12, Sect. 1.3]).

We obtain three possible cases corresponding to Case 1, Case 3.1, and Case 3.3, respectively. The velocity profiles are analogous to the ones in Fig. 3, where the possible constant intervals are reduced to one point (the rigid parts disappear).

In Case 1 Eq. (2.6) with constraint (2.7) gives

$$(\eta\lambda^2 - 1)f - 2\eta_2 v_0(1+\lambda)^2 > 0. \tag{2.20}$$

We remark that this case cannot hold if $(\eta_2/\eta_1)\lambda^2 < 1$, that is, if the viscosity of fluid 1 is larger than the viscosity of fluid 2.

In Case 3.1 we have from (2.14) and (2.15)

$$(\eta\lambda^2 + 2\eta\lambda + 1)f - 2\eta_2 v_0(1+\lambda)^2 < 0. \tag{2.21}$$

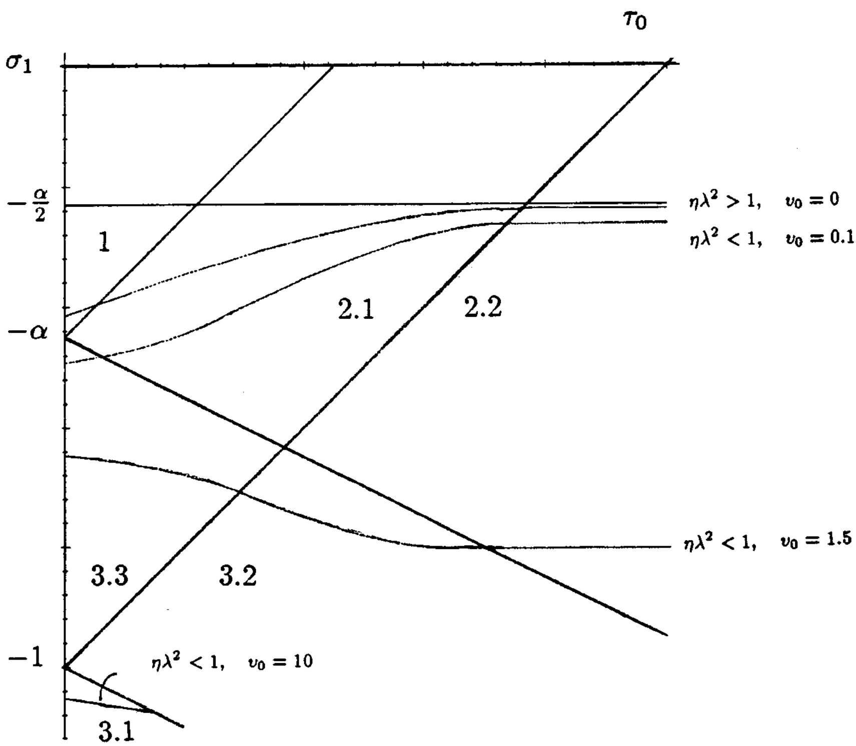

FIG. 2. $\sigma(1)$ versus τ_0.

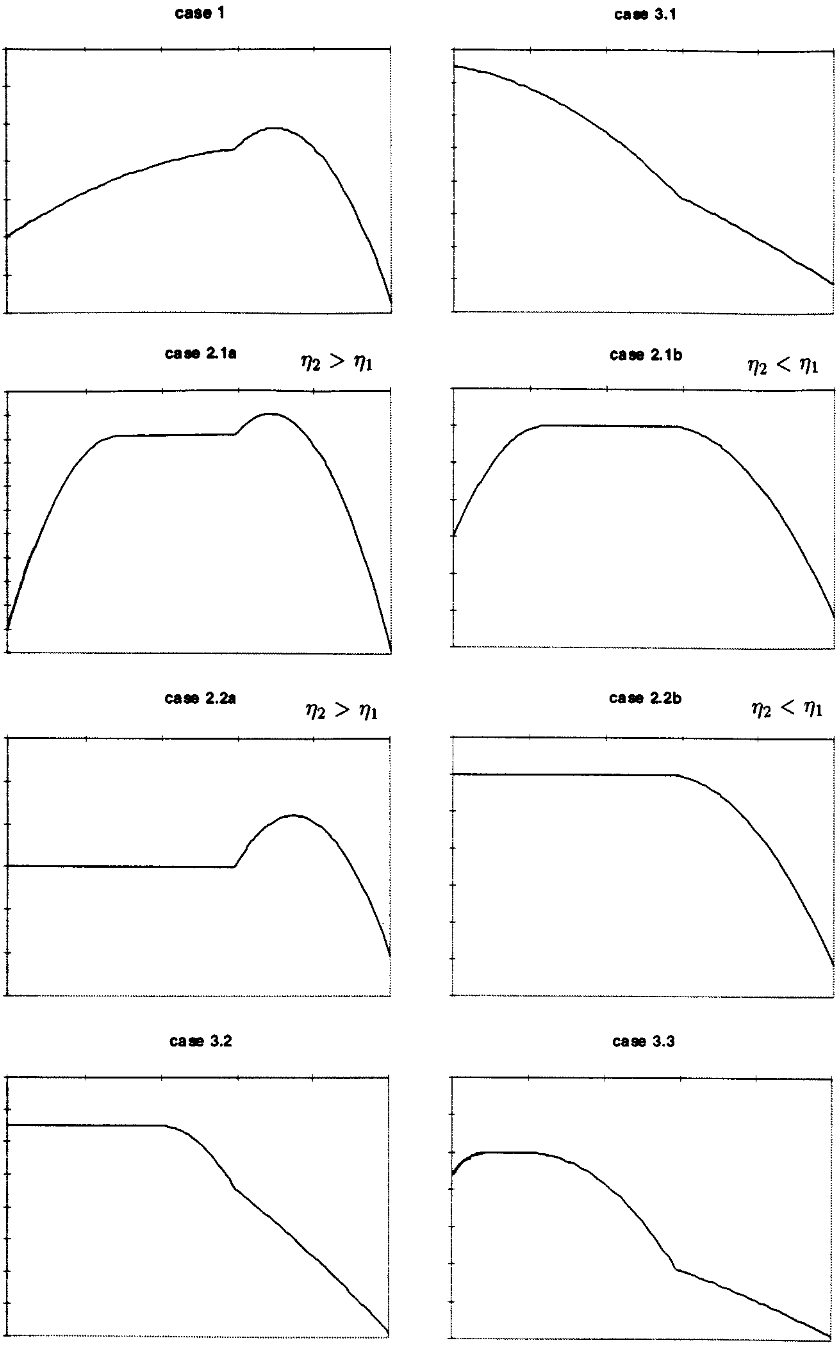

FIG. 3. Velocity profiles.

Case 3.3 holds if

$$\begin{gathered}(\eta\lambda^2 - 1)f - 2\eta_2 v_0(1 + \lambda^2) > 0,\\ (\eta\lambda^2 + 2\eta\lambda + 1)f - 2\eta_2 v_0(1 + \lambda)^2 < 0.\end{gathered} \tag{2.22}$$

3. STEADY SYMMETRIC CASE

If we consider axisymmetrical solutions, the assumption $\sigma(0) = f + \sigma_1 = 0$ gives a further relationship between the two parameters f and σ_1. This guarantees for any f and τ_0 a unique stationary solution.

For $x > 0$, Eq. (2.1) reduces to $\sigma(x) = -fx$.

Referring again to Fig. 1, in the case $\sigma(1/(1 + \lambda)) \geq -\tau_0$, we have that all the Bingham fluid is rigid. This case corresponds to previous Case 2.2, with $\sigma(1/(1 + \lambda)) < 0$.

Hence from (2.11) we obtain the expression of v_0 as a function of f,

$$v_0 = \frac{f}{2\eta_1}\left[1 - \left(\frac{1}{1 + \lambda}\right)^2\right], \tag{3.1}$$

with the condition obtained imposing $\sigma(1/(1 + \lambda)) \geq -\tau_0$

$$f \leq \tau_0(1 + \lambda). \tag{3.2}$$

The expression for v_∞ is then

$$v_\infty(x) = \begin{cases} \dfrac{f}{2\eta_1}\left[1 - \left(\dfrac{1}{1 + \lambda}\right)^2\right], & 0 \leq x \leq \dfrac{1}{1 + \lambda},\\[2ex] \dfrac{f}{2\eta_1}(1 - x^2), & \dfrac{1}{1 + \lambda} < x \leq 1. \end{cases} \tag{3.3}$$

If $\sigma(1/(1 + \lambda)) < -\tau_0$ the Bingham fluid presents a rigid zone of width $s_\infty = \tau_0/f$, in contact with a fluid one close to the newtonian fluid, as in Case 3.2.

Equation (2.16) becomes

$$v_0 = \frac{f}{2\eta_2}\left[\left(\frac{1}{1 + \lambda} - \frac{\tau_0}{f}\right)^2 + \eta\left(1 - \left(\frac{1}{1 + \lambda}\right)^2\right)\right], \tag{3.4}$$

which holds, recalling (2.13), if

$$f > \tau_0(1 + \lambda). \tag{3.5}$$

The explicit expression for v_∞ in this case is the following:

$$v_\infty(x) = \begin{cases} \dfrac{f}{2\eta_2}\left[\left(\dfrac{1}{1+\lambda} - \dfrac{\tau_0}{f}\right)^2 + \eta\left(1 - \left(\dfrac{1}{1+\lambda}\right)^2\right)\right], & 0 \le x \le \dfrac{\tau_0}{f}, \\ \dfrac{f}{2\eta_2}\left\{\left[\left(\dfrac{1}{1+\lambda}\right)^2 - x^2\right] - \dfrac{2\tau_0}{f}\left(\dfrac{1}{1+\lambda} - x\right)\right. \\ \qquad \left. + \eta\left(1 - \left(\dfrac{1}{1+\lambda}\right)^2\right)\right\}, & \dfrac{\tau_0}{f} < x < \dfrac{1}{1+\lambda}, \\ \dfrac{f}{2\eta_1}(1 - x^2), & \dfrac{1}{1+\lambda} < x < 1. \end{cases} \tag{3.6}$$

4. TIME DEPENDENT SOLUTIONS

Let us consider the evolution problem defined in Section 1, looking for symmetric solutions.

For $x > 0$ we have

$$\rho_i v_{it} - \eta_i v_{ixx} = f, \qquad \text{in } D_{iT},\ i = 1,2, \tag{4.1}$$

$$v_1\left(\frac{1}{1+\lambda}, t\right) = v_2\left(\frac{1}{1+\lambda}, t\right), \qquad t > 0, \tag{4.2}$$

$$\eta_1 v_{1x}\left(\frac{1}{1+\lambda}, t\right) = \eta_2 v_{2x}\left(\frac{1}{1+\lambda}, t\right) - \tau_0, \qquad t > 0, \tag{4.3}$$

$$v_{2x}(s(t), t) = 0, \qquad t > 0, \tag{4.4}$$

$$(\rho_2 v_{2t}(s(t), t) - f)s(t) = -\tau_0, \qquad t > 0, \tag{4.5}$$

$$s(0) = s_0, \qquad v_i(x, 0) = v_{i0}(x), \qquad i = 1,2, \qquad s_0 \le x \le 1, \tag{4.6}$$

with $0 < s_0 < 1/(1+\lambda)$.

Equation (4.3) expresses the continuity of the stress at $x = 1/(1+\lambda)$ if $s(t) < 1/(1+\lambda)$.

The functions $v_{i0}(x)$ in (4.6) satisfy the following assumptions:

$$v_{i0} \in C^3, \qquad v_{i0} \ge 0, \qquad v'_{i0} \le 0, \qquad v''_{i0} \le 0, \tag{4.7}$$

and the second-order compatibility conditions.

We recall that we are looking for nonnegative classical solutions such that $v_{2x} < 0$ in D_{2T}.

Concerning local existence and uniqueness of the solution of problem (4.1)–(4.6) we refer to [6] and [11].

In particular we note that classical methods can be applied to the problem satisfied by the function $z = v_t$,

$$\rho_i z_{it} - \eta_i z_{ixx} = 0, \qquad \text{in } D_{iT}, i = 1, 2, \tag{4.8}$$

$$z_1(1, t) = 0, \qquad z_1\left(\frac{1}{1+\lambda}, t\right) = z_2\left(\frac{1}{1+\lambda}, t\right), \qquad t > 0, \tag{4.9}$$

$$\eta_1 z_{1x}\left(\frac{1}{1+\lambda}, t\right) = \eta_2 z_{2x}\left(\frac{1}{1+\lambda}, t\right), \qquad t > 0, \tag{4.10}$$

$$z_2(s(t), t) = \frac{1}{\rho_2}\left(f - \frac{\tau_0}{s(t)}\right), \qquad t > 0, \tag{4.11}$$

$$z_{2x}(s(t), t) = \frac{1}{\eta_2}\frac{\tau_0}{s(t)}\dot{s}(t), \qquad t > 0, \tag{4.12}$$

$$z_i(x, 0) = \frac{1}{\rho_i}(\eta_i v_i'' + f), \qquad 0 \le x \le 1. \tag{4.13}$$

This is a Stefan problem in D_{2T} coupled by means of diffraction conditions on $x = 1/(1 + \lambda)$ with a standard parabolic problem in D_{1T}. To prove regularity for the solution of problem (4.8)–(4.13), we need hypotheses on the second derivative of the initial data (see (4.7)).

In order to study the continuation of the solution, and its behaviour, we can follow the methods of [4], noting that problem (4.1)–(4.6) can be transformed into a free boundary problem with Cauchy data prescribed in $s(t)$ in the zone occupied by the Bingham fluid for the function defined as follows:

$$C_2(x, t) = -\frac{\rho_2}{\eta_2}\int_{s(t)}^{x} d\xi \int_{s(t)}^{\xi}\left(z_2(\zeta, t) - \frac{f}{\rho_2}\right) d\zeta. \tag{4.14}$$

The function C_2 satisfies a problem like the reaction diffusion problem for the concentration of oxygen in a living tissue (see [5] and [7]).

We remark that

$$C_2(x, t) = v_2(s(t), t) - v_2(x, t). \tag{4.15}$$

Recalling the results of [7] we have that the solution could either exist for any T or become extinct or blow up. Note that it could happen also that the model loses its sense because the constraint $v_{2x} < 0$ is violated at some time.

Moreover we have to exclude that $s(t)$ vanishes at some time (i.e., the extinction of the rigid zone), because at that time we would have a maximum for v, where the stress would be zero, inside the fluid domain.

We can state that only one of the following cases can happen:

(a) there exists a global solution;

(b) $\exists \bar{t}$ such that the whole region occupied by the Bingham fluid is rigid (extinction of the solution in $D_{2\bar{t}}$);

(c) $\exists t_0$ such that the constraint $v_{2x} < 0$ is violated on $x = 1/(1+\lambda)$ at $t = t_0$.

In order to characterize cases (a)–(c), we have first to exclude the occurrence of blow-up and the disappearance of the rigid zone. This will be done by means of the following estimates and by means of a comparison with suitable supersolutions.

PROPOSITION 4.1. *Under assumption* (4.7) *we have*

$$v_i \geq 0, \qquad in\ \overline{D}_{iT}, \tag{4.16}$$

$$v_{ix} < 0, \qquad in\ D_{iT} \cup \left\{ x = \frac{1}{1+\lambda}, 0 < t < T \right\}, \tag{4.17}$$

$$v_{it} < \max\left\{ \frac{f}{\rho_1}, \frac{f}{\rho_2} \right\}, \qquad in\ D_{iT}. \tag{4.18}$$

Moreover, in the case $\rho_1 < \rho_2$, *if we assume also*

$$v_{1xx}(x,0) < \left(\frac{\rho_1}{\rho_2} - 1 \right) \frac{f}{\eta_1} < 0, \tag{4.19}$$

we have

$$v_{it} < \frac{f}{\rho_2}, \qquad in\ D_{iT}. \tag{4.20}$$

Proof. The estimates can be easily obtained from the maximum principle. Here we have denoted by T the maximum existence time of the solution, both in the case of global solution and in the case of existence up to a finite time (either $T = \bar{t}$ for case (b) or $T = t_0$ for case (c)). ∎

Remark 4.1. Inequalities (4.18) and (4.20) provide estimates of v_{ixx} depending on the density of the two fluids. Then

$$\begin{aligned} &\text{if } \rho_1 \geq \rho_2, \qquad v_{2xx} < 0 \qquad \text{in } D_{2T},\\ &\text{if } \rho_1 < \rho_2, \qquad v_{1xx} < 0 \qquad \text{in } D_{1T}. \end{aligned} \tag{4.21}$$

If $\rho_1 < \rho_2$ and (4.19) holds

$$v_{ixx} < 0, \qquad \text{in } D_{iT}. \tag{4.22}$$

COROLLARY 4.1. *Under hypotheses* (4.7), *blow-up of the solution cannot happen.*

Proof. Recalling [8], the occurrence of blow-up is related to the onset of a negative set for the function C_2 defined in (4.14); however, in this case estimate (4.17) implies $C_2 > 0$ in D_{2T}. ∎

Now we will construct supersolutions and subsolutions in order to characterize cases (a)–(c). The existence of supersolutions will enable us also to exclude the extinction of the rigid zone.

Let $(\bar{v}_i, \bar{s}, \bar{T})$ and $(\underline{v}, \underline{s}, \underline{T})$ be the solutions of problem (4.1)–(4.6) corresponding to the data $(\bar{v}_i(x,0), \bar{s}(0), \bar{f})$ and $(\underline{v}_i(x,0), \underline{s}(0), \underline{f})$, respectively.

PROPOSITION 4.2. *Suppose that*

$$\bar{s}(0) > \underline{s}(0), \qquad \bar{v}_{ix}(x,0) > \underline{v}_{ix}(x,0), \tag{4.23}$$

and either

(i) $\rho_1 \geq \rho_2$, $\bar{f} \leq \underline{f}$

or

(ii) $\rho_1 < \rho_2$, $\bar{f} = \underline{f}$.

Then

$$\bar{s}(t) > \underline{s}(t), \qquad 0 < t < \min(T_1, T_2) = \bar{T}, \tag{4.24}$$

$$\underline{v}_i(x,t) \leq \bar{v}_i(x,t), \qquad \textit{in } \tilde{D}_{i\bar{T}}, \tag{4.25}$$

$$\bar{v}_{ix}(x,t) \geq \underline{v}_{ix}(x,t), \qquad \textit{in } \tilde{D}_{i\bar{T}}, \tag{4.26}$$

where

$$\tilde{D}_{1\bar{T}} = \left\{ \frac{1}{1+\lambda} < x < 1, 0 < t < \bar{T} \right\},$$

$$\tilde{D}_{2\bar{T}} = \left\{ \bar{s}(t) < x < \frac{1}{1+\lambda}, 0 < t < \bar{T} \right\}.$$

Proof. The result follows using the methods of [2] applying the maximum principle to the functions $\Delta w_i = \bar{v}_{ix} - \underline{v}_{ix}$ in $\tilde{D}_{iT}$, taking into account the conditions satisfied on $x = 1/(1+\lambda)$:

$$\eta_1 \, \Delta w_1\left(\frac{1}{1+\lambda}, t\right) = \eta_2 \, \Delta w_2\left(\frac{1}{1+\lambda}, t\right),$$

$$\eta_1 \, \Delta w_{1x}\left(\frac{1}{1+\lambda}, t\right) = \frac{\rho_1}{\rho_2} \eta_2 \, \Delta w_{2x}\left(\frac{1}{1+\lambda}, t\right) + \left(\bar{f} - \underline{f}\right)\left(\frac{\rho_1}{\rho_2} - 1\right).$$

∎

The proposition above enables us to compare the solution of (4.1)–(4.6) with suitable stationary solutions.

Let $(\underline{v}_\infty, \underline{s}_\infty)$ and $(\bar{v}_\infty, \bar{s}_\infty)$ be the stationary solutions defined in (3.6) corresponding to $\underline{f}$ and $\bar{f}$, respectively, with $\bar{f} < \underline{f}$.

Let (v, s, T) the solution of (4.1)–(4.6).

PROPOSITION 4.3. *Suppose that* $\rho_1 \geq \rho_2$. *Suppose that* (4.7) *holds and*

$$\underline{s}_\infty < s(0) < \bar{s}_\infty, \qquad \underline{v}'_{i\infty}(x) < v_{ix}(x,0) < \bar{v}'_{i\infty}(x), \qquad \bar{f} \leq f \leq \underline{f}. \quad (4.27)$$

Then

$$\begin{gathered} \underline{s}_\infty < s(t) < \bar{s}_\infty, \qquad \bar{v}_{i\infty}(x) < v_i(x,t) < \underline{v}_{i\infty}(x), \\ \underline{v}'_{i\infty}(x) < v_{ix}(x,t) < \bar{v}'_{i\infty}(x). \end{gathered} \quad (4.28)$$

Proof. Follows immediately from the previous proposition. ∎

PROPOSITION 4.4. *Suppose that* $\rho_1 < \rho_2$. *Suppose that* (4.7) *and* (4.27) *hold and*

$$\frac{\rho_1}{\rho_2}\underline{v}''_{1\infty}(x) < v_{1xx}(x,0) < \left(\frac{\rho_1}{\rho_2} - 1\right)\frac{f}{\eta_1} + \frac{\rho_1}{\rho_2}\bar{v}''_{1\infty}(x), \quad (4.29)$$

$$\underline{v}''_{2\infty}(x) < v_{2xx}(x,0) < \bar{v}''_{2\infty}(x). \quad (4.30)$$

Then

$$\underline{s}_\infty < s(t) < \bar{s}_\infty, \qquad \bar{v}_{i\infty}(x) < v_i(x,t) < \underline{v}_{i\infty}(x) < v_{ix}(x,t) < \bar{v}'_{i\infty}(x). \quad (4.31)$$

Proof. Let us first consider the comparison with $(\underline{v}_\infty, \underline{s}_\infty)$.

Noting that $s(0) > \underline{s}_\infty$, then $s(t) > \underline{s}_\infty$ at least up to a time $t = t_0$. Suppose that $s(t_0) = \underline{s}_\infty$. Then the function $z_i = v_{it}$ satisfies problem (4.8)–(4.13) in D_{it_0}.

Set $M = (1/\rho_2)(\underline{f} - f) > 0$. Then, using maximum principle, from assumptions on the data we get $z_i(x,t) > -M$ for $t < t_0$, and $z_2(s(t_0), t_0)$ would be a minimum; hence, noting that $\dot{s}(t_0) < 0$ we get a contradiction with (4.12). Hence $s > \underline{s}_\infty$ and the lower estimates for v_{ixx}. The reverse inequalities can be proved in a similar manner.

Other estimates in (4.31) follow as consequences. ∎

PROPOSITION 4.5. *Suppose that assumptions* (4.7) *hold. Then*

$$s(t) > \frac{\tau_0}{F_1} > 0 \tag{4.32}$$

with

$$F_1 > \max\left\{ \frac{\tau_0}{s_0}, \eta_i \| v_{ixx}(x,0) \| \right\}. \tag{4.33}$$

Proof. The stationary solution (3.6) corresponding to $f = F_1$ satisfies the hypotheses of Propositions 4.3 and 4.4. ∎

Looking for a possible upper bound for s we need more hypotheses on the data

PROPOSITION 4.6. *Suppose* (4.7) *holds,* $f > \tau_0(1 + \lambda)$, *and*

$$\max_x \{\eta_i v_{ixx}(x,0)\} \equiv -a_i \leq -\tau_0(1+\lambda). \tag{4.34}$$

Then

$$s(t) < \frac{\tau_0}{F_2} \leq \frac{1}{1+\lambda}, \tag{4.35}$$

with

$$\tau_0(1+\lambda) \leq F_2 \leq \min\left\{ \frac{\tau_0}{s_0}, a_i \right\}. \tag{4.36}$$

Proof. If $\rho_1 \geq \rho_2$, then for any $a_i \geq \tau_0(1 + \lambda)$ we can choose F_2 as in (4.34) and consider the stationary solution corresponding to F_2. Then the hypotheses of Proposition 4.3 are verified and (4.35) holds.

If $\rho_1 < \rho_2$, in order to verify the hypotheses of Proposition 4.4, we have to suppose

$$a_1 \geq -\left(\frac{\rho_1}{\rho_2} - 1 \right) f + \frac{\rho_1}{\rho_2} \tau_0(1+\lambda), \qquad a_2 \geq \tau_0(1+\lambda).$$

∎

The previous results guarantee that the following theorem holds.

THEOREM 4.1. *Suppose that* (4.7) *and* (4.34) *hold. Suppose that* $f \geq \tau_0(1+\lambda)$. *Then there exists a unique solution for any* t.

In the case of global existence we can give a further estimate:

PROPOSITION 4.7. *Under the hypotheses of Theorem* 4.1 *we have*

$$\int_0^{\infty}\left(\frac{\tau_0}{s(t)} - f\right) dt < +\infty. \tag{4.37}$$

Proof. Inequality (4.37) can be obtained from the Green's identity

$$\int_{\partial D_{iT}} \rho_i u_i v \, dx + \eta_i (u_{ix} v - u_i v_x) \, dt = 0,$$

setting $u_i = z_i$ and $v = x$.

Noting that, from (4.32) and (4.35), $s_\infty \equiv \lim_{\tau \to \infty} s(t)$ exists and that $z(x,t)$ tends to 0 uniformly as $t \to \infty$, we obtain

$$\frac{f}{2}(1 - s_\infty^2) = \frac{\eta_2}{\rho_2}\int_0^{\infty}\left(\frac{\tau_0}{s} - f\right) dt - \frac{\tau_0}{1+\lambda} - \eta_2 v_2(s_0, 0) + f$$
$$+ (\eta_2 - \eta_1)\frac{f}{2\eta_1}\left[1 - \left(\frac{1}{1+\lambda}\right)^2\right].$$

∎

Recall that, if $f < \tau_0(1+\lambda)$, the corresponding stationary solution is given by (3.3) and $s_\infty \equiv 1/(1+\lambda)$. Then (4.37) cannot be satisfied.

Therefore we have

PROPOSITION 4.8. *Suppose that* (4.7) *holds and that, if* $\rho_1 < \rho_2$, (4.19) *holds. Suppose that* $f < \tau_0(1+\lambda)$. *Then there exists a time* t_0 *such that* $s(t_0) = 1/(1+\lambda)$.

Proof. If $f < \tau_0(1+\lambda)$ then case (a) cannot hold. Moreover case (c) is excluded because v_{2xx} stays negative (see Remark 4.2). ∎

Under the hypotheses of the previous proposition, the solution exists again for any t, with the whole region occupied by the Bingham fluid that stays rigid. In fact if we consider the problem for $t \geq t_0$, it is always possible to find a stationary solution defined by (3.3), such that $v_1 > v_{1\infty}$ in D_{1t_0T}.

Assuming further hypotheses on the data, we can give some monotonicity results.

PROPOSITION 4.9. *Suppose that the assumptions of Proposition* 4.3 *or* 4.4 *are satisfied. we have that*

(i) *if* $v_{ixxx}(x,0) < 0$ *then*

$$a(t) < \dot{s}(t) < 0, \qquad \textit{for any } t > 0, \tag{4.38}$$

$$v_{ixxx}(x,t) < 0, \qquad \textit{in } D_{iT}, \tag{4.39}$$

where $a(t)$ *is a negative function which is bounded for any* t;

(ii) *if* $v_{ixxx}(x,0) > 0$ *and* $v_{ixx}(x,0) < -\tau_0/\eta_i x$ *then*

$$0 < \dot{s}(t) < \frac{\eta_2}{\rho_2 s}, \qquad t > 0, \tag{4.40}$$

$$v_{ixx}(x,t) < -\frac{\tau_0}{\eta_i x}, \qquad v_{ixxx}(x,t) > 0, \qquad \textit{in } D_{iT}. \tag{4.41}$$

Proof. We obtain estimates (4.38) and (4.39) as done in [2]. Concerning case (ii) we apply the maximum principle to the function $v_i = z_i - (1/\rho_i)(f - \tau_0/x)$. ∎

Let us now investigate case (c).

If $\rho_1 < \rho_2$, for suitable initial data not satisfying condition (4.19), then v_{2x} can vanish at $(1/(1+\lambda), t_0)$. Then at time $t = t_0$, our model loses its validity and we need regularization in order to study the behaviour of the solution after t_0.

5. REGULARIZATION PROBLEM

Suppose that $\rho_1 < \rho_2$ and that there exists a time t_0 such that $v_{2x}(1/(1+\lambda), t_0) = 0$, then at this time the stress at $x = 1/(1+\lambda)$ reaches the threshold value.

We formulate a regularized free boundary problem with a new free boundary $x = \Sigma(t)$ appearing at $1/(1+\lambda)$, at time t_0. Then an internal rigid zone appears: it is separated from the fluid zone by this new free boundary and it is in contact with the newtonian fluid [4].

For the sake of simplicity we shift the time origin to $t = t_0$. Let

$$D_{2T} = \left\{(x,t): 0 < s(t) < x < \Sigma(t) < \frac{1}{1+\lambda}, 0 < t < T\right\},$$

$$D_{1T} = \left\{(x,t) \ \frac{1}{1+\lambda} < x < 1, 0 < t < T\right\},$$

$$\rho_i v_{it} = \eta_i v_{ixx} + f, \qquad \text{in } D_{iT}, \tag{5.1}$$

$$v_1(1,t) = 0, \qquad t > 0, \tag{5.2}$$

$$v_2(\Sigma(t),t) = v_1\left(\frac{1}{1+\lambda}, t\right), \qquad t > 0, \tag{5.3}$$

$$\left(\frac{1}{1+\lambda} - \Sigma(t)\right)\left[v_{2t}(\Sigma(t),t) - \frac{f}{\rho_2}\right] = \frac{1}{\rho_2}\left[\tau_0 + \eta_1 v_{1x}\left(\frac{1}{1+\lambda}, t\right)\right], \tag{5.4}$$

$$v_{2x}(\Sigma(t),t) = 0, \qquad t > 0, \tag{5.5}$$

$$v_{2x}(s(t),t) = 0, \qquad t > 0, \tag{5.6}$$

$$v_{2t}(s(t),t) = \frac{1}{\rho_2}\left(f - \frac{\tau_0}{s(t)}\right), \qquad t > 0, \tag{5.7}$$

$$v_i(x,0) = v_{i0}(x), \qquad 0 \le x \le 1, \qquad s(0) = s_1, \qquad \Sigma(0) = \frac{1}{1+\lambda}. \tag{5.8}$$

Equation (5.4) is the rigid motion equation for the zone between Σ and $1/(1+\lambda)$, where we took into account the continuity of the stress on $x = 1/(1+\lambda)$.

Existence and uniqueness of the solution of problem (5.1)–(5.8) can be proved again by means of classical methods, studying the problem satisfied by $z_i = v_{it}$.

According to the results of the previous section, the assumptions on the data are

$$0 < s_1 < \frac{1}{1+\lambda},$$

$$v_i(x,0) > 0, \qquad v_{ix}(x,0) < 0, \qquad 0 < v_{it}(x,0) < \frac{f}{\rho_1}, \tag{5.9}$$

$$\exists \bar{x}: \qquad v_{2xx}(x,0) < 0, \qquad \text{for } s_1 < x < \bar{x},$$

$$v_{2xx}(x,0) > 0, \qquad \text{for } \bar{x} < x < \frac{1}{1+\lambda}.$$

Moreover, in order to obtain $\dot{\Sigma}(0) < 0$, we need also

$$z_{2x}(x,0) > 0, \qquad \text{in } \delta - \frac{1}{1+\lambda} < x < \frac{1}{1+\lambda}, \tag{5.10}$$

for some $0 < \delta < 1/(1+\lambda)$.

In fact, assuming z_{2x} continuous at $x = 1/(1+\lambda)$, we get, at least for a small time t, $\dot{\Sigma}(t) < 0$, recalling

$$z_{2x}(\Sigma(t),t) = -\frac{\dot{\Sigma}}{\eta_2}(\rho_2 z_2(\Sigma(t),t) - f).$$

We conclude by giving some estimates following from the maximum principle that hold in D_{iT}:

$$v_1(x,t) > 0, \qquad v_2(x,t) > v_2\left(\frac{1}{1+\lambda}, 0\right) > 0, \tag{5.11}$$

$$v_{ix}(x,t) < 0, \tag{5.12}$$

$$0 < v_{it}(x,t) < \frac{f}{\rho_1}, \tag{5.13}$$

$$s(t) > 0. \tag{5.14}$$

A detailed analysis of this case could be performed using the methods of [3].

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