UNIVERSITÀ DEGLI STUDI DI FIRENZE

FACOLTÀ DI SCIENZE MATEMATICHE, FISICHE E NATURALI

TESI DI DOTTORATO IN FISICA

DYNAMICAL IMPLICATIONS OF COSMOLOGICAL THEORIES WITH DISSIPATIVE MATTER AND TORSION EFFECTS IN NON-EINSTEINIAN SPACE-TIMES

Settore Scientifico-Disciplinare: FIS/02

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Introduction

The Einstein Equations constitute a well established dynamical scheme to represent the gravitational phenomena. This geometrical approach finds a convincing confirmation both an a theoretical end experimental point of view. However, a wide number of attempts to extend such an Einstenian formulation were done over nine decades.

The first evidence for a generalization of the theory was originally proposed by Einstein himself. It concerns the fact that the field equations link two very different objects: the curvature tensor and the energy momentum of the matter source. The absence of a unique origin for this two ingredients is to be considered as a possible hint for new gravitational physics. In particular, the representation of the matter sources by macroscopic properties is a proper choice to deal with the gravitational interaction, but it opens significant questions about the microscopic features underling this averaging methods. Indeed, a microscopic interaction between particles and gravity could involve, as suggested in recent decades, the necessity to upgrade the Riemannian geometry to extended features. In this thesis work, we will address the point of view presented above by analyzing both the geometrical and matter aspects.

In Chapter 1, the discussion of a generalized cosmological dynamics, ables to account for dissipative effects, is analyzed firstly in the context the matter-dominated era of the Universe and, secondly, during the early phases of isotropic and Quasi-Isotropic cosmological models. In this respect, we will treat the very early-Universe evolution and the asymptotic gravitational collapse, by means of an hydrodynamical approach to the description of viscous properties of the cosmological fluid. Such an approach is required in view of the extreme regime the cosmological fluid feels during the considered phases.

In Chapter 2, from the point of view of extended geometry, we discuss the role of torsion as both a macroscopic and microscopic property of the space-time. In the microscopic sector, we arrive to formulate a gauge theory which allows to recognize on-shell (by means of the field equations) the contortion tetrad-projections as the gauge potentials.

The results of this two investigations in cosmology and in fundamental physics gave rise to a number of promising issue which can get light on some open questions on both the research fields. As we will discuss in detail in the relative Chapters, our analysis gives a clear picture of the physical insight contained in the addressed generalization of the Einstein Theory. The scientific reliabilities of our investigation and its links with the preexisting literature can be recognized from the published material on this topics.

Publication list:

1. NC and G. Montani, in *Proc. of III Stueckelberg Workshop*, in preparation.
   `New Issues in Gravitational Instability.`
2. NC, O.M. Lecian and G. Montani, in *Proc. of III Stu. Workshop*, in preparation.
   `A Novel Approach to Lorentz Gauge Theory.`
3. NC and G. Montani, submitted to *Int. J. Mod. Phys. D*, Nov. 2008.
   `Jean Instability in Presence of Dissipative Effects.`
4. NC, O.M. Lecian and G. Montani, *Mod. Phys. Let. A*, in press.
   `Fermion Dynamics by Internal and Space-Time Symmetries.`
5. NC and G. Montani, *Int. J. Mod. Phys. D* **17**(6), 881 (2008).
   `Study of the Quasi-Isotropic Solution Near the Cosmological Singularity in Presence of Bulk Viscosity.` [arXiv:0711.1952]
6. NC and G. Montani, *Int. J. Mod. Phys. A* **23**(8), 1248 (2008).
   `On the Role of Viscosity in Early Cosmology.` [arXiv:0801.3368]
7. NC, O.M. Lecian and G. Montani, *Int. J. Mod. Phys. A* **23**(8), 1282 (2008).
   `Lorentz Gauge Theory and Spinor Interaction.` [arXiv:0801.4242]
8. NC, O.M. Lecian, G. Montani, *Ann. Fond. L. deBroglie* **32**(2/3), 281 (2007).
   `Macroscopic and Microscopic Paradigms for the Torsion Field: from the Test-Particle Motion to a Lorentz Gauge Theory.` [arXiv:0711.3538]
9. NC and G. Montani, *AIP Conf. Proc.* **966**, 241 (2007).
   `Gravitational Stability and Bulk Cosmology.` [arXiv:0710.0313]

NOTATION:

Greek indices $\mu,\ \nu,\ \rho,\ \sigma,\ \lambda,\ \tau,\ \upsilon,\ \varepsilon$ run over the four coordinate labels in a general coordinate system 0, 1, 2, 3 or $t,\ x,\ y,\ z$.

The signature is set $[+,\ -,\ -,\ -]$ unless otherwise indicated.

Latin indices $a,\ b,\ c,\ d,\ e,\ f$ run over the tetrad labels 0, 1, 2, 3.

Only the indices $\alpha,\ \beta,\ \gamma$ run over three spatial coordinate labels 1, 2, 3 or $x,\ y,\ z$.

Repeated indices are summed unless otherwise indicated.

A comma or a semicolon between indices denotes a derivative or a covariant derivative, respectively.

The symbol ${A_\mu}^{[a}\,{B_\nu}^{b]}$ denotes the anti-symmetrization whit respect to ab and ${A_\mu}^{(a}\,{B_\nu}^{b)}$ denotes the symmetrization, respectively.

A dot $(\dot{\ })$ over any quantity denotes the total time derivative of that quantity and the symbol ∇ denotes the usual 3-dimensional Nabla Operator.

Cartesian 3-vectors are indicated by boldface type and their components are labeled ***only*** by $\alpha,\ \beta,\ \gamma$.

Units are used such that $c = \hbar = 1$ unless otherwise indicated.

Abbreviations are used: DOF indicates “degrees of freedom”; WRT indicates “whit respect to”; LHS indicates “left hand side”; RHS indicates “right hand side”; eq. indicates “equation; eqs. indicates “equations”.

The symbol $[x, y]$ denotes the commutator and $[x, y]_+$ the anti-commutator between x and y, respectively.

1 Dissipative Cosmologies

1.1 General statements

The *Cosmological Standard Model* (CSM) [1] well describes many parts of the Universe evolution and it takes into account the Friedmann-Lemaître-Robertson-Walker (FLRW) metric as the highest symmetric background. The FLRW metric is based on the assumption of homogeneity and isotropy of the Universe and it also assumes that the spatial component of the metric can be time dependent (in particular, proportional to the so called *scale factor*). In this respect, considering the mean energy-density at large scales, *i.e.*, greater than 100 Mpc, the Universe tends to an homogeneous distribution. On the other hand, observations at small scales show a very inhomogeneous and anisotropic matter- and energy-distribution.

The isotropic hypothesis of the Universe, stated by the *Cosmological Principle* [2] is indeed not based on the large-scale observations but on the strong isotropy of the Cosmic Microwave Background Radiation (CMBR). In cosmology, CMBR is a form of electromagnetic radiation filling the Universe [3]: the space between the stars and galaxies is not black but there is an almost isotropic glow, not coming from the agglomerates. This glow is strongest in the microwave region of the radio spectrum and corresponds to a relic radiation comes out from the very early Universe. The CMBR has a thermal black-body spectrum at temperature $T \sim 2.725\,\mathrm{K}$ with fluctuations of order $\mathcal{O}(10^{-4})$. This way, the spectrum peaks in the microwave range frequency of $\mathcal{O}(160\ GHz)$. The CMBR was discovery in 1964 by A. Penzias and R. Wilson [4] and further physical characterization are obtained in [5, 6].

The CMBR is well explained by CSM. The very early stages of the Universe evolution, after the Big Bang singularity, are characterized by a very hight temperature and a uniform glow derived from its red-hot fog of hydrogen plasma. During the expansion, Universe grew cooler, both the plasma itself and the radiation filling it. When the Universe reached a cool enough temperature, stable atoms could form.

Such atoms could no longer absorb the thermal radiation and the Universe became transparent. In particular, the CMBR shows a spatial power spectrum contains small anisotropies which vary with the size of the region examined. As a result, in cosmology, this radiation is considered to be the best evidence for the Big Bang CSM model. Moreover, the CSM is confirmed by the primordial-nucleosynthesis prediction for the light elements, which is in agreement with direct observations.

The crucial dichotomy between the isotropy of region at red-shift $z_{rs} \sim 10^3$ and the extreme irregularity of the recent Universe, $z_{rs} \ll 1$, is at the ground of the interest in the study of the perturbative gravitational instability for the structure formation. In this respect, the study of the cosmological-perturbation evolution can be separated in two distinct regimes, characterized by different values of the *density contrast* δ. This quantity is defined as the ratio of the density perturbations $\delta\rho$ over the background density ρ_0, *i.e.*,

$$\delta = \delta\rho \,/\, \rho_0 \,.$$

In correspondence of δ much less than unity, the linear regime is addressed, on the other hand the non-linear one occurs as soon as $\delta > 1$, giving rise to the effective structure formation. Despite the approximate hypotheses, the linear regime provides interesting predictive informations also at low red-shift, since an analytical description can be addressed to study the growth of the density contrast.

As matter of fact, we underline that the study of the perturbation dynamics in the radiation-dominated early Universe requires a pure relativistic treatment, in order to correlate the matter fluctuations with the geometrical ones. On the other hand, the evolution during the matter-dominated era can be consistently described using the Newtonian-approximation picture, as soon as sub-horizon sized scales are treated. In this scheme, the fundamental result of the density-perturbation analysis is the so-called *Jeans Mass*, which is the threshold value for the fluctuation masses to condense generating a real structure. If masses greater than the Jeans Mass are addressed, density perturbations begin to diverge as function of time giving rise to the gravitational collapse [7, 8]. Since density is assumed to be homogeneous, the concept of the Jeans Mass can be supported by the *Jeans Length*. Such a value defines the threshold scale over which perturbations gravitationally condense.

In this work, we present a study of the effects induced by the presence of viscosity on the gravitational instability and on the structure formation, in the linear regime described above. The physical motivations for introducing viscosity into the cosmological-perturbation dynamics are due to the fact that, as soon as the Newtonian regime can be addressed, the gravitational collapse can induce hight matter density values such that a dissipative analysis results to be necessary. Moreover, the primordial Universe can be naturally characterized by viscosity in view of the very large mean densities (much greater than the nuclear one) reached in the limit towards the initial Big Bang. Although the physical description of such stages is very difficult, several studies in literature [9, 10, 11] promote the idea that Gluon Plasma, at very hight temperatures, show strong viscous properties.

Our analysis treats the Newtonian-approximated cases, in which the viscosity directly affects the *Jeans Mechanism*, and the pure relativistic limit dealing with the primordial Universe near the Big Bang. The stating point corresponds to the viscous modification of the Euler Equations and of the ideal-fluid energy-momentum tensor, respectively, in order to describe the background dynamics (as discussed in Section 1.2). The second step corresponds to a first-order perturbative theory in order to get the fundamental equations governing the gravitational collapse, *i.e.*, the density-contrast time evolution. In particular, two regimes can be reached:

$\delta \to 0$: the background density ρ_0 grows more rapidly than perturbations $\delta\rho$. A single structure is generated in the gravitational collapse.

$\delta \to \infty$: perturbations $\delta\rho$ grow more rapidly than the background density ρ_0. The sub-structure fragmentation scheme occurs.

In the Newtonian scheme, three different cases are treated in presence of viscosity:
Section 1.3 - The standard Jeans Model (uniform and static background)
Section 1.4 - The gas-cloud condensation (spherically-symmetric collapsing backg.)
Section 1.5 - The expanding Universe (expanding matter-dominated Universe backg.)

As a result, we show how the presence of viscous effects oppose the density-contrast growth, strongly contrasting the structure formation in the *top-down* mechanism, mainly associated with the hot dark matter phenomenology [12, 13]. Such a scheme is based on the idea that perturbation scales, contained within a collapsing gas cloud, start to collapse (forming sub-structures) because their mass overcomes the decreasing Jeans value of the background system. The resulting effect of such a gravitational instability consists of a progressive enhancement of the density contrast associated to the perturbation sub-scales.

In the three cases, the larger are the viscous contributions, the larger is the damping of the density-contrast evolution. In fact, if a gravitational-collapsing structure is addressed (*i.e.*, of mass greater than the Jeans one), in presence of viscosity the density contrast (associated to a given perturbation) progressively decreases and the fragmentation is suppressed. The main merit of this works is to be determined in having traced a possible scenario for fragmentation processes in presence of viscosity. We infer that the unfavored nature of the top-down mechanism, appearing when a viscous trace is present, can survive also in the non-linear regime when dissipative effects play surely an important role in the structure formation.

The pure relativistic analysis treats two different cases:

Section 1.6 - The Quasi-Isotropic Model.

Section 1.7 - The pure isotropic Universe.

Near the Cosmological Singularity, the isotropic nature of the Universe corresponds to a class of solutions of the Einstein Equations containing three physically arbitrary functions of the space coordinates. In the case of a radiation-dominated Universe, such a class was found by E.M. Lifshitz and I.M. Khalatnikov in 1963 [14]. In the original work, the *Quasi-Isotropic* (QI) *Model* is treated as a Taylor expansion of the 3-metric tensor in powers of the synchronous time. In this work, we fix the attention on the relevance of dealing with viscous properties of the cosmological fluid approaching the Big Bang singularity. For this purpose, we investigate the Einstein Equations under the assumptions proper of the QI Model. We separate zeroth- and first-order terms into the 3-metric tensor and the whole analysis follows this scheme of approximation. In the search for a self-consistent solution, we prove the existence of a QI Solution, which has a structure analogous to that provided by in the original work. In particular, we find that such a solution exists only if when viscosity remains smaller than a certain critical value. Finally, in determining the density-contrast evolution, strong analogies about the damping of density perturbations in the Newtonian limit, are founded.

A particular case of the QI Model corresponds to the FLRW pure isotropic approach where the 3-metric tensor Taylor expansion is addressed only at the zeroth-order. Aim of the work is to investigate the effects that viscosity has on the stability of such an isotropic Universe, *i.e.*, the dynamics of cosmological perturbations is analyzed when viscous phenomena affect the zeroth- and first-order evolution of the system. We consider a background corresponding to a FLRW model filled with ultra-relativistic viscous matter and then we develop a perturbative theory which generalizes the "Landau School" works [15, 14] to the presence of viscosity. Though the analysis is performed for the case of a flat model, nevertheless it holds in general, as soon as the perturbation scales remain much smaller than the Universe radius of curvature. As issue of our analysis, we find that two different dynamical regimes appear when viscous effects are taken into account and the transition from one regime to the other one takes place when the viscosity overcomes a given threshold value. However, in both these stages of evolution, the Universe results to be stable as it expands; the effect of increasing viscosity is that the density contrast begins to decrease with increasing time when viscous effect is over the threshold. It follows that a real new feature arises WRT the standard analysis, when the collapsing point of view is addressed. In fact, as far as viscosity remains below the threshold value, the isotropic Universe approaches the initial Big Bang with vanishing density contrast and its stability is preserved in close analogy to the non-viscous behavior. But if the viscous effect overcomes its critical value, then the density contrast explodes asymptotically to the singularity and the isotropic Universe results unstable approaching the initial singularity. In the non-viscous analysis, this same backward in time instability takes place only when tensor perturbations (gravitational waves) are taken into account, since their amplitude increases backward as the inverse of the cosmic scale factor.

The new feature induced by viscosity consists of having instability simply in correspondence to scalar perturbations induced by fluctuations in the matter filling the Universe. The cosmological interest in such instability of the primordial Universe (towards scalar perturbations) comes out reversing the picture from collapse to expansion and taking into account the time reversibility of the Einstein Equations. In fact, if the early Universe does not emerge from the Planck era peaked around the FLRW geometry (indeed a good degree of generality in its structure is predicted either by classical and quantum argumentation [16]), then it can not reach (according to our analysis) an homogeneous and isotropic stage of evolution before the viscous effect become sufficiently small.

1.2 Viscous processes in the fluid dynamics

The physical motivation in dealing with dissipative dynamics is related to the fact that both the extreme regime of a gravitational collapse and the very early stages of the Universe evolution are characterized by a thermal history which can not be regarded as settled down into the equilibrium. Indeed, at sufficiently high temperatures, the cross sections of the microphysical processes, responsible for the thermal equilibrium, decay like $\mathcal{O}(1/T^2)$ and they are no longer able to restore the equilibrium during the expansion. Thus, we meet stages where the expansion has an increasing rate and induces non-equilibrium phenomena in the matter compression and rarefaction.

The average effect of having a microphysics, unable to follow the fluid expansion by equilibrium stages, results into dissipative processes appropriately described by the presence of ***bulk viscosity***.

In what follows, we will discuss, in some details, how to introduce dissipative effects both in the Newtonian dynamics and in the pure relativistic limit, considering an homogeneous and isotropic picture.

Viscous effects in the Newtonian picture In order to describe the Newtonian evolution of a fluid, we introduce here the Eulerian Equations governing the fluid parameters, *i.e.*, the density ρ, the local 3-velocity $\mathbf{v}$ (of components v_α) and the pressure p, in presence of a gravitational potential Φ.

Adiabatic ideal fluids are governed, in Newtonian regime, by the following set of equations [17]: the *Continuity Equation*, which guarantees the energy conservation

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0\ , \tag{1.2.1}$$

the *Euler Equation*, which ensures the momentum conservation

$$\rho\,\partial_t\mathbf{v} + \rho\,(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p - \rho\,\nabla\Phi\ , \tag{1.2.2}$$

while pressure and density are linked by the *Equation of State* (EoS):

$$p = p\,(\rho)\ . \tag{1.2.3}$$

In this picture, the sound speed is defined by the relation $v_s^2 = \delta p/\delta\rho$.

Let us now introduce the effects of the energy dissipation during the motion of the fluid, due to the thermodynamical non-reversibility and to internal friction (we neglect the thermal conductivity). To obtain the motion equations for a viscous fluid, we have to include some additional terms in the ideal fluid description. The Continuity Equation is derived by the time evolution of the matter density and by the mass conservation law. This way, it remains valid for any kind of fluid.

Euler Equation, in absence of the gravitational field, rewrites

$$\partial_t(\rho v_\alpha) = -\partial^\beta\,\Pi_{\alpha\beta}\;, \tag{1.2.4}$$

where $\Pi_{\alpha\beta}$ denotes the momentum-flux energy tensor. If ideal fluids are addressed, we deal with completely reversible transfer of momentum, obtaining the expression: $\Pi_{\alpha\beta} = p\,\delta_{\alpha\beta} + \rho\,v_\alpha v_\beta$. Viscosity is responsible for an additional term $\tilde{\sigma}_{\alpha\beta}$ due to another irreversible momentum transfer, where non-vanishing velocity gradients are present. For a viscous fluid we get

$$\Pi_{\alpha\beta} = p\,\delta_{\alpha\beta} + \rho\,v_\alpha v_\beta - \tilde{\sigma}_{\alpha\beta} = -\sigma_{\alpha\beta} + \rho\,v_\alpha v_\beta\;, \qquad \sigma_{\alpha\beta} = -p\,\delta_{\alpha\beta} + \tilde{\sigma}_{\alpha\beta}\;, \tag{1.2.5}$$

where $\sigma_{\alpha\beta}$ is the *stress tensor* and $\tilde{\sigma}_{\alpha\beta}$ is called the *viscous stress-tensor*.

The general form of $\tilde{\sigma}_{\alpha\beta}$ can be derived by a qualitative analysis of the velocity gradients in presence of uniform rotation and volume changes of the fluid. The most general form of the viscous stress tensor is [17]

$$\tilde{\sigma}_{\alpha\beta} = \vartheta\,(\partial_\beta v_\alpha + \partial_\alpha v_\beta - \tfrac{2}{3}\,\delta_{\alpha\beta}\partial_\gamma v^\gamma) + \zeta\,\delta_{\alpha\beta}\,\partial_\gamma v^\gamma\;, \tag{1.2.6}$$

where the coefficients ϑ e ζ are not dependent on velocity (the fluid is isotropic and its properties must be described only by scalar quantities) and the term proportional to the ϑ coefficient vanishes for the contraction over α and β. Here, the coefficient ϑ is called *shear viscosity* while ζ denotes *bulk viscosity* and they are both positive quantities.

Using the Continuity Equation, the ideal fluid Euler Equation rewrites

$$\rho(\partial_t v_\alpha + v_\beta\,\partial^\beta v_\alpha) = -\partial_\alpha p\;,$$

and the motion equation of a viscous fluids can now be obtained by adding the expression $\partial^\beta\tilde{\sigma}_{\alpha\beta}$ to the RHS of the equation above, obtaining

$$\rho(\partial_t v_\alpha + v_\beta\,\partial^\beta v_\alpha) = -\partial_\alpha p + \partial^\beta[\vartheta\,(\partial_\beta v_\alpha + \partial_\alpha v_\beta - \tfrac{2}{3}\,\delta_{\alpha\beta}\partial^\gamma v_\gamma)] + \partial_\alpha(\zeta\,\partial^\gamma v_\gamma)\;.$$

The viscous coefficients are not constant and we will express their dependence on the state parameters of the fluid. If we assume ϑ to be negligible (as we will discuss later), the Euler Equation takes the following form

$$\rho\,\partial_t\mathbf{v} + \rho\,(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p\, - \zeta\,\nabla(\nabla\cdot\mathbf{v}) = 0\;, \tag{1.2.7}$$

which is the well-known *Navier-Stokes Equation*.

This analysis is developed without considering the gravitational field, which has to be introduced in the Euler Equation as usual. We have also to consider the equation describing the gravitational field itself: the *Poisson Equation*. Let us now recall the set of motion equations in the case of an adiabatic viscous fluid:

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0\;, \tag{1.2.8a}$$

$$\rho\,\partial_t\mathbf{v} + \rho\,(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p\, - \zeta\,\nabla(\nabla\cdot\mathbf{v}) + \rho\,\nabla\Phi = 0\;, \tag{1.2.8b}$$

$$\nabla^2\Phi - 4\pi G\rho = 0\;, \tag{1.2.8c}$$

such a system is the starting point to analyze the gravitational instability in the Newtonian approximation picture.

Energy-momentum tensor viscous corrections in Einstein General Relativity

In order to discuss the pure relativistic limit of the cosmological gravitational instability, let us now introduce dissipative effects in the matter source term of the Einstein Equations.

The energy-momentum tensor (EMT) of a *perfect fluid* is standardly defined as [17]:

$$T^{(P)}_{\mu\nu} = (p+\rho)u_\mu u_\nu - p\,g_{\mu\nu}\;, \tag{1.2.9}$$

where ρ is the energy density and u_μ defines the 4-velocity.

Viscous corrections generate additional terms to the above expression. The EMT of a viscous fluid reads

$$T_{\mu\nu} = (p+\rho)u_\mu u_\nu - p\,g_{\mu\nu} + \tau_{\mu\nu}\;. \tag{1.2.10}$$

Let us now discuss in some details the concept of the 4-velocity u^μ. In relativistic mechanics, an energy flux necessarily leads to a mass flux. This way, the definition of the velocity in terms of the mass-flux density has no direct meaning. The velocity is therefore defined by the condition that, in the proper frame of any fluid element, the momentum of the latter vanish and its energy can be characterized in terms of the other thermodynamical quantities by the same expression as when dissipative processes are absent. Hence, in the proper frame, the components τ_{00} and $\tau_{0\alpha}$ of the tensor $\tau_{\mu\nu}$ are zero. Since, in such a frame the 3-velocity u^{α} vanish also, we obtain, in any frame, the equation

$$\tau_{\mu\nu}\, u^{\mu} = 0 \; . \tag{1.2.11}$$

The required form of the tensor $\tau_{\mu\nu}$ can be established from the law if increasing entropy. This law must be contained in the motion equations,

$$T^{\nu}_{\mu,\,\nu} = 0 \; ,$$

in fact the condition of constant entropy enters the dynamics of an ideal fluid [18]. The Continuity Equation of the fluid results to be

$$(nu^{\mu})_{,\,\mu} = 0 \; ,$$

where n is the particle number defining the particle-flux density vector

$$n_{\mu} = n\, u_{\mu} + \upsilon_{\mu} \; , \tag{1.2.12}$$

modified for the dissipative term υ_{μ}. Using the expression above and by multiplying the motion equations for u^{μ}, one can get

$$u^{\mu}\, T^{\nu}_{\mu,\,\nu} \;=\; T(\sigma u^{\mu})_{,\,\mu} \,+\, \bar{\mu}\, \upsilon^{\mu}{}_{,\,\mu} \,+\, u^{\mu}\; \tau^{\nu}_{\mu,\,\nu} \; ,$$

where $\bar{\mu} = (\rho + p - Ts)/n$ denotes the relativistic chemical potential satisfying the thermodynamical equation

$$d\bar{\mu} = (1/n)dp - (s/n)dT \; ,$$

and s is the entropy density. Finally, using the relation $\tau_{\mu\nu}\, u^{\mu} = 0$, one can write such equation as

$$\left(su^{\mu} - \frac{\bar{\mu}}{T}\, \upsilon^{\mu}\right)_{,\,\mu} = -\upsilon^{\mu} \left(\frac{\bar{\mu}}{T}\right)_{,\,\mu} + \frac{\tau^{\nu}_{\mu}}{T}\, u^{\mu}{}_{,\,\nu} \; . \tag{1.2.13}$$

The LHS term must be the 4-divergence of the entropy flux and the RHS term the increase in entropy due to dissipative effects. The entropy-flux density vector writes

$$s^{\mu} = su^{\mu} - (\bar{\mu}/T)\, \upsilon^{\mu} \; ,$$

and $\tau_{\mu\nu}$ and υ^μ must be linear functions of the gradients of velocity and thermodynamical quantities, such as to make the RHS of eq. (1.2.13) necessarily positive. This condition, together with eq. (1.2.11), uniquely defines the form of the symmetric tensor $\tau_{\mu\nu}$:

$$\begin{aligned}\tau_{\mu\nu} = &-\vartheta\big(u_{\mu,\,\nu} + u_{\nu,\,\mu} - u_\nu u^\sigma u_{\mu,\,\sigma} - u_\mu u^\sigma u_{\nu,\,\sigma}\big) + \\ &+ \big(\zeta - \tfrac{2}{3}\vartheta\big) u^\sigma_{\ ,\,\sigma}\big(g_{\mu\nu} - u_\mu u_\nu\big) \ .\end{aligned} \tag{1.2.14}$$

Here ϑ and ζ are the viscosity coefficients as in the non-relativistic analysis. In this limit, the $\tau_{\alpha\beta}$ components reduce to the ones of the 3-dimensional viscous stress-tensor $\tilde{\sigma}_{\alpha\beta}$.

The viscous corrections are derived by using a flat Minkowskian metric. Let us now apply the *General Covariance Principle* [19] and write the EMT in presence of a gravitational field, *i.e.*, in curved space-time. Hence, we replace the ordinary derivative with the covariant ones and we consider the generic metric tensor $g_{\mu\nu}$. The EMT rewrites now

$$\begin{aligned}T_{\mu\nu} = (p+\rho)\, u_\mu u_\nu - p\, g_{\mu\nu} + (\zeta - \tfrac{2}{3}\,\vartheta)\, u^\rho_{\ ;\,\rho}\, (g_{\mu\nu} - u_\mu u_\nu) + \\ +\vartheta\, (u_{\mu;\,\nu} + u_{\nu;\,\mu} - u_\nu u^\rho u_{\mu;\,\rho} - u_\mu u^\rho u_{\nu;\,\rho}) \ .\end{aligned} \tag{1.2.15}$$

As soon as the shear viscosity ϑ can be negligible, $T_{\mu\nu}$ assumes a simplified form in terms of the so-called *bulk pressure* $\tilde{p}$:

$$T_{\mu\nu} = (\tilde{p} + \rho) u_\mu u_\nu - \tilde{p}\, g_{\mu\nu} \ , \tag{1.2.16a}$$

$$\tilde{p} = p - \zeta\, u^\rho_{\ ;\,\rho} \ . \tag{1.2.16b}$$

It is worth noting that the effect of bulk viscosity is to generate an negative pressure term beside the thermostatic one.

Characterization of the bulk viscosity The viscous effects discussed in the previous paragraphs are summarized by two different kind of viscosity: shear viscosity ϑ and bulk viscosity ζ. The Newtonian motion equations and the viscous EMT source, introduced above, describe the unperturbed dynamics on which develop a first-order perturbative theory in order to study the evolution of small fluctuations generated on the background.

As already discussed, in this work, we are aimed to analyze isotropic (or almost isotropic) and homogeneous cosmological models. In this respect, we can safely neglect shear viscosity in the unperturbed dynamics. In fact, in such models, there is no displacement of matter layers WRT each other, in the zeroth-order motion, and this kind of viscosity represents the energy dissipation due to this effect. Indeed, in presence of small inhomogeneities, such effects should be taken into account, in principle. However, in this work, we are aimed to studying the behavior of scalar density perturbations, in order to analyze the evolution of the density contrast. In this respect, volume changes of a given mass scale are essentially involved and, therefore, we concentrate our attention to bulk viscosity effects only.

In fact, we expect that the non-equilibrium dynamics of matter compression and rarefaction is more relevant than friction among the different layers and bulk viscosity outcomes as a phenomenological issue inherent to the difficulty for a thermodynamical system to follow the equilibrium configuration. It is worth noting that such viscous contributions can not dominate the fluid evolution because of their thermodynamical perturbative origin. Nevertheless, we are interested in those regimes where such effects are not at all negligible.

We underline that, in the pure relativistic analysis, we will fix our attention on the relevance of dealing with bulk viscous properties of the cosmological fluid approaching the Big Bang. Since, asymptotically near the singularity, the volume of the Universe has a very fast time variation, we expect the bulk viscous effects naturally arise in the dynamics. Furthermore, a detailed discussion regarding the motivation for neglecting shear viscosity, in the relativistic regime, is addressed in Section 1.6.1.

The bulk-viscosity coefficient ζ is assumed to be not constant and we want now to discuss how to express its dependence on the state parameters of the fluid. The presence of non-equilibrium phenomena during the fluid volume-expansion can be phenomenologically described by such kind of viscosity. It is worth noting that the analysis of a microphysics, unable to follow the expansion by equilibrium stages, is a very intriguing but complicated problem. In particular, a pure kinetic theory approach [20, 2, 21] concerning the cosmological fluid and the results describing viscosity become not applicable. In this respect, we follow the line of the fundamental viscous-cosmology analysis due to the "Landau School" [22, 23, 24], implementing the so-called *hydrodynamical description* of the fluid. Hence, we assume that an arbitrary state is consistently characterized by the particle-flow vector and the EMT alone [25] and viscosity is fixed by the macroscopic parameters which govern the system evolution.

In the homogeneous models ζ depends only on time, and therefore, the most natural choice is to take it as a power-law in the energy density of the fluid (for a detailed discussions see [22]). Using such a phenomenological assumption, we express bulk viscosity in the from

$$\zeta = \zeta_0\, \rho^s \,, \tag{1.2.17}$$

where ζ_0 is a constant parameter, which defines the intensity of the viscous effects, and s is a dimensionless constant.

1.3 Analysis of the Jeans Mechanism in presence of viscous effects

The Universe is uniform at big scales but many concentrations are presented at small ones, *e.g.*, galaxies and clusters, where the mass density is larger than the Universe mean-density. These mass agglomerates are due to the gravitational instability: if density perturbations are generated in a certain volume, the gravitational forces act contracting this volume, allowing a gravitational collapse. The only forces which contrast such gravitational contraction are the pressure ones, which act in order to maintain uniform the energy density.

The Jeans Mechanism analyzes what are the conditions for which density perturbations become unstable to the gravitational collapse. In particular a threshold value for the perturbation mass is founded: the so-called *Jeans Mass*. If density fluctuations of mass greater than the Jeans one are addressed, they asymptotically diverge in time generating the gravitational collapse.

In what follows, we generalize such a model including the effects of bulk viscosity into the dynamics[1] in order to analyze the perturbation evolution and possible modification to the Jeans Mass. As results, the fluctuation dynamics is founded to be damped by viscous processes and the top-down mechanism of structure formation is suppressed. In such a scheme, the Jeans Mass remain unchanged also in presence of viscosity.

The Jeans Model [7] is based on a Newtonian approach and the effects of the expanding Universe are neglected. The fundamental hypothesis of such an analysis is a static and uniform solution for the zeroth-order dynamics

$$\mathbf{v}_0 = 0 \;, \qquad \rho_0 = const. \;, \qquad p_0 = const. \;, \qquad \Phi_0 = const. \tag{1.3.1}$$

Of course, this assumption contradicts the Poisson Equation, but we follow the original Jeans analysis imposing the so-called "Jeans swindle" [8, 2]. We underline that our study will focus on Universe stages when the mean density is very small: in particular the recombination era, after decoupling. This way, the effects of bulk viscosity on the unperturbed dynamics can be consistently neglected in view of its phenomenological behavior.

[1] NC and G. Montani, "Jeans Instability in Presence of Viscous Effects", submitted to *Int. J. Mod. Phys. D*, Nov. 2008.

Review on the Jeans Model In the standard Jeans Model [7], a perfect fluid background is assumed. Setting $\zeta = 0$ in the Newtonian motion equations (1.2.8), one gets the following system

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \, , \tag{1.3.2a}$$

$$\rho \, \partial_t \mathbf{v} + \rho \, (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p + \rho \, \nabla \Phi = 0 \, , \tag{1.3.2b}$$

$$\nabla^2 \Phi - 4\pi G \rho = 0 \, . \tag{1.3.2c}$$

Let now add small fluctuations to the unperturbed solution (1.3.1):

$$\rho = \rho_0 + \delta\rho \, , \qquad p = p_0 + \delta p \, , \qquad \Phi = \Phi_0 + \delta\Phi \, , \qquad \mathbf{v} = \mathbf{v}_0 + \delta\mathbf{v} \, . \tag{1.3.3}$$

Furthermore, only adiabatic perturbations are treated and the sound speed is defined as $v_s^2 = \delta p / \delta\rho$. Substituting such expressions in the system (1.3.2), and neglecting second-order terms, one gets the following set of equations

$$\partial_t \delta\rho + \rho_0 \, \nabla \cdot \delta\mathbf{v} = 0 \, , \tag{1.3.4a}$$

$$\rho_0 \, \partial_t \delta\mathbf{v} + v_s^2 \, \nabla \delta\rho + \rho_0 \, \nabla \delta\Phi = 0 \, , \tag{1.3.4b}$$

$$\nabla^2 \delta\Phi - 4\pi G \delta\rho = 0 \, . \tag{1.3.4c}$$

After standard manipulation, one differential equation for the density perturbations can be derived:

$$\partial_t^2 \delta\rho - v_s^2 \, \nabla^2 \delta\rho = 4\pi G \, \rho_0 \, \delta\rho \, . \tag{1.3.5}$$

To study the properties of $\delta\rho$, we now consider plane-wave solutions of the form

$$\delta\rho \, (\mathbf{r}, t) = A \, e^{i\omega t - i\mathbf{k}\cdot\mathbf{r}} \, , \tag{1.3.6}$$

where ω and $\mathbf{k}$ ($k = |\mathbf{k}|$) are the angular frequency and the wave number, respectively. This way, one can obtain the following dispersion relation

$$\omega^2 = v_s^2 k^2 - 4\pi G \, \rho_0 \, . \tag{1.3.7}$$

In this scheme, two different regimes are present: if $\omega^2 > 0$ a pure time oscillatory-behavior for density perturbations is obtained. While if $\omega^2 < 0$, the fluctuations exponentially grow in time, in the $t \to \infty$ asymptotic limit (*i.e.*, we choose the negative imaginary part of the angular-frequency solution) and the gravitational collapse is addressed since also the density contrast $\delta = \delta\rho/\rho_0$ diverges. The condition $\omega^2 = 0$

defines the so-called *Jeans Scale* K_J and the Jeans Mass M_J (which is the total mass in a sphere of radius $R = \pi/K_J$). Such threshold quantities read

$$K_J = \rho_0 \sqrt{\frac{4\pi G \rho_0}{v_s^2}} \;, \qquad M_J = \frac{4\pi}{3}\left(\frac{\pi}{K_J}\right)^3 \rho_0 = \frac{\pi^{5/2}\, v_s^3}{6G^{3/2}{\rho_0}^{1/2}} \;. \tag{1.3.8}$$

Let us now analyze in some details the two regimes.

(1.) In the case $M < M_J$ (*i.e.*, $\omega^2 > 0$), $\delta\rho$ behave like two progressive sound waves, with constant amplitude, propagating in the $\pm\mathbf{k}$ directions with velocity

$$v_w = v_s \sqrt{(1 - (K_J/k)^2)} \;. \tag{1.3.9}$$

In the limit $k \to \infty$, the propagation velocity approaches the value v_s, and fluctuations behave like pure sound waves. On the other hand, if $k \to K_J$, stationary waves are addressed (*i.e.*, $v_w = 0$).

(2.) In the case $M > M_J$ (*i.e.*, $\omega^2 < 0$), density perturbations evolve like stationary waves with a time dependent amplitude. In particular, choosing the negative imaginary part of the solution for ω, the wave amplitude exponentially explodes, generating the gravitational collapse.

Jeans Mechanism in presence of bulk viscosity Let us now analyze how viscosity can affect the gravitational-collapse dynamics. As already discussed, the only viscous process we address in an homogeneous and isotropic model is bulk viscosity and we are able to neglect such kind of viscosity in the unperturbed dynamics, which results to be described by the static and uniform solution (1.3.1).

We now start by adding the usual small fluctuations to such a solution, as eq. (1.3.3), and in treating bulk-viscosity perturbations, we use the expansion $\zeta = \bar{\zeta} + \delta\zeta$ where

$$\bar{\zeta} = \zeta(\rho_0) = \zeta_0 \rho_0^s = const. \;, \qquad \delta\zeta = \delta\rho\; (\partial\zeta/\partial\rho) + ... = \zeta_0\, s\, \rho_0^{s-1}\, \delta\rho\; + ... \;. \tag{1.3.10}$$

Substituting all fluctuations in the Newtonian motion equations (1.2.8):

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \;, \tag{1.3.11a}$$

$$\rho\, \partial_t \mathbf{v} + \rho\, (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p \,-\, \zeta\, \nabla(\nabla \cdot \mathbf{v}) + \rho\, \nabla\Phi = 0 \;, \tag{1.3.11b}$$

$$\nabla^2\Phi - 4\pi G \rho = 0 \;, \tag{1.3.11c}$$

we get the first-order motion equations of the model

$$\partial_t \delta\rho + \rho_0 \nabla \cdot \delta\mathbf{v} = 0 \;, \qquad (1.3.12a)$$

$$\rho_0\, \partial_t \delta\mathbf{v} + v_s^2\, \nabla\delta\rho + \rho_0\, \nabla\delta\Phi - \bar\zeta\, \nabla(\nabla\cdot\delta\mathbf{v}) = 0 \;, \qquad (1.3.12b)$$

$$\nabla^2\delta\Phi - 4\pi G\, \delta\rho = 0 \;. \qquad (1.3.12c)$$

With some little algebra, one can obtain an unique equation for density perturbations, describing the dynamics of the gravitational collapse:

$$\rho_0\, \partial_t^2\delta\rho - \rho_0\, v_s^2 \nabla^2\, \delta\rho - \bar\zeta\, \nabla^2\, \partial_t\delta\rho = 4\pi G \rho_0^2\, \delta\rho \;. \qquad (1.3.13)$$

Using the linearity of the equation above, a decomposition in Fourier expansion can be performed. This way, plane waves solutions (1.3.6) can be addressed, obtaining a generalized dispersion relation

$$\rho_0\, \omega^2 - \, i\, \bar\zeta\, k^2\, \omega + \rho_0 (4\pi G\rho_0 - v_s^2 k^2) = 0 \;. \qquad (1.3.14)$$

As in the standard Jean Model, the nature of the angular frequency is responsible of two different regimes for the density-perturbation evolution. The dispersion relation has the solution

$$\omega = i\, \frac{\bar\zeta k^2}{2\,\rho_0} \pm \sqrt{\bar\omega} \;, \qquad \bar\omega = -\frac{k^4\bar\zeta^2}{4{\rho_0}^2} + v_s^2 k^2 - 4\pi G\rho_0 \;, \qquad (1.3.15)$$

thus we obtain the time exponential-regime for $\bar\omega \leqslant 0$ and a damped oscillatory regime for $\bar\omega > 0$. It's worth noting that the pure oscillatory regime of the ideal fluid Jeans Mechanism is lost. The equation $\bar\omega = 0$ admits the solutions K_1 and K_2 which read

$$K_{\frac{1}{2}} = \frac{\sqrt{2}\,\rho_0 v_s}{\bar\zeta} \Big(1 \mp \sqrt{1 - \Big(\frac{K_J\bar\zeta}{\rho_0 v_s}\Big)^2}\;\Big)^{\frac{1}{2}} \;, \qquad K_1, K_2 > 0, \quad K_1 < K_2 \;. \qquad (1.3.16)$$

The existence of such solutions gives rise to a constraint on the viscosity coefficient:

$$\bar\zeta \leqslant \zeta_c = \rho_0 v_s / K_J \;. \qquad (1.3.17)$$

An estimation in the recombination era[2] after decoupling, yields to the value

$$\zeta_c = 7.38 \cdot 10^4\; g\, cm^{-1}\, s^{-1} \;,$$

[2]The parameters are set as follows: the usual barotropic relation $p = c^2 \rho_0^{\gamma} / \tilde\rho^{\gamma-1}$ is assumed and the constant $\tilde\rho$ can be derived from the expression expression M_J (1.3.8). Universe is dominated by matter and we can impose the values: $M_J \sim 10^6 M_\odot$, $\gamma = 5/3$, $\rho_c = 1.879\, h^2 \cdot 10^{-29}\; g\, cm^{-3}$, $h = 0.7$, $z = 10^3$ and $\rho_0 = \rho_c\, z^3 = 0.92 \cdot 10^{-20}\; g\, cm^{-3}$. Using these quantities one finds $\tilde\rho = 9.034 \cdot 10^{-7}\; g\, cm^{-3}$, $v_s = 8.39 \cdot 10^5\; cm\, s^{-1}$ and the threshold value ζ_c.

and confronting this threshold with usual viscosity (*e.g.*, $\bar{\zeta}^{Hydr.} = 8.4{\cdot}10^{-7} g\, cm^{-1}\, s^{-1}$), we can conclude that the range $\bar{\zeta} \leqslant \zeta_c$ is the only of physical interest. Finally we obtain: $\bar{\omega} \leqslant 0$ for $k \leqslant K_1$, $K_2 \leqslant k$ and $\bar{\omega} > 0$ for $K_1 < k < K_2$.

Let us now analyze the density-perturbation exponential solutions in correspondence of $\bar{\omega} \leqslant 0$:

$$\delta\rho \sim e^{w\,t}\;, \qquad w = -\frac{\bar{\zeta} k^2}{2\rho_0} \mp \sqrt{-\bar{\omega}}\;. \tag{1.3.18}$$

To obtain the structure formation, the amplitude of such stationary waves must grow for increasing time. The exponential collapse for $t \to \infty$ is addressed, choosing the (+)-sign solution, only if $w > 0$, *i.e.*, $k < K_J$ with $K_J < K_1 < K_2$. As a result, we show how the structure formation occurs only if $M > M_J$, as in the standard Jeans Model.

The viscous effects do not alter the threshold value of the Jeans Mass, but they change the perturbation evolution and the pure oscillatory behavior is lost in presence of dissipative effects. In particular, we get two distinct decreasing regimes:

(1.) For $K_1 < k < K_2$ (*i.e.*, $\bar{\omega} > 0$), we obtain a damped oscillatory evolution of perturbations:

$$\delta\rho \sim e^{-\frac{\zeta_0 k^2}{2\rho_0}\,t}\,\cos\left(\sqrt{\bar{\omega}}\,t\right)\,, \tag{1.3.19}$$

(2.) For $K_J < k < K_1$ and $K_2 < k$, density perturbations exponentially decrease as

$$\delta\rho \sim e^{w\,t}\;, \qquad w < 0\;, \tag{1.3.20}$$

in the limit $t \to \infty$.

1.3.1 Implication for the top-down mechanism

As shown above, since the pure oscillatory regime does not occurs, we deal with a decreasing exponential or a damped oscillatory evolution of perturbations. This allows to perform a qualitative analysis of the top-down fragmentation scheme [1], *i.e.*, the comparison between the evolution of two structures: one collapsing agglomerate with $M \gg M_J$ and an internal non-collapsing sub-structure with $M < M_J$. If this picture is addressed, the sub-structure mass must be compared with a decreasing Jeans Mass since the latter is inversely proportional to the collapsing agglomerate background mass. This way, as soon as such a Jeans Mass reaches the sub-structure one, the latter begins to condense implying the fragmentation. In the standard Jeans Model, this mechanism is always allowed since the amplitude for perturbations characterized by $M < M_J$ remains constant in time. On the other hand, the presence of decreasing fluctuations in the viscous model requires a discussion on the effective damping and an the efficacy of the top-down mechanism. Of course, such an analysis contrasts the hypothesis of a constant background density, but it can be useful to estimate the strength of the dissipative effects.

We now study two cases for different values of the bulk-viscosity coefficient: $\bar{\zeta} \ll 1$ and $\bar{\zeta} > 1$. In this analysis, a perturbative validity-limit has to be set: we suppose $\delta\rho/\rho_0 \sim 0.01$ as the limit of the model and we use the recombination era parameters (see footnote [2]), in particular the initial time of the collapse is defined as the beginning of the matter-dominated Universe, *i.e.*, $t_0 = t_{MD} = 1.39 \cdot 10^{13}\, s$.

(1.) In correspondence of a very small viscosity coefficient, Fig.1, we consider

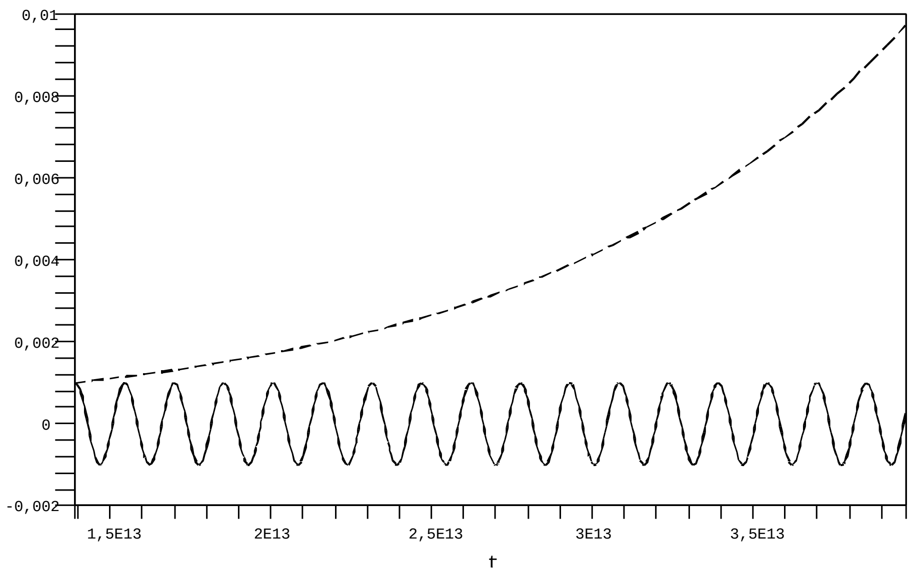

Figure 1.1: Case $\bar{\zeta} = 10^{-5}\, g\, cm^{-1}\, s^{-1}$. Galaxy density contrast: δ_G - $M_G = 10^{12}\, M_\odot$ - (dashed line). Sub-structure density contrast δ_S - $M_S = 10\, M_\odot$ - (normal line).

a decreasing structure of mass $M_S = 10\, M_\odot$ within a collapsing galaxy with mass $M_G = 10^{12}\, M_\odot$, the Jeans Mass is $M_J = 10^6\, M_\odot$. The sub-structure wave number K_S is in the region $K_1 < K_S < K_2$ and density perturbations evolve like eq. (1.3.19). Fluctuations have to be imposed small at the initial time t_0, this way, we consider density contrasts (δ_G for the galaxy and δ_S for the sub-structure) of $\mathcal{O}(10^{-3})$. In this scheme, the galaxy starts to collapse and the validity limit is reached at $t^* = 6.25 \cdot 10^{13}$.

As a result, in Fig.1 we can show how the sub-structure survives in the oscillatory regime during the background collapse until the threshold time value t^*. Thus, we can conclude that, if the viscous damping is sufficiently small, the top-down fragmentation can occur.

(2.) Let us now discuss the case $\bar{\zeta} > 1$, Fig.2, by changing the sub-structure mass, which is now $M_S = M_\odot$. Here, the viscosity coefficient is greater than one and the

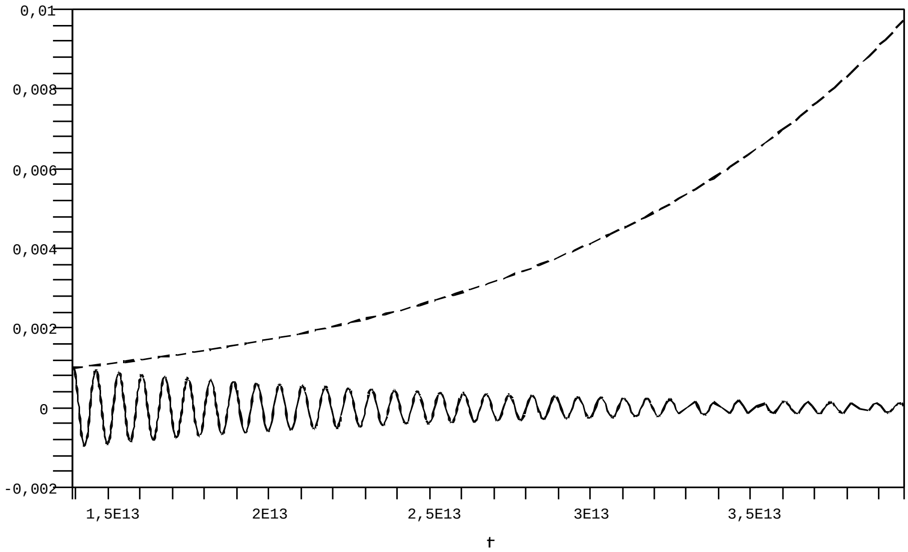

Figure 1.2: Case $\bar{\zeta} = 14\, g\, cm^{-1}\, s^{-1}$. Galaxy density contrast: δ_G - $M_G = 10^{12}\, M_\odot$ - (dashed line). Sub-structure density contrast δ_S - $M_S = M_\odot$ - (normal line).

damping effects is stronger. In fact, when the galaxy density-contrast reaches the threshold value $\delta_G = 0.1$, we obtain $\delta_S = 10^{-5}$. The top-down mechanism for structure formation results to be unfavored by the presence of strong viscous effects: the damping becomes very strong and the sub-structure vanishes during the agglomerate evolution.

1.4 Newtonian spherically symmetric gas cloud fragmentation

In a work by C. Hunter [26], a specific model for a gas-cloud fragmentation was addressed and the behavior of sub-scales density perturbations, outcoming in the extreme collapse, was analytically described. The hypotheses on which this model is based are the homogeneity and the spherical symmetry, respectively, of the collapsing cloud that starts at rest its fall. Furthermore, it is assumed that pressure forces are negligible in the unperturbed dynamics and therefore a real notion of Jeans mass is not required in this approach. According to this scheme, the Lagrangian and Eulerian formulations of the zeroth- and first-order dynamics are developed, respectively. The result of this analysis shows that the density contrast grows, approaching the singularity, inducing a fragmentation process of the basic flow. It is outlined that first-order pressure effects do not influence the perturbations behavior considering an isothermal-like politropic index γ (*i.e.* for $1 \leqslant \gamma < 4/3$). On the other hand, such effects increase as γ runs from $4/3$ to the adiabatic value $5/3$. In particular, the case $\gamma = 5/3$, represents an exception being characterized by a density contrast which remains constant asymptotically to the singularity.

In what follows, we investigate how the above picture is modified by including, in the gas-cloud dynamics, the presence of bulk-viscosity effects[3]. In this respect, we generalize the Lagrangian evolution by taking into account the force acting on the collapsing shell as a result of the negative pressure connected to the viscosity. We construct such an extension requiring that the asymptotic dynamics of the collapsing cloud is not qualitatively affected by the presence of viscosity [27]. In particular, we analytically integrate the dynamics in correspondence to the constitutive equation for the viscosity coefficient (1.2.17), where the exponent is assumed to be $s = 5/6$. Then, we face the Eulerian motion of the inhomogeneous perturbations living within the cloud. The resulting viscous dynamics is treated in the asymptotic limit to the singularity. As a result, we show that the density contrast behaves, in the isothermal-like collapse, as in the non-viscous case. On the other hand, the perturbation damping increases monotonically as γ runs from $4/3$ to $5/3$. In fact, for such adiabatic-like

[3] NC and G. Montani, "Gravitational Stability and Bulk Cosmology", *AIP Conf. Proc.* **966**, 241 (2007).

case, we show that the density contrast asymptotically vanishes and no fragmentation processes take place in the cloud, when the viscous corrections are sufficiently large. In particular, we observe the appearance of a threshold value for the scale of the collapsing perturbations depending on the values taken by the parameters ζ_0 and $\gamma \in (4/3, 5/3]$; such a viscous effect corresponds to deal with an analogous of the Jeans Length, above which perturbations are able to collapse. However such a threshold value does not ensure the diverging behavior of density contrast which takes place, in turn, only when a second (greater) critical length is overcome.

Since, in the extreme collapse, it is expected that viscous processes are relevant, our analysis suggests that the top-down scheme of structure formation can be deeply influenced when non-equilibrium features of the dynamics arise. According to our study, if such viscous effects are sufficiently intense, the final system configuration is not a fragmented cloud as a cluster of sub-structures but simply a single object (a black hole, in the present case, because pressure forces are assumed negligible). Furthermore, we discuss why the choice $s = 5/6$ has a physical meaning in the viscous dynamics: we show that for $s > 5/6$ the background evolution would be asymptotically affected by viscosity which would acquire a non-perturbative character. On the other hand, for $s < 5/6$ no modifications occur WRT the dynamics of the non-viscous density contrast.

Review on the non-viscous cloud fragmentation We recall here the hydrodynamical analysis of a spherically symmetric non-viscous gas-cloud collapse. This model was firstly proposed by C. Hunter in 1962 [26] where he supposed that the gas cloud becomes unstable WRT its own gravitation and begin to condense. The collapsing cloud is assumed to be the dynamical background on which studying, in a Newtonian regime, the evolution of density perturbations generated on this basic flow. In the Hunter model, the unperturbed flow was supposed to be homogeneous, spherically symmetric and initially at rest. Furthermore the gravitational forces are assumed to be very much greater than the pressure ones, which are therefore neglected in the zeroth-order analysis. In such an approach, the gas results to be unstable since there are no forces which can contrast the collapse, and the condensation starts immediately.

The basic flow is governed by the Lagrangian motion equation of a spherically symmetric gas distribution which collapses under the only gravitational action. Assuming that the initial density of the cloud is constant in space, the dynamics reads

$$\frac{\partial^2 r}{\partial t^2} = -\frac{GM}{r^2} \ , \tag{1.4.1}$$

where the origin O is taken at the center of the gas, r is the radial distance, G the gravitational constant and M the mass of the gas inside a sphere of radius r. In what follows, we shall suppose that the gas was at a distance a from O in correspondence to the initial instant t_0; this distance a identifies a fluid particle and will be used as a Lagrangian independent variable so $r = r(a,t)$. Provided that particles do not pass trough each other, the mass M inside a sphere of radius r is not time dependent and is a function of a only; using the integral form

$$M(a) = \int_0^r dr'\, 4\pi\rho r'^2 = \int_0^a da'\, 4\pi\rho^*(a')a'^2 \ , \tag{1.4.2}$$

where ρ is the gas cloud density and $\rho^* = \rho(t_0)$ the initial one, we get the relation

$$\rho r^2 \frac{\partial r}{\partial a} = \rho^* a^2 \ . \tag{1.4.3}$$

A first integration of (1.4.1) yields the expression of the radial velocity $v_0 = \partial r/\partial t$, which reads

$$v_0 = -[2GM\,(1/r - 1/a)]^{\frac{1}{2}} \ , \tag{1.4.4}$$

where we considered the negative solution in order to obtain a collapse. Let us now introduce the parametrization

$$r = a\cos^2\beta \ , \tag{1.4.5}$$

where $\beta = \beta(t)$ is a time-dependent function such that $\beta(t_0) = 0$ and $\beta(0) = \pi/2$, since we choose the origin of time to have $t = 0$ when $r = 0$ and t_0 takes negative values. We assume ρ_0 to be uniform and we are now able to integrate eq. (1.4.4) to get the following relation between β and t and the expression of the initial time t_0:

$$\beta + \tfrac{1}{2}\sin 2\beta = \tfrac{\pi}{2} + t\sqrt{\tfrac{8}{3}\pi\rho_0 G} \ , \tag{1.4.6}$$

$$t_0 = -\sqrt{3\pi\,/\,32\rho_0 G} \ . \tag{1.4.7}$$

It is more convenient to use an Eulerian representation of the flow field. To this end, using relation (1.4.5) and (1.4.3), we obtain the unperturbed radial velocity v_0 and the basic flow density ρ_0, respectively. Furthermore, solving the Poisson Equation for the gravitational potential Φ, we get the unperturbed solutions describing the background motion; all these quantities take the explicit forms

$$\mathbf{v}_0 = [v_0, 0, 0] \, , \qquad v_0 = -2r\dot{\beta}\tan\beta \, , \tag{1.4.8a}$$

$$\rho_0 = \bar{\rho}\cos^{-6}\beta \, , \tag{1.4.8b}$$

$$\Phi_0 = -2\pi\bar{\rho}G\left(a^2 - r^2/3\right)\cos^{-6}\beta \, , \tag{1.4.8c}$$

where the non-radial components of velocity must vanish since we are considering a spherical symmetry.

The first-order perturbations to the basic flow (higher orders analysis was made by Hunter in two later articles [28, 29]) are investigated in the non-viscous Newtonian limit of system (1.2.8), *i.e.*,

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0 \, , \tag{1.4.9a}$$

$$\rho\,\partial_t\mathbf{v} + \rho\,(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p + \rho\,\nabla\Phi = 0 \, , \tag{1.4.9b}$$

$$\nabla^2\Phi - 4\pi G\rho = 0 \, . \tag{1.4.9c}$$

The gas is furthermore assumed to be barotropic, *i.e.*, the pressure depends only by the background density ρ_0. In this model, zeroth-order solutions (1.4.8) are already verified since the pressure gradient, in the homogeneity hypothesis, vanishes and the pressure affects only the perturbative dynamics.

Let us now investigate first-order fluctuations around unperturbed solutions, *i.e.*, we replace the perturbed quantities: $(\mathbf{v}_0 + \delta\mathbf{v})$, $(\rho_0 + \delta\rho)$, $(\Phi_0 + \delta\Phi)$ and also $(p_0 + \delta p)$ where $p_0 = p\,(\rho_0)$. Substituting these solutions in eq. (1.4.9b) and taking the **rot** of the final expression, one gets, linearizing in the perturbed quantities, an equation for the vorticity $\delta\mathbf{w} = \nabla\times\delta\mathbf{v}$, which stands

$$\dot{\delta\mathbf{w}} = -\nabla\times[\delta\mathbf{w}\times\mathbf{v}] \, . \tag{1.4.10}$$

Using spherical coordinates $[r(a,t), \theta, \varphi]$ we are able to build the solutions for the three components of the vorticity, getting

$$\delta\mathbf{w} = \left[\, l\,\cos^{-4}\beta + h \, , \quad m\,\cos^{-4}\beta \, , \quad n\,\cos^{-4}\beta \,\right] \, . \tag{1.4.11}$$

Here l, m, n are arbitrary functions of the new variables $[a, \theta, \varphi]$ (the radial coordinate transforms like (1.4.5)) which must satisfy the relation $\nabla\cdot\delta\mathbf{w} = 0$ [4] and h is physically

[4] We remember that in any coordinates system the relation $div\,\mathbf{rot} \equiv 0$ stands.

irrelevant since it represents a static distribution of the $\delta\mathbf{w}$ first component in the space.

We are now able to find a solution for the perturbed velocity using the vorticity expression; one can always consider a solution of the form

$$\delta\mathbf{v} = \mathbf{V}\cos^{-2}\beta + \nabla\Psi \ , \tag{1.4.12}$$

where Ψ, $\mathbf{V}$ are arbitrary functions of the coordinates and for $\mathbf{V}$ we assume a restriction gives by the relation $\nabla\cdot\mathbf{V} = 0$.

Let us now write three equations for the perturbed quantities $\delta\mathbf{v}$, $\delta\rho$ and $\delta\Phi$. Substituting last expression for the velocity fluctuations in the Newtonian system and eliminating the variable r trough the relation (1.4.5) we get

$$\partial_t\delta\rho - 6\,\delta\rho\,\dot{\beta}\tan\beta + \bar{\rho}\cos^{-10}\beta\; D^2\Psi = 0 \ , \tag{1.4.13a}$$

$$\dot{\Psi} + \delta\Phi + \frac{v_s^2}{\bar{\rho}}\cos^6\beta\,\delta\rho = 0 \ , \tag{1.4.13b}$$

$$D^2\delta\Phi - 4\pi G\cos^4\beta\,\delta\rho = 0 \ . \tag{1.4.13c}$$

Here time differentiation is taken at some fixed co-moving radial coordinate, v_s is the sound speed given by $v_s^2 = \delta p/\delta\rho$ and D^2 is the Laplace operator as written in co-moving coordinates.

A single second-order differential equation for $\delta\rho$ can be obtained from the set of eqs. (1.4.13). This final equation is as follow

$$\partial_t\left(\cos^{10}\beta\;\partial_t\delta\rho - 6\;\sin\beta\;\cos^9\beta\,\dot{\beta}\,\delta\rho\right) - 4\pi G\bar{\rho}\;\cos^4\beta\,\delta\rho = {v_s}^2\;\cos^6\beta\;D^2\delta\rho \ . \tag{1.4.14}$$

In order to study the temporal evolution of density perturbations, we assume to expand $\delta\rho$ in plane waves of the form

$$\delta\rho(\mathbf{r},t) = \rho_1(t)\;e^{-i\mathbf{k}\cdot\mathbf{r}} \ , \tag{1.4.15}$$

where $1/k$ (with $k = |\mathbf{k}|$) represents the initial length scale of the considered fluctuation. We shall now express the thermostatic pressure as a function of the basic-flow density by using the barotropical law

$$p_0 = \kappa\,\rho_0^\gamma \ , \tag{1.4.16}$$

where κ, γ are constants and $1 \leqslant \gamma \leqslant 5/3$. By this expression, we are able to distinguish a set of different cases related to different values of the politropic index

γ. The asymptotic value $\gamma = 1$ represents an isothermal behavior of the gas cloud and corresponds to a constant sound speed v_s; the case $\gamma = 5/3$ describes, instead, an adiabatic behavior and it will be valid when changes are taking place so fast that no heat is transferred between elements of the gas. We can suppose that intermediate values of γ will describe intermediate types of behavior between the isothermal and adiabatic ones.

The temporal evolution of density perturbations is governed by eq. (1.4.14); this equation can not be solved in general but we can determine the asymptotic behavior of solutions for the final part of the collapse as $(-t) \to 0$. In this limit, we develop up to the first-order the eq. (1.4.6) which, once integrated, gives the time dependence of the parameter β. For small β, we are able to approximate $sin\beta \approx 1$ in order to obtain the relation

$$\cos\beta^3 = \sqrt{6\pi G\bar{\rho}}\ \ (-t)\ . \tag{1.4.17}$$

In this approach, we determine the asymptotic temporal evolution of the basic-flow unperturbed density (1.4.8b) which now reads

$$\rho_0 \sim (-t)^{-2}\ . \tag{1.4.18}$$

Substituting this expression in eq. (1.4.14), together with eqs. (1.4.15) and (1.4.16), we get the following asymptotic equation for the final part of the collapse

$$(-t)^2\,\ddot{\rho}_1 - \frac{16}{3}(-t)\,\dot{\rho}_1 + \left[4 + \frac{v_0^2\,k^2\,(-t)^{8/3-2\gamma}}{(6\pi G\bar{\rho})^{\gamma-1/3}}\right]\ \rho_1 = 0\ , \tag{1.4.19}$$

where $v_0^2 = \kappa\gamma\bar{\rho}^{\,\gamma-1}$. A complete solution of this equation involves Bessel functions and reads

$$\rho_1 = (-t)^{-13/6}\ \left[C_1\,J_n\big[q(-t)^{4/3-\gamma}\big]\ + C_2\,Y_n\big[q(-t)^{4/3-\gamma}\big]\right]\ , \tag{1.4.20}$$

where the parameters n and q are

$$n = 5/\ 6(4/3-\gamma)\ , \tag{1.4.21a}$$

$$q = -v_0 k(6\pi\bar{\rho}G)^{1/6-\gamma/2}\ /\,(4/3-\gamma)\ . \tag{1.4.21b}$$

In order to study the asymptotic evolution of this solution, we shall analyze the cases $1 \leqslant \gamma < 4/3$ and $4/3 < \gamma \leqslant 5/3$ separately, since Bessel functions have different limits connected to the magnitude of their argument. In the asymptotic limit to the singularity, the isothermal-like case is characterized by a positive time exponent inside Bessel functions so $qt^{4/3-\gamma} \ll 1$, on the other hand, in the adiabatic-like behavior we obtain $qt^{4/3-\gamma} \gg 1$.

(1.) **Isothermal-like Case**: For $1 \leqslant \gamma < 4/3$ Bessel functions J and Y behave like power-laws of the form $J_n(x) \sim x^{+n}$, $Y_n(x) \sim x^{-n}$, for $x \ll 1$. By this approximation, the solution (1.4.20) assumes the the following asymptotic form

$$\rho_1^{ISO} \sim (-t)^{-3} \,, \tag{1.4.22}$$

which holds for all the isothermal-like γ values. This result implies that density perturbations grow to infinity as $(-t) \to 0$. Let us now study the asymptotic behavior of the density contrast $\delta = \rho_1/\rho$. It is immediate to see that for all the values of γ in the interval $[1, 4/3)$, the density contrast asymptotically diverges like

$$\delta^{ISO} \sim (-t)^{-1} \,, \tag{1.4.23}$$

implying that perturbations grow more rapidly than the back-ground density favoring the fragmentation of the basic structure independently on the value of the politropic index.

(2.) **Adiabatic-like Case**: For $4/3 < \gamma \leqslant 5/3$, the argument of Bessels becomes much gather than unity and they assume oscillating behaviors like $J_n(x) \sim x^{-1/2}\cos(x)$, $Y_n(x) \sim x^{-1/2}\sin(x)$, for $x \gg 1$. The solution (1.4.20) asymptotically reads

$$\rho_1^{ADB} \sim (-t)^{\gamma/2-17/6} \, {}^{\cos}_{\sin}\left[\frac{v_0 k(-t)^{4/3-\gamma}}{(4/3-\gamma)(6\pi G\bar{\rho})^{\gamma/2-1/6}}\right] \,, \tag{1.4.24}$$

and therefore perturbations oscillate with ever increasing frequency and amplitude. In this case, the density contrast assumes the form

$$\delta^{ADB} \sim (-t)^{\frac{\gamma}{2}-\frac{5}{6}} \,, \tag{1.4.25}$$

and it is outlined how perturbations, for intermediate stages as $4/3 < \gamma < 5/3$, collapse before that the basic flow completes the condensation (*i.e.*, $\gamma/2 - 5/6 < 0$) and the fragmentation of the background fluid is favored. On the other hand, if the gas cloud behaves adiabatically (*i.e.*, $\gamma = 5/3$), perturbations remain of the same order as the basic-flow density (1.4.18). We can conclude that, in this adiabatic-like case, pressure forces become progressively strong during the collapse as γ increases having a stabilizing effect which prevents that density perturbations grow in amplitude WRT the unperturbed flow. An intermediate type of behavior exists for $\gamma = 4/3$, in this case the disturbances grow like $(-t)^{-13/6}$.

Unperturbed viscous dynamics We now aim to discuss a model in order to build the motion equations of a spherically symmetric and uniform gas cloud, including the corrections due to the presence of dissipative processes; the hypothesis that the fluid is initially at rest already stands here. The Lagrangian (1.4.1) describes a spherical shell which collapses under the gravitational action. In such an approach, the shell results co-moving with the collapsing background. This implies that there are no displacements between parts of fluid WRT ones other since we assume an homogeneous and isotropic flow. Dissipative processes are therefore related to the presence of bulk viscosity and we can safely neglect shear viscosity since it is connected with processes of relative motion among different parts of the fluid.

In order to include bulk-viscosity effects into the dynamics, we introduce the bulk pressure (see eq. (1.2.16b))

$$\tilde{p} = p - \zeta_0 \rho^s \, u^{\mu}_{\;;\mu} \; , \tag{1.4.26}$$

where $u_\mu = (1, \mathbf{0})$ is the shell co-moving 4-velocity. In the Newtonian limit we consider, the metric can be assumed as a flat Minkowskian one expressed in the usual spherical coordinates $[t,\, r,\, \theta,\, \phi]$ and the metric determinant g becomes $g = -r^4 sin^2\theta$. In this case, for the 4-divergence $u^{\mu}_{\;;\mu}$ we immediately obtain

$$u^{\mu}_{\;;\mu} = 2\dot{r}/r \; . \tag{1.4.27}$$

Considering the basic-flow density as

$$\rho_0 = M/(\tfrac{4}{3}\pi r^3) \; , \tag{1.4.28}$$

and the pressure force acting on the collapsing shell of the form

$$F_{\tilde{p}} = \tilde{p}\, 4\pi r^2 \; , \tag{1.4.29}$$

the Lagrangian motion equation for a viscous fluid reads now

$$\frac{\partial^2 r}{\partial t^2} = -\frac{GM}{r^2} - \frac{C}{r^{3s-1}}\frac{\partial r}{\partial t} \; , \tag{1.4.30}$$

where $C = 8\pi\zeta_0 \left(3M/4\pi\right)^s$.

The equation above must be integrated to obtain the evolution of the unperturbed radial velocity v_0 and density ρ_0. In order to compare our viscous analysis WRT the Hunter case, let us now require that the viscosity does not affect the final form of the velocity [27] and, for instance, it should be yet proportional to $1/\sqrt{r}$ (see (1.4.4)). Substituting an expression of the form

$$v_0 = B/\sqrt{r} \tag{1.4.31}$$

into eq. (1.4.30) we see that, in correspondence to the choice $s = 5/6$, it is again a solution as soon as we take the following identification

$$B = C - \sqrt{C^2 + 2GM}\ , \tag{1.4.32}$$

where B assumes only negative values. Although this dynamics is analytically integrable only for the particular value $s = 5/6$, the obtained behavior $v \sim 1/\sqrt{r}$ remains asymptotically valid as $r \to 0$ if the condition $s < 5/6$ is satisfied.

Using such a solution we are able to build an explicit form of the quantity β defined by eq. (1.4.5); differentiating this relation WRT time and taking into account eq. (1.4.31), we obtain a differential equation for the variable β which admits the solution

$$\cos\beta^3 = 3A\,(-t)\ , \tag{1.4.33}$$

where A is defined to be $A = -B/2a^{3/2}$. The Eulerian expressions (1.4.8) of the unperturbed quantities hold here since they are derived simply from relations (1.4.5) and (1.4.3); the effects of bulk viscosity in the zeroth-order analysis are summarized by the new time dependence (1.4.33) of the parameter β which implies a different dynamics for the basic flow.

First-order perturbative theory The zeroth-order motion of a viscous basic flow which collapses under the action of its own gravitation was discussed above. We shall now suppose that small disturbances appear on this field. Perturbations are investigated in the Newtonian limit starting from the system (1.2.8), *i.e.*,

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0\ , \tag{1.4.34a}$$

$$\rho\,\partial_t\mathbf{v} + \rho\,(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p - \zeta\,\nabla(\nabla\cdot\mathbf{v}) + \rho\,\nabla\Phi = 0\ , \tag{1.4.34b}$$

$$\nabla^2\Phi - 4\pi G\rho = 0\ . \tag{1.4.34c}$$

Let us now investigate first-order fluctuations around the unperturbed solutions, *i.e.*, we replace the perturbed quantities: $(\mathbf{v}_0+\delta\mathbf{v})$, $(\rho_0+\delta\rho)$, $(\Phi_0+\delta\Phi)$ and $(p_0+\delta p)$ in eqs. (1.4.34). Taking into account the parameterization (1.4.5) and the bulk-viscosity expansion (1.3.10), *i.e.*,

$$\bar{\zeta} = \zeta(\rho_0) = \zeta_0 \rho_0^s \,, \qquad \delta\zeta = \delta\rho\,(\partial\zeta/\partial\rho) + ... = \zeta_0\, s\, \rho_0^{s-1}\,\delta\rho\; + ... \,. \tag{1.4.35}$$

eq. (1.4.34b) rewrites

$$\partial_t \delta\mathbf{v} + \nabla(\mathbf{v}_0 \cdot \delta\mathbf{v}) + (\nabla \times \delta\mathbf{v}) \times \mathbf{v}_0 = -\nabla\delta\Phi - \frac{v_s^2}{\rho}\nabla\delta\rho + \frac{\bar{\zeta}}{\rho_0}\nabla(\nabla\cdot\delta\mathbf{v}) \,. \tag{1.4.36}$$

As in the non-viscous case, we shall now apply the **rot** operator to this equation in order to get first the solution for the vorticity $\delta\mathbf{w}$ and then the expression for the velocity perturbations $\delta\mathbf{v}$. Indeed, using the vectorial identity $\mathbf{rot}[\nabla f] = 0$ (which holds for each scalar function f), all terms in the RHS of eq. (1.4.36) vanish under this operation. In particular, the term due to the viscous correction disappears from this equation because $\bar{\zeta}$ is, by assumption, a space-independent function. This way, we reach the eq. (1.4.10) for the vorticity which yields the same solution (1.4.12) as in the non-viscous case.

We now build the equations for the perturbed quantities $\delta\mathbf{v}$, $\delta\rho$ and $\delta\Phi$. Substituting the expression (1.4.12) into the first-order perturbed Eulerian motion (1.4.36), we obtain (using eq. (1.4.5) and the conformal spherical coordinates $[a, \theta, \varphi]$), the equation

$$\bar{\rho}\,\dot{\Psi} + \bar{\rho}\,\delta\Phi + \bar{\rho}\,v_s^2 \cos^6\beta\,\delta\rho - \bar{\zeta}\cos^2 D^2\Psi = 0 \,, \tag{1.4.37}$$

which corresponds to the viscous generalization of eq. (1.4.13b). The other perturbed equations maintain their own forms (1.4.13a) and (1.4.13c) also in the viscous case.

Our analysis follows in order to build an unique equation which describes the evolution of density perturbations. By using the procedure developed in the non-dissipative approach, we get now the first-order perturbative equation

$$\begin{aligned} &\partial_t(\cos^{10}\beta\,\partial_t\delta\rho - 6\sin\beta\cos^9\beta\dot{\beta}\delta\rho) - 4\pi G\bar{\rho}\cos^4\beta\delta\rho = \\ &\quad = (v_s{}^2\,\cos^6\beta - 6\,\tfrac{\bar{\zeta}}{\bar{\rho}}\,\sin\beta\cos^{11}\beta\,\dot{\beta})\;D^2\delta\rho + \tfrac{\bar{\zeta}}{\bar{\rho}}\,\cos^{12}\beta\;D^2\,\partial_t\delta\rho \,. \end{aligned} \tag{1.4.38}$$

Here time differentiation is taken at some fixed co-moving radial coordinate.

In order to study the temporal evolution of density perturbations, let us now factorize perturbations $\delta\rho$ in plane waves by the formula (1.4.15) and use the barotropic relation $p = \kappa\rho^\gamma$. According to these assumptions, we are able to write the asymptotic form of eq. (1.4.38), near the end of the collapse as $(-t) \to 0$. In this case, the quantity $cos\beta$ is given by (1.4.33), *i.e.*,

$$\cos\beta = (3A)^{1/3}\,(-t)^{1/3}\,, \tag{1.4.39}$$

and, asymptotically, we can make the approximation $sin\beta \approx 1$ in order to obtain an equation which generalizes eq. (1.4.19) in presence of viscosity;

The background motion equations were derived for a particular value of the viscosity parameter: $s = 5/6$. Substituting the basic-flow density given by eq. (1.4.8b) in the standard expression of the bulk viscosity (1.4.35), we obtain

$$\bar{\zeta} = \zeta_0 \bar{\rho}^{\;5/6}\, cos^{-5}\beta\,. \tag{1.4.40}$$

With these assumptions, eq. (1.4.38) now reads

$$(-t)^2\,\ddot{\rho}_1 - \left[\frac{16}{3} - \frac{\lambda}{3A}\right](-t)\,\dot{\rho}_1 + \\ + \left[\frac{14}{3} - \frac{4\pi G\bar{\rho}}{9A^2} + \frac{v_0^2\,k^2(-t)^{8/3-2\gamma}}{(3A)^{2\gamma-2/3}} - \frac{2\lambda}{3A}\right]\rho_1 = 0\,, \tag{1.4.41}$$

where $v_0^2 = \kappa\gamma\bar{\rho}^{\,\gamma-1}$ and the viscous parameter λ is given by

$$\lambda = \zeta_0\,(\bar{\rho})^{\,-1/6}\,k^2\,. \tag{1.4.42}$$

1.4.1 Density-contrast viscous evolution

A complete solution of eq. (1.4.41) involves Bessel functions of first- and second-species, *i.e.*, J and Y, respectively, and it explicitly reads

$$\rho_1 = C_1\,G_1(t) + C_2\,G_2(t)\,, \tag{1.4.43}$$

where C_1, C_2 are integration constants and the functions G_1 and G_2 are defined to be

$$G_1(t) = (-t)^{-\frac{13}{6}+\frac{\lambda}{6A}}\;J_n\!\left[q(-t)^{4/3-\gamma}\right]\,, \tag{1.4.44}$$

$$G_2(t) = (-t)^{-\frac{13}{6}+\frac{\lambda}{6A}}\;Y_n\!\left[q(-t)^{4/3-\gamma}\right]\,, \tag{1.4.45}$$

having set the Bessel parameters n and q as

$$n = [A^2 - 2\lambda A + \lambda^2 + 16\pi G\bar{\rho}]^{\frac{1}{2}}\;/\;(6A(4/3-\gamma))\,,$$

$$q = -kv_0(3A)^{1/3-\gamma}\;/\;(4/3-\gamma)\,.$$

We now aim, in order to study the asymptotic evolution of the solution (1.4.43), to analyze the cases $1 \leqslant \gamma < 4/3$ and $4/3 < \gamma \leqslant 5/3$ separately, since Bessel functions have different limits connected to the magnitude of their argument.

(1.) **Isothermal-like Case**: In this regime, an asymptotic form of functions G can be found as follow

$$G^{ISO}_{1,2} = c_1 \ (-t)^{-\frac{13}{6}+\frac{\lambda}{6A}\pm(\frac{4}{3}-\gamma)n} \tag{1.4.47}$$

where c_1 and c_2 are constants quantities. The condition which implies the density perturbation collapse is that at least one of G functions diverges as $(-t) \to 0$. An analysis of time exponents yields that G_1 explodes if $\lambda < 7A - 2\pi G\bar{\rho}/3A$ but, on the other hand, G_2 is always divergent for all λ. These results imply that, in the isothermal case, perturbations always condense.

Let us now compare this collapse with the basic flow one; the background density evolves like $\rho_0 \sim cos^{-6}\beta$ (see eq. (1.4.8b)) that is, using eq. (1.4.33),

$$\rho_0 \sim (-t)^{-2} \ . \tag{1.4.48}$$

In the non-dissipative case ($\lambda = 0$), perturbations grow more rapidly WRT the background density involving the fragmentation of the basic flow independently on the value of γ; in presence of viscosity the density contrast assumes the asymptotic form

$$\delta^{ISO} \sim (-t)^{-\frac{1}{6}+\frac{\lambda}{6A}-\frac{1}{6A}\sqrt{[A^2-2\lambda A+\lambda^2+16\pi G\bar{\rho}]}} \ . \tag{1.4.49}$$

Here the exponent is always negative and it does not depend on γ, this implies that δ^{ISO} diverges as the singularity is approached and real sub-structures are formed involving the basic flow fragmentation. This issue means that the viscous forces do not have enough strength to contrast an isothermal perturbations collapse in order to form of an unique structure.

(2.) **Adiabatic-like Case**: For $4/3 < \gamma \leqslant 5/3$, J and Y assume an oscillating behavior. In this regime functions G read

$$G^{ADB}_{1,2} = \tilde{c}_{1,2} \ {}^{\cos}_{\sin}\left[q(-t)^{4/3-\gamma}\right] \ (-t)^{\frac{\gamma}{2}-\frac{17}{6}+\frac{\lambda}{6A}} \ , \tag{1.4.50}$$

where $\tilde{c}_{1,2}$ are constants. Following the isothermal approach, we shall now analyze the time power-law exponent in order to determine the collapse conditions. G functions diverge, involving perturbations condensation, if the parameter λ is less than a threshold value: this condition reads $\lambda < 17A - 3A\gamma$ (for a given value of the index γ). Expressing λ in function of the wave number (1.4.42), we outline, for a fixed viscous parameter ζ_0, a constraint on k which is similar to the condition appearing in the Jeans Model. The threshold value for the wave number is given by the relation

$$K_C^2 = (17A - 3\gamma A)\bar{\rho}^{\ 1/6} / \zeta_0 \tag{1.4.51}$$

and therefore the condition for the density-perturbation collapse, *i.e.*, $\delta\rho^{ADB} \to \infty$, reads $k < K_C$, recalling that, in the Jeans Model for a static background, the condition for the collapse is $k < K_J = [4\pi G\rho_0 / v_s^2]^{1/2}$. It is to be remarked that, in absence of viscosity ($\zeta_0 = 0$), expression (1.4.51) diverges implying that all perturbation scales can be conducted to the collapse. On the other hand, if we consider perturbations of fixed wave number, they decrease as $(-t) \to 0$ for $\lambda > 17A - 3A\gamma$. Thus, for each k, there is a value of the bulk-viscosity coefficient over which the dissipative forces contrast the formation of sub-structures.

If $k < K_C$, perturbations oscillate with ever increasing frequency and amplitude. For a non-zero viscosity coefficient, the density contrast evolves like

$$\delta^{\,ADB} \sim (-t)^{\frac{\gamma}{2} - \frac{5}{6} + \frac{\lambda}{6A}} \,. \tag{1.4.52}$$

A study of the time exponent yields a new threshold value. If $\lambda < 5A - 3A\gamma$, *i.e.*, the viscosity is enough small, sub-structures form; on the other hand, when the parameter ζ_0, or the wave number k, provides a λ-term overcoming this value, the perturbations collapse is so much contrasted that no fragmentation process occurs. By other words, if $\lambda > 5A - 3A\gamma$, we get $\delta^{\,ADB} \to 0$, *i.e.*, for a given γ there is a viscous coefficient ζ_0 enough large ables to prevents the sub-structure formation.

It is remarkable that in the pure adiabatic case, $\gamma = 5/3$, dissipative processes, of any magnitude order, contrast the fragmentation because, while the Jeans-like Length survives, the threshold value for sub-structure formation approaches infinity. We can conclude that, in the case $4/3 < \gamma \leqslant 5/3$, the fragmentation in the top-down scheme is deeply unfavored by the presence of bulk viscosity which strongly contrasts the density-perturbation collapse.

Remarks on the zeroth-order cloud dynamics We now clarify why the choice $s = 5/6$ is appropriate to a consistent treatment of the asymptotic viscous collapse. We start by observing that bulk viscous effects can be treated in a predictive way only if they behave as small corrections to the thermodynamical system. In this respect, we have to require that the asymptotic collapse is yet appropriately described by the non-viscous background flow. As soon as we recognize that eq. (1.4.30) can be rewritten as follows

$$v_0 \frac{\partial v_0}{\partial r} + \frac{C}{r^{3\nu-1}}\, v_0 = -\frac{GM}{r^2} \,, \tag{1.4.53}$$

it is easy to infer that, in the asymptotic limit as $r \to 0$, the non-viscous behavior $v \sim {}^1\!/\!\sqrt{r}$ is preserved only if $s \leqslant {}^5\!/\!_6$. In fact, in correspondence to this restriction, the viscous correction, behaving like $\mathcal{O}(r^{-3s+1/2})$, is negligible WRT the leading order $\mathcal{O}(r^{-2})$ when the singularity is approached; therefore, the request that the viscosity is a small correction implies the choice $s \leqslant {}^5\!/\!_6$.

On the other hand, if we take $s = {}^5\!/\!_6 - \Delta$, with $\Delta > 0$, the perturbation equation in the viscous case (1.4.41) rewrites as

$$\begin{aligned} &(-t)^2\, \ddot{\rho}_1 - \left[\frac{16}{3} - \frac{\lambda}{(3A)^{1-2\Delta}}(-t)^{2\Delta}\right](-t)\,\dot{\rho}_1 + \\ &+ \left[\frac{14}{3} - \frac{4\pi G\bar{\rho}}{9A^2} + \frac{v_0^2\,k^2}{(3A)^{2\gamma-2/3}}(-t)^{8/3-2\gamma} - \frac{2\lambda}{(3A)^{1-2\Delta}}(-t)^{2\Delta}\right]\rho_1 = 0\ . \end{aligned} \tag{1.4.54}$$

If we deal with the adiabatic-like case, it is immediate to verify that, as $(-t) \to 0$, the viscous terms in $\dot{\rho}_1$ and in ρ_1, respectively, are negligible and the dynamics matches asymptotically the non-viscous results (apart from non-relevant features). In the isothermal-like case the viscous term in $\dot{\rho}_1$ is again negligible, but for $\Delta < {}^4\!/\!_3 - \gamma$ the one appearing in ρ_1 could now dominate. However as $(-t) \to 0$ both these terms provide higher-order corrections WRT the constants in ρ_1 and eq. (1.4.54) reduces to an equation whose solution overlaps the non-viscous behavior (we remark that, in the case $s < 5/6$, the viscous parameter asymptotically disappears from the background dynamics too).

Matching together the above considerations for the zeroth- and first-order, respectively, we infer that $s = {}^5\!/\!_6$ is the only physical value which does not affect the background dynamics but makes important the viscous corrections in the asymptotic behavior of the density contrast.

Validity of the Newtonian approximation Since our analysis addresses Newtonian dynamics, when the cloud approaches the extreme collapse, it is relevant to precise the conditions which ensure the validity of such a scheme. The request that the shell corresponding to the radial coordinate r lives in the Newtonian paradigm leads to impose that it remains greater than its own *Schwarzschild Radius*, *i.e.*,

$$r(t) \gg 2GM(a)\ , \tag{1.4.55}$$

where $M(a) = \bar{\rho}\,({}^{4}\!/\!{}_{3}\pi a^3)$ and by eq. (1.4.5) together with the solution (1.4.33) we reache the inequality

$$(-t) \gg -\tfrac{2}{3}\left(\tfrac{8}{3}\pi G\bar{\rho}\right)^{3/2}\left[8\pi\zeta_0\bar{\rho}^{\,s} - \sqrt{(8\pi\zeta_0\bar{\rho}^{\,s})^2 + \tfrac{8}{3}\pi G\bar{\rho}\,a^{3-6s}}\ \right]^{-1} a^{9/2-3s}\ . \tag{1.4.56}$$

Once fixed the fundamental parameters a, $\bar{\rho}$ and ζ_0, the above constraint on the time variable states up to which limit a shell remains appropriately described by the Newtonian approach.

About the dynamics of a physical perturbation scale $l = (2\pi/k)cos\beta^2$ (here $cos\beta^2$ plays the same role of a cosmic scale factor), its Newtonian evolution is ensured by the linear behavior, as soon as, condition (1.4.56) for the background holds. More precisely a perturbations scale is Newtonian if its size is much smaller than the typical space-time curvature length, but for a weak gravitational field this requirement must have no-physical relevance. To explicit such a condition, we require that the physical perturbation scale is much greater than its own *Schwarzschild Radius*, which leads to the inequality $k \gg \chi(-t)^{-1/3}$, where $\chi = \left[{}^{4}\!/\!{}_{3}(2\pi)^3 G\bar{\rho}(3A)^{-2/3}\right]^{1/2}$; combining this result with the inequality (1.4.56), we arrive to the following constraint

$$k \gg \frac{2\pi}{(3A)^{8/3}\,a}\ , \tag{1.4.57}$$

being

$$A = -{}^{1}\!/\!{}_{2}\,a^{3s-3/2}\left[8\pi\zeta_0\bar{\rho}^{\,s} - \sqrt{(8\pi\zeta_0\bar{\rho}^{\,s})^2 + {}^{8}\!/\!{}_{3}\pi G\bar{\rho}\,a^{3-6s}}\ \right]\ . \tag{1.4.58}$$

The condition (1.4.57) tells us which modes are Newtonian within the shell whose initial radius takes the value a.

1.5 Generalization of the Jeans Model to the expanding Universe

In this Section, we discuss the generalization to the expanding-Universe background of the Jeans Mechanism [2, 21, 30]. In this model, the unperturbed background solution is assumed to be static and Uniform and we are now aimed to consider the Universe evolution as the real zeroth-order flow. The equations which describe the homogeneous- and isotropic-Universe dynamics are the well-known *Friedmann Equations*. Such equations are derived by Einstein Equations using a perfect fluid EMT as the matter source of the gravitational field. It's worth remarking that, we can safely address a first-order Newtonian scheme for astrophysical models, as soon as we treat problems in which the energy density is dominated by non-relativistic particles and in which the linear scales involved are small compared with the characteristic scale of the Universe.

As soon as the Universe background dynamics is considered, the Newtonian evolution of the density contrast outlines how the Jeans Mass is already the threshold value for the gravitational collapse. In fact, δ diverges only if the perturbation mass is grater than the Jeans one. Furthermore, different modes appears in the dynamics but they simply vanish during the Universe expansion. In what follows, we include bulk viscous effects to such an analysis[5]. The viscosity is addressed in the first-order analysis and, as in the standard Jeans Model, the key value of the Jeans Mass is not affected but viscous processes modify the evolution of perturbations. In particular, we show how bulk viscosity damps the density-contrast evolution, suppressing the sub-structure formation as in the Jeans Mechanism.

The unperturbed dynamics: the FLRW model The homogeneous and isotropic Universe is described by the FLRW line element

$$ds^2 = dt^2 - a^2(t)\, d\ell^2 \,, \tag{1.5.1}$$

where $a(t)$ is the scale factor of the Universe. If we consider the matter-dominated era, the background is described by an EoS so that $p \sim 0$ ($p \ll \rho$).

[5] NC and G. Montani, "Jeans Instability in Presence of Viscous Effects", submitted to *Int. J. Mod. Phys. D*, Nov. 2008.

As in the Jeans Model, we here use the power-law (1.2.17) to describe bulk viscosity, *i.e.*, $\zeta = \zeta_0 \rho^s$, which results to be proportional to a positive power of the energy density in the matter-dominated scheme we consider. This way, being the matter density very small, we can consistently neglect viscosity in the unperturbed dynamics.

The zeroth-order solution corresponds to the evolution of an homogeneous and isotropic Universe filled with the source: $T_\mu{}^\nu = \mathrm{diag}\,[\,\rho,\,-p,\,-p,\,-p\,]$. The dynamics equations are the energy-momentum conservation law $T^\nu_{\mu;\,\nu} = 0$ (for $\mu = 0$), written in a co-moving frame,

$$\dot{\rho} \,+\, 3\,\frac{\dot{a}}{a}\,(\rho + p) = 0\;, \tag{1.5.2}$$

and the Friedmann Equation

$$\dot{a}^2 \,+\, \mathcal{K} \,=\, \frac{8\pi G}{3}\,\rho\, a^2\;, \tag{1.5.3}$$

where $\mathcal{K} = const.$ is the curvature factor. In this picture, the unperturbed solutions are, setting $p = p_0 = 0$,

$$\rho_0 = \bar{\rho}\Big(\,\frac{a_0}{a}\,\Big)^3\;, \qquad \mathbf{v}_0 = \mathbf{r}\,\frac{\dot{a}}{a}\;, \qquad \nabla\Phi_0 = \tfrac{4}{3}\,\mathbf{r}\,\pi G \rho_0\;, \tag{1.5.4}$$

where $\bar{\rho}$ and a_0 are dimensional constants, $\mathbf{r}$ ($r = |\mathbf{r}|$) denotes the radial coordinate vector and, of course, $a(t)$ satisfies (1.5.3). The solutions $\mathbf{v}_0$ and Φ_0 are derived from the Continuity Equation (1.2.8a) and the Poisson Equation (1.2.8c) respectively, while the Navier-Stokes Equation results to be satisfied since the Friedmann Equations hold.

To obtain the time dependence of the parameters involved in the model, we limit our analysis to early times, since fluctuations arise from the recombination era and, furthermore, the Jeans Mass is so small for recent times that it is of little interest [21]. This way, the study is restricted to scale factors satisfy the condition $a(t) \ll a_0$, so that $\dot{a}^2 \gg 1$, $8\pi\rho a^2/3 \gg 1$ and we can use the zero-curvature solution without loss of generality. Setting $\mathcal{K} = 0$ in the cosmological equation (1.5.3) and using the solution for ρ_0 (1.5.4), one can get the following time dependence

$$a \sim t^{2/3}\;, \qquad \rho_0 = \frac{1}{6\pi G t^2}\;. \tag{1.5.5}$$

The study of the gravitational instability is characterized by the evolution of the density contrast and, in particular, of the small fluctuations. In this respect, we underline that $v_s^2 = \delta p/\delta\rho$ takes account for first-order terms and we have to explicitly write its time dependence during the Universe expansion. For a general specific heat ratio γ, we can assume that the pressure varies as ρ_0^γ and the speed of sound is find to be

$$v_s \sim \; t^{1-\gamma} \; . \tag{1.5.6}$$

Such solutions characterize the background dynamics of the expanding Universe. It's worth noting that, in this generalization, the unperturbed dynamics is now a real solution of the zeroth-order equations and we do not have to apply the "Jeans swindle" static-solution assumption.

Review of the non-dissipative case We want now to study the behavior of the density contrast, in absence of dissipative effects. Since we consider small scales, *i.e.*, $r \ll a$ $(r/a = 0)$, as the fluid motion equations one can assume the non-viscous Newtonian equations, see (1.2.8),

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0 \; , \tag{1.5.7a}$$

$$\rho\,\partial_t\mathbf{v} + \rho\,(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p + \rho\,\nabla\Phi = 0 \; , \tag{1.5.7b}$$

$$\nabla^2\Phi - 4\pi G\rho = 0 \; . \tag{1.5.7c}$$

Performing now the usual perturbative theory, the resulting first-order motion-equations are

$$\partial_t\delta\rho + 3\,\frac{\dot a}{a}\,\delta\rho + \frac{\dot a}{a}\,(\mathbf{r}\cdot\nabla)\delta\rho + \rho_0\;\nabla\cdot\delta\mathbf{v}\; = 0 \; , \tag{1.5.8a}$$

$$\rho_0\,\partial_t\delta\mathbf{v} + \rho_0\,\frac{\dot a}{a}\,\delta\mathbf{v} + \rho_0\,\frac{\dot a}{a}\,(\mathbf{r}\cdot\nabla)\delta\mathbf{v} + v_s^2\;\nabla\,\delta\rho + \rho_0\,\nabla\,\delta\Phi = 0 \; , \tag{1.5.8b}$$

$$\nabla^2\delta\Phi - 4\pi G\delta\rho\; = 0 \; , \tag{1.5.8c}$$

where the relation $\delta p = v_s^2\,\delta\rho$ has been used. The equations above are spatially homogeneous [2], so one can address the usual plane-wave solutions of the form

$$\delta\rho(\mathbf{r},t) = \rho_1(t)\,e^{\frac{i\,\mathbf{r}\cdot\mathbf{q}}{a}} \; , \qquad \delta\mathbf{v}(\mathbf{r},t) = \mathbf{v}_1(t)\,e^{\frac{i\,\mathbf{r}\cdot\mathbf{q}}{a}} \; , \qquad \delta\Phi(\mathbf{r},t) = \Phi_1(t)\,e^{\frac{i\,\mathbf{r}\cdot\mathbf{q}}{a}} \; . \tag{1.5.9}$$

The factor $1/a(t)$ represents the wave-length reduction dues to the Universe expansion: $q = |\mathbf{q}|$ is the co-moving weave number, being $k = q/a$ the physical one. To complete our analysis, in the limit $r/a = 0$, it is convenient to decompose the time depending velocity fluctuations $\mathbf{v}_1$ into two part: one transversal and one parallel to the $\mathbf{q}$ direction, respectively:

$$\mathbf{v}_1(t) = \mathbf{v}_1^\perp \; + \; i\mathbf{q}\,\epsilon \; , \qquad \mathbf{q}\cdot\mathbf{v}_1^\perp = 0 \; , \qquad \epsilon = -\tfrac{i}{q^2}(\mathbf{q}\cdot\mathbf{v}_1) \; . \tag{1.5.10}$$

It is also useful to express ρ_1 in terms of the density contrast, *i.e.*, $\rho_1(t) = \rho_0\delta$, obtaining the following system

$$\dot{\mathbf{v}}_1^{\perp} + \frac{\dot{a}}{a}\,\mathbf{v}_1^{\perp} = 0\,, \qquad (1.5.11a)$$

$$\dot{\epsilon} + \frac{\dot{a}}{a}\,\epsilon - \Big(\frac{4\pi G\rho_0 a}{q^2} - \frac{v_s^2}{a}\Big)\,\delta = 0\,, \qquad (1.5.11b)$$

$$\dot{\delta} - \frac{q^2}{a}\,\epsilon = 0\,. \qquad (1.5.11c)$$

A simple algebraic analysis of the first-order dynamics shows that two different types of normal modes arise. The *Rotational Modes* are described by $\mathbf{v}_1^{\perp}$ and simply decay as $\mathbf{v}_1^{\perp}(t) \sim 1/a$ during the Universe expansion. On the other hand, the *Compressional Modes* are characterized by ϵ e δ and require a more interesting analysis. Such modes are described by the equation

$$\ddot{\delta} + \frac{2\dot{a}}{a}\,\dot{\delta} + \Big(\frac{v_s^2\,q^2}{a^2} - 4\pi G\rho_0\Big)\,\delta = 0\,, \qquad (1.5.12)$$

which reduces to the Jeans dispersion-relation (1.3.7) as soon as we set $a = const.$ and consider the physical wave number k. Taking into account the zeroth-order time dependence (1.5.5) and (1.5.6), one finds that the solution of eq. (1.5.12) involves Bessel functions. As already discussed, such special functions have different behavior corresponding to small or large proper argument. If the argument is large, *i.e.*, much greater than one, the density contrast oscillates, on the other hand, it evolves like

$$\delta \sim t^{-1/6\pm 5/6}\,, \qquad (1.5.13)$$

as soon as the Bessel argument is much less than unity. The condition which separates the two regimes, implying the gravitational collapse in the limit $t \to \infty$ (of course choosing the positive solution for δ), can be write [2] as

$$v_s^2\,q^2/a^2 \lesssim 6\pi G\rho_0\,, \qquad (1.5.14)$$

which is substantially the same as the Jeans condition derived from (1.3.7). It worth underling that the standard Jeans condition is perfectly recast if the parameter γ of (1.5.6) is set to the value 4/3.

In conclusion, we can infer that the dynamics of the expanding Universe does not modify (substantially) the value of the Jeans Mass which remains the threshold to address the gravitational collapse of structures.

1.5.1 Bulk-viscosity effects on the density-contrast dynamics

We are now aimed to introducing bulk-viscosity effects into the dynamics. As discussed above, such a dissipative effect can be consistently neglected from the zeroth-order analysis, since we are dealing with a matter-dominated Universe. This way, the unperturbed background on which develop the perturbative theory corresponds to the solution (1.5.4) of a Friedmann Universe.

Adding small fluctuations to the Newtonian system (1.2.8), *i.e.*,

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0\ , \quad (1.5.15a)$$

$$\rho\,\partial_t\mathbf{v} + \rho\,(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p - \zeta\,\nabla(\nabla\cdot\mathbf{v}) + \rho\,\nabla\Phi = 0\ , \quad (1.5.15b)$$

$$\nabla^2\Phi - 4\pi G\rho = 0\ , \quad (1.5.15c)$$

and neglecting second-order terms, we get the following set of equations

$$\partial_t\delta\rho + 3\,\frac{\dot a}{a}\,\delta\rho + \frac{\dot a}{a}\,(\mathbf{r}\cdot\nabla)\delta\rho + \rho_0\nabla\cdot\delta\mathbf{v} = 0\ , \quad (1.5.16a)$$

$$\rho_0\partial_t\delta\mathbf{v} + \rho_0\frac{\dot a}{a}\,\delta\mathbf{v} + \rho_0\frac{\dot a}{a}\,(\mathbf{r}\cdot\nabla)\delta\mathbf{v} + v_s^2\nabla\delta\rho + \rho_0\nabla\delta\Phi - \bar\zeta\nabla(\nabla\cdot\delta\mathbf{v}) = 0\ , \quad (1.5.16b)$$

$$\nabla^2\delta\Phi - 4\pi G\delta\rho = 0\ , \quad (1.5.16c)$$

where the relation $\delta p = v_s^2\,\delta\rho$ has been used and we recall that

$$\bar\zeta = \zeta(\rho_0)\ ,$$

see eq. (1.3.10). As in the non-dissipative case, the plane-wave expansion (1.5.9) for the fluid parameters can be addressed and, using the hypothesis $r/a\sim 0$, the system above reduces to:

$$\dot\rho_1 + 3\,\frac{\dot a}{a}\,\rho_1 + \frac{i\,\rho_0}{a}\,(\mathbf{q}\cdot\mathbf{v}_1) = 0\ , \quad (1.5.17a)$$

$$\dot{\mathbf{v}}_1 + \frac{\dot a}{a}\,\mathbf{v}_1 + \frac{i\,v_s^2}{a\,\rho_0}\,\mathbf{q}\,\rho_1 - 4\pi i\,Ga\,\rho_1\,\frac{\mathbf{q}}{q^2} + \frac{\bar\zeta}{a^2\,\rho_0}\,\mathbf{q}\,(\mathbf{q}\cdot\mathbf{v}_1) = 0\ . \quad (1.5.17b)$$

Let us now follow the standard analysis and use the decomposition (1.5.10) in order to compare our results WRT the non-dissipative ones. We finally get

$$\dot{\mathbf{v}}_1^{\perp} + \frac{\dot a}{a}\,\mathbf{v}_1^{\perp} = 0\ , \quad (1.5.18a)$$

$$\dot\epsilon + \Big(\frac{\dot a}{a} + \frac{\bar\zeta\,q^2}{\rho_0\,a^2}\Big)\,\epsilon - \Big(\frac{4\pi G\rho_0 a}{q^2} - \frac{v_s^2}{a}\Big)\,\delta = 0\ , \quad (1.5.18b)$$

$$\dot\delta - \frac{q^2}{a}\,\epsilon = 0\ . \quad (1.5.18c)$$

The Rotational Modes are not affected by viscosity. In fact, they are governed by eq. (1.5.18a) which has the solution

$$\mathbf{v}_1^{\perp}(t) \sim 1/a \;, \tag{1.5.19}$$

as in the non-viscous analysis presented above.

The Compressional Modes Compressional Modes are influenced by the presence of viscosity. In particular, combining together (1.5.18b) and (1.5.18c), we get an equation which generalizes the compressional equation (1.5.12). It reads

$$\ddot{\delta} + \Big(2\,\frac{\dot{a}}{a} + \frac{\bar{\zeta}q^2}{\rho_0\,a^2}\Big)\,\dot{\delta} + \Big(\frac{v_s^2\,q^2}{a^2} - 4\pi G\rho_0\Big)\,\delta = 0 \;. \tag{1.5.20}$$

This is the fundamental equation which governs the evolutions of the density contrast on an expanding Universe. Let us now write explicitly the time dependence of the parameters involved in the model. The zeroth-order analysis still remains valid in presence of viscosity and we can address expressions (1.5.5) and (1.5.6), as soon as we restrict the study to early times, so that $a(t) \ll a_0$. Furthermore, using the power-law relation (1.2.17) for the bulk-viscosity coefficient, one easily finds

$$\bar{\zeta} = \bar{\zeta}_0\, t^{-2s} \;, \qquad\qquad \bar{\zeta}_0 = \zeta_0/(6\pi G)^s \;. \tag{1.5.21}$$

With the help of these expressions, we can isolate two constants in the eq. (1.5.20), which finally rewrites

$$\ddot{\delta} + \left[\frac{4}{3\,t} + \frac{\chi}{t^{2(s-1/3)}}\right]\dot{\delta} + \left[\frac{\Lambda^2}{t^{2\gamma-2/3}} - \frac{2}{3\,t^2}\right]\delta = 0 \;, \tag{1.5.22}$$

where the constants χ and Λ are

$$\chi = \frac{t^{2(s-1/3)}\,\bar{\zeta}q^2}{\rho_0\,a^2} \;, \qquad\qquad \Lambda = \frac{t^{\gamma-1/3}\,v_s q}{a} \;. \tag{1.5.23}$$

This equation can not be analytically solved in general. As in the previous Section, let us now discuss the case $s = 5/6$. Indeed, this case is the only of physical interest since it deals with the maximum effect that bulk viscosity has without dominating the dynamics, in view of its non-equilibrium perturbative characterization. In fact, in the collapsing limit as $t \rightarrow \infty$, if $s > 5/6$ the viscous term proportional to χ results to be of higher order and dominant. On the other hand, it can be neglected in eq. (1.5.22), if $s < 5/6$. Substituting this value in the equation above, one gets the following integrable expression

$$\ddot{\delta} + \left[\frac{4}{3} + \chi\right]\frac{\dot{\delta}}{t} + \left[\frac{\Lambda^2}{t^{2\gamma-2/3}} - \frac{2}{3\,t^2}\right]\delta = 0\,. \tag{1.5.24}$$

The solutions are

$$\delta(t) = t^{-\frac{1}{6}-\frac{\chi}{2}}\left[C_1\,J_n\Big(\frac{\Lambda t^{-\bar{\gamma}}}{\bar{\gamma}}\Big) + C_2\,Y_n\Big(\frac{\Lambda t^{-\bar{\gamma}}}{\bar{\gamma}}\Big)\right]\,, \tag{1.5.25}$$

where J_n and Y_n denote Bessel functions and

$$n = -\sqrt{25 + 6\chi + 9\chi^2}\,/\,6\bar{\gamma}\,, \qquad \bar{\gamma} = \gamma - 4/3\,. \tag{1.5.26}$$

These functions oscillate for $t \ll \Lambda^{1/\bar{\gamma}}$, while for $t \gg \Lambda^{1/\bar{\gamma}}$ the density-contrast solution (1.5.25) evolves like

$$\delta \sim t^{-1/6\,-\,\chi/2\,\mp\,\bar{\gamma}n}\,. \tag{1.5.27}$$

A simple analysis of the exponent of such solutions shows how it is always positive, for all values of the viscous parameter χ, as soon as we choose the $(-)$-sign solution. This behavior corresponds to a gravitational collapse, if we consider the asymptotic limit $t \to \infty$. The threshold value which separates the different regimes, implying the growth of the density contrast, is defined by the relation $t > \Lambda^{1/\bar{\gamma}}$ which, using (1.5.5), corresponds to the Jeans condition (1.3.7):

$$v_s^2\,q^2/a^2 \lesssim 6\pi G\rho_0\,, \tag{1.5.28}$$

as in the non-dissipative case. We remark that such solutions will apply only after the recombination, with $4/3 < \gamma \leqslant 5/3$. In fact, in correspondence of $\gamma = 4/3$, the solutions (1.5.25) show a singular behavior and eq. (1.5.22) requires a different treatment.

As in the standard Jeans Model, the key value of the Jeans Mass is not affected by bulk viscosity, *i.e.*, gravitational collapses for $\delta \to \infty$ are addressed if

$$k < K_J^* = \sqrt{\frac{6\pi G\rho_0}{\bar{\gamma}^2 v_s^2}}\,. \tag{1.5.29}$$

The effect of viscous processes is to modify the evolution of perturbations. In fact, comparing expression (1.5.27) WRT the non-dissipative behavior of growing density contrast $\delta \sim t^{2/3}$, see (1.5.13), one can show that the relation

$$-1/6 - \chi/2 - \bar{\gamma}n < 2/3 \tag{1.5.30}$$

is always verified. We can conclude that the effect of bulk viscosity is to damp the density-contrast evolution, suppressing the structure formation as in the Jeans Mechanism.

1.6 Quasi-Isotropic Model in presence of bulk viscosity

The isotropic nature of the Universe corresponds to a class of the gravitational solutions which involve three physically arbitrary coordinate functions. Such a class was found by E.M. Lifshitz and I.M. Khalatnikov in 1963 [14], addressing a radiation-dominated Universe, and then generalized to an arbitrary fluid state equation in [31]. Earlier extensions of this Quasi-Isotropic (QI) scheme were provided in [32, 33, 34, 35], where different evolutionary stages of the Universe are characterized.

The QI Model corresponds to a Taylor expansion of the 3-metric tensor in powers of the synchronous time. However, further investigation outlined the necessity of treating generic power-law components of the 3-metric. In what follows[6,7], we generalize the original work by Lifshitz and Khalatnikov to the presence of bulk viscosity since, asymptotically to the singularity, the Universe volume has a very fast time variation and we naturally expect that viscous effects arise. Our aim is to determine the conditions on the viscosity intensity which allows for the existence of a QI regime for the radiation-dominated Universe.

As already discussed, general analyses of the Universe behavior in presence of bulk viscosity characterize such a coefficient as a power-law of the energy density, *i.e.*, $\zeta = \zeta_0\, \rho^s$. As far as this phenomenological *ansatz* is referred to the early Universe, it is easy to realize that the choice $s = 1/2$ prevents dominating viscous effects. On the other hand, simple considerations, as well as the analysis presented in the works by J.D. Barrow [36, 37, 38], indicate that the case $s < 1/2$ leads to negligible contributions of the viscosity to the asymptotic regime towards the Big Bang. As a consequence, in studying the singularity physics in the QI Model, the most appropriate form of the power-law is $\zeta = \zeta_0 \sqrt{\rho}$.

In the QI non-viscous scheme, after setting the form of the 3-metric expansion, the integration of the Einstein Equations is performed in order to obtain a solution for the energy density, the density contrast and the 3-velocity of the perfect fluid filling the space-time. In order to include bulk viscous effects into the dynamics, we investigate

[6] NC and G. Montani, "Study of the Quasi-Isotropic Solution Near the Cosmological Singularity in Presence of Bulk Viscosity", *Int. J. Mod. Phys. D* **17**(6), 881 (2008).

[7] NC and G. Montani, "On the Role of Viscosity in Early Cosmology", *Int. J. Mod. Phys. A* **23**(8), 1248 (2008).

the Einstein Equations under the assumptions proper of the QI Model. We separate zeroth- and first-order terms into the 3-metric tensor and the whole analysis follows this scheme of approximation. In the search for a self-consistent solution, we make use of the hydrodynamics equations, in view of fixing the form of the energy density. As a result, we prove the existence of a QI Solution, which has a structure analogous to that provided by Khalatnikov, Kamenshchik and Starobinsky [31]. Of course, in our solution the power-law for the leading 3-metric term is sensitive to the viscosity parameter ζ_0. In particular, we show how the solution exists only if when ζ_0 remains smaller than a threshold value. Finally, the density contrast and its dependence on ζ_0 are determined. This behavior confirms and generalizes the result obtained in [39] about the damping of density perturbations by the viscous correction.

Review on the LK-QI Solution In 1963, E.M. Lifshitz and I.M. Khalatnikov [14] first proposed the so-called QI Solution. This model is based on the idea that the space contracts maintaining linear-distance changes with the same time dependence order by order (*i.e.*, a Taylor expansion of the 3-metric is addressed). In this approach, the Friedmann solution becomes a particular case of a larger class of solutions existing only for space filled with matter [40].

The metric evolution is strongly characterized by the matter EoS. For an ultra-relativistic perfect fluid, characterized by an EoS so that, $p = \rho/3$, the spatial metric assumes the form $\gamma_{\alpha\beta} \sim a_{\alpha\beta}\, t$, asymptotically as $t \to 0$ (the cosmological singularity is set by convention in $t = 0$), where $a_{\alpha\beta}$ are assigned functions of the spatial coordinates. As a function of time, the 3-metric is expandable in powers of t. The QI Solution is formulated in a synchronous system (*i.e.*, $g_{0\alpha} = 0$, $g_{00} = -1$), which is not strictly a co-moving one. The line element writes as[8]

$$ds^2 = -dt^2 + \gamma_{\alpha\beta}(t, x^\gamma) dx^\alpha dx^\beta \,, \tag{1.6.1}$$

with a spatial metric of the form

$$\gamma_{\alpha\beta} = t\, a_{\alpha\beta} + t^2\, b_{\alpha\beta} + \dots \,, \qquad \gamma^{\alpha\beta} = t^{-1}\, a^{\alpha\beta} - \; b^{\alpha\beta} \,, \tag{1.6.2}$$

[8]The whole analysis of this Section is devoted to compare, step by step, the dissipative effects WRT the Landau-School analysis. In this respect, we follow the notation signature $[-, +, +, +]$ ($u_\mu u^\mu = -1$) of the original work by Lifshitz and Khalatnikov. As you can see in eq. (1.6.6), this choice requires a sign-modification also in the EMT expression. We also use units so that $8\pi G = 1$.

where $a^{\alpha\beta}$ is defined as $a^{\alpha\beta}a_{\beta\gamma} = \delta^\alpha_\gamma$; furthermore, the relation $b^\alpha_\beta = a^{\alpha\gamma}b_{\gamma\beta}$ is ensured by the scheme of approximation.

The Einstein Equations in the synchronous system assume the form [15]

$$R^0_0 = \tfrac{1}{2}\,\kappa^\alpha_{\alpha,0} + \tfrac{1}{4}\,\kappa^\beta_\alpha\kappa^\alpha_\beta = \; T^0_0 - \tfrac{1}{2}T \;, \tag{1.6.3a}$$

$$R^0_\alpha = \tfrac{1}{2}\,(\kappa^\beta_{\beta;\,\alpha} - \kappa^\beta_{\alpha;\,\beta}) = \; T^0_\alpha \;, \tag{1.6.3b}$$

$$R^\beta_\alpha = \tfrac{1}{2\sqrt{\gamma}}\,(\sqrt{\gamma}\,\kappa^\beta_\alpha)_{,0} + P^\beta_\alpha = \; T^\beta_\alpha - \tfrac{1}{2}T\delta^\beta_\alpha \;, \tag{1.6.3c}$$

where the extrinsic curvature tensor $\kappa_{\alpha\beta}$ and its contractions read

$$\kappa_{\alpha\beta} = \gamma_{\alpha\beta,\,0} = a_{\alpha\beta} + 2\,t\;b_{\alpha\beta} \;, \tag{1.6.4a}$$

$$\kappa^\beta_\alpha = \gamma^{\beta\delta}\,\kappa_{\alpha\delta} = t^{-1}\;\delta^\beta_\alpha + \; b^\beta_\alpha \;, \tag{1.6.4b}$$

$$\kappa \; = (\ln\sqrt{\gamma})_{,0} = 3\,t^{-1} \; + \; b \;, \tag{1.6.4c}$$

and

$$\gamma = \det(\gamma_{\alpha\beta}) \sim t^3(1+tb)\det(a_{\alpha\beta}) \;. \tag{1.6.5}$$

Matter is described by an ultra-relativistic perfect fluid EMT

$$T^{(P)}_{\mu\nu} = (p+\rho)\,u_\mu u_\nu + p\,g_{\mu\nu} = \tfrac{1}{3}\,\rho\,(4u_\mu u_\nu + g_{\mu\nu}) \;, \tag{1.6.6}$$

which provides the following identities

$$T^0_0 = \tfrac{1}{3}\,\rho\,(-4u_0^2+1) \;, \quad T^0_\alpha = \tfrac{4}{3}\,\rho\,u_\alpha u^0 \;, \quad T^\beta_\alpha = -\tfrac{4}{3}\,\rho\,u_\alpha u^\beta \;, \quad T = 0 \;. \tag{1.6.7}$$

Calculating the LHS of eq. (1.6.3a) and eq. (1.6.3b) up to zeroth-order, *i.e.*, $\mathcal{O}(1/t^2)$, and first-order, *i.e.*, $\mathcal{O}(1/t)$, in $1/t$, we rewrite them respectively as

$$-\frac{3}{4\,t^2} + \frac{b}{2\,t} = \frac{\rho}{3}\,(-4u_0^2+1) \;, \qquad \frac{1}{2}\,(b_{;\,\alpha} - b^\beta_{\alpha;\,\beta}) = -\frac{4\,\rho}{3}\;u_\alpha u_0 \;. \tag{1.6.8}$$

Because of the identity $-1 = u_\mu u^\mu \sim -u_0^2 + t^{-1}u_\alpha u_\beta\,a^{\alpha\beta}$, it is immediate to see that $\rho \sim t^{-2}$ and $u_\alpha \sim t^2$; hence, in the asymptotic limit $t \to 0$, $u_0^2 \simeq 1$ $(u_0 = -1)$. From the first of eq. (1.6.8), one can find the first two terms of the energy density expansion, while, from the second equation, the leading term of the velocity arises

$$\rho = \frac{3}{4\,t^2} - \frac{b}{2\,t} \;, \qquad u_\alpha = \frac{t^2}{2}\,(b_{;\,\alpha} - b^\beta_{\alpha;\,\beta}) \;. \tag{1.6.9}$$

Because of eqs. (1.6.9), the expression for the density contrast δ can be found as first- and zeroth-order energy-density ratio, *i.e.*,

$$\delta \; = \; -\tfrac{2}{3}\,b\;t \;. \tag{1.6.10}$$

This behavior implies that, as expected in the cosmological standard model, the zeroth-order term of energy density diverges more rapidly than the perturbations and the singularity is naturally approached with a vanishing density contrast in this scenario.

Besides the solutions for ρ and u_α, one has to consider the pure spatial components of Gravitational Equations, *i.e.*, eq. (1.6.3c). Up to first approximation, the Ricci tensor can be written as $P^\beta_\alpha = \bar{P}^\beta_\alpha / t$, where $\bar{P}^\beta_\alpha$ is constructed by the constant 3-tensor $a_{\alpha\beta}$. The terms of order t^{-2} automatically cancel out, while those proportional to t^{-1} give

$$\bar{P}^\beta_\alpha \;+\; \tfrac{3}{4}\, b^\beta_\alpha \;+\; \tfrac{5}{12}\, b\, \delta^\beta_\alpha = 0 \; . \tag{1.6.11}$$

Performing the trace of this equation, a relation between the quite arbitrary six functions $a_{\alpha\beta}$ and the coefficients $b_{\alpha\beta}$ from the next-to-leading term of expansion can be determined: $b^\beta_\alpha = -{}^4\!/_3 \bar{P}^\beta_\alpha + {}^5\!/_{18}\, \bar{P}\, \delta^\beta_\alpha$. It is worth reminding that, in the asymptotic limit $t \to 0$, the matter distribution becomes homogeneous because ρ approaches a value independent of b.

Now, using the Ricci identity $\bar{P}^\beta_{\alpha;\,\beta} = {}^1\!/_2\, \bar{P}_{;\,\beta}$, the useful relation $b^\beta_{\alpha;\,\beta} = {}^7\!/_9\, b_{;\,\alpha}$ can be determined; this gives the final expression for the 3-velocity distribution as

$$u_\alpha = \frac{t^2}{9}\; b_{;\,\alpha} \; . \tag{1.6.12}$$

This result implies that, in this approximation, the 3-velocity is a gradient field of a scalar function fixed by the non perturbed metric $a_{\alpha\beta}$. As a consequence, the curl of the velocity vanishes and no rotations take place into the fluid.

Finally, it must be observed that the metric (1.6.2) allows for an arbitrary 3-space coordinate transformation and the solution above contains only $6 - 3 = 3$ arbitrary space functions arising from $a_{\alpha\beta}$. A particular choice of this functions, those which correspond to the space of constant curvature ($\bar{P}^\beta_\alpha \sim \delta^\beta_\alpha$), can reproduce the pure isotropic and homogeneous model.

1.6.1 Generalized Quasi-Isotropic line element

In order to generalize the QI Solution of the Einstein Equations for the presence of dissipative effects into the evolution of the energy source, we deal with a more complex (no longer in integer powers) form of the 3-metric (1.6.2) [32, 33, 31]. In this respect, we take the spatial metric of the form

$$\gamma_{\alpha\beta} = t^x \, a_{\alpha\beta} + t^y \, b_{\alpha\beta} \,, \qquad \gamma^{\alpha\beta} = t^{-x} \, a^{\alpha\beta} - t^{y-2x} \, b^{\alpha\beta} \,. \tag{1.6.13}$$

Here, the constraints for the space contraction (*i.e.*, $x > 0$), and for the consistence of the perturbative scheme (*i.e.*, $y > x$) have to be imposed for the proper development of the model. In this approach, the extrinsic curvature and its contractions read

$$\kappa_{\alpha\beta} = x\, t^{x-1} \, a_{\alpha\beta} + y\, t^{y-1} \, b_{\alpha\beta} \,, \tag{1.6.14a}$$

$$\kappa_\alpha^\beta = x\, t^{-1} \, \delta_\alpha^\beta + (y-x)\, t^{y-x-1} \, b_\alpha^\beta \,, \tag{1.6.14b}$$

$$\kappa \;= 3x\, t^{-1} \; + (y-x)\, t^{y-x-1} \, b \,, \tag{1.6.14c}$$

furthermore, we calculate the following useful relation

$$(\ln \sqrt{\gamma})_{,0} = \tfrac{1}{2} \, \kappa = \tfrac{3}{2} \, x t^{-1} + \tfrac{1}{2} \, (y-x) \, t^{y-x-1} \, b \,. \tag{1.6.15}$$

We are now able to write down the final form of the Ricci-tensor components contained in the Einstein Equations (1.6.3). These new expressions allow us to generalize the original QI approach. Our aim is to obtain constraints and relations for the exponents x, y in order to guarantee the existence of the solution of our model. They explicitly read

$$R_0^0 = -\frac{3x(2-x)}{4t^2} \; + \; (y-x)(y-1)\frac{b}{2t^{2-y+x}} \,, \tag{1.6.16a}$$

$$R_\alpha^0 = (b_{\,;\,\alpha} - b^\beta_{\alpha;\,\beta})\frac{y-x}{2t^{1-y+x}} \,, \tag{1.6.16b}$$

$$R_\alpha^\beta = \frac{x(3x-2)}{4t^2} \, \delta_\alpha^\beta \; + \; \frac{(y-x)(2y+x-2)}{4t^{2-y+x}} \, b_\alpha^\beta \, + \\ + \; \frac{(y-x)x}{4t^{2-y+x}} \, b \, \delta_\alpha^\beta \; + \; \frac{\bar{P}_\alpha^\beta}{t^x} \; + \; \frac{P_\alpha^{*\beta}}{t^{2x-y}} \,. \tag{1.6.16c}$$

We note that in eq. (1.6.16c), $\bar{P}_\alpha^\beta$ represents the 3-dimensional Ricci tensor constructed by the metric $a_{\alpha\beta}$. On the other hand, the higher-order term $P_\alpha^{*\beta}$ denotes the part of P_α^β containing the 3-tensor $b_{\alpha\beta}$.

The form of energy density in the viscous approach In the QI Solution, the Universe is assumed, according to the CSM, to be described by the EMT of an ultra-relativistic perfect fluid. In connection with the development of new cosmological models, the discovery of the cosmic acceleration suggests matter to play an essential role at different stages of cosmological evolution and it can obey very different EoS [41]. Thus, corrections in this sense to the original formulation of the QI Model can be useful in this new context.

In this work, we treat the immediate generalization of LK scheme considering the presence of dissipative processes within the fluid dynamics. As discussed in Section 1.2, using the different signature, the new EMT reads now (see eq. (1.2.16))

$$T_{\mu\nu} = (\tilde{p}+\rho)u_\mu u_\nu + \tilde{p}\, g_{\mu\nu} = \tfrac{1}{3}\,\rho\,(4u_\mu u_\nu + g_{\mu\nu}) - \zeta\, u^\rho_{\;;\rho}(u_\mu u_\nu + g_{\mu\nu})\,, \qquad (1.6.17)$$

$$\tilde{p} = p - \zeta\, u^\rho_{\;;\rho}\,, \qquad (1.6.18)$$

where, of course, $p = \rho/3$ and $\zeta = \zeta_0\,\rho^s$ (see eq. (1.2.17)).

Let us now write the expressions of the components of EMT (1.6.17) up to higher-order corrections as

$$T^0_0 = -\frac{\rho}{3}\,(4u_0^2 - 1) + \zeta_0\rho^s\, u^\mu_{\;;\mu}\,(u_0^2 - 1)\,, \qquad (1.6.19a)$$

$$T \;= -3\,\zeta_0\,\rho^s\, u^\mu_{\;;\mu}\,, \qquad (1.6.19b)$$

$$T^\beta_\alpha = \frac{\rho}{3}\,(4u_\alpha u^\beta + \delta^\beta_\alpha) - \zeta_0\rho^s\, u^\mu_{\;;\mu}\,(u_\alpha u^\beta + \delta^\beta_\alpha)\,, \qquad (1.6.19c)$$

$$T^0_\alpha = \tfrac{4}{3}\,\rho\, u_\alpha u^0 - \zeta_0\rho^s\, u^\mu_{\;;\mu}\, u_\alpha u^0\,, \qquad (1.6.19d)$$

where the divergence of the 4-velocity reads

$$u^\mu_{\;;\mu} = (\ln\sqrt{\gamma})_{,0} = \tfrac{3}{2}\,x t^{-1} + \tfrac{1}{2}\,(y-x)\,t^{y-x-1}\,b\,. \qquad (1.6.20)$$

Here we assume, as in the non-viscous case, the relation: $u_0^2 \simeq 1$ (with $u_0 = -1$), whose consistence must be verified *a posteriori* comparing the time behavior of the quantities involved in the model. Taking into account expressions (1.6.19a) (1.6.19b), we can recast now the Einstein Equation (1.6.3a) in the form

$$-\frac{3x(2-x)}{4t^2} + (y-x)(y-1)\frac{b}{2t^{2-y+x}} = -\rho + \frac{9\,x}{4\,t}\,\zeta_0\rho^s + \frac{3(y-x)}{4\,t^{1-y+x}}\,\zeta_0\rho^s\,b\,. \quad (1.6.21)$$

In what follows, we fix the value

$$s = 1/2\,, \qquad (1.6.22)$$

in order to deal with the maximum effect that bulk viscosity can have without dominating the dynamics. As already discussed, the notion of this kind of viscosity corresponds to a phenomenological issue of perturbations to the thermodynamical equilibrium [23]. In this sense, we remark that, if $s > 1/2$, the dissipative effects become dominant and non-perturbative. Moreover, if we assume the viscous parameter $s < 1/2$, the dynamics of the early Universe is characterized by an expansion via a power-law $a(t) \sim t^{2/3\gamma}$ starting from a perfect fluid Friedmann singularity at $t = 0$ (here γ is identify by the relation $p = (\gamma - 1)\,\rho$). After this first stage of evolution, where viscosity does not affect at all the dynamics, the Universe inflates out of our approximation scheme (*i.e.*, in the limit $t \to \infty$) to a viscous deSitter solution which is characterized by $a(t) \sim e^{H_0 t}$, where $H_0 = \sqrt{\rho_0}/3 = 1/3(\zeta_0\sqrt{3}/\gamma)^{1/(1-2s)}$ [37, 38].

Since, in this work, we deal with the asymptotic limit $t \to 0$, we only treat the case $s = 1/2$ in order to quantitatively include dissipative effects in the primordial dynamics. From eq. (1.6.21), if $s = 1/2$, we expand the energy density ρ in Taylor series:

$$\rho = \frac{e_0}{t^2} + \frac{e_1\, b}{t^{2-y+x}}\,, \qquad \sqrt{\rho} = \frac{\sqrt{e_0}}{t}\left(1 + \frac{e_1\, b}{2e_0}\, t^{y-x}\right)\,, \tag{1.6.23}$$

where the constants e_0, e_1 have to be determined combining the 00-component of Gravitational Equations with the hydrodynamical ones comparing the terms order by order, as treated below. We remark that, only for the case $s = 1/2$, all terms coming on the LHS and the RHS respectively of eq. (1.6.21) result to have the same time behavior up to first-order because of eq. (1.6.23).

Comments on the adopted paradigm We here discuss in some details the hypotheses at the ground of our analysis of the QI viscous Universe dynamics. In particular, we investigated some peculiar features of the very early evolution (near the cosmological singularity) since their presence leads to a specific treatment of the viscous phenomena.

(1.) It is well known [1] the crucial role played in cosmology by the *microphysical horizon*, as far as the thermodynamical equilibrium is concerned. In the isotropic Universe, such a quantity is fixed by the inverse of the expansion rate,

$$\ell_h = H^{-1} \equiv (a/\dot{a})\,, \tag{1.6.24}$$

a being the scale factor of the Universe and the dot identifies time derivatives, and it gives the characteristic scale below which the elementary-particle interactions are able to preserve the thermal equilibrium of the system. Therefore, if the mean free-path of particles ℓ is greater than the microphysical horizon (*i.e.*, $\ell > H^{-1}$), no real notion of thermal equilibrium can be recovered at the micro-causal scale. If we indicate by n the number density of particles and by σ the averaged cross section of interactions, then the mean free-path of the ultra-relativistic cosmological fluid (in the early Universe, the particle velocity is very close to speed of light) takes the form

$$\ell \sim 1/n\sigma \, . \tag{1.6.25}$$

Interactions mediated by massless gauge bosons are characterized by the cross section $\sigma \sim \alpha^2 T^{-2}$ ($g = \sqrt{4\pi\alpha}$ being the gauge coupling strength) and the physical estimation $n \sim T^3$ leads to the result $\ell \sim 1/\alpha^2 T$ [1]. During the radiation-dominated era $H \sim T^2/m_{Pl}$, so that

$$\ell \sim \frac{T}{\alpha^2 m_{Pl}} \, H^{-1} \, . \tag{1.6.26}$$

Therefore in the case of $T \gtrsim \alpha^2 m_{Pl} \sim \mathcal{O}(10^{16} GeV)$, *i.e.*, during the earliest epoch of pre-inflating Universe, the interactions above are effectively "frozen out" and they are not able to maintain or to establish thermal equilibrium. To complete this consideration we remark that, at temperatures grater than $\mathcal{O}(10^{16} GeV)$, the contributions to the estimation above due to the mass term of the gauge bosons can be ruled out for all known and proposed perturbative interactions.

As a consequence of this non-equilibrium configuration of the causal regions characterizing the early Universe, most of the well-established results about the kinetic theory [2, 20, 21] concerning the cosmological fluid nearby equilibrium become not applicable. Indeed all these analysis are based on the assumption to deal with a finite mean free-path of the particles and, in particular, results about the characterization of viscosity are established when pure collisions among particles are retained. However, when the mean free-path is grater than the micro-causal horizon (which, in the pre-inflating Universe, coincides with the cosmological horizon), ℓ can be taken of infinite magnitude for any physical purpose.

The fundamental analysis of the viscous cosmology is due to the Landau School [22, 23, 24]; since they were aware of these difficulties for a consistent kinetic theory, such an analysis was essentially based on an hydrodynamical approach. A real notion of the *hydrodynamical description* can be provided by assuming that an arbitrary state is adequately specified by the particle-flow vector and the EMT alone [25]. In particular, the entropy flux has to be expressed as a function of these two hydrodynamical variables without additional parameters. Following this point of view, the viscosity effects are treated on the ground of a thermodynamical description of the fluid, *i.e.*, the viscosity coefficients are fixed by the macroscopic parameters which govern the system evolution. In this respect, the most natural choice is to take such a (shear and bulk) viscosity coefficients as power-laws in the energy density of the fluid (for a detailed discussions see [22]). Such a phenomenological assumption can be reconciled, for some simple cases, with a relativistic kinetic theory approach [42], especially in the limits of small and large energy densities.

Addressing the hydrodynamical approach, we are lead to retain the same EoS which would characterize the corresponding ideal fluid. This fact is supported by idea that the viscosity effects provide only small corrections to the thermodynamical setting of the system. As clarified above, in the present analysis, we deal with the case in which bulk viscous corrections are of the same order of the perfect fluid contributions, in order to maximize their influence in the Universe dynamics. Nevertheless, since we are treating an ultra-relativistic thermodynamical system, which is very weakly interacting on the micro-causal scale, there are well-grounded reasons to describe it by the EoS so that $p = \rho/3$.

(2.) Another important point concerning the ground assumptions of our model, is why the shear viscosity ϑ is not addressed in the present scheme. Indeed, this kind of viscosity accounts for the friction forces acting between different portions of viscous fluid. Therefore, as far as the isotropic character of the Universe is retained, the shear viscosity must not provide any contributions, as discussed in [23]. On the contrary, the rapid expansion of the early Universe suggests that an important contribution comes out from the bulk viscosity as an averaged effect of a quasi-equilibrium evolution.

Indeed, our present analysis deals with small inhomogeneous corrections to the background FLRW metric. Thus, at first-order in our solution, shear viscosity should be, in principle, included into the dynamics. In this sense, it is shown in [22] that, if the bulk-viscosity coefficient behaves like $\zeta \sim \rho^s$, then the corresponding shear one behaves as $\vartheta \sim \rho^m$, where m must satisfy the constraint condition $m \geqslant s + 1/2$. Here we treat the case $s = 1/2$, thus getting $m \geqslant 1$ for the ϑ coefficient. This issue is incompatible with the symmetries and the approximations here addressed. In fact, the shear viscosity provides, among others, an equivalent contribution to the bulk one, since the EMT of the viscous fluid contains the term

$$T_{\mu\nu} \sim \, ... \, - \, (\zeta - \tfrac{2}{3}\vartheta)\, u^{\rho}_{\ ;\rho}(u_\mu u_\nu + g_{\mu\nu}) \, + \, ... \, . \tag{1.6.27}$$

We now observe that, at zeroth-order, $u^\rho_{\ ;\rho} \sim \mathcal{O}(1/t)$, while the first-order correction in the energy density behaves like $\mathcal{O}(1/t^x)$ and we will show the relation $1 \leqslant x < 2$ in the next Section. Since the request $x \geqslant 1$ comes out from the zeroth-order analysis, which by isotropy is independent of the shear contribution, we can conclude that, for our model, the shear viscosity would produce the inconsistency associated to the term $\mathcal{O}(1/t^{mx+1})$. The point is that the request $mx+1 \geqslant 2$ would make such a contribution dominant in the model, against the basic assumption. Thus, to include shear viscosity in a QI Model, we should choose the case $s < 1/2$ which is out of the aim of this paper since it is devoted to maximize the bulk effects in a coherent cosmological dynamics.

(3.) To conclude, we would like to discuss the question concerning the implementation of a causal thermodynamics for our cosmological model. Indeed, the hydrodynamical theory of a viscous fluid is applicable only when the spatial and temporal derivatives of the velocity of the matter are small [25, 43]. This condition is necessarily violated in the asymptotic limit near the cosmological singularity. This way, viscous fluids would have to be described by using a relaxation equation similar to the Maxwell Equation in the theory of viscoelasticity [24].

In this scheme, the EMT assumes the form

$$T_{\mu\nu} = \rho\, u_\mu u_\nu + (p+\sigma)\,(g_{\mu\nu} + u_\mu u_\nu)\ , \tag{1.6.28}$$

where p denotes the thermostatic pressure and σ is the *bulk-stress density*. In the very early Universe, the relation between σ and the relaxation time τ_0 reads as follow

$$\sigma \ + \ \dot\sigma\,\tau_0 \ = \ \zeta\, u^\rho_{\ ;\rho}\ . \tag{1.6.29}$$

The relaxation time can be expressed as $\tau_0/\zeta \ \sim \ 1/\rho$: this physical assumption follows from the fact that the transverse-wave velocity in matter has finite (non-zero) magnitude in the case of large values of ρ [24].

In this scheme, we are able to express the time dependence of τ_0. Since, at leading order, $\rho \sim 1/t^2$, we obtain, using the standard power-law for the bulk-viscosity coefficient, the following behavior for the relaxation time $\tau_0 \sim t^{2-2s}$. In our model, we deal with the case $s = 1/2$ which yields $\tau_0 \sim t$ and, if we address a power-law dependence on σ (according the structure of the solution) such as $\dot\sigma \sim \sigma/t$ [24], relation (1.6.29) rewrites as

$$\sigma = \tilde\zeta_0\, \rho^s\, u^\rho_{\ ;\rho}\ . \tag{1.6.30}$$

From this analysis, we can apply the standard expression for the bulk viscous hydrodynamic taking into account the reparameterization $\zeta_0 \to \tilde{\zeta}_0$ of the bulk coefficient.

The considerations above allow us to regard the subtle paradigm of the causal thermodynamics, having in mind that it would affect only qualitative details of our analysis, but it could not alter the validity of our results.

1.6.2 The solutions

The 00-component of Einstein Equations (1.6.3a) has been used to obtain the qualitative expression for the energy density ρ, when the matter filling the space is described by a viscous fluid EMT. We now match eq. (1.6.21) rewritten as

$$\begin{aligned}&\big[-\tfrac{3}{4}\,x\,(2-x)+\,e_0\,-\tfrac{9}{4}\,\zeta_0\,x\,\sqrt{e_0}\,\big]\;t^{-2}+\\&+\big[\tfrac{1}{2}(y-x)(y-1)+\,e_1\,-\tfrac{9}{8}\,\zeta_0\,x\,e_1\,e_0^{-1/2}-\tfrac{3}{4}(y-x)\,\zeta_0\,\sqrt{e_0}\big]\;b\,t^{y-x-2}=0\;,\end{aligned} \tag{1.6.31}$$

with the hydrodynamical ones $T^{\nu}_{\mu;\,\nu}=0$. It is worth noting that, in the non-viscous case ($\zeta_0 = 0$), the energy-density solution is determined without exploiting the hydrodynamical equations, as in [31], since ρ directly comes out from the 00-component of Gravitational Equations. In our approximation (u_α is neglected WRT u_0), the EMT conservation law provides the equation

$$\rho_{,0}+(\ln\sqrt{\gamma})_{,0}\;\big[\tfrac{4}{3}\,\rho-\zeta_0\rho^s(\ln\sqrt{\gamma})_{,0}\big]=0\;, \tag{1.6.32}$$

which can be simplified as follows

$$\begin{aligned}&\big[2e_0(x-1)-\tfrac{9}{4}\,\zeta_0\,x^2\sqrt{e_0}\,\big]\;t^{-3}+\\&\quad+\big[e_1\big(b(y-x-2)+2xb-\tfrac{9}{8}\,\zeta_0 x^2 b\,e_0^{-1/2}\;\big)+\\&\qquad+\tfrac{2}{3}(y-x)\,b\,e_0-\tfrac{3}{2}\,x(y-x)\,\zeta_0\,b\sqrt{e_0}\,\big]\;t^{y-x-3}=0\;.\end{aligned} \tag{1.6.33}$$

Eq. (1.6.31) and eq. (1.6.33) have to be combined together and solved order by order in the expansion in $1/t$ (in the asymptotic limit $t \to 0$). Since for the coherence of the solution we impose $y > x$, by solving the leading-order identities we get

$$x=\frac{1}{1-\frac{3\sqrt{3}}{4}\,\zeta_0}\;,\qquad\qquad e_0=\tfrac{3}{4}\,x^2\;. \tag{1.6.34}$$

The parameter ζ_0 has here the restriction $\zeta_0 \leqslant 4/3\sqrt{3}$ in order to satisfy the condition $x > 0$. This way the exponent of the metric power-law x runs from 1 (which corresponds to the non viscous limit $\zeta_0 = 0$) to ∞ [39, 31]. We remark that this constraint on ζ_0 arises from a zeroth-order analysis and defines the existence of a viscous Friedmann-like model, in which the early Universe has to expand with positive powers of time.

Comparing now the two first-order identities (which involve the terms proportional to t^{y-x-2} and t^{y-x-3}), we easily get an algebraic equation for the y parameter

$$y^2 - y(x+1) + 2x - 2 = 0 \ . \tag{1.6.35}$$

The solutions are $y = 2$, $y = x - 1$[9]. Obviously the second one does not ensure the condition $y > x$; hence the first-order correction to the 3-metric is characterized by the following values

$$y = 2 \ , \qquad e_1 = -\tfrac{1}{2}\,x^3 + 2x^2 - 2x \ . \tag{1.6.36}$$

It is immediate to see that, in the non viscous case $\zeta_0 = 0$, we obtain $x = 1$, $e_0 = 3/4$, $e_1 = -1/2$, which reproduce the energy-density solution (1.6.9).

By guaranteeing the consistence of the model, we now narrow the validity of the parameter x to the values which satisfy the constraint $x < y$. Thus, from (1.6.34), the QI Solution exists only if

$$\zeta_0 < \zeta_0^* = \frac{2}{3\sqrt{3}} \ , \tag{1.6.37}$$

i.e., the viscosity is sufficiently small. For values of the viscous parameter ζ_0 that overcome the critical one (ζ_0^*), the QI expansion in the asymptotic limit as $t \to 0$ can not be addressed, since perturbations would grow more rapidly than the zeroth-order terms. As one will recognize in the next Section, the study of the perturbation dynamics in a pure isotropic picture will yield a very similar asymptotic behavior when viscous effects are taken into account [39]. The Friedmann-singularity scheme is preserved only if we deal with limited values of the viscosity parameter, in particular we obtain the condition $\zeta_0^{(iso)} < \zeta_0^*/3$: this constraint is physically motivated if we consider, as it is, the Friedmann model as a particular case of the QI Solution.

Comments on the total pressure sign The solution of the unperturbed dynamics gives rise to the expression of the metric exponent x in terms of the

[9] We remark that in [31] (see eq. (34) and eq. (35) therein) this solution is found by imposing the consistence of the $\alpha\beta$-Einstein Equation and not as a pure dynamical condition derived by the solution of the perturbed hydrodynamical equation.

viscous parameter ζ_0 and to the zeroth-order expression of the energy density, which reads

$$\rho = \frac{3x^2}{4t^2} + \dots \, . \tag{1.6.38}$$

In order to characterize the effective expansion of the early Universe, let us now recall the expression of the total pressure $\tilde{p}$ (1.6.18) at leading-order:

$$\tilde{p} = \tfrac{1}{3}\,\rho + \tfrac{3}{2\,t}\,\zeta_0\,\sqrt{\rho}\,x \, , \tag{1.6.39}$$

where we have used the 4-divergence (1.6.20) truncated at zeroth-order. By using these identities, the condition $\tilde{p} \geqslant 0$ yields the inequality

$$\zeta_0 \leqslant \zeta_0^*/2 \, , \tag{1.6.40}$$

which strengths the constraint (1.6.37) and restricts the x-domain to $[1, 4/3]$.

The request to deal with a positive (at most zero) total pressure is consistent with the idea that bulk viscosity must not drastically change the standard dynamics of the isotropic Universe. In this respect, we address the domain $\zeta_0^* \leqslant \zeta_0^*/2$ as a physical restriction on the initial conditions for the existence of a well grounded QI Solution.

We here rewrite the expression of the energy density in order to analyze the density-contrast evolution. In presence of bulk viscosity, ρ assumes the form

$$\rho = \frac{3\,x^2}{4\,t^2} - \frac{(x^3/2 - 2x^2 + 2x)\,b}{t^x} \, , \tag{1.6.41}$$

and, hence, the density contrast δ can be written as

$$\delta \;=\; -\tfrac{8}{3}\,(x/4 + 1/x - 1)\,b\,t^{2-x} \, . \tag{1.6.42}$$

Since x runs from 1 to 2 as the viscosity increases towards its critical value, we note that the density contrast evolution is strongly damped by the presence of dissipative effects which act on the perturbations. In this sense, we remark that bulk viscosity can damp the evolution of perturbations forward in time. This behavior implies that the density contrast approaches the singularity, *i.e.*, $\delta = 0$, more weakly as $t \to 0$ when the viscosity runs to ζ_0^*. In correspondence with this threshold value the density contrast remains constant in time and hence it must be excluded by the possible ζ_0 choices.

The relation for the velocity and the 3-metric The 00-component of Einstein Equations provides the solution for the energy density; to perform a complete analysis of the QI Model and to verify the consistence of our approximations, we now investigate the solutions of the 0α-components of the Gravitational Equations and the spatial $\alpha\beta$- ones.

Imposing the condition $s = 1/2$, the Einstein Equation (1.6.3b) reads

$$\frac{y-x}{2\,t^{1-y+x}}\left(b_{;\alpha} - b^{\beta}_{\alpha;\beta}\right) = \frac{4}{3}\,\rho\,u_\alpha - \zeta_0\sqrt{\rho}\,u_\alpha\left(\frac{3x}{2t} + \frac{(y-x)b}{2\,t^{1-y+x}}\right)\,. \tag{1.6.43}$$

Substituting (1.6.41) in the last equation, we get the following expression for the velocity, up to the leading-order of expansion (here in particular we neglect terms of order $\mathcal{O}(t^{-1})$ and $\mathcal{O}(t^{1-x})$):

$$u_\alpha = \frac{2-x}{2x}\,(b_{,\alpha} - b^{\beta}_{\alpha;\beta})\;t^{3-x}\,. \tag{1.6.44}$$

It is worth noting that, in our generalization, the assumption $u_0^2 \simeq 1$ is well verified, since we immediately see that $u_\alpha u^\beta \sim t^{6-3x}$, which can be neglected in the 4-velocity contraction $u_\mu u^\mu = -1$; hence the approximated hydrodynamical equation (1.6.33) is still self-consistent using this expression of u_α.

Let us now write down eq. (1.6.3c): here, the first two leading-orders of the RHS are $\mathcal{O}(t^{-2})$ and $\mathcal{O}(t^{-x})$ respectively only if $x < 2$, like in our scheme; hence $u_\alpha u^\beta$ is neglected, as seen before, $\mathcal{O}(t^{-2})$ terms cancel each other, while those proportional to t^{-x} give the following equation (which generalize eq. (1.6.11))

$$\bar{P}^{\beta}_{\alpha} \;+\; A\,b^{\beta}_{\alpha} \;+\; B\,b\,\delta^{\beta}_{\alpha} \;+\; C\,\delta^{\beta}_{\alpha} = 0\,, \tag{1.6.45}$$

where the quantities A, B, C are defined as

$$A = \tfrac{1}{4}\,(4-x^2)\,,\quad B = \tfrac{1}{6}(2x-1)(x-2)^2 - \tfrac{1}{4}\,x(x-2)\,,\quad C = -\tfrac{1}{6}(2-x)(x-1)\,, \tag{1.6.46}$$

respectively. Taking the trace of (1.6.45), we obtain the relation $(A+3B)\,b = -\bar{P} - 3C$ which provides the following equation

$$2A\,b^{\beta}_{\alpha;\beta} = (A+B)\,b_{,\alpha}\,, \tag{1.6.47}$$

when combined with the Ricci 3-tensor relation $\bar{P}^{\beta}_{\alpha;\beta} = 1/2\,\bar{P}_{;\beta}$.

Therefore we are now able to write down the final form of the 3-velocity related to the perturbed metric-tensor trace b :

$$u_\alpha = \frac{2-x}{4xA}\,(A-B)\;t^{3-x}\,b_{,\alpha}\,. \tag{1.6.48}$$

As it can be easily checked, the solution here constructed matches the non-viscous one (1.6.12) if we set $\zeta_0 = 0$ and it is completely self-consistent up to the first two orders in time. As in the original analysis, the present model contains only three physically arbitrary functions of the spatial coordinates, *i.e.*, the six functions $a_{\alpha\beta}$ minus three DOF ruled out by fixing suitable space coordinates. The only remaining free parameter of the model is viscous one, ζ_0.

1.7 The pure FLRW isotropic model

In this Section, we investigate the effects that bulk viscosity has on the stability of the pure isotropic Universe[10,11]. This scheme corresponds to a particular case of the QI Model previously discussed. In particular, the Taylor expansion of the 3-metric results here to be truncated only to the zeroth-order and, hence, the analysis of the gravitational instability requires a standard perturbative theory. In this respect, the dynamics of cosmological perturbations is analyzed when viscous phenomena affect the zeroth- and first-order evolution of the system. We consider a background corresponding to a FLRW model filled with ultra-relativistic viscous matter, whose coefficient ζ corresponds to the choice $s = 1/2$ and then we develop a perturbative theory which generalizes the E.M. Lifshitz works [14, 44] to the presence of bulk viscosity. Though the analysis is performed for the case of a flat model, nevertheless it holds in general as soon as the perturbation scale remains much smaller than the Universe radius of curvature. In this respect, we deal with perturbations such that $\eta q \ll 1$ ($2\pi/q$ being the size of the coordinate scale and η the conformal time variable). Since the dynamics we consider holds near the singularity for $\eta \ll 1$, then we make allowance for arbitrarily large values of q and therefore the condition for the general validity $q \gg 2\pi|\mathcal{K}|^{1/2}$ (we recall that $\mathcal{K}$ indicates the FLRW curvature parameter) can be always fulfilled.

As result, the analytic expression of the density contrast shows that, for small values of the parameter ζ_0, its behavior is not significantly different from the non-viscous one derived by Lifshitz [14]. But as soon as ζ_0 overcomes a critical value, the growth of the density contrast is suppressed forward in time by viscosity and the stability of the Universe is favored in the expanding picture. On the other hand, in such a regime, the asymptotic approach to the initial singularity (taken at $t = 0$) is deeply modified by the apparency of significant viscosity in the primordial thermal bath, *i.e.*, the isotropic and homogeneous Universe admits an unstable collapsing picture. In our model, this feature regards also scalar perturbations while in the non-viscous case it appears only for tensor modes. Since a reliable estimation [1] fixes the appearance

[10] NC and G. Montani, "Gravitational Stability and Bulk Cosmology", *AIP Conf. Proc.* **966**, 241 (2007).

[11] NC and G. Montani, "On the Role of Viscosity in Early Cosmology", *Int. J. Mod. Phys. A* **23**(8), 1248 (2008).

of thermal bath into the equilibrium below temperatures $\mathcal{O}(10^{16}GeV)$ and this limit corresponds to the pre-inflationary age, our result supports the idea that an isotropic universe outcomes only after a vacuum phase transition settled down.

Perturbative Theory to the Einstein Equations In order to describe the temporal evolution of the energy-density small fluctuations, we develop a perturbative theory on the Einstein Equations. We limit our work to the study of space regions having small dimensions compared with the scale factor of the Universe a [15]. According to this approximation, we can consider a 3-dimensional Euclidean (time dependent) metric as the spatial component of the background line element

$$ds^2 = dt^2 - a^2\left(dx^2 + dy^2 + dz^2\right) . \tag{1.7.1}$$

In linear approximation, perturbed Einstein Equations write as

$$\delta R^{\nu}_{\mu} - \tfrac{1}{2}\delta^{\nu}_{\mu}\delta R = 8\pi G \delta T^{\nu}_{\mu} \ , \tag{1.7.2}$$

where the term δT^{ν}_{μ} represents the perturbation of the EMT. The perturbations of the Ricci tensor δR^{ν}_{μ} can be written in terms of metric perturbations

$$h^{\nu}_{\mu} = -\delta g^{\nu}_{\mu} \ , \tag{1.7.3}$$

starting from the general expression for the perturbed curvature tensor [15], *i.e.*,

$$\delta R^{\sigma}_{\mu\nu\rho} = \tfrac{1}{2}\left(h^{\sigma}_{\mu;\,\rho;\,\nu} + h^{\sigma}_{\rho;\,\mu;\,\nu} - h^{;\,\sigma}_{\mu\rho;\,\nu} - h^{\sigma}_{\mu;\,\nu;\,\rho} - h^{\sigma}_{\nu;\,\mu;\,\rho} + h^{;\,\sigma}_{\mu\nu;\,\rho}\right) . \tag{1.7.4}$$

For convenience, let us now introduce a new temporal variable η, set by the relation

$$dt = a\, d\eta \ , \tag{1.7.5}$$

and use the symbol ($'$) for its derivatives; we moreover impose, without loss of generality, that the synchronous reference system is still preserved in the perturbations scheme

$$h_{00} = h_{0\alpha} = 0 \ . \tag{1.7.6}$$

With the assumptions above, the perturbations of the Ricci tensor and of the curvature scalar read:

$$\delta R_0^0 = -\frac{1}{2a^2}\, h'' - \frac{a'}{2a^3}\, h' \;, \tag{1.7.7a}$$

$$\delta R_0^\alpha = \frac{1}{2a^2}\, (h^{,\alpha\,\prime} - h_\beta^{\alpha,\beta\,\prime})\,, \tag{1.7.7b}$$

$$\begin{aligned}\delta R_\alpha^\beta = &-\frac{1}{2a^2}\,(h_{\alpha,\gamma}^{\gamma,\beta} + h_{\gamma,\alpha}^{\beta,\gamma} - h_{\alpha,\gamma}^{\beta,\gamma} - h_{,\alpha}^{,\beta}) + \\ &- \frac{1}{2a^2}\, h_\alpha^{\beta\,\prime\prime} - \frac{a'}{a^3}\, h_\alpha^{\beta\,\prime} - \frac{a'}{2a^3}\, h'\, \delta_\alpha^\beta \;,\end{aligned} \tag{1.7.7c}$$

$$\delta R = -\frac{1}{a^2}\,(h_{\alpha,\gamma}^{\gamma,\alpha} - h_{,\gamma}^{,\gamma}) - \frac{1}{a^2}\, h'' - \frac{3a'}{a^3}\, h' \;. \tag{1.7.7d}$$

By using these expressions, we are able to rewrite the LHS of Einstein Equations (1.7.2) through the metric perturbations h_β^α.

Dynamical Representation of Perturbations Since we use an Euclidean background metric (1.7.1), we can expand the perturbations in plane waves of the form $e^{i\,\mathbf{q}\cdot\mathbf{r}}$, where $\mathbf{q}$ (of components q_α ($q = |\mathbf{q}|$)) is the dimensionless co-moving wave vector being the physical one $\mathbf{k} = \mathbf{q}/a$ ($k = |\mathbf{k}|$). Here we investigate the gravitational stability properly described by the behavior of the energy-density perturbations expressible only by a scalar function; in this sense we have to choose a scalar representation of the metric perturbations [14, 15]. Such a picture is made by the scalar harmonics

$$Q = e^{i\mathbf{q}\cdot\mathbf{r}} \;, \tag{1.7.8}$$

from which the following tensors

$$Q_\alpha^\beta = \tfrac{1}{3}\delta_\alpha^\beta Q \;, \qquad P_\alpha^\beta = [\tfrac{1}{3}\delta_\alpha^\beta - \tfrac{q_\alpha q^\beta}{q^2}]Q \;, \tag{1.7.9}$$

can be constructed. We can now express the time dependence of the gravitational perturbations through two functions $\lambda(\eta)$, $\mu(\eta)$ and write the tensor h_α^β in the form

$$h_\alpha^\beta = \lambda(\eta) P_\alpha^\beta + \mu(\eta) Q_\alpha^\beta \;, \qquad h = \mu(\eta) Q \;. \tag{1.7.10}$$

Review of the Lifshitz analysis In the standard analyses the Universe is assumed, in its primordial expansion, to behave like a perfect fluid. This hypothesis can be expressed writing the EMT tensor in the form (1.2.9), *i.e.*,

$$T_\mu^{\;\nu\,(P)} = (\rho + p) u_\mu u^\nu \, - \, p\,\delta_\mu^\nu \;, \tag{1.7.11}$$

where u_μ is its 4-velocity, expressed in the co-moving system we consider

$$u^0 = 1/a \qquad u^\alpha = 0 \; . \tag{1.7.12}$$

Using the synchronous character of the perturbed metric, perturbations to the above EMT write

$$\delta T_0^0 = \delta\rho \; , \qquad \delta T_0^\alpha = a\,(p_0 + \rho_0)\,\delta u^\alpha \; , \qquad \delta T_\alpha^\beta = -\delta_\alpha^\beta\, v_s^2 \delta\rho \; , \tag{1.7.13}$$

where the standard expansions $\rho = \rho_0 + \delta\rho$ and $p = p_0 + \delta p$ are used and δu^α is the 3-velocity perturbation (being $v_s^2 = \delta p/\delta\rho$). In this scheme, using the expression above for the 4-velocity and eq. (1.7.6), one obtain the relation

$$\delta u^0 = 0 \; . \tag{1.7.14}$$

Let us now consider the primordial stages of the Universe expansion, *i.e.*, $\eta \ll 1$, when the radiation-like density dominates the matter one. The EoS is $p_0 = \rho_0/3$, from which the relations (for a flat Universe $\mathcal{K} = 0$) arise

$$\rho_0 = C a^{-4} \; , \qquad a = a_1 \eta \; , \qquad v_s^2 = 1/3 \; , \tag{1.7.15}$$

where C is an integration constant and $a_1 = \sqrt{8\pi G C/3}$. Such expression generalize to conformal time the standard ones (1.5.4). In this approximation, we can obtain the basic equations which describe the temporal evolution of the perturbations. Expressing eq. (1.7.7) through the representation (1.7.10) and using expressions (1.7.13) in the form

$$\delta T_\alpha^\beta = -\delta_\alpha^\beta v_s^2 \delta T_0^0 \; , \tag{1.7.16}$$

the perturbed Einstein Equations give, for $\alpha \neq \beta$ and for contraction over these indexes, two equations for the metric perturbations, respectively

$$\lambda'' + \frac{2}{\eta}\,\lambda' - \frac{q^2}{3}(\lambda + \mu) = 0 \; , \qquad \mu'' + \frac{3}{\eta}\,\mu' + \frac{2q^2}{3}(\lambda + \mu) = 0 \; . \tag{1.7.17}$$

Furthermore, taking the 00-components of (1.7.2), we can express the energy density directly from the adopted functions λ and μ, in the form

$$\delta\rho = \frac{Q}{24\pi G a^2}\left[q^2(\lambda + \mu) + \frac{3a'}{a}\,\mu'\right] \; . \tag{1.7.18}$$

Among the solutions, there are some which can be removed by a simple transformation of the reference system (compatible with its synchronous character) and therefore they do not represent any real physical change in the metric. The corresponding expression for the metric perturbations can be established, *a priori*, through a coordinates transformation [15], taking into account the constraint (1.7.6):

$$\bar{h}_{\alpha}^{\beta} = f_{0,\alpha}^{;\beta} \int \frac{d\eta}{a} + \frac{a'}{a^2} f_0 \, \delta_{\alpha}^{\beta} + \left(f_{\alpha}^{;\beta} + f_{,\alpha}^{\beta} \right) \; , \tag{1.7.19}$$

where f_0, f_α are arbitrary (small) functions of the coordinates.

In the assumption $\eta\, q \ll 1$, the eq. (1.7.17) admit, to the leading-order, the solutions

$$\lambda = \frac{3C_1}{\eta} + C_2 \; , \qquad \mu = -\frac{2q^2}{3} C_1 \, \eta + C_2 \; , \tag{1.7.20}$$

where the fictitious solutions (1.7.19), which in our ultrarelativistic approach assume the form $\lambda - \mu = const$ ($f_0 = 0$, $f_\alpha = P_\alpha$) and $\lambda + \mu \sim 1/\eta^2$ ($f_0 = Q$, $f_\alpha = 0$), are excluded. The final expressions for the gravitational perturbations and for the density contrast $\delta = \delta\rho/\rho_0$ can be obtained substituting this solutions in eq. (1.7.10) and (1.7.18)

$$h_{\alpha}^{\beta} = \frac{3C_1}{\eta} P_{\alpha}^{\beta} + C_2 (Q_{\alpha}^{\beta} + P_{\alpha}^{\beta}) \tag{1.7.21}$$

$$\delta = \frac{q^2}{9} (C_1 \eta + C_2 \eta^2) Q \; . \tag{1.7.22}$$

Here the constants C_1, C_2 must satisfy the conditions expressing the smallness of the perturbations at the moment η_0 when they arise; assuming that harmonics Q are of the unity order magnitude, the inequalities $\lambda \ll 1$, $\mu \ll 1$ give the constraints $C_1 \ll \eta_0 \ll 1$ and $C_2 \ll 1$.

The expression of the cosmological perturbation (1.7.22) contains terms which increase, in an expanding Universe, proportionally to positive powers of the scale-factor $a = a_1 \eta$. This expansion can not, nevertheless, imply the gravitational instability: if we consider the magnitude order $\eta \sim 1/q$, the conditions satisfied by the constants C_1, C_2 imply that the density perturbation remains small even in the higher-order of approximation. This behavior of the cosmological fluctuation yields the gravitational stability of the primordial Universe; the only stability we can found in a non-viscous Universe [14] is provided by the tensor perturbations h_{β}^{α} and takes place approaching backward the Big Bang.

Unperturbed viscous cosmology As already discussed, the presence of dissipative processes within the Universe dynamics, as it is expected at temperatures above

$\mathcal{O}(10^{16}GeV)$, can be expressed by an additional term in the standard ideal fluid EMT (see eqs. (1.2.16)):

$$T_{\mu\nu} = (\tilde{p} + \rho)u_\mu u_\nu - \tilde{p}\, g_{\mu\nu}\,, \qquad \tilde{p} = p - \zeta\, u^\rho_{;\rho}\,, \tag{1.7.23}$$

Furthermore, in a co-moving system the 4-velocity can be expressed as $u^0 = {}^1\!/_a,\ u^\alpha = 0$ and the viscous pressure $\tilde{p}$ assumes the form

$$\tilde{p} = p - 3\,\zeta_0\,\rho^s\,\frac{a'}{a^2}\,. \tag{1.7.24}$$

Let us now consider the earlier stages of a flat Universe corresponding to $\eta \ll 1$. The Universe zeroth-order dynamics is described by the energy conservation equation and the Friedmann one (see eq. (1.5.2) and eq. (1.5.3)), written using the conformal time, *i.e.*,

$$\rho' + 3\,\frac{a'}{a}\,(\,\rho + \tilde{p}\,) = 0\,, \qquad \frac{a'}{a^2} \;=\; \sqrt{{}^8\!/_3\,\pi G\rho}\,. \tag{1.7.25}$$

As discussed in the previous Section, in this analysis, we assume $s = {}^1\!/_2$, in order to deal with the maximum effect that bulk viscosity can have without dominating the Universe dynamics since it corresponds to a phenomenological issue of perturbations to the thermodynamical equilibrium [36, 37]. The solutions of the zeroth-order dynamics, for $s = {}^1\!/_2$ and $p = p_0 = \rho_0/3$, assume the form

$$\rho_0 = C a^{-(2+2\omega)}\,, \qquad a = a_1\,\eta^{1/\omega}\,, \qquad \omega = 1 - \chi\,\zeta_0\,, \tag{1.7.26}$$

being C an integration constant, $\chi = \sqrt{54\pi G}$ and $a_1 = (8\omega^2\pi CG/3)^{1/2\omega}$. We also obtain the relation

$$\tilde{p}_0 = \frac{\rho_0}{3} - 3\,\zeta_0\,\rho_0^s\,\frac{a'}{a^2}\,. \tag{1.7.27}$$

Since we consider an expanding Universe, the factor a must increase with positive power of the temporal variable (*i.e.*, $\omega > 0$) thus we obtain the constraint

$$0 \leqslant \zeta_0 < {}^1\!/\chi\,, \tag{1.7.28}$$

which ensures this feature.

Perturbative theory in the viscous case Let us now perturb the viscous EMT. Using the synchronous character of the perturbed metric we get the following expressions

$$\delta T^0_0 = \delta\rho\,, \quad \delta T^\alpha_0 = a\,(\tilde{p}_0 + \rho_0)\,\delta u^\alpha\,, \tag{1.7.29a}$$

$$\delta T^\beta_\alpha = \delta^\beta_\alpha\left[-\Sigma^2\delta\rho + \zeta\left(\delta u^\gamma_{,\gamma} + h'/2a^2\right)\right]\,, \tag{1.7.29b}$$

where

$$\Sigma^2 \equiv v_s^2 - 3\zeta_0\, s\rho_0^{s-1}\, a'/a^2 \; . \tag{1.7.30}$$

The presence of viscosity does not influence the expression of the Ricci tensor and its perturbations, thus we can still keep expressions (1.7.7) and use the perturbed form of the EMT to build up the equations which describe the dynamics of h^β_α and $\delta\rho$. It is convenient to choose, as final equations, the ones obtained from the Einstein ones for $\alpha \neq \beta$ and for contraction over α and β, which read respectively

$$\left(h^{\gamma,\beta}_{\alpha,\gamma} + h^{\beta,\gamma}_{\gamma,\alpha} - h^{\beta,\gamma}_{\alpha,\gamma} - h^{,\beta}_{,\alpha}\right) + h^{\beta\,\prime\prime}_\alpha + \tfrac{2a'}{a}\, h^{\beta\,\prime}_\alpha = 0\,, \tag{1.7.31}$$

$$\begin{aligned} &\tfrac{1}{2}\left(h^{\gamma,\alpha}_{\alpha,\gamma} - h^{,\gamma}_{,\gamma}\right)\left(1+3\Sigma^2\right) + h'' + \\ &\quad + \frac{a'}{a}\left(2 + 3\Sigma^2 - 12\pi G\frac{a}{a'}\,\zeta\right) h' + \\ &\qquad - \frac{3\zeta}{2a(\tilde{p}_0+\rho_0)}\left(h^{,\alpha\,\prime}_{,\alpha} - h^{\gamma,\alpha\,\prime}_{\alpha,\gamma}\right) = 0\,. \end{aligned} \tag{1.7.32}$$

Taking the 00-component of Gravitational Equations, the expression of the density perturbations (1.7.18) can be addressed, *i.e.*,

$$\delta\rho = \frac{1}{16\pi G a^2}\left(h^{\gamma,\alpha}_{\alpha,\gamma} - h^{,\alpha}_{,\alpha} + \tfrac{2a'}{a}\, h'\right)\,. \tag{1.7.33}$$

Furthermore the form of fictitious solutions (1.7.19) is the same also in presence of dissipative processes because they are founded by a transformation of synchronous reference system.

Substituting in eq. (1.7.31) and eq. (1.7.32) the zeroth-order solutions (1.7.26) and the scalar representation of the metric perturbations (1.7.10), we can get, respectively, two equations for λ, μ which read

$$\lambda'' + \frac{2}{\omega\eta}\,\lambda' - \frac{q^2}{3}\,(\lambda+\mu) = 0\,, \tag{1.7.34}$$

$$\begin{aligned} &\mu'' + \left(\frac{2+3\Sigma^2}{\omega\eta}\right)\mu' - \left(\frac{12\pi\sqrt{C}G\zeta_0}{a_1^{1+\omega}\,\eta^{1+1/\omega}}\right)\mu' + \\ &\qquad + \tfrac{q^2}{3}\,(\lambda+\mu)\left(1+3\Sigma^2\right) + \frac{q^2\zeta_0\eta\,(\mu'+\lambda')}{4\sqrt{C}/3a_1^\omega - 3\zeta_0/\omega} = 0\,. \end{aligned} \tag{1.7.35}$$

1.7.1 The problem of the singularity

Let us now study the gravitational-collapse dynamics of the primordial Universe near the initial Big Bang, in the limit $\eta \ll 1$. As in Lifshitz work [14], we analyze the case of perturbation scales sufficiently large to use the approximation $\eta q \ll 1$. In our scheme, eqs. (1.7.34) and (1.7.35) admit asymptotic analytic solutions for the functions λ and μ; in the leading-order λ takes the form

$$\lambda = \frac{C_1}{\eta^{2/\omega-1}} + C_2 \,, \tag{1.7.36}$$

where C_1, C_2 are two integration constants. Substituting this expression in eq. (1.7.35) we get, in the same order of approximation, the behavior of the function μ as

$$\mu = \frac{\tilde{C}_1}{\eta^{1/\omega-3}} + C_2 \,, \tag{1.7.37}$$

where we have excluded the non-physical solutions (1.7.19) as written in the form $\lambda - \mu = const$. The constant $\tilde{C}_1$ is given by the expression $\tilde{C}_1 = {}^{A}/_{B}(3 - {}^{1}/_{\omega})$, A and B being constants having the form

$$A = \frac{C_1 q^2}{3}\left(1 + 3\Sigma^2\right) + \frac{C_1(1-2/\omega)q^2\zeta_0}{4\sqrt{C}/3a_1^{\omega} - 3\zeta_0/\omega} \,, \qquad B = \frac{12\pi\sqrt{C}G\zeta_0}{a_1^{1+\omega}} \,.$$

Let us now write the final form of perturbations, pointing out their temporal dependence in the viscous Universe. The metric perturbations (1.7.10) become

$$h_\alpha^\beta = \frac{C_1}{\eta^{2/\omega-1}} P_\alpha^\beta + \frac{\tilde{C}_1}{\eta^{1/\omega-3}} Q_\alpha^\beta + C_2\left(Q_\alpha^\beta + P_\alpha^\beta\right) \,, \tag{1.7.38}$$

and, by (1.7.33) and (1.7.26), the density contrast reads

$$\delta = F[C_1\eta^{3-2/\omega} + C_2\eta^2 + C_3\eta^{3-1/\omega} + \tilde{C}_1\eta^{5-1/\omega}], \tag{1.7.39}$$

where $C_3 = 3A/q^2\omega B$ and $F = \omega^2 Q q^2/9$.

As in the non viscous case, we have now to impose the conditions expressing the smallness of perturbations at a given initial time η_0. The inequalities $h_\alpha^\beta \ll 1$ and $\delta \ll 1$ yield only two fundamental constraints for the integration constants:

$$C_1 \ll \eta_0^{2/\omega-1} \,, \qquad C_2 \ll 1 \,, \tag{1.7.40}$$

for any ω-value within the interval $(0,\, 1]$. Furthermore, we find an additional condition which involves the wave number q and the integration constant C; in particular a rough estimation for $\omega < 1/3$ of the inequalities $\tilde{C}_1 \ll \eta_0^{1/\omega-3}$ and $C_3 \ll \eta_0^{1/\omega-3}$ yields the condition $q \ll (GC\eta_0)^{-1/2\omega}$ which ensures the smallness of the cosmological perturbations.

Using the hypothesis $\eta \ll 1$, we can get the asymptotic form of the corrections to the cosmological background. The exponents of the variable η can be positive or negative according to the value of the viscous parameter $\omega(\zeta_0)$. This behavior produces two different regimes of the density-contrast evolution:

***(1.)* Case** $0 \leqslant \zeta_0 < {}^1\!/_{3\chi}$: Here perturbations increase forward in time. This behavior corresponds qualitatively to the same picture of the non-viscous Universe (obtained setting $\zeta_0 = 0$) in which the expansion can not, nevertheless, imply the gravitational instability: if we consider the magnitude order $\eta \sim 1/q$, the constraints on C_1, C_2 imply that δ remains small even in the higher order of approximation. This behavior yields the gravitational stability of the primordial Universe.

***(2.)* Case** ${}^1\!/_{3\chi} < \zeta_0 < {}^1\!/_{\chi}$: In this regime, the density contrast is suppressed behaving like a negative power of η. When the density contrast results to be increasing, the presence of viscosity induces a *damping* of the perturbation evolution in the direction of the expanding Universe, so the cosmological stability is fortified since the leading η powers are smaller than the non-viscous ones obtained setting $\zeta_0 = 0$.

In this case, density fluctuations decrease forward in time but the most interesting result is the instability which the isotropic and homogeneous Universe acquires in the direction of the collapse toward the Big Bang. For $\zeta_0 > {}^1\!/_{3\chi}$ the density contrast diverges approaching the cosmological singularity, *i.e.*, for $\eta \to 0$. In this regime, scalar perturbations destroy asymptotically the primordial Universe symmetry. The dynamical implication of this issue is that an isotropic and homogeneous stage of the Universe can not be generated, from generic initial conditions, as far as the viscosity becomes smaller than the critical value

$$\zeta_0^{(iso)} = 1/3\chi \,. \tag{1.7.41}$$

It worth underlining that this threshold value, by considering suitable units such that $8\pi G = 1$ in the Einstein Equations, can be rewritten as $\zeta_0^{(iso)} = 2/9\sqrt{3}$. This conditions corresponds to

$$\zeta_0^{(iso)} = \tfrac{1}{3}\,\zeta_0^* \,, \tag{1.7.42}$$

where ζ_0^* is the validity threshold of the QI Model, see eq. (1.6.37). In this respect, we underline that the perturbed FLRW model, here proposed, corresponds to a special case of the QI general analysis and this characterization is summarized by the constraint above.

1.8 Concluding remarks

In this Chapter, we have discussed the dynamics of the gravitational collapses both in Newtonian approximation and in the pure relativistic limit in presence of bulk viscosity. All the models proposed have been analyzed in a first-order perturbative regime starting from the background-fluid evolution and then adding small non-homogeneous fluctuations into the dynamics. The viscosity has been expressed, following an hydrodynamical description of the fluid, as a power-law of the the energy (or matter) density and the effects on the Newtonian motion equations and on the EMT source have been analyzed.

Five different cases have been studied: three distinct unperturbed solutions of the Newtonian dynamics, where viscosity has been assumed to affect only the first-order analysis, and two pure relativistic models.

— In Section 1.3, the standard Jeans Model with a static and uniform background-solution has been analyzed. The main result, in dealing with the viscous generalization, has been to show how bulk viscosity damps the density contrast evolution, maintaining unchanged the threshold value of the Jeans Mass. Such an effect suppresses the sub-structure formation in the top-down fragmentation mechanism. In particular, a new decreasing regime for perturbations has been found. The presence of such a behavior has induced to the study of the top-down scheme for small and strong viscous effects. In the first case, the density-perturbation amplitude of a sub-structure remains substantially constant during the main structure collapse. On the other hand, if viscous effects are sufficiently strong, the sub-structure vanish in the linear perturbative regime, unfavoring the fragmentation.

— In Section 1.4, our analysis outlined how the presence of bulk viscosity induces a deep modification of the extreme gravitational collapse relative to an uniform, spherically-symmetric and dust-like gas cloud. While the isothermal-like collapse is characterized by sub-structure formation even when viscous effects are taken into account, the adiabatic-like one undergoes an opposite asymptotic regime as soon as the viscosity become sufficiently intense. Though bulk viscosity does not affect (by hypothesis) the extreme collapse of the background flow, nevertheless its presence changes drastically the dynamics of perturbations which are damped at the point to generate vanishing density contrasts. Thus, in the adiabatic case, the fate of a collapsing cloud is sensitive to the viscous effects by itself induced. In particular, bulk viscosity is able to restore a kind of Jeans length for the cloud perturbations; scales above this threshold begin to collapse but, if below the second threshold, no sub-structures formation takes place.

— In Section 1.5, the effects induced by the presence of bulk viscosity have been analyzed in the generalization of the Jeans Model, in treating an expanding Universe background. In this case, the static and uniform background solution for the unperturbed evolution, proper of the Jeans analysis, has been generalized by the dynamics of an homogeneous and isotropic Friedmann Universe. In this scheme, a Jeans-like relation has been obtained and a considerable damping of the density-contrast growth has been found.

— In Section 1.6, the QI Solution in the asymptotic limit near cosmological singularity has been studied. The investigation started from the modification of the Einstein Equations, induced by a viscous matter term and then proceeded by the integration of the new Gravitational Equations matched together with the hydrodynamical ones, order by order in the $1/t$ expansion. As a main result, we have shown that the QI Solution exists only for particular values of the bulk-viscosity coefficient. When the dissipative effects become too relevant, we are not able to construct the solution following the line of the original Lifshitz-Khalatnikov model. In fact, when the viscosity coefficient approaches a threshold value, the approximation scheme breaks down and the model becomes non self-consistent. By requiring that the viscosity parameter be under its critical value, we have also outlined how the behavior of the density contrast is deeply influenced by the presence of bulk viscosity. In fact, as far as dissipative effects are taken into account, the density-contrast contraction ($\delta \to 0$ as $t \to 0$), is damped until remaining constant if such a parameter assumes its critical value.

We conclude by stressing that our result is relevant near the singularity, where the volume of the QI Universe changes rapidly and as a consequence, the cosmological fluid has to follow this rapid variation by subsequent stages of thermal equilibrium. Then bulk viscosity emerges from the average non-equilibrium effects and it is expected to be increasingly relevant, when the singularity is approached.

— In Section 1.7, the main issue of our investigation is to have shown that the isotropic Universe acquires, backward in time, a regime of instability corresponding to sufficiently high values of the viscous parameter. Such a window of instability implies that, if the Universe was born sufficiently far from the homogeneous and isotropic stage, then bulk viscosity (*i.e.*, the absence of a stable thermal equilibrium) works against isotropization mechanisms and the inflation becomes the scenario from which a FLRW geometry arises (at least on a given scale). The explanation of this result is in the real physical meaning of the bulk viscosity: such viscous effects come out from the difficulty that microphysics finds to restore the thermal equilibrium against the rapid Universe expansion. As a natural consequence of this physical context, bulk viscosity makes unfavored the establishment of an homogeneous stage from a general cosmological dynamics. On the other hand in a FLRW Universe, already settled down, we expect that, as we find, the viscous effects depress the density contrast because the particles inside the inhomogeneous fluctuations undergo dissipative processes which frozen the growth of the structures.

Despite of the reliable feature of our results, the present investigation, as well as the whole previous literature on this subject, relies on a phenomenological ground; in fact the description of the viscous effects is based on the constitutive equation relating the viscosity coefficient to a power-law of the system energy-density. This statement appears well-grounded, but nevertheless it requires to be carefully considered in a precise derivation of the viscosity coefficient from a real kinetic theory of matter. We will address for such a point in a further investigation, which will be aimed to yield an upgrading of the present cosmological issue.

2 Torsion Effects in Non-Einsteinian Space-Time

2.1 General statements

Torsion represents the most natural extension of General Relativity (GR) and it attracted interest over the years in view of its link with the fundamental properties of particle motion.

The torsion field was taken into consideration chiefly by É. Cartan [45, 46, 47]. The usual version of Einstein-Cartan Theory (ECT) [48, 49] is based on the standard Einstein-Hilbert Action, where the scalar curvature is a function of both metric and torsion. From variational principles, field equations are obtained in presence of matter and it can be pointed out that, in such a theory, torsion is not really a dynamical field in the same sense as the metric field. Recent studies on the coupling of torsion with spin matter are those in [48, 49, 50, 51, 52, 53]. In the U^4 theory [48], torsion corresponds to the rotation gauge potential, and it is related to the intrinsic angular momentum of matter. In Poincaré Gauge Theory (PGT) [54, 55, 56], torsion and bein vectors are the gauge fields that account for local Poincaré transformations. These descriptions predict a non-propagating torsion field, so that only a contact interaction is expected, because the equations of motion are algebraic rather than differential.

After reviewing the most popular approaches to torsion gravity, we will propose a microscopic and macroscopic paradigm to describe the role of the torsion field, as far as a propagating feature of the resulting dynamics is concerned. In both these schemes, the dynamics of torsion will acquire particular features that imply interesting perspectives about it detection.

The two proposals deal with distinct schemes: a macroscopic approach, based on the construction of suitable potentials for the torsion field, and a microscopic one, which relies on the identification of torsion with the fields which enter the dynamics of a generalized gauge theory picture of the Lorentz Group. We analyze in some details both points of view and their implications on the coupling between torsion and matter. In particular, in the macroscopic case, we analyze the test-particle motion to determine the physical trajectories. On the other hand, in the microscopic scheme, a study of the coupling between torsion and fermion fields is performed.

In Section 2.2, a general overview on several approaches to torsion field is discussed. After introducing the basic concepts and definitions, an analysis of the Einstein metric gravity and of the ECT is addressed. Then we focus the attention on the propagating description of the torsion field including torsion potentials into the dynamics. Such (classical) macroscopic features are at the ground of our subsequent analysis. A discussion on the several gauge approaches to gravity follows. In particular, we consider the ordinary tetradic formalism of gravity, PGT and teleparallel theory as the main microscopic approaches. At the end of this Section, we introduce an analysis of the early Universe in presence of torsion as a possible link between the dissipative cosmologies developed in the previous Chapter.

In Section 2.3, the macroscopic approach is developed by some assumptions about the form of the torsion tensor: the completely anti-symmetric and trace part of the tensor are considered to derive from two local torsion potential. As original result, the motion equation of test particles are determined as the *Autoparallels* and both the non-relativistic limit of these trajectories and of the tidal effects show that the torsion trace potential enters all the equations in the same way as the gravitational one.

In Section 2.4, propagating torsion will be also derived form a microscopic point of view. In fact, we propose a gauge theory of the group $SO(3,1)$ on flat Minkowskian space-time which allow us to identify new connections with torsion, as soon as we postulate to direct generalize the picture on curved space-time. The comparison of First- and Second-Order Approaches will be explained in the linearized regime, where the role of the gravitational field as a source for torsion will be compared with the spin-current term of the Second-Order Formalism. An analysis of the effects of the new connections in flat space-time is also addressed giving a modification of the well-known *Pauli Equation*.

Note - During this Chapter, we denote with the symbol ($\tilde{}$) all torsion-dependent tensor quantities.

2.2 The torsion field: an overview

As well known, in non-flat spaces the concept of the parallel transport of vector fields needs the introduction of connections which also define the covariant derivative. By means of connections, we can define the equation of the parallel transport as follows: on a general 4-dimensional manifold M^4, given a curve $\gamma(t)$ passing for a point $P \in M^4$, the parallel transported vector of the vector field $V^\rho(P)$ along $\gamma(t)$ is the solution of the equation

$$\frac{dV^\rho}{dt} = -\tilde{\Gamma}^\rho_{\mu\nu} V^\nu \dot{\gamma}^\mu \ , \tag{2.2.1}$$

where the $\tilde{\Gamma}^\rho_{\mu\nu}$'s denote *general affine connections*. Moreover, the general *covariant derivative* $\tilde{\nabla}_\mu$ of a tensor field $V^\rho_\nu(x)$ is defined as

$$\tilde{\nabla}_\mu V^\rho_\nu = \partial_\mu V^\rho_\nu + \tilde{\Gamma}^\rho_{\mu\sigma} V^\sigma_\nu - \tilde{\Gamma}^\sigma_{\mu\nu} V^\rho_\sigma \ , \tag{2.2.2}$$

where ∂_μ indicates the ordinary partial derivative. In fact, in order to compute the derivative of a vector, it must be evaluated at two different space-time points and it is therefore necessary to transport the displaced vector back to its original position for comparison. In particular, if the vector V^ρ is parallel transported along the infinitesimal dx^μ, the change due to this transport is given by $-\tilde{\Gamma}^\rho_{\mu\nu} V^\nu dx^\mu$, which leads to the correct definition of the covariant derivative. In this respect, one may define the *curvature tensor* as the result of parallel transporting a vector V^ρ around a closed path ξ^μ,

$$\Delta V^\rho = \tfrac{1}{2} \, V^\nu \, \tilde{R}^\rho_{\sigma\mu\nu} \oint \xi^\mu dx^\sigma \ . \tag{2.2.3}$$

Let us now suppose to transport the infinitesimal vector l^ρ along m^ρ and compare that WRT transporting m^ρ along l^ρ. We define the vector $A^\rho = l^\rho + m^\rho - \tilde{\Gamma}^\rho_{\mu\nu} l^\mu m^\mu$ and the vector $B^\rho = m^\rho + l^\rho - \tilde{\Gamma}^\rho_{\mu\nu} m^\mu l^\mu$. Their difference is $C^\rho = 2\tilde{\Gamma}^\rho_{[\mu\nu]} l^\mu m^\mu$. One can easily realize that the vectors A_ρ and B_ρ do not form a close parallelogram if $\tilde{\Gamma}^\rho_{[\mu\nu]} \neq 0$ [48, 57]. The non-closure of parallelograms in space-time is due to the anti-symmetric part of general affine connections which define the ***torsion*** tensor

$$\mathcal{T}^\rho_{\cdot\mu\nu} = \tilde{\Gamma}^\rho_{[\mu\nu]} \ . \tag{2.2.4}$$

In general, connections $\tilde{\Gamma}^\rho_{\mu\nu}$ are non-tensor quantities, on the other hand, their anti-symmetric part transforms like a tensor, as fas as the most general metric-compatible form of connections are concerned.

Because of this property, the presence of a torsion field denies the Equivalence Principle of its importance; indeed, we are not referring to the equivalence between inertial and gravitational mass, which is preserved since the theory remains geometric, but to the formulation of the Equivalence Principle [58] according to which, once defined an inertial frame in a point, the physical laws are the same as those of special relativity. In presence of torsion, the latter, behaving like a tensor, can not be set to zero by a convenient coordinate choice. Therefore, since we expect torsion to be source of some "force", it is not possible to define an inertial frame in any point, which is a necessary condition for the applicability of the principle.

The metric tensor, connections and the Einstein tensor Let us now introduce a metric defined by the square modulus of a vector V^ρ as

$$\|V\|^2 = g_{\mu\nu} V^\mu V^\nu \ , \tag{2.2.5}$$

here $g_{\mu\nu}$ denotes the symmetric metric tensor defining the square of the infinitesimal interval ds as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \ . \tag{2.2.6}$$

It is possible to establish a relation between connections, torsion and metric tensor of the form [57, 59]

$$\tilde{\Gamma}^\rho_{\mu\nu} = \tfrac{1}{2}\, g^{\rho\sigma} \left[\partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu} + \partial_\nu g_{\mu\sigma}\right] + \tfrac{1}{2}\left[\mathcal{T}^{\rho}_{\cdot\mu\nu} - \mathcal{T}^{\rho}_{\mu\cdot\nu} - \mathcal{T}^{\rho}_{\nu\cdot\mu}\right] = \Gamma^\rho_{\mu\nu} + \mathcal{K}^{\rho}_{\cdot\mu\nu} \ . \tag{2.2.7}$$

Here $\Gamma^\rho_{\mu\nu}$ denote the *Christoffel Symbols* (which are symmetric in the lower two indices) and $\mathcal{K}^{\rho}_{\cdot\mu\nu}$ identifies the ***contortion*** tensor defined as

$$\mathcal{K}^{\rho}_{\cdot\mu\nu} = \tfrac{1}{2}\left[\, \mathcal{T}^{\rho}_{\cdot\mu\nu} - \mathcal{T}^{\rho}_{\mu\cdot\nu} - \mathcal{T}^{\rho}_{\nu\cdot\mu}\,\right] \ , \tag{2.2.8}$$

and it is anti-symmetric in the last two indices. A space endowed with affine connections (2.2.7) is called Einstein-Cartan (EC) Space U^4. In such a space, using the definition of connections (2.2.7), one can write the curvature tensor [57, 59] in presence of torsion, it reads

$$\tilde{R}^\sigma_{\mu\nu\rho} = \partial_\nu\, \tilde{\Gamma}^\sigma_{\mu\rho} - \partial_\rho\, \tilde{\Gamma}^\sigma_{\mu\nu} + \tilde{\Gamma}^\sigma_{\gamma\nu}\, \tilde{\Gamma}^\gamma_{\mu\rho} - \tilde{\Gamma}^\sigma_{\gamma\rho}\, \tilde{\Gamma}^\gamma_{\mu\nu} \ . \tag{2.2.9}$$

Such a curvature, can be easily expressed through the Riemannian tensor $R^\sigma_{\mu\nu\rho}$ (curvature tensor depending only on metric), the covariant derivative ∇_μ (torsionless case of eq. (2.2.2)) and contortion as

$$\tilde{R}^{\sigma}_{\mu\nu\rho} = R^{\sigma}_{\mu\nu\rho} + \nabla_{\nu}\,\mathcal{K}^{\sigma}_{\cdot\mu\rho} - \nabla_{\rho}\,\mathcal{K}^{\sigma}_{\cdot\mu\nu} + \mathcal{K}^{\sigma}_{\cdot\gamma\nu}\,\mathcal{K}^{\gamma}_{\cdot\mu\rho} - \mathcal{K}^{\sigma}_{\cdot\gamma\rho}\,\mathcal{K}^{\gamma}_{\cdot\mu\nu}\;. \tag{2.2.10}$$

Similar formulas can be written for the Ricci tensor and for the scalar curvature with torsion:

$$\tilde{R}_{\mu\rho} = \tilde{R}^{\nu}_{\mu\nu\rho} = R_{\mu\rho} + \nabla_{\sigma}\,\mathcal{K}^{\sigma}_{\cdot\mu\rho} - \nabla_{\rho}\,\mathcal{K}^{\sigma}_{\cdot\mu\sigma} + \mathcal{K}^{\sigma}_{\cdot\gamma\sigma}\,\mathcal{K}^{\gamma}_{\cdot\mu\rho} - \mathcal{K}^{\sigma}_{\cdot\mu\gamma}\,\mathcal{K}^{\gamma}_{\cdot\sigma\rho}\;, \tag{2.2.11}$$

(it is worth remark that it is not symmetric) and

$$\tilde{R} = g^{\mu\rho}\,\tilde{R}_{\mu\rho} = R + 2\,\nabla^{\sigma}\,\mathcal{K}^{\mu}_{\cdot\sigma\mu} - \mathcal{K}^{\sigma}_{\mu\sigma\cdot}\,\mathcal{K}^{\mu\gamma\cdot}_{\gamma} + \mathcal{K}_{\mu\gamma\sigma}\,\mathcal{K}^{\mu\sigma\gamma}\;. \tag{2.2.12}$$

We underline that $R_{\mu\rho}$ and R are the Riemannian quantity derived by the curvature tensor $R^{\sigma}_{\mu\nu\rho}$ which is constructed in the same way as eq. (2.2.9) but using Christoffel Symbols as connections. In this scheme, the Einstein tensor in presence of torsion is defined according the standard picture as

$$\tilde{G}_{\mu\nu} = \tilde{R}_{\mu\nu} - \tfrac{1}{2}\,g_{\mu\nu}\,\tilde{R}\;, \tag{2.2.13}$$

and, by such a definition, one can show [48] that the anti-symmetric part of the Einstein tensor is related to torsion field by the following relation

$$\tilde{G}_{[\mu\nu]} = (\nabla_{\rho} + 2\mathcal{T}^{\sigma}_{\rho\sigma\cdot})\;\mathcal{T}^{\rho}_{\mu\nu\cdot}\;. \tag{2.2.14}$$

The non-metricity tensor It is worth noting that more general affine connections can be implemented. We underline that, in order to maintain the correct behavior of the covariant derivative (2.2.2), *any* tensor $A^{\rho}_{\nu\mu}$ can be added to connections [59]. A particular choice corresponds to define the affine-connection coefficients as

$$\overset{*}{\Gamma}{}^{\rho}_{\mu\nu} = \tilde{\Gamma}^{\rho}_{\mu\nu} + \tfrac{1}{2}\left[\mathcal{Q}^{\rho}_{\mu\nu\cdot} - \mathcal{Q}^{\rho}_{\nu\mu\cdot} + \mathcal{Q}^{\rho}_{\cdot\nu\mu}\right]\;, \tag{2.2.15}$$

where we have introduced the tensor of *non-metricity* defined as

$$\mathcal{Q}^{\cdot\nu\rho}_{\mu} = \overset{*}{\nabla}_{\mu}\,g^{\nu\rho}\;, \tag{2.2.16}$$

here $\overset{*}{\nabla}_{\mu}$ denotes the covariant derivative $\overset{*}{\nabla}_{\mu}\,V^{\rho} = \partial_{\mu}V^{\rho} + \overset{*}{\Gamma}{}^{\rho}_{\mu\nu}\,V^{\nu}$. We remark that non-metricity does not preserve lengths and angles under parallel displacement. To conclude, we summarize the space characterization is presence of the tensor quantities introduced above

$$\boxed{\text{General Linear space } L^4} \xrightarrow{\mathcal{Q}_{\mu\nu\rho}=0} \boxed{\text{Einstein-Cartan space } U^4} \xrightarrow{\mathcal{T}_{\mu\nu\rho}=0} \boxed{\text{Riemann space } V^4}$$

2.2.1 Einstein metric gravity: the Einstein-Hilbert Action

Guiding principles in the development of the Lagrangian density for the gravitational field are the Equivalence Principle and the General Covariance. The latter imposes the action be invariant under diffeomorphisms, *i.e.*, general coordinate transformations, while the former states that, by a coordinate transformation, the metric tensor can always be reduced to a Minkowskian one locally, thus first derivatives of the metric can be made to vanish in any local region. Therefore, if combined together, they forbid the existence of a sensible action for the gravitational field with only first-order derivatives. Hence, second-order derivatives have to be contained in the Lagrangian, but only trough a surface term, to avoid the appearance of third-order derivatives in the equations of motion.

Let us consider a 4-dimensional *torsionless* space-time manifold endowed with a metric $g_{\mu\nu}$, the simplest Lagrangian satisfying the above mentioned properties is the Einstein-Hilbert (EH) [19, 58, 60] one,

$$\mathcal{L}_{EH} = \sqrt{-g}\; R\;, \tag{2.2.17}$$

where g denotes the metric tensor determinant and R is the torsionless scalar curvature discussed above and it is expressed only with the Christoffel Symbols. Using such a Lagrangian density, we write down the well-known EH Action

$$\mathcal{S}_{EH} = -\tfrac{1}{2}\textstyle\int d^4x\sqrt{-g}\, R\;. \tag{2.2.18}$$

By varying the action WRT the metric tensor, the Einstein Equations come out

$$\delta\mathcal{S}_{EH} = -\tfrac{1}{2}\textstyle\int d^4x\,(R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R)\delta g^{\mu\nu}\;. \tag{2.2.19}$$

once we should require the variation of the metric and of its first derivatives vanish on the boundary [61, 62]. Hence, in general, a term is added to the Lagrangian density, in order to cancel the surface piece.

This approach corresponds to the Second-Order Formalism where the metric tensor is treated like an independent field and variation WRT $g_{\mu\nu}$ are performed. Indeed, the additional equations we obtain from the variation of the EH Action WRT $\Gamma^{\rho}_{\mu\nu}$ imply them be equal to Christoffel Connections.

2.2.2 Einstein-Cartan Theory and non-dynamical torsion

Completely neglected in the first formulation of the theory of GR by Einstein, the introduction of torsion was later implemented by Einstein himself [63], A.S. Eddington [64], E. Schrödinger [65] and principally É. Cartan [45, 46, 47], who connects torsion with the spin angular-momentum.

In the original ECT, the geometric Lagrangian density is assume to be composed by the curvature scalar $\tilde{R}$, generalizing the EH Action in presence of torsion, and the matter Lagrangian is taken into account simply through the minimal coupling rule: $\eta_{\mu\nu} \to g_{\mu\nu}$, $\partial_\mu \to \tilde{\nabla}_\mu$. The minimal substitution can be applied to matter field only, but not to gauge fields of internal symmetry groups [48]. This way, the gravitational action corresponds to the EH Action written in the EC Space, *i.e.*,

$$\begin{aligned} \mathcal{S}_{EC} = \tilde{\mathcal{S}}_{EH} = -\tfrac{1}{2}\int d^4x\sqrt{-g}\,\tilde{R} = \\ = -\tfrac{1}{2}\int d^4x\sqrt{-g}\;\, g^{\nu\rho}\,\delta^\mu_\sigma\,\left(\partial_\mu\tilde{\Gamma}^\sigma_{\nu\rho} - \partial_\nu\tilde{\Gamma}^\sigma_{\mu\rho} - \tilde{\Gamma}^\epsilon_{\mu\rho}\tilde{\Gamma}^\sigma_{\nu\epsilon} + \tilde{\Gamma}^\epsilon_{\nu\rho}\tilde{\Gamma}^\sigma_{\mu\epsilon}\right)\,. \end{aligned} \quad (2.2.20)$$

Being φ the matter field, after the minimal coupling procedure generating the total Lagrangian density $\mathcal{L} = \mathcal{L}(\varphi,\, \partial_\mu\varphi,\, g,\, \partial g,\, \mathcal{T})$, one can define the usual EMT through eq. (2.2.19) as

$$T^{\mu\nu} = \delta\mathcal{L}/\delta g_{\mu\nu}\;. \quad (2.2.21)$$

In the same way, one can suppose [48] to define an analogous quantity related to the torsion (or contortion) field, *i.e.*,

$$s_\rho^{\cdot\mu\nu} = \delta\mathcal{L}/\delta\mathcal{T}^\rho_{\cdot\mu\nu}\;. \quad (2.2.22)$$

Such a tensor in constructed from the matter fields φ but may also depend on metric and torsion [59]. If Dirac fermion minimally coupled to torsion are considered, $s_\rho^{\cdot\mu\nu}$ corresponds to the spin energy-potential.

Considering now the variational principle $\delta(\mathcal{S}_{EC} + \mathcal{S}_M + \mathcal{S}_T) = 0$, where $\mathcal{S}_M$ and $\mathcal{S}_T$ are defined as

$$\mathcal{S}_M = \tfrac{1}{2}\int d^4x\sqrt{-g}\,\mathcal{L}_M\;, \qquad \mathcal{S}_T = \tfrac{1}{2}\int d^4x\sqrt{-g}\; s_\rho^{\cdot\mu\nu}\,\mathcal{T}^\rho_{\cdot\mu\nu}\;, \quad (2.2.23)$$

here $\mathcal{L}_M = \mathcal{L}_M(\varphi,\, \partial_\mu\varphi)$ denotes the matter Lagrangian density, one obtains, according to the previous expressions, the following variations

$$\delta\mathcal{S}_M = \tfrac{1}{2}\int d^4x\sqrt{-g}\;T^{\mu\nu}\;\delta g_{\mu\nu}\;, \quad (2.2.24)$$

$$\delta\mathcal{S}_T = \tfrac{1}{2}\int d^4x\sqrt{-g}\; s_{\rho}^{\cdot\mu\nu}\;\delta\mathcal{T}^{\rho}_{\cdot\mu\nu}\;, \tag{2.2.25}$$

and the field equation of the system can be derived. Using the definition (2.2.21), one obtains

$$G^{\mu\nu} - (\nabla_\rho + 2\mathcal{T}^{\sigma}_{\rho\sigma\cdot})\,[\bar{\mathcal{T}}^{\mu\nu\rho} + \bar{\mathcal{T}}^{\rho\mu\nu} + \bar{\mathcal{T}}^{\rho\nu\mu}] = T^{\mu\nu}\;, \tag{2.2.26a}$$

$$\mathcal{T}_{\mu\nu\rho} = 2\,(s_{\rho[\mu\nu]} + s_{[\mu}\,g_{\nu]\rho})\;, \tag{2.2.26b}$$

where $s_\mu = s_{\rho}^{\cdot\mu\rho}$ and $\bar{\mathcal{T}}^{\mu\nu\rho}$ is the modified torsion tensor defined as

$$\bar{\mathcal{T}}^{\mu\nu\rho} = \mathcal{T}^{\mu\nu\rho} + \mathcal{T}^{\nu\cdot\sigma}_{\sigma}\,g^{\mu\rho} - \mathcal{T}^{\mu\cdot\sigma}_{\sigma}\,g^{\nu\rho}\;, \tag{2.2.27}$$

and, of course, $G_{\mu\nu}$ is the torsionless case of eq. (2.2.13). In vacuum eq. (2.2.26b) give the results $\mathcal{T}_{\mu\nu\rho} = 0$. One can easily see that torsion is proportional to the spin energy-potential and in vacuum it vanishes. In this picture, torsion obeying to an algebraic equation, instead of a differential one, and it acquires a non-propagating dynamics. Torsion is inextricably bound to matter and cannot propagate through the vacuum as a wave or via any interaction of non-vanishing range. At the same time, we can underline that, because of such a character, one is able to substitute everywhere spin for torsion and cast out effectively torsion from the formalism. In particular, using eq. (2.2.25) and (2.2.26b), torsion leads to the *contact spin-spin interaction* which can be expressed by the classical potential $V(s) \sim s^2$.

It is worth noting that we can relate an analogous quantity as $s_{\rho}^{\cdot\mu\nu}$ but related to the contortion field. In this case, one can recognized it as the proper spin angular-momentum tensor. Denoting such a tensor with $\tau^{\rho}_{\mu\nu}$, after some manipulation, one can recast eq. (2.2.26b) in the form

$$\mathcal{T}^{\rho}_{\mu\nu} + \delta^{\rho}_{\mu}\mathcal{T}^{\sigma}_{\nu\sigma\cdot} - \delta^{\rho}_{\nu}\mathcal{T}^{\sigma}_{\mu\sigma\cdot} = \tau^{\rho}_{\mu\nu}\;. \tag{2.2.28}$$

2.2.3 Propagating torsion: the torsion potentials

Since, in the first instance, it is reasonable to expect torsion to behave as any other interaction field, *i.e.*, propagating into vacuum, the non-dynamical torsion feature of the U^4 theory is unsatisfactory and, in the following, possible theories to overcome this problem are discussed.

(1.) Following the Brans-Dicke [57] analysis, one can perform the transformation $G^{\mu\nu} \to \phi G^{\mu\nu}$, using the dimensionless scalar field ϕ. In this picture, using the EH

Action expressed above, the vacuum torsion equation reads

$$\mathcal{T}_{\mu\nu\rho} = (1/\phi)\ \phi_{[,\mu}\ g_{\nu]\rho}\ . \tag{2.2.29}$$

As soon as a Lagrangian with the usual kinetic terms for the scalar field is assumed, ϕ results to be a propagating fields, then the torsion field itself propagates.

(2.) As another example, one can introduce the alternative Lagrangian

$$\mathcal{S}_G = -\tfrac{1}{2}\int d^4x\sqrt{-g}\,(\tilde{R} + C_1\ \tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu})\ , \tag{2.2.30}$$

where C_1 is a constant. The use of quadratic terms is expected in gauge theories and is required for renormalization [66, 67]. For these reasons, the expression above would be included in more general and complete formulations. The equations for the torsion field are obtained by varying the torsion and by keeping the metric tensor fixed. They read [68]

$$\bar{\mathcal{T}}^{\mu\nu\rho} = C_1\ \mathcal{R}^{\rho[\mu\nu]}\ , \tag{2.2.31}$$

where

$$\mathcal{R}^{\mu\nu\rho} = -(\nabla^\rho + 2\mathcal{T}^{\rho\sigma\cdot}_{\sigma})\tilde{R}^{\mu\nu} + g^{\mu\rho}(\nabla_\epsilon + 2\mathcal{T}^{\ \sigma}_{\epsilon\sigma\cdot})\tilde{R}^{\epsilon\nu} + 2\tilde{R}^{\nu}_{\epsilon}\ \mathcal{T}^{\rho\epsilon\mu}\ . \tag{2.2.32}$$

Since the curvature tensor contains the first derivative of the torsion, equations above show that torsion, in vacuum, obeys second-order differential equations and therefore it propagates.

(3.) A more complete form for the action has been examined in [69]. The generalization of eq. (2.2.30) can be written as

$$\mathcal{S}_G = -\tfrac{1}{2}\int d^4x\sqrt{-g}\,(\tilde{R} + C_1\tilde{R}^2 + C_2\ \tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu} + C_3\ \tilde{R}_{\mu\nu}\tilde{R}^{\nu\mu})\ . \tag{2.2.33}$$

The resulting field equation for the torsion field is analogous the the one obtained in the previous case and also here torsion propagates.

(4.) Another physical approach is based in the idea that torsion is derivable from a scalar potential [70, 71, 72, 73, 74, 75, 76]. As an example, one can consider

$$\mathcal{T}^{\rho}_{\mu\nu} = \phi_{,\mu}\delta^{\rho}_{\nu} - \phi_{,\nu}\delta^{\rho}_{\mu} \tag{2.2.34}$$

and define the source according to the following expression

$$\delta\mathcal{S}_T = -\tfrac{1}{2}\int d^4x\sqrt{-g}\,\rho\ \delta\phi\ . \tag{2.2.35}$$

Variations WRT the potential ϕ yield the equation

$$\tilde{\nabla}_\mu \tilde{\nabla}^\mu \, \phi = -48\rho \, . \tag{2.2.36}$$

Such an expression shows, again, that torsion behaves like a dynamical field. The reason that torsion propagates is due to the fact that, when it is assumed to be derived from a potential, the Lagrangian (the curvature scalar) contains products bilinear in the first derivative of the potential and therefore the field equations are of second differential order. The advantage of this approach is that we may retain the curvature scalar as the Lagrangian and are not forced into adopting much more complicated quadratic Lagrangians and their associated equations.

Many other approaches for propagating torsion are present in literature, see [57, 77] and references therein. In particular, more general actions can describe a dynamical torsion field and they result to be quadratic in such a field. A discussion and an application to cosmology is addressed in Section 2.2.5. This approach is analyzed, among others, in [78, 79, 80, 81, 56].

In what follows we focus the attention on the particular approach of the torsion potentials since we aim to implement such a formalism to construct the motion equations for a test particle.

The torsion potentials Torsion is a three-index tensor, anti-symmetric in the first two indices; according to group theory, it can be decomposed in a *completely anti-symmetric part*, a *trace part* and a third part with no special symmetry properties [82]. In our analysis, we here consider only the first two terms and we assume they to be derived from the exterior derivative of two *potentials* [83, 76],

$$B_{\mu\nu\rho} \equiv \mathcal{T}_{[\mu\nu\rho]} = \nabla_{[\mu} V_{\nu\rho]} \, , \tag{2.2.37a}$$

$$\mathcal{T}^{(tr)}_{\mu\nu\rho} = \tfrac{1}{3}(g_{\nu\rho}\partial_\mu\phi - g_{\mu\rho}\partial_\nu\phi) \, , \tag{2.2.37b}$$

where $V_{\mu\nu}(x)$ is an anti-symmetric tensor, while $\phi(x)$ is a scalar (of course $\tilde{\nabla}_\mu$ is defined by eq. (2.2.2)). These potentials play a role analogous to that of metric in the symmetric part of connections .

In this picture, contortion and connections rewrite as

$$\mathcal{K}_{\mu\nu\rho} = B_{\mu\nu\rho} + 2\,\mathcal{T}^{(tr)}_{\mu\nu\rho}\ , \qquad (2.2.38a)$$

$$\tilde{\Gamma}_{\mu\nu\rho} = \Gamma_{\mu\nu\rho} + \partial_{[\mu}V_{\nu\rho]} + \tfrac{2}{3}((\partial_\rho\phi)g_{\mu\nu} - (\partial_\nu\phi)g_{\mu\rho})\ , \qquad (2.2.38b)$$

respectively. As already discussed, the introduction of the potential for the anti-symmetric part of torsion [75, 84] has its main motivation in obtaining a propagating field in vacuum. As far as the expression (2.2.37b) for the trace part is concerned, it is worth noting that the same expression is addressed in [75] but in a different scenario. In fact, in Hojman *et al.* article, such a term is introduced to get a coupling of torsion to electromagnetic field which do not break gauge symmetry. As already mentioned, another mechanism to obtain propagation can be developed; it consists, in analogy to Yang-Mills theory, in introducing square terms in curvature and torsion in the EH Action. Here we make the different choice of using torsion potentials which, we believe, has these advantages: *(i)* the simplicity of the EH Action is preserved as soon as the minimal substitution $\Gamma_{\mu\nu\rho} \to \Gamma_{\mu\nu\rho} + \mathcal{K}_{\mu\nu\rho}$ is addressed; *(ii)* both Riemannian connections and torsion are similarly treated since as the former is derived from metric, the latter is derived from potentials; *(iii)* in the limit of small and slow varying ϕ, the total action is equivalent to the low-energy limit of string-theory Lagrangian, as already mentioned in [57] (and reference therein), suggesting torsion potentials to be a necessary ingredient in more general theories.

Field equations for the torsion potential To calculate field equations, we now introduce the usual EH Action (2.2.20), *i.e.*,

$$\tilde{\mathcal{S}}_{EH} = -\tfrac{1}{2}\int d^4x\sqrt{-g}\ \ g^{\nu\rho}\ \delta^\mu_\sigma\ \left(\partial_\mu\tilde{\Gamma}^\sigma_{\nu\rho} - \partial_\nu\tilde{\Gamma}^\sigma_{\mu\rho} - \tilde{\Gamma}^\epsilon_{\mu\rho}\tilde{\Gamma}^\sigma_{\nu\epsilon} + \tilde{\Gamma}^\epsilon_{\nu\rho}\tilde{\Gamma}^\sigma_{\mu\epsilon}\right)\ . \qquad (2.2.39)$$

Using torsion potentials, such an expression can be split up in its Riemannian part plus torsion-depending terms

$$\tilde{\mathcal{S}}_{EH} = -\tfrac{1}{2}\int d^4x\sqrt{-g}\ (R - B^{\mu\nu\rho}B_{\mu\nu\rho} - \tfrac{2}{3}(\partial_\mu\phi)^2)\ . \qquad (2.2.40)$$

We obtain field equations by variational principles: variations WRT $g_{\mu\nu}$, $V_{\mu\nu}$ and ϕ yield, respectively,

$$R^{\mu\nu} - \tfrac{1}{2}\, g^{\mu\nu} R + \tfrac{1}{2}\, g^{\mu\nu} B^{\rho\sigma\epsilon} B_{\rho\sigma\epsilon} - 3 B^{\mu\sigma\epsilon} B^{\nu}_{\sigma\epsilon} + \\ + \tfrac{8}{3} \left(\tfrac{1}{2}\, g^{\mu\nu} (\partial_\rho \phi)^2 - g^{\mu\rho} g^{\nu\sigma} (\partial_\rho \phi)(\partial_\sigma \phi) \right) = 0 \ , \qquad (2.2.41)$$

$$\nabla_\mu B^{\mu\nu\rho} = 0 \ , \qquad (2.2.42)$$

$$\nabla_\mu g^{\mu\nu} \partial_\nu \phi = 0 \ . \qquad (2.2.43)$$

The first of equations above consists of the (Riemannian) Einstein tensor, as in GR, plus four terms all quadratic in the torsion potentials. This way, if we are interested in solving the first-order dynamics for little values of torsion potentials, we can neglect such quadratic terms and fall back in the GR field equations; the resulting metric can be replaced in eq. (2.2.42) and eq. (2.2.43) to find, at first-order, the torsion potentials.

One can easily check that, in eqs. (2.2.42) and (2.2.43), the goal of a propagating description for torsion has been achieved obtaining two second-order *PDE*'s for both potentials.

To conclude, we write down the gauge transformations for the tensor potential

$$V_{\mu\nu} \to V'_{\mu\nu} = V_{\mu\nu} + \nabla_\mu Y_\nu - \nabla_\mu Y_\nu \ , \qquad (2.2.44)$$

by which, setting Y_ν such that $\nabla_\mu V'^{\mu}_{\nu} = 0$, it's easy to see that eq. (2.2.42) rewrites as

$$\nabla_\rho \nabla^\rho V'_{\mu\nu} - R^{\sigma}_{\rho\mu\nu} V'^{\rho}_{\sigma} + R_{\mu\sigma} V'^{\sigma}_{\nu} + R_{\nu\sigma} V'^{\sigma}_{\mu} - R^{\sigma}_{\rho\nu\mu} V'^{\rho}_{\sigma} = \Delta_{DR}(V'_{\mu\nu}) = 0 \ , \qquad (2.2.45)$$

where Δ_{DR} is the deRham operator which generalize the Laplace operator in non-flat spaces. It is easy to show that a field $V'_{\mu\nu}$ obeying eq. (2.2.45) is characterized by only one polarization in the iperplane normal to its propagation direction, *i.e.*, only one DOF . It is worth noting that, as far as eq. (2.2.43) is concerned, a massless Klein-Gordon field equation is recovered, so that the potential ϕ can be considered as a geometrical manifestation of this field.

2.2.4 Gauge approach to gravity

Gauge theories describe all physical interactions, but the gravitational one. Many attempts to construct a gauge model of gravitation exist, in particular the papers by

Utiyama [50] and by Kibble [51] were the starting points for various gauge approaches to gravitation. As a result, PGT [85, 86, 52, 48, 87, 54, 55, 56, 53, 88, 89] is a generalization of the Einstein scheme of gravity, in which not only the EMT, but also the spin of matter plays a dynamical role when coupled to spin connections, in a non-Riemannian space-time. Anyway, up to now, neither PGT nor other gauge approaches to the gravitational interaction have led to a consistent quantum scheme of the gravitational field [90].

As we will discuss, the role of fermion is very important in GR and, to include spinor fields consistently, it is necessary to extend the framework of the standard theory of gravitation, as already realized by Hehl et al. [48]: this necessity is strictly connected with the non existence in GR of an independent concept of spin angular-momentum for physical fields. The ECT accounts for both mass and spin of matter as sources of the gravitational field and represent a description of gravity which is more suitable than GR from a microscopical point of view. In fact, fundamental interactions other than gravity are usually described within a theoretical framework where symmetries and conservation laws are properly encoded. In GR, contrastingly, matter can be described by point particles, fluids and light rays. This fundamental difference notwithstanding, spin effects are negligible for macroscopic matter, so that the observational predictions of ECT are regarded as the same as GR, from a phenomenological point of view [55]. Furthermore, ECT is a special case of PGT which is much more general and encompasses also propagating spin connections.

2.2.4.1 Tetradic formalism and spin connections

In what follows, we want to analyze the internal symmetries of the space-time in the standard tetradic approach: the usual orthonormal basis $e_\mu^{\ a}$ (tetrads) for the local Minkowskian tangent space-time is introduced for a 4-dimensional manifold.

The gauge freedom of the ordinary metric gravity corresponds to the invariance under diffeomorphisms [19], *i.e.*, the General Covariance Principle. In this respect, we underline that 10 metric fields enters the dynamics and the diffeomorphism invariance reduces such 10 metric components to 2 DOF . On the other hand, if the tetrad formulation of gravity is addressed, the gauge freedom of diffeomorphisms is maintained under world-indices ($_\mu$) transformations but another, independent, gauge invariance appears considering the Lorentz tetrad-indices ($_a$) transformations. Tetrad vectors exhibits new DOF related to *independent rotations not specified by the metric structure*: this allows to consider 16 fields for the model. It is worth noting that 6 gauge fields can be fixed using 6 new first-class constraints derived by new gauge invariance and, as discussed above, the maintained diffeomorphism invariance reduces the metric components to 2 DOF (*i.e.*, the graviton). In this picture, the dynamics of the gravitational field reduces to that of tetrads.

In the tetrad formalism, the relations between tetrads and the metric $g_{\mu\nu}$ are

$$g_{\mu\nu} = \eta_{ab}\, e_\mu^{\ a}\, e_\nu^{\ b}\,, \qquad e_\mu^{\ a}\, e^\mu_{\ b} = \delta^a_b\,, \qquad e_\mu^{\ a}\, e^\nu_{\ a} = \delta^\nu_\mu\,, \tag{2.2.46}$$

where η_{ab} is the local Minkowski metric (by which one can raise or lower tetrad indices) and the tetrad projection of a generic tensor results to be

$$V_a^b = e^\mu_{\ a} e_\nu^{\ b}\, V_\mu^\nu\,. \tag{2.2.47}$$

Local Lorentz transformations usually act on the tetrad basis as

$$e_\mu^{\ a} \xrightarrow{L} \Lambda^a_b\; e_\mu^{\ b}\,, \tag{2.2.48}$$

where Λ^a_b denotes the Lorentz matrix. It worth noting that an the infinitesimal local Lorentz transformations can be defined as

$$\Lambda_a^b = \delta_a^b + \epsilon_a^b\,, \tag{2.2.49}$$

using the infinitesimal Lorentz rotational parameter $\epsilon^a_b(x)$. Under such a transformation, tetrads behave like

$$e_\mu^{\ a} \xrightarrow{L} e_\mu^{\ a} + \epsilon^a_b e_\mu^{\ b}\,. \tag{2.2.50}$$

Given $e_\mu^{\ a}$, the metric tensor $g_{\mu\nu}$ is uniquely determined and all metric properties of the space-time are expressed by the tetrad fields, accordingly, but the converse is not true: there are infinitely many choices of the local basis that reproduce the same metric tensor, because of the local Lorentz invariance.

In the tetrad formalism, starting from the definition of the geometric covariant derivative (in the torsionless case, it is implemented using the Christoffel Symbols $\Gamma^\rho_{\mu\nu}$), one can define the projected covariant derivative of a vector field

$$\nabla_a V^b = \partial_a V^b \;+\; \Gamma^b_{ac}\, V^c\,. \tag{2.2.51}$$

Writing $\nabla_a V^b = e^\mu_{\ a} e_\nu^{\ b} \nabla_\mu V^\nu$, one deduces the relation

$$\Gamma^c_{ab} = e^\mu_{\ [a} e_\nu^{\ b} e^\nu_{\ c]} \Gamma^\rho_{\mu\nu} - e^\mu_{\ b} e^{\ c}_{[\mu,a]}\,. \tag{2.2.52}$$

After some little algebra [57], one can recast such a results, in terms of the so-called *spin connections* $\omega_\mu{}^{ab} = e_\mu{}^c\,\Gamma^{ab}_{c}$, written as

$$\omega_\mu{}^{ab} = e^{\nu a}\,\nabla_\mu e_\nu{}^b = e_\mu{}^c\,\gamma^{ba}{}_c\,, \tag{2.2.53}$$

where $\gamma^{ba}{}_c$ are the *Ricci rotation coefficients* [15] defined by the relation

$$\gamma_{abc} = e^\mu{}_c e^\nu{}_b\,\nabla_\mu e_{\nu a}\,, \tag{2.2.54}$$

here ∇_μ denotes the usual coordinate covariant derivative.

It is worth noting that the introduction of the tetrad formalism is related to the presence of spinor fields in the dynamics, since spinors transform like a particular representation S of the Lorentz Group (LG), *i.e.*, $\psi \to S\psi$. The covariance of the spin derivative $\partial_\mu\psi$ is ensured by the same spin connections $\omega_\mu{}^{ab}$ [57]. This way, one is able to define a Lorentz covariant derivative, *i.e.*, $\partial_\mu \to D^{(S)}_\mu$

$$D^{(S)}_\mu = \partial_\mu + \Gamma^{(S)}_\mu\,, \qquad\qquad \Gamma^{(S)}_\mu = \tfrac{1}{2}\,\omega_\mu{}^{ab}\,\Sigma_{ab}\,, \tag{2.2.55}$$

here Σ_{ab} are the generators of the LG defined considering the spin-$1/2$ representation so that

$$\Sigma_{ab} = \tfrac{i}{2}[\gamma_a, \gamma_b]\,. \tag{2.2.56}$$

On the other hand, spin connections $\omega_\mu{}^{ab}$ are introduced to restore the correct Dirac algebra in curved (torsionless) space-time, *i.e.*,

$$D^{(S)}_\mu\,\gamma^\nu = 0\,, \qquad\qquad \Gamma^{(S)}_\mu = -\tfrac{1}{4}\,\gamma^\rho\nabla_\mu\gamma_\rho = \tfrac{1}{2}\,\omega_\mu{}^{ab}\,\Sigma_{ab}\,. \tag{2.2.57}$$

By other words, a treatment of spinors in curved space-time can leads to the introduction of those connections which reflect the covariance under the LG. In fact, when spinor fields are taken into account, their transformations under the local Lorentz symmetry imply that the Dirac Equation is endowed with non-zero spin connections, even in flat space-time.

Comment on the spinor fields: topology The problem of spinor fields on flat space-time is well established [91] and it gives rise to a consistent formulation of the Dirac Equation. The analysis of the fermion dynamics in non-inertial Minkowski frame treating, the Lorentz transformation as frame-preserving diffeomorphisms, is discussed in [92, 93]. On the other hand, in Riemannian curved space-time, without torsion, the Dirac fermion dynamics is treated in standard tetrad gravity, see [94] and reference therein, but without a complete constraint theory. In particular, in [95], it has been shown that spinning particles (in a semi-classical picture the γ matrices are described by the Grassmann variables) do not couple with torsion since the constraint algebra is not close in presence of torsion field.

The definition of spinor fields on flat Minkowski space-time is motivated by the fact that the latter are described by a spinor representation of the LG. However, the LG does not act in a natural way on a curved space-time, so clearly this characteristic property of spinor fields can not be carried over in a direct manner to curved spaces. Thus, the notion of fermions requires a particular treatment and the definition of the so-called *spin bundle*, since there is no natural action of the full group of diffeomorphisms of spinor fields.

Let us start from the basic notions[1]. A *manifold* is a topological space which looks locally like $\mathbb{R}^m$, but not necessarily so globally [96]. By introducing a chart, we give a local Euclidean structure to a manifold, which enables us to use the conventional calculus of several variables. A fibre bundle is, so to speak, a topological space which looks locally like a direct product of two topological spaces. Many theories in physics, such as general relativity and gauge theories, can be described naturally in terms of fibre bundles. In fact, physical fields are assumed to be geometrically represented by sections of fiber bundles functorially associated with some jets prolongation of the relevant principal bundle by means of left actions of Lie groups on manifolds, usually tensor spaces. Such an approach enables to functorially define the Lie derivative of physical fields with respect to gauge-natural lifts of (prolongations of) infinitesimal principal automorphisms of the underlying principal bundle.

The basic idea is to start with the notion of the *tangent bundle* TM, defined as the sum over all point p in M of all the tangent spaces T_pM of an m-dimensional manifold M. The manifold M over which TM is constructed is called the *base space*. Let us now define the chart U_i as an open covering of M, on which we consider the coordinates x^μ. The space TU_i, defined according the previous notation, result to be $2m$-dimensional differentiable manifold and we are naturally led to the concept of projection. In the context of the theory of fibre bundles, T_pM is called the *fibre* at point p. It is obvious by construction, that if $M = \mathbb{R}^m$, the tangent bundle itself is

[1]Only for this paragraph the notation does not follow global notations already introduced.

expressed as a direct product $\mathbb{R}^m \times \mathbb{R}^m$. However, this is not always the case and the non-trivial structure of the tangent bundle measures the topological non-triviality of M.

In this scheme, the fibre coordinates are rotated by an element of $GL(m, \mathbb{R})$ whenever we change the coordinates. Such a group is called the *structure group* of TM. This way, fibres are interwoven together to form a tangent bundle, which consequently may have quite a complicated topological structure.

The tangent bundle is an example of a more general framework called *fibre bundle*. A (differentiable) fibre bundle consists of the following elements: *(i)* a differentiable manifold E called the *total space*; *(ii)* a differentiable manifold M called the base space; *(iii)* a differentiable manifold F of dimension k called the fibre; *(iv)* a projection $\pi : E \to M$; *(v)* a Lie Group G called the structure group; *(vi)* a set of open covering U_i of M with a diffeomorphism $\phi_i : U_i \times F \to \pi^{-1}U_i$ such that $\pi\phi_i(p, f) = p$; *(vii)* setting $\phi_i(p, f) = \phi_{i,p}$, the map $\phi_{i,p} : F \to F_p$ is a diffeomorphism.

Considering not only a single chart, on $U_i \cap U_j$, we can define useful the functions $t_{ij}(p) = \phi_{i,p}^{-1}\phi_{j,p} : F \to F$ to be an element of the structure group G. Then ϕ_i and ϕ_j are related by a smooth map t_{ij} as $\phi_j(p, f) = \phi_i(p, t_{ij}f)$. Functions t_{ij} are called the *transition functions*. Strictly speaking, the definition of a fibre bundle should be independent of the special covering U_i of M. If all the transition functions can be taken to be identity maps, the fibre bundle is called a *trivial bundle*, which is the direct product $M \times F$. Given a fibre bundle, the possible set of transition functions is obviously far from unique. This way, let U_i be a covering of M we can define ϕ_i and $\tilde{\phi}_i$ as two sets of diffeomorphism giving rise to the same fibre bundle. In this respect, we can set $\tilde{t}_{ij}(p) = \tilde{\phi}_{i,p}^{-1}\tilde{\phi}_{j,p}$.

A spinor field on M is a *section* of a spin bundle. A section is defined by $s : M \to E$ and it is a smooth map which satisfied $\pi s : M \to M$, *i.e.*, the identity map. Since $GL(m, \mathbb{R})$ has no spinor representation, we need to introduce an orthonormal frame bundle whose structure group is $SO(m)$. The presence spin bundle tells us whether a manifold admits a spin or not. Let TM be a tangent bundle with $\dim M = m$ and the structure group G is taken to be $O(m)$. If, furthermore, M is orientable, G can be reduced down to $SO(m)$. The set of $\tilde{t}_{ij}$ defines a spin bundle $PS(M)$ over M, and M is said to admit a spin structure (of course, M may admit many spin structures depending on the choice of $\tilde{t}_{ij}$).

It is interesting to note that not all manifolds admit suitable spin structures. Nonadmittance of spin structures is measured by the topological invariant known as Second Stieffel-Whitney Class w_2. In particular, the condition $w_2 = 0$ on an orientable manifold M, is necessary and sufficient to ensure the existence of a spin structure on M, but we do not want to address a detailed discussion about this (see [96, 19]).

In conclusion, one can infer that, let TM be the tangent bundle over an *orientable* manifold M so that each fibers is diffeomorphic to the proper LG, there *exists a spin bundle* over M if and only if the transition functions satisfy the relation $\tilde{t}_{ij}\tilde{t}_{jk}\tilde{t}_{ki} = 1$. A spinor may then be defined as a point in the fiber of the spin bundle. By other words, a more physical interpretation of this condition can be given as follows: a space-time-orientable manifold M admits a spin structures if and only if on any closed 2-surface L in M there exists a set of $n-1$ continuous fields of tangent vectors to M, linearly independent at every point of L. In this case, no obstructions to spin structures can occur [97, 98].

We want to remark that, in what follows, we use the standard treatment of the spinor fields, addressed at the begin of this Section, without enters the details of the well-grounded topology approach.

Structure Equations The picture derived by using spin connections (2.2.53), suggests in appearance the description of gravity as a gauge model [99, 100]. As discussed above, spin connections are a suitable bein projection of Ricci rotation coefficients,

$$\omega_\mu^{\ ab} = e_\mu^{\ c}\,\gamma^{ba}_{\ \ c}\,, \tag{2.2.58}$$

and this formalism leads to the following definition of the curvature tensor:

$$R_{\mu\nu}^{\ \ ab} = \partial_\nu\omega_\mu^{\ ab} - \partial_\mu\omega_\nu^{\ ab} + \mathcal{F}^{ab}_{cdef}\omega_\mu^{\ cd}\omega_\nu^{\ ef}\,, \tag{2.2.59}$$

which is the *I Cartan Structure Equation* and $\mathcal{F}^{ab}_{cdef}$ are the LG structure constants. The EH Action consists of the lowest-order non-trivial scalar combination of the Riemann curvature tensor and the tetrad fields, *i.e.*,

$$\mathcal{S}_G(e,\omega) = -\tfrac{1}{2}\int \det(e)\, d^4x\ \ e_a^{\ \mu}e_b^{\ \nu}R_{\mu\nu}^{\ \ ab}\,. \tag{2.2.60}$$

Variation WRT connections leads to the *II Cartan Structure Equation*,

$$\partial_\mu e_\nu^{\ a} - \partial_\nu e_\mu^{\ a} - \omega_\mu^{\ ab}e_{\nu b} + \omega_\nu^{\ ab}e_{\mu b} = 0\,, \tag{2.2.61}$$

which links the tetrad fields to the spin connections: the solutions have, of course, the form (2.2.58). Furthermore, variation WRT tetrads, leads to the dynamical Einstein field equations. This approach corresponds to the First-Order Formalism where the tetrad vectors and connections are treated like independent fields.

Gauge model? Since ${\omega_\mu}^{ab} = {e_\mu}^c {\gamma^{ba}}_c$, it is worth underlining that such connections behave like ordinary vectors under general coordinate transformations (*i.e.*, world transformations). In the standard approach, spin connections transform like *Lorentz gauge vectors* under infinitesimal local Lorentz transformations $\Lambda_a^b = \delta_a^b + \epsilon_a^b$,

$$ {\omega_\mu}^{ab} \xrightarrow{L} {\omega_\mu}^{ab} - \partial_\mu \epsilon^{ab} + \tfrac{1}{4}\mathcal{F}^{ab}_{cdef}\epsilon^{cd}{\omega_\nu}^{ef} \tag{2.2.62} $$

and the Riemann tensor is preserved by such a change; therefore, in flat space-time, one can deal with non-zero gauge connections, but a vanishing curvature. In both flat and curved space-time, the connections ${\omega_\mu}^{ab}$ exhibit the right behavior to play the role of Lorentz gauge fields and GR exhibits the features of a gauge theory. On the other hand, the presence of the tetrad field, introduced by the General Covariance Principle, is an ambiguous element for the gauge paradigm. This scenario would be appropriate if the theory were based on two independent degrees of freedom. *Since spin connections ${\omega_\mu}^{ab}$ can be uniquely determined as functions of tetrad fields, this correlation opens a puzzle in the interpretation of these connections as the only fundamental fields of the gauge scheme.*

2.2.4.2 Poincaré Gauge Theory

The first paper that formulates gravitation as a gauge theory was the work by R. Utiyama in 1956 [50]. It is sometimes argued that gravity is already a gauge theory of the group of diffeomorphisms [101], but the first attempt at making gravity a local gauge theory in more modern sense was made in such a work by Utiyama. As is the ordinary tetrad approach to gravity, Utiyama assumed the gauge group as Lorentz one. As previously discussed, by going to the tetrad, one assumes that the effect of the Lorentz transformation is to rotate such a base. However, nowadays we see some difficulties with the details. In order to relate the gauge connections to the affine ones, Utiyama essentially assumed the affine connection to be symmetric. Moreover, his conservation law seems only to contain orbital angular-momentum, but the biggest problem is this: the LG relates to orbital angular momentum while, in GR, the source is the EMT. A major improvement was made by T.W.B Kibble [51] who solved these problems by taking the underlying symmetry group to be the inhomogeneous LG, or the Poincaré Group. In fact, after Kibble's work, Utiyama (with Fukuyama) used the inhomogeneous Lorentz invariance to show that a symmetric second rank tensor was required as the gauge field [102].

In what follows, we analyze the proposal to connect the presence of torsion with the local nature of the Poincaré symmetry. PGT can be described from both a gauge and a geometrical point of view and particular attention will be payed to the physical meaning of field equations, which predict a contact interaction, *i.e.*, a non-propagating gauge field.

Global Poincaré transformations Let us start by considering that the only space where the Poincaré generators are defined is the flat Minkowski one. We implement now an infinitesimal global Poincaré transformation, including the translation ε^μ,

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu_\nu\, x^\nu + \varepsilon^\mu \,, \tag{2.2.63}$$

and the consequent transformation law for spinor fields

$$\psi(x) \xrightarrow{P} (1 + \varepsilon^\mu \partial_\mu + \tfrac{1}{2}\,\epsilon^{\mu\nu}\,\Sigma_{\mu\nu})\,\psi(x) \,, \tag{2.2.64}$$

where the $\Sigma_{\mu\nu}$ are the generators of the LG and, of course, ∂_μ corresponds to the translation operator. If the matter Lagrangian density is assumed to depend on the spinor field and on its derivatives only, *i.e.*, $\mathcal{L} = \mathcal{L}(\psi,\, \partial_\mu\psi)$, and if the motion equations are assumed to hold, the conservation law

$$\partial_\mu J^\mu = 0 \,, \tag{2.2.65}$$

is found, where

$$J^\mu = \tfrac{1}{2}\,\epsilon^{\nu\rho} M^\mu_{\nu\rho} - \varepsilon^\nu T^\mu_\nu \,. \tag{2.2.66}$$

Here the canonical EMT and angular-momentum tensor are defined, according to the analysis discussed in the ECT (see eq. (2.2.21)), as

$$T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial\psi_{,\mu}}\,\partial_\nu\psi - \delta^\mu_\nu\mathcal{L} \,, \tag{2.2.67}$$

$$M^\mu_{\nu\rho} = (x_\nu T^\mu_\rho - x_\rho T^\mu_\nu) + \frac{\partial\mathcal{L}}{\partial\psi_{,\mu}}\,\Sigma_{\nu\rho}\psi \,, \tag{2.2.68}$$

respectively. Because the parameters in eq. (2.2.66) are constant, according to the Nöether Theorem, conservation laws for the energy-momentum current and for the angular-momentum current, together with the related charges, are established (if the integration on the boundaries of the 3-space brings vanishing contributions):

$$\partial_\mu T^\mu_\nu = 0 \quad \Rightarrow \quad P^\nu = \int d^3x\, T^{0\nu} \ , \tag{2.2.69}$$

$$\partial_\mu M^\mu_{\nu\rho} = 0 \quad \Rightarrow \quad M_{\nu\rho} = \int d^3x\, M^0_{\nu\rho} \ . \tag{2.2.70}$$

Gauge approach In the original analysis, Kibble [51] consider the non-symmetric nature of the affine connection introducing torsion in the space-time and shows that spin gives rise to an anti-symmetric part. Kibble work and that of Sciama [49] are discussed in more detail by Hehl *et al.* who give the most comprehensive formulation of a local PGT of gravity in [48].

When transformations are locally implemented, *i.e.*, the parameters $\epsilon^{\mu\nu}$ and ε^μ are functions of space and time, eqs. (2.2.66)-(2.2.70) do not hold any more. In order to maintain invariance, the ordinary partial derivative must be replaced by the gauge covariant derivative,

$$\partial_a \rightarrow \hat{D}_a = e^\mu{}_a(\partial_\mu + \tfrac{1}{2}\hat{\Gamma}_\mu{}^{ab}\Sigma_{ab}) \ . \tag{2.2.71}$$

This way, the Poincaré Group has the four translation operators and six rotation operators. Tetrads $e^\mu{}_a$ become the translation gauge potential and $\hat{\Gamma}_\mu{}^{ab}$ are the rotation gauge potential.

The Lagrangian density depends on the covariant derivative of the fields, instead of the ordinary one, $\mathcal{L} = \mathcal{L}(\psi, \hat{D}_a\psi)$. Covariant derivatives (2.2.71) do not commute, but satisfy the commutation relations

$$[\hat{D}_a, \hat{D}_b] = e^\mu{}_a e^\nu{}_b (\tfrac{1}{2}\, F_{\mu\nu}{}^{cd}\, \Sigma_{cd} - F_{\mu\nu}{}^c \hat{D}_c) \ , \tag{2.2.72}$$

where $F_{\mu\nu}{}^{ab}$ are the Lorentz rotation field strength defined, in according to (2.2.59) (here the $\hat{\Gamma}_\mu{}^{ab}$'s play the role of the spin connections $\omega_\mu{}^{ab}$) as

$$F_{\mu\nu}{}^{ab} = R_{\mu\nu}{}^{ab} \ . \tag{2.2.73}$$

The quantities $F_{\mu\nu}{}^c$ are the translation field strength defined trough the torsion field as

$$-F_{\mu\nu}{}^c = \mathcal{T}_{\mu\nu}{}^c \ . \tag{2.2.74}$$

The gauge covariant translation operator changes the commutation relations WRT the Minkowski space results; in particular the algebra does not close, as shown above. Covariant energy-momentum and spin currents can be found, in analogy with the global case. In particular, we underline that the expression of the translation field strength as torsion in eq. (2.2.74) leads to the same analysis of the ECT dealing with a contact spin-spin interaction proper of a non-propagating torsion.

The comparison of gauge model WRT the tetrad formalism of gravity leads to the identification of the rotational gauge fields $\hat{\Gamma}_{\mu}{}^{ab}$, which accounts for local Lorentz transformations, with the spin connections $\omega_{\mu}{}^{ab}$, and of the fields $e^{\mu}{}_{a}$, which describe translations, with the components of the tetrad field. This way, the identifications of the Lorentz field strength with curvature and that of the translation field strength with torsion, are straightforward.

As outlined in the U^4 theory by Hehl *et. al* [48], it is possible to infer the inadequacy of special relativity to describe the behavior of matter fields under local Poincaré transformations. Global Poincaré transformations preserve distances between events and the metric properties of neighboring matter fields: comparing field amplitudes before performing the transformation, and then transforming the result, or comparing the transformed amplitudes of the fields is equivalent. This property is known as *rigidity condition*, as matter fields behave as rigid bodies under this kind of transformations. On the contrary, it can be shown that the action of local Poincaré transformations can be interpreted as an irregular deformation of matter fields, thus predicting different phenomenological evidences for the field and for the transformed one. The compensating gauge fields $e_{\mu}{}^{a}$ and $\hat{\Gamma}_{\mu}{}^{ab}$, introduced to restore local invariance, describe geometrical properties of the space-time.

Other gauge theories In a work by K. Hayashi and T. Shirafuji [103], they further examined PGT, but addressing the notion of quadratic Lagrangians. They considered the irreducible decompositions (under the LG) of the torsion tensor as follows. The trace of the torsion is defined as

$$\mathcal{T}_{\mu} = \mathcal{T}^{\rho}_{\mu\rho\cdot}\ . \tag{2.2.75}$$

Using the Young table method they also define the traceless part

$$t_{\mu\nu\rho} = \mathcal{T}_{\mu\nu\rho} + \mathcal{T}_{\nu\rho\mu} - \tfrac{1}{3}(\mathcal{T}_{\nu}\, g_{\mu\rho} + \mathcal{T}_{\mu}\, g_{\nu\rho}) + \tfrac{2}{3}\mathcal{T}_{\rho}\, g_{\mu\nu}\ , \tag{2.2.76}$$

and anti-symmetric part as

$$a_\sigma = \frac{1}{3}\,\epsilon_{\sigma\mu\nu\rho}\mathcal{T}^{\mu\nu\rho}\ . \tag{2.2.77}$$

The most general Lagrangian quadratic in these irreducible parts of the torsion can be addressed as follows

$$\mathcal{L}_T = A\ t_{\mu\nu\rho}t^{\mu\nu\rho} + 4B\ \mathcal{T}_\mu\mathcal{T}^\mu + C\ a_\mu\, a^\mu\ . \tag{2.2.78}$$

They repeat this procedure for the curvature scalar getting five terms quadratic in the curvature tensor, or combinations and contractions of such quantities. Adding the scalar invariant $\tilde{R}$ a ten parameter Lagrangian for the PGT has been obtained. Advantages of this general framework are that torsion propagates, and the use of terms quadratic in the field strength mimics conventional gauge theory.

2.2.4.3 Teleparallelism

An interesting limit of PGT is Weitzenböck or teleparallel geometry, defined by the requirement

$$R_{\mu\nu}{}^{ab} = 0\ . \tag{2.2.79}$$

Teleparallel geometry (see, for example, [54] for a hand-on review and all the references therein) can be interpreted, to some extents, as complementary to Riemannian: curvature vanishes and torsion remains to characterize the parallel transport. The physical interpretation of such a geometry relies on the fact that there is a one-parameter family of teleparallel Lagrangians which is empirically equivalent to GR [104, 88].

Lagrangian and field equations Within this framework, the gravitational field is described by tetrads $e_\mu{}^a$ and Lorentz connections $\omega_\mu{}^{ab}$, where (2.2.79) has to be taken into account. For our purposes, it is useful to consider the class of Lagrangians quadratic in torsion, *i.e.*,

$$\mathcal{L}_{TP} = b\ \mathcal{L}_T + \lambda^{\mu\nu}{}_{ab}R_{\mu\nu}{}^{ab} + \mathcal{L}_{Matter}\ , \tag{2.2.80}$$

where $\lambda^{\mu\nu}{}_{ab}$ are Lagrange multipliers introduced to ensure condition (2.2.79) in the variational formalism and $\mathcal{L}_T$ is now defined as

$$\mathcal{L}_T = A\mathcal{T}_{abc}\mathcal{T}^{abc} + B\mathcal{T}_{abc}\mathcal{T}^{bac} + C\mathcal{T}_a\mathcal{T}^a = \beta_{abc}\mathcal{T}^{abc}\ , \tag{2.2.81}$$

where

$$\beta_{abc} = a(A\mathcal{T}_{abc} + B\mathcal{T}_{[bac]} + C\eta_{a[b}\mathcal{T}_{c]}) \ , \tag{2.2.82}$$

here a, b, A, B, C are constant parameters and η_{ab} denotes the Lorentz metric. Variations of (2.2.80) WRT ${e_\mu}^a$, ${\omega_\mu}^{ab}$ and ${\lambda^{\mu\nu}}_{ab}$ lead to the following field equations [54]:

$$\begin{aligned} 4\nabla_\rho(b{\beta_a}^{\mu\rho}) - 4b\beta^{bc\mu}\mathcal{T}_{bca} + {e^\mu}_a b\mathcal{L}_T &= {T^\mu}_a \ , && (2.2.83a)\\ 4\nabla_\rho{\lambda^{\mu\rho}}_{ab} - 8b{\beta^\mu}_{[ab]} &= {\tau^\mu}_{ab} \ , && (2.2.83b)\\ {R_{\mu\nu}}^{ab} &= 0 \ , && (2.2.83c) \end{aligned}$$

where ${\tau^\mu}_{ab}$ is the spin current introduced in the ECT. Eq. (2.2.83c) ensures (2.2.79) from variational principles, on the other hand, eq. (2.2.83a) is a dynamical equation for ${e^\mu}_a$. The only role of (2.2.83b) is to determine the Lagrange multipliers ${\lambda^{\mu\nu}}_{ab}$ and the non-uniqueness of such coefficients is related to an extra gauge freedom in the theory. In fact, the gravitational Lagrangian (2.2.80) is, by construction, invariant under the local Poincaré transformations and, up to a four-divergence, under the transformations [105]

$$\delta{\lambda^{\mu\nu}}_{ab} = \nabla_\rho{\varepsilon^{\mu\nu\rho}}_{ab} \ , \tag{2.2.84}$$

where the gauge parameter ${\varepsilon^{\mu\nu\rho}}_{ab} = -{\varepsilon^{\mu\nu\rho}}_{ba}$ is completely anti-symmetric. The λ-transformations can be recasted in

$$\delta{\lambda^{\alpha\beta}}_{ab} = \nabla_0{\varepsilon^{\alpha\beta}}_{ab} + \nabla_\gamma{\varepsilon^{\alpha\beta\gamma}}_{ab} \ , \qquad \delta{\lambda^{0\beta}}_{ab} = \nabla_\gamma{\varepsilon^{\beta\gamma}}_{ab} \ , \tag{2.2.85}$$

where ${\varepsilon^{\mu\nu}}_{ab} = {\varepsilon^{\mu\nu0}}_{ab}$ (the invariance of eq. (2.2.83b) follows directly from ${R_{\mu\nu}}^{ab} = 0$). One can show that the only independent parameters of the λ symmetry are ${\varepsilon^{\alpha\beta}}_{ab}$, so that the six parameters ${\varepsilon^{\alpha\beta\gamma}}_{ab}$ are not independent of ${\varepsilon^{\alpha\beta}}_{ab}$ and can be completely discarded, leaving 18 independent gauge parameters, which can be used to fix 18 multipliers ${\lambda^{\mu\nu}}_{ab}$, whereupon the remaining 18 multipliers are determined by the independent field equations (2.2.83b) (at least locally). The gauge structure of such a one-parameter teleparallel theory is believed to be still problematic.

Orthonormal frames If a manifold is paralellizable (which is a quite strong topological restriction), the vanishing of curvature implies that the parallel transport is path independent, so that the resulting tetrads are globally well defined. In such an orthonormal teleparallel frame, the connection coefficients vanish:

$$\omega_\mu{}^{ab} = 0\ . \tag{2.2.86}$$

This construction is not unique, but it defines a class of orthonormal frames, related to each other by global Lorentz transformations. In such a frame, the covariant derivative reduces to the partial one and the torsion takes the simple form:

$$\mathcal{T}_{\mu\nu}{}^a = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a\ . \tag{2.2.87}$$

Eq. (2.2.86) defines a particular solution of the condition $R_{\mu\nu}{}^{ab} = 0$. Since a local Lorentz transformation of the tetrad fields induces a non-homogeneous change in the connection, *i.e.*,

$$e_\mu{}^a \to \Lambda^a_c e_\mu{}^c \qquad \Rightarrow \qquad \omega_\mu{}^{ab} \to \Lambda^a_c \Lambda^b_d\, \omega_\mu{}^{cd} + \Lambda^a_c \partial_\mu \Lambda^{bc}\ , \tag{2.2.88}$$

it follows that the general solution of $R_{\mu\nu}{}^{ab} = 0$ has the form $\omega_\mu{}^{ab} = \Lambda^a_c \partial_\mu \Lambda^{bc}$. Thus, the choice (2.2.86) breaks local Lorentz invariance, and represents a gauge fixing condition.

Discussion In eq. (2.2.80), the teleparallel condition is ensured by the presence of the Lagrange multiplier. Eq. (2.2.83b) merely serves to determine the multiplier, while the non-trivial dynamics is completely contained in eq. (2.2.83a). So far, teleparallel theory (on parallelizable manifolds) may also be described by imposing the gauge condition (2.2.86) directly in the action. The resulting theory is defined in terms of the tetrad fields only and may be thought of as the gauge theory of translations.

The consistency of teleparallel gravity when spinning matter is taken into account has also been discussed within the framework of the teleparallel limit of PGT [106, 107]. In [106], an inconsistency, due to frame dependence, was illustrated to arise for every gauge theory of the Poincaré Group that admits a teleparallel limit in the absence of spinning matter. Furthermore, in [107], a restricted class of transformations was found, according to which the frame invariance of the gravitational Lagrangian does not lead to inconsistencies, even as far as Standard Model for Particles are concerned, and experimental aspects were analyzed.

2.2.5 Torsion and cosmology: outlooks

An advanced topic concerning the Cosmological Dynamics is the analysis dealing with the presence of the torsion field during early Universe evolution (as discussed in [59]) and the study of possible links with the effects induced by dissipative processes discussed in the previous Chapter. In fact, introducing a non-vanishing average spin axial current J^μ and a related torsion field in the relativistic dynamics of the early Universe, a generalized Friedmann Equation for the primordial evolution can be derived. For particular values of the parameters related to torsion field, the primordial singularity can be prevented. In this respect, a parallel study of the modified cosmological equations, in presence of viscous effects, can be performed.

It is worth noting that a "classical action of torsion" can be used only in some special sense. In the ECT, torsion does not have dynamics and therefore can only lead to the contact interaction between spins. On the other hand, the spin of the particle is essentially quantum characteristic. Therefore, the classical torsion can be understood only as the result of a semi-classical approximation in some quantum theory.

Let us now suppose that in the early Universe, due to quantum effects of matter, the average spin axial current does not vanish

$$J^\mu = \langle \bar{\psi} \gamma^5 \gamma^\mu \psi \rangle \; . \tag{2.2.89}$$

Furthermore, torsion is assumed to be completely anti-symmetric and we define the pseudotrace axial vector

$$S^\mu = \epsilon^{\sigma\nu\rho\mu} \, \mathcal{T}_{\sigma\nu\rho} \; . \tag{2.2.90}$$

The EC Action, with this additional current is [108]

$$\mathcal{S}_{EC} = \int d^4x \sqrt{-g} \, \left[-\tfrac{1}{2} \left(R + \kappa \, S_\mu \, S^\mu \right) + S_\mu \, J^\mu \right] \; . \tag{2.2.91}$$

The arbitrary coefficient κ has been included into the dynamics, but it could be suitably included into the definition of the global current (2.2.89). Torsion does not have its own dynamics and, on shell, it simply reads

$$S^\mu = J^\mu \, / \, \kappa \; . \tag{2.2.92}$$

Replacing expression (2.2.92) back into the action (2.2.91), one gets

$$\mathcal{S}_{EC} = \int d^4x \sqrt{-g} \, \left[-\tfrac{1}{2} \, R + \tfrac{1}{2\kappa} \, J_\mu \, J^\mu \right] \; . \tag{2.2.93}$$

For the sake of simplicity, let us consider the conformally flat metric

$$g_{\mu\nu} = \eta_{\mu\nu}\, a^2(\eta) \ ,$$

where $\eta_{\mu\nu}$is the Minkowsky metric and η is the conformal time. Using eq. (2.2.89), the current J^μ has to be replaced by $J^\mu = a^{-4}\ \hat{J}^\mu$, where $\hat{J}^\mu$ is constant. By the definition

$$\tfrac{4}{3\kappa}\ \eta_{\mu\nu} \hat{J}^\mu \hat{J}^\nu = K = const. \ , \tag{2.2.94}$$

one can get the action and the corresponding motion equation for the scale factor a. They read, respectively

$$\mathcal{S} = -3 \textstyle\int d\eta \int d^3x \left[(\nabla a)^2 - K/a^2 \right] \ , \qquad\qquad \frac{d^2 a}{d\eta^2} = \frac{K}{a^3} \ . \tag{2.2.95}$$

The last equation can be rewritten in terms of physical time t, where $a(\eta)d\eta = dt$:

$$a^2 \ddot{a} + a\dot{a}^2 = K a^{-3} \ . \tag{2.2.96}$$

By standard manipulation, the integral solving this equation is

$$\int \frac{a^2\, da}{\sqrt{Ca^2 - K}} = t - t_0 \ , \tag{2.2.97}$$

where C is the integration constant.

The integral above has different solutions depending on the signs of K and C. According to this fact, different cases can be addressed:

(1.) $K > 0$, spin current time-like: Analyzing eq. (2.2.97) one can show that $C > 0$ and $a(t)$ has minimal value

$$a > a_0 = \sqrt{K/C} > 0 \ . \tag{2.2.98}$$

As a result, the presence of the global time-like spinor current, in the ECT, *prevents the singularity.* Indeed, since such a global spinor current can appear only as a result of some quantum effects, one can consider this as an example of quantum elimination of the Big Bang singularity.

The final explicit solution of eq. (2.2.97) reads

$$arccosh \left(\sqrt{\tfrac{C}{K}}\, a \right) + a \sqrt{a^2 - \tfrac{K}{C}} = \tfrac{2C^{3/2}}{K} \left(t - t_0\right) \ . \tag{2.2.99}$$

In the limit $t \to \infty$ the asymptotic behavior $a \sim t^{2/3}$ is reached. The importance of torsion is seen only at small distances and times and for the scale factor comparable to $a_0 = \sqrt{K/C}$. At this scale torsion prevents singularity and provides the cosmological solution with bounce.

(2.) $K < 0$, spin current space-like: For any value of C, singularities occur. If $C > 0$, the solution is

$$\frac{a}{C} \sqrt{1 + \frac{C}{|K|} a^2} \; - \; \frac{|K|^{1/2}}{C^{3/2}} \ln \left[\sqrt{\frac{C}{|K|}}\, a + \sqrt{1 + \frac{C}{|K|}\, a^2} \right] = 2\,(t - t_0)\;, \tag{2.2.100}$$

while in case of negative C the the solution reads

$$-\frac{a}{|C|} \sqrt{1 - \left|\frac{C}{K}\right|\, a^2} \; + \; \left|\frac{K}{C^3}\right|^{1/2} \; arcsin \left(\sqrt{\left|\frac{C}{K}\right|}\, a \right) = 2\,(t - t_0)\;, \tag{2.2.101}$$

while if $C = 0$ one gets

$$a(t) = \Big[3\,|K|\,(t - t_0) \Big]^{1/3} \sim t^{1/3}\;. \tag{2.2.102}$$

(3.) $K = 0$, spin current light-like: In this case $C > 0$ and the solution is

$$a(t) = \Big[2\,\sqrt{C}\,(t - t_0) \Big]^{1/2} \sim t^{1/2}\;, \tag{2.2.103}$$

which is, of course, exactly the same solution as one meets in the theory without torsion.

2.3 Propagating torsion: effects on the gravitational potential

Working in the Lagrangian framework and using a geometric theory in vacuum with propagating torsion, we establish the principle of minimal substitution to derive test-particle motion equation. In particular, we obtain, as result[2], that they move along Autoparallels. We then calculate the analogous of the geodesic deviation for these trajectories and analyze their behavior in the non-relativistic limit, showing that a part of the torsion field has a phenomenology which is indistinguishable from that of the gravitational Newton field

In this analysis, we follow the torsion-potential approach to describe a propagating torsion. In this respect, we recall that the anti-symmetric and trace parts of the torsion tensor are considered as derived from local potential fields: a tensor quantity $V_{\mu\nu}(x)$ and a scalar one $\phi(x)$. In presence of torsion and requiring that the non-metricity $Q_{\mu\nu\rho}$ be vanishing, affine connections write as eq. (2.2.7), *i.e.*,

$$\tilde{\Gamma}_{\mu\nu\rho} = \Gamma_{\mu\nu\rho} + \mathcal{K}_{\mu\nu\rho} \; , \tag{2.3.1}$$

where, we remind that $\Gamma_{\mu\nu\rho}$ are the Christoffel Symbols and $\mathcal{K}_{\mu\nu\rho}$ is defined using eq. (2.2.8). Expressing now the torsion field trough the potentials as in eqs. (2.2.37), connections read

$$\tilde{\Gamma}_{\mu\nu\rho} = \Gamma_{\mu\nu\rho} + \partial_{[\mu} V_{\nu\rho]} + \tfrac{2}{3}((\partial_\rho\phi) g_{\mu\nu} - (\partial_\nu\phi) g_{\mu\rho}) \; . \tag{2.3.2}$$

2.3.1 Test-particle motion

The problem of determining the test-particle motion equations can be approached by several point of view. In particular, the one proposed by A. Papapetrou [109] consists in obtaining the equations of motion from the conservation law of the EMT. According to us this approach has some unsatisfactory aspects. First of all, some ambiguities are generated concerning the derivation of the conservation law since it can be evaluated both using the Nöether theorem and Ricci identities, but, in presence of torsion, the results can be different [110]. Secondly, once the conservation law is obtained, we

[2]NC, O.M. Lecian and G. Montani, "Macroscopic and Microscopic Paradigms for the Torsion Field: from the Test-Particle Motion to a Lorentz Gauge Theory", *Ann. Fond. L. deBroglie* **32**(2/3), 281 (2007).

have to explicitly write the expression of the EMT which is rather difficult, especially in the case of presence of non-Riemannian quantities as torsion. In fact, the EMT probably depends on spin and it is not clear to give a semi-classical expression of it, being spin a purely quantum quantity.

Another approach to test-particle motion equations is developed by S. Hojman [111] and consists in defining all compatible scalar quantities to be involved in the test-particle action and then the equations of motion are obtained by variations WRT the particle coordinates. This approach has one of the same unsatisfactory aspects of the previous one since, taking into account the test-particle spin, we need again to address a semi-classical expression for the spin-depending part of the action.

Furthermore, equations of motion can be derived using the shortest-path principle, assuming that the test-particle trajectory from a point A to another point B corresponds to the least length among all the curves joining A with B. Although this method seems simple and appreciable, it is completely regardless of the presence of torsion because this property of the space does not contribute to the length of a path and does not appear in the motion equation of any test particle.

The minimal substitution In view of the consideration above, the presence of a tensor quantity as torsion, which has a role in the parallel transport of vector fields in space, should have some effects on the motion and therefore the correct method is to implement the minimal substitution $({}^{d}/{}_{d\tau}) \to ({}^{\tilde{\nabla}}/{}_{d\tau})$.

According to this rule, the motion equations in curved space are derived from that of special relativity

$$\frac{du^\mu}{d\tau} = 0 \,, \tag{2.3.3}$$

where u^μ denotes the 4-velocity, for which ${}^{\tilde{\nabla} u^\mu}/{}_{d\tau} = 0$ is obtained. Using eq. (2.3.2), such an expression can be rewritten as

$$\frac{\tilde{\nabla} u^\rho}{d\tau} = \frac{du^\rho}{d\tau} + \Gamma^\rho_{\mu\nu} u^\mu u^\nu + \tfrac{2}{3}\, g^{\rho\sigma}(g_{\mu\nu}\partial_\sigma\phi - g_{\mu\sigma}\partial_\nu\phi)u^\mu u^\nu = 0 \,. \tag{2.3.4}$$

When we discuss the preferred curves in presence of torsion, we must distinguish two different classes of curves, both of which reduce to the *Geodesics* in correspondence of the torsionless limit of standard GR. *(i)* The *Autoparallels* (straightest lines), described by eq. (2.3.4), are curves whose tangent vector is parallelly transported along itself. Note that only the symmetric part (but torsion dependent) of connections enters (2.3.4). *(ii)* The *Extremal Curves* (shortest or longest lines) are those curves which are of extremal length WRT the metric of the manifold. The length between two points depends only on the metric field and not on the torsion tensor. Therefore, the differential equation for the extremals can be derived from $\delta\int ds = 0$ exactly in the corresponding Riemannian space obtaining the Geodesics,

$$\frac{du^\rho}{d\tau} + \Gamma^\rho_{\mu\nu} u^\mu u^\nu = 0\ . \tag{2.3.5}$$

In a U^4 space the Autoparallels and the Geodesics coincide if and only if the torsion is totally anti-symmetric. The Autoparallels (2.3.4) are the simplest generalization of the flat-space motion equation, which is suitable to take into account torsion or other non-Riemannian quantities.

New action principle and non-holonomic map This approach is proposed in [112, 113] and is based on the idea that it is possible to introduce a new action principle such that, starting from a modified action

$$\mathcal{S}^M = -\tfrac{M}{2}\int_{\tau_1}^{\tau_2} d\tau\ \dot{x}^2\ , \tag{2.3.6}$$

where τ is the proper time, Autoparallels are obtained as the right trajectories.

The key point is that a space-time with torsion can be obtained by a non-holonomic mapping from a flat space-time. We refer to such a mapping when the object of non-holonomity,

$$\Omega_{\mu\nu}{}^a = \partial_{[\mu}\, e_{\nu]}^{\ a}\ , \tag{2.3.7}$$

does not vanish. This quantity measures the non-commutativity of the tetrad basis and enters the definition of the tetrad projection of affine connections as [57, 48]

$$\tilde{\Gamma}_{abc} = -\Omega_{abc} + \Omega_{bca} - \Omega_{cab} - \mathcal{K}_{abc}\ . \tag{2.3.8}$$

In this scheme, the relation between the old paths, *i.e.*, $x^a(\tau)$ and the new one, *i.e.*, $q^\mu(\tau)$, can be written in the following integral form

$$q^\mu(\tau) = q^\mu(\tau_1) + \int_{\tau_1}^{\tau} d\tau' e^\mu{}_a(q(\tau'))\dot{x}^a(\tau')\ , \tag{2.3.9}$$

where $e^\mu{}_a(q(\tau'))$ represents the non-holonomic mapping. As already discussed, the space-time is characterized by open (non-close) parallelograms; as a consequence, variations of test-particle trajectories cannot be performed keeping $\delta x^a(\tau)$ vanishing at endpoints. In fact, the variation $\delta^S q^\mu(\tau)$, images of $\delta x^a(\tau)$ under a non-holonomic mapping, are generally non-vanishing. This way, they can be chosen to be zero at the initial point but then they are non-vanishing at the final point. This behavior is due to torsion. In this scheme, the variation associated to $q^\mu(\tau)$ assume the form:

$$\delta^S q^\mu(\tau) = \int_{\tau_1}^{\tau} d\tau' \big[\big(\delta^S e^\mu{}_a(q(\tau'))\big)\dot{x}^a(\tau') + e^\mu{}_a(q(\tau'))\delta\dot{x}^a(\tau')\big] \ . \qquad (2.3.10)$$

Let us now take into account the auxiliary non-holonomic variation, defined as

$$\bar{\delta} q^\mu(\tau) \equiv e^\mu{}_a(q(\tau))\delta x^a(\tau) \ , \qquad (2.3.11)$$

which, differently from $\delta^S q^\mu(\tau)$, vanishes at endpoints and forms closed paths in the q-space. We can now evaluate the relation

$$\begin{aligned} \tfrac{d}{d\tau}\,\delta^S q^\mu(\tau) &= \big(\delta^S e^\mu{}_a(q(\tau))\big)\dot{x}^a(\tau) + e^\mu{}_a(q(\tau))\delta\dot{x}^a(\tau) = \\ &= \big[\delta^S e^\mu{}_a(q(\tau))\big]\dot{x}^a(\tau) + e^\mu{}_a(q(\tau))\tfrac{d}{d\tau}\big[e_\nu{}^a(q(\tau))\bar{\delta} q^\nu(\tau)\big] \ , \end{aligned} \qquad (2.3.12)$$

which, substituting the expressions

$$\delta^S e^\mu{}_a = -\tilde{\Gamma}^\mu_{\lambda\nu}\delta^S q^\lambda e^\mu{}_a \ , \qquad \tfrac{d}{d\tau} e_\nu{}^a = \tilde{\Gamma}^\mu_{\lambda\nu}\dot{q}^\lambda e_\mu{}^a \ , \qquad (2.3.13)$$

can be rewritten as

$$\tfrac{d}{d\tau}\delta^S q^\mu = -\tilde{\Gamma}^\mu_{\lambda\nu}\delta^S q^\lambda \dot{q}^\nu + \tilde{\Gamma}^\mu_{\lambda\nu}\dot{q}^\lambda \bar{\delta} q^\nu + \tfrac{d}{d\tau}\bar{\delta} q^\mu \ . \qquad (2.3.14)$$

Introducing the parameter $\delta^S b^\mu$, *i.e.*, the difference between $\delta^S q^\mu$ and $\bar{\delta} q^\mu$, the equation above reads

$$\tfrac{d}{d\tau}\delta^S b^\mu = -\tilde{\Gamma}^\mu_{\lambda\nu}\delta^S b^\lambda \dot{q}^\nu + 2\mathcal{T}^\mu_{\lambda\nu}\dot{q}^\lambda \bar{\delta} q^\nu \ . \qquad (2.3.15)$$

The variation of the action (2.3.6), written in terms of the new paths $q^\mu(\tau)$, under $\delta^S q^\mu = \bar{\delta} q^\mu + \delta^S b^\mu$ reads now

$$\begin{aligned} \delta^S \mathcal{S}^M &= \delta^S\big(-\tfrac{M}{2}\int_{\tau_1}^{\tau_2} d\tau\, g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu\big) = \\ &= -M\int_{\tau_1}^{\tau_2} d\tau\big(g_{\mu\nu}\dot{q}^\nu\delta^S\dot{q}^\mu + \tfrac{1}{2}\partial_\mu g_{\lambda\rho}\delta^S q^\mu\,\dot{q}^\lambda\dot{q}^\rho\big) \ . \end{aligned} \qquad (2.3.16)$$

Using the relation $[\delta^S, {}^{d}/_{d\tau}] = 0$, which follows from (2.3.9), we can integrate the $\delta^S q-$term and, by the identity $\partial_\mu g_{\nu\lambda} = \tilde{\Gamma}_{\mu\nu\lambda} + \tilde{\Gamma}_{\mu\lambda\nu}$, we get

$$\begin{aligned} \delta^S \mathcal{S}^M = -M\int_{\tau_1}^{\tau_2} d\tau\big[&- g_{\mu\nu}\big(\ddot{q}^\nu + \Gamma^\nu_{\lambda\rho}\dot{q}^\lambda\dot{q}^\rho\big)\bar{\delta} q^\mu + \\ &+ \big(g_{\mu\nu}\dot{q}^\nu\tfrac{d}{d\tau}\delta^S b^\mu + \tilde{\Gamma}_{\mu\lambda\rho}\delta^S b^\mu\dot{q}^\lambda\dot{q}^\rho\big)\big] \ . \end{aligned} \qquad (2.3.17)$$

It is straightforward now to obtain first the motion equation in absence of torsion, *i.e.*, $\delta^S b^\mu(\tau) \equiv 0$,

$$\ddot{q}^\nu + \Gamma^\nu_{\lambda\rho}\, \dot{q}^\lambda \dot{q}^\rho = 0 \ , \tag{2.3.18}$$

that corresponds to the Geodesics (2.3.5). On the other hand, taking into account the torsion field, with the help of (2.3.15), we can get the Autoparallels (2.3.4) in presence of torsion

$$\ddot{q}^\nu + \tilde{\Gamma}^\nu_{\lambda\rho}\, \dot{q}^\lambda \dot{q}^\rho = 0 \ . \tag{2.3.19}$$

Autoparallels from a modified action As demonstrated by Papapetrou in [109] the Autoparallel motion can be derived from the EMT ($T_{\mu\nu}$) conservation law at the lowest order of a multipole expansion around the world line. We now give a possible modification of the test-particle action, such that this result could be partially obtained. To this end, we assume the test-particle action of the form

$$\mathcal{S}^M = \int d\tau\, g_{\mu\nu} u^\mu u^\nu\, e^{-\phi/4} \ , \tag{2.3.20}$$

where ϕ correspond to the torsion scalar potential, see eq. (2.3.2). Taking into account the generic identity

$$\delta\mathcal{S} = \int d^4x \sqrt{-g}\, \left({}^gT^{\mu\nu}\, \delta g_{\mu\nu} + {}^\phi T\, \delta\phi\right) \ , \tag{2.3.21}$$

we now calculate the action variations WRT $g_{\mu\nu}$ and ϕ, respectively:

$$\begin{aligned} {}^gT^{\mu\nu} &= \frac{\delta\mathcal{S}^M}{\delta g_{\mu\nu}} = \int \frac{d\tau}{\sqrt{-g}}\ u^\mu u^\nu e^{-\phi/4}\, \delta(x - x_0) \ , \\ {}^\phi T &= \frac{\delta\mathcal{S}^M}{\delta\phi} = -\frac{1}{4}\int \frac{d\tau}{\sqrt{-g}}\ g_{\mu\nu}\, u^\mu u^\nu e^{-\phi/4}\, \delta(x - x_0) \ . \end{aligned} \tag{2.3.22}$$

Following the work by Hammond [114], we consider the motion of a test particle, which negligibly perturbs the background geometry in which it lives and we start from the identity

$$(\sqrt{-g}\, {}^gT^{\mu\nu})_{,\nu} = \sqrt{-g}\ {}^gT^{\mu\nu}_{\ ;\nu} - \sqrt{-g}\, \Gamma^\mu_{\rho\sigma}\ {}^gT^{\rho\sigma} \ . \tag{2.3.23}$$

Let us now integrate the last expression over a volume dV, where the test-particle contribution to the EMT is the only non-negligible one. Taking into account the Bianchi Identity, one can derive the conservation law [114]

$${}^gT^{\mu\nu}_{\ ;\nu} = \frac{8}{3}\, \partial^\mu\phi\ \ {}^\phi T \ , \tag{2.3.24}$$

and discarding all surface terms, we get

$$\tfrac{d}{u^0 d\tau}\int dV\sqrt{-g}\;\; {}^{g}T^{\mu 0} = \tfrac{8}{3}\,\partial^{\mu}\phi\,\int dV\sqrt{-g}\;\; {}^{\phi}T - \;\Gamma^{\mu}_{\rho\sigma}\,\int dV\sqrt{-g}\;\; {}^{g}T^{\rho\sigma}\;. \tag{2.3.25}$$

By (2.3.22), this identity can be rewritten in the following form

$$\frac{du^{\rho}}{d\tau} = -\Gamma^{\rho}_{\mu\nu}u^{\mu}u^{\nu} - \tfrac{2}{3}\,g^{\rho\sigma}(\partial_{\sigma}\phi)g_{\mu\nu}u^{\mu}u^{\nu}\;, \tag{2.3.26}$$

and, if we multiply the LHS and the RHS of this equation by u_{ρ}, we obtain the identity

$$0 = u_{\rho}\,\partial^{\rho}\phi\;. \tag{2.3.27}$$

Taking into account the Autoparallels (2.3.4), we immediate recognize that it matches the results (2.3.26) and (2.3.27).

2.3.2 Non-relativistic limit and the role of torsion potentials

On the basis of the minimal-substitution rule we have introduced, test particles are found to follow Autoparallel trajectories (2.3.4). Such trajectories can be rewritten as

$$\frac{d^2x^{\rho}}{d\tau^2} = -\Gamma^{\rho}_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} - \mathcal{K}^{\rho}_{\cdot\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}\;. \tag{2.3.28}$$

We remind that the anti-symmetric part of the torsion contribution vanishes; it only contributes as a source for the metric through (2.2.41). In what follows, we will study the non-relativistic limit of Autoparallels and we will calculate the analogous of the *Geodesic Deviation* in order to characterize the role of torsion in the *Tidal Forces*.

Non-relativistic limit of Autoparallels In order to calculate the non-relativistic limit, the following hypotheses can be stated:
(i) the *3*-velocity is much smaller than c, so we can assume $u^{\alpha} \ll 1$;
(ii) the gravitational field and torsion potential ϕ are static and weak.
Since we want to keep only first order terms, by virtue of these assumptions, we will neglect all second-order terms in the quantities above. After some calculations, we obtain the Autoparallels as

$$\frac{du_{\alpha}}{dt} = -\tfrac{1}{2}\,\partial_{\alpha}h_{00} - \tfrac{2}{3}\,\partial_{\alpha}\phi\;, \tag{2.3.29}$$

where we have introduced the metric perturbation

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \ , \tag{2.3.30}$$

$\eta_{\mu\nu}$ being the Minkowsky metric. Now we recall that, in GR, we get the expression

$$\frac{du_\alpha}{dt} = -\tfrac{1}{2}\,\partial_\alpha h_{00} \ , \tag{2.3.31}$$

which allows one to identify h_{00} with the gravitational potential Φ,

$$\tfrac{1}{2}\,h_{00} = \Phi \ . \tag{2.3.32}$$

As one can see from eq. (2.3.29), the "force" due to the torsion potential is present in the same form of the gravitational field h_{00}; in addition, as for the order we are interested in, and reminding of the supposed field's static nature, eq. (2.2.43) for the field ϕ reduces to

$$\Delta\phi(\mathbf{x}) = 0 \ , \tag{2.3.33}$$

which recasts the gravitational field one

$$\Delta h_{00}(\mathbf{x}) = 4\pi\rho \ . \tag{2.3.34}$$

Deviation of Autoparallels Since test particles move along Autoparallels, we are able to calculate the relative acceleration between two such objects. Assuming two particles initially very close to each other, we obtain the expression

$$\frac{\nabla^2 s^\rho}{d\tau^2} = -R^\rho_{\mu\nu\sigma}\, s^\mu u^\nu u^\sigma + -\mathcal{K}^\rho_{\cdot\sigma\nu}\big(\tfrac{ds^\nu}{d\tau}\, u^\sigma + \tfrac{ds^\sigma}{d\tau}\, u^\nu\big) - \big(\nabla_\mu \mathcal{K}^\rho_{\cdot\sigma\nu}\big)\, s^\mu u^\sigma u^\nu \ . \tag{2.3.35}$$

Here s^μ is an infinitesimal vector representing the relative displacement between the two particles. Substituting in the equation above the expression of the contortion tensor (2.2.38a), we get

$$\begin{aligned}\frac{\nabla^2 s^\rho}{d\tau^2} = -R^\rho_{\sigma\nu\mu} s^\sigma u^\mu u^\nu - \tfrac{2}{3}\big[\delta^\rho_\mu\,(\partial_\nu\phi) + g^{\rho\epsilon} g_{\mu\nu}\,(\partial_\epsilon\phi)\big]\big(\tfrac{ds^\nu}{d\tau} u^\mu + \tfrac{ds^\mu}{d\tau} u^\nu\big) + \\ -\tfrac{2}{3}\big[\delta^\rho_\mu \nabla_\sigma\,(\partial_\nu\phi) + g^{\rho\epsilon} g_{\mu\nu} \nabla_\sigma\,(\partial_\epsilon\phi)\big] s^\sigma u^\mu u^\nu \ .\end{aligned} \tag{2.3.36}$$

This equation represents the generalization of the Geodesic Deviation

$$\frac{\nabla^2 s^\rho}{d\tau^2} = -R^\rho_{\sigma\nu\mu} s^\sigma u^\mu u^\nu \ , \tag{2.3.37}$$

of standard GR to a theory with torsion. Once again, we note that the completely anti-symmetric part of torsion contributes to the field equation only as a source.

In order to perform the non-relativistic analysis, we still keep the hypotheses *(i)* and *(ii)* above and we furthermore state that:
(iii) the 4-velocity can be written as $u^\rho \sim (1,0,0,0)$;
(iv) the particle accelerations are compared at the same time, *i.e.*, $s^0 = {}^{ds^0}/_{d\tau} = 0$.
Within this scheme, substituting the expansion (2.3.30), only terms containing $h_{\mu\nu}$ or ϕ as factors multiplied times s^α are non-negligible. This way, eq. (2.3.36) reduces to

$$\frac{d^2 s^\alpha}{dt^2} \simeq -R^\alpha_{\beta 00}\, s^\beta - \tfrac{2}{3}\, \eta^{\alpha\beta}\, s^\gamma\, \partial_{\kappa\beta}\phi\ , \tag{2.3.38}$$

this way, the Tidal Field becomes

$$\mathcal{G}_\alpha = -R_{\beta 00\alpha}\, s^\beta - \tfrac{2}{3}\, \delta^\beta_\alpha\, s^\gamma\, \partial_\gamma \partial_\beta \phi\ . \tag{2.3.39}$$

From the non-relativistic limit of GR, we can identify

$$R_{\beta 00\alpha} = \partial_\beta \partial_\alpha \Phi\ , \tag{2.3.40}$$

where Φ is the gravitational potential. The final expression for the Tidal Field writes as follows:

$$\mathcal{G}_\alpha = -s^\beta\, \partial_\alpha \partial_\beta \Phi - \tfrac{2}{3}\, s^\beta\, \partial_\alpha \partial_\beta \phi\ . \tag{2.3.41}$$

We can conclude that, in the non-relativistic limit, torsion produces a Tidal-Force effect analogous to that produced by the gravitational field.

It is worth noting that, since the fields h_{00} and ϕ (in the non-relativistic limit) obey the Poisson *PDE*'s (2.3.33) and (2.3.34) and enter eq. (2.3.29) and eq. (2.3.41) in the same way, it is impossible to distinguish the effect of the torsion field from that of the gravitational one. This fact, together with the small intensity of torsion forces, makes them even more difficult to be detected.

2.4 The microscopic role of torsion

This Section is devoted to introduce a gauge theory of the group $SO(3,1)$ in order to describe, on shell, a dynamical torsion[3,4,5]. Using the tetradic formalism, in Section (2.2.4.1), we have introduced the spin connections ω_μ^{ab} as functions of tetrads, *i.e.*,

$$\omega_\mu{}^{ab} = e^{\nu a}\nabla_\mu e_\nu{}^b \, . \tag{2.4.1}$$

As already discussed, this correlation yields to a non-suitable interpretation of these connections as real gauge fields of $SL(2,C)$.

In *flat Minkowskian space-time*, the study appears well grounded since an appropriate description of the LG can be addressed. As a result, we formulate a model in which gauge fields of the group $SO(3,1)$ of spin-1, denoted by $A_\mu{}^{ab}$, are added to the spin connections $\omega_\mu{}^{ab}$ and new general connections,

$$\bar{\omega}_\mu{}^{ab} = \omega_\mu{}^{ab} + A_\mu{}^{ab} \, , \tag{2.4.2}$$

enters the dynamics. In the case of flat space-time, spin connections can be chosen to vanish and the effects of connections $A_\mu{}^{ab}$ (*which do not depend on tetrads*) on one-electron atom spectral lines are discussed.

If a *curved space-time* is addressed, we postulate the direct generalization of the picture described above. In this scheme, the function of the $\omega_\mu{}^{ab}$'s is to restore the Dirac algebra, as in eq. (2.2.57). On the other hand, the connections $A_\mu{}^{ab}$ are treated in order to find a relation to the contortion field. Indeed, an identification can be only stated *a posteriori* using the field equations. In this respect, we underline that further analyses can be developed to relate the dynamics of the propagating torsion addressed in [59, 73, 77, 75, 111, 57] to the $A_\mu{}^{ab}$ Lagrangian. In particular, if the quadratic torsion Lagrangians can be stated in terms of a Yang-Mill one.

Since, in our approach, the introduction of the gauge model is related to the fact that spinors behave as a representation of the LG, translations are not included in

[3] NC, O.M. Lecian and G. Montani, "Fermion Dynamics by Internal and Space-Time Symmetries", *Mod. Phys. Let. A*, in press.

[4] NC, O.M. Lecian and G. Montani, "Lorentz Gauge Theory and Spinor Interaction", *Int. J. Mod. Phys. A* **23**(8), 1282 (2008).

[5] NC, O.M. Lecian, G. Montani, "Macroscopic and Microscopic Paradigms for the Torsion Field: from the Test-Particle Motion to a Lorentz Gauge Theory", *Ann. Fond. L. deBroglie* **32**(2/3), 281 (2007).

this gauge picture. In this respect, it is worth recalling that the teleparallel theory of gravity can be treated physically as a gauge theory of translations. In fact, teleparallel gravity can be understood within the framework of metric-affine gravitational theories [82] and it is picked up from such other models by reducing the affine symmetry group to the translation subgroup, *i.e.*, by imposing vanishing curvature and non-metricity.

2.4.1 Spinors and SO(3,1) gauge theory on flat space-time

Let us now analyze the formulation of the gauge model in a flat Minkowskian space-time. The choice of flat space is due to the fact that the Riemann curvature tensor vanishes and, consequently, the usual spin connections ${\omega_\mu}^{ab}$ can be set to zero choosing the gauge ${e_\mu}^a = {\delta_\mu}^a$ (in general, the ${\omega_\mu}^{ab}$'s are allowed to be non-vanishing quantities). The introduction of the $SO(3,1)$ connections ${A_\mu}^{ab}$, leads to the identification

$$\bar{\omega}_\mu{}^{ab} = {A_\mu}^{ab} \, . \tag{2.4.3}$$

In a 4-dimensional flat space-time, the metric tensor reads $g_{\mu\nu} = \eta_{ab} {e_\mu}^a {e_\nu}^b$ and spin-1/2 fields are described by the usual Lagrangian density

$$\mathcal{L}_F = \tfrac{i}{2}\, \bar{\psi}\gamma^a e^\mu{}_a \partial_\mu \psi - \tfrac{i}{2}\, e^\mu{}_a \partial_\mu \bar{\psi}\gamma^a \psi \, . \tag{2.4.4}$$

Let us now consider an infinitesimal $SO(3,1)$ local transformation $S = S(\Lambda(x))$, described by the anti-symmetric parameter $\epsilon^a_b(x)$:

$$S \;=\; I - \tfrac{i}{4}\, \epsilon^{ab}\, \Sigma_{ab} \, , \tag{2.4.5}$$

$$\Sigma_{ab} = \tfrac{i}{2}\, [\gamma_a, \gamma_b] \, , \qquad [\Sigma_{cd}, \Sigma_{ef}] = i \mathcal{F}^{ab}_{cdef}\, \Sigma_{ab} \, . \tag{2.4.6}$$

In analogy with the formalisms of particle physics and renormalization techniques [115, 116], a suitable coupling constant could be attributed to the symmetry. Anyhow, because of the technical character of this analysis, here we prefer follow the notation of the great majority of the works [19, 59]. Nonetheless, it is worth remembering that such a coupling constant should be very small, as this kind of interaction has not been detected experimentally yet [117]. For some issues related to the use of such a coupling constant, see also [118].

The transformations (2.4.5) act on the spinor in the standard way:

$$\psi(x) \to S\, \psi(x) \, , \qquad \bar{\psi}(x) \to \bar{\psi}(x)\, S^{-1} \, , \tag{2.4.7}$$

and γ matrices are assumed to transform like vectors, *i.e.*,

$$S\,\gamma^a\,S^{-1} = (\Lambda^{-1})^a_b\,\gamma^b\;. \qquad (2.4.8)$$

The gauge model In this approach, when $\omega_\mu{}^{ab}=0$, the gauge invariance is restored by the substitution of a new covariant derivative, *i.e.*, $\partial_\mu \to D^{(A)}_\mu$, in the Lagrangian (2.4.4):

$$D^{(A)}_\mu\psi = (\partial_\mu - \tfrac{i}{4}\,A_\mu)\,\psi = (\partial_\mu - \tfrac{i}{4}\,A_\mu{}^{ab}\,\Sigma_{ab})\,\psi\;, \qquad (2.4.9)$$

which behaves correctly as $\gamma^\mu D^{(A)}_\mu\psi \to S(\Lambda)\gamma^\mu D^{(A)}_\mu\psi$. In fact, the new gauge field $A_\mu = A_\mu{}^{ab}\Sigma_{ab}$ transforms under the following law: $A_\mu \to S\,A_\mu\,S^{-1} - 4i\,S\,\partial_\mu\,S^{-1}$ and the connections

$$A_\mu{}^{ab} \neq \omega_\mu{}^{ab}(e_\nu{}^c) \qquad (2.4.10)$$

behave like

$$A_\mu{}^{ab} \to A_\mu{}^{ab} - \partial_\mu\epsilon^{ab} + 4\mathcal{F}^{ab}_{cdef}\;\epsilon^{ef}\,A_\mu{}^{cd}\;, \qquad (2.4.11)$$

i.e., as natural Yang-Mill fields associated to the $SO(3,1)$ group. A Lagrangian associated to the gauge connections can be constructed by the introduction of the gauge field strength

$$F_{\mu\nu}{}^{ab} = \partial_\mu A_\nu{}^{ab} - \partial_\nu A_\mu{}^{ab} + \tfrac{1}{4}\mathcal{F}^{ab}_{cdef}A_\mu{}^{cd}A_\nu{}^{ef}\;, \qquad (2.4.12)$$

which is not invariant under gauge transformations, as usual in Yang-Mills gauge theories, but the gauge invariant Lagrangian for the model

$$\mathcal{L}_A = -\tfrac{1}{4}\;F_{\mu\nu}{}^{ab}F^{\mu\nu}{}_{ab}\;, \qquad (2.4.13)$$

can be introduced: in flat a space-time, the only real dynamical fields are the new connections.

In this scheme the total Lagrangian density $\mathcal{L}_{tot} = \mathcal{L}_F(D^{(A)}_\mu\psi) + \mathcal{L}_A$, reads

$$\mathcal{L}_{tot} = \tfrac{i}{2}\;\bar\psi\gamma^a e^\mu{}_a\partial_\mu\psi - \tfrac{i}{2}\;e^\mu{}_a\partial_\mu\bar\psi\gamma^a\psi + \tfrac{1}{8}\,e^\mu{}_c\,\bar\psi\,[\gamma^c,\Sigma_{ab}]_+\,A_\mu{}^{ab}\,\psi\; - \tfrac{1}{4}F_{\mu\nu}{}^{ab}F^{\mu\nu}{}_{ab}\;. \qquad (2.4.14)$$

An interaction term is of course generated and the interaction Lagrangian density can be equivalently written as

$$\mathcal{L}_{int} = -J^\mu{}_{ab}\,A_\mu{}^{ab}\;, \qquad (2.4.15)$$

with

$$J^\mu{}_{ab} = -\tfrac{1}{4}\,\epsilon^{cd}_{ab}\,e^\mu{}_c\,j^{(ax)}_d\;, \qquad j^{(ax)}_d = \bar\psi\,\gamma_5\gamma_d\,\psi\;, \qquad (2.4.16)$$

where $j^{(ax)}_d$ denotes the spin axial current, since we can evaluate the following relation

$$[\gamma^c,\Sigma_{ab}]_+ = 2\,\epsilon^c_{abd}\,\gamma_5\,\gamma^d. \qquad (2.4.17)$$

Field equations The total action of the model is derived by $\mathcal{L}_{tot}$ and it reads

$$ {}^{(Flat)}\mathcal{S}_{tot} = \int \det(e) d^4x \, \big(\tfrac{i}{2} \, \bar{\psi}\gamma^a e^\mu{}_a \partial_\mu \psi - \tfrac{i}{2} \, e^\mu{}_a \partial_\mu \bar{\psi}\gamma^a \psi + - J^\mu{}_{ab} \, A_\mu{}^{ab} - \tfrac{1}{4} F_{\mu\nu}{}^{ab} F^{\mu\nu}{}_{ab}\big) \ . \tag{2.4.18} $$

It is straightforward to verify that this expression naturally fits all the features of a Yang-Mills gauge description. In fact, the covariant derivative (2.4.9) assures invariance under a gauge transformation for the spinor part of the action, and the term (2.4.13) also is invariant under such transformations. According to this picture, it will be natural to obtain the typical field equations of a Yang-Mills theory. Furthermore, it is worth remarking that the introduction of different irreducible pieces of $F_{\mu\nu}{}^{ab}$ with different weights would spoil such gauge description. Since we are dealing with flat space-time, tetrad vectors are not dynamical fields, but only projectors from the target space to the general physical space, then they will appear only in the expression of the invariant volume of the space-time and in scalar products: no variation WRT them will be needed for field equations. Actually the only real dynamical field are the connections $A_\mu{}^{ab}$. In fact, if, in analogy with GR, the curvature saturated on bein vectors is considered as an action for the model, a trivial theory is obtained.

Variation of eq. (2.4.18) WRT tetrad fields would provide the total EMT accounting for the dynamics and interactions of the vector field $A_\mu{}^{ab}$ and the spinor fields, respectively. Variation WRT new connections leads to the dynamical equations

$$ D^{(A)}_\mu F^{\mu\nu}{}_{ab} = J^\nu{}_{ab} \ , \tag{2.4.19} $$

which are the Yang-Mills Equations for the non-Abelian gauge field on flat space-time. The source of this gauge field is the conserved spin-density of the fermion matter. Variation WRT the spinor fields, leads to the usual Dirac interaction equations for the spinor field and for the adjoint field.

Field equations illustrate that the dynamics for a spinor field in an accelerated frame differs from the standard Dirac dynamics for the spinor-gauge field interaction term, *i.e.*, spinor fields are not free fields any more. For the analysis of the Dirac Equation in non-inertial systems in flat space-time, see also [92].

2.4.2 The generalized Pauli Equation

The aim of this Section is investigating the effects that the gauge fields $A_\mu{}^{ab}$ can generate in a *flat space-time*. In particular, we treat the interaction between connections $A_\mu^{\ ab}$ and the 4-spinor ψ of mass m, in order to generalize the well-known Pauli Equation, which corresponds to the motion equation of an electron in presence of an electro-magnetic field [115, 59].

The implementation of the gauge model in flat space, *i.e.*, $\partial_\mu \to D_\mu^{(A)}$ leads to the fermion Lagrangian density

$$\mathcal{L}_F = \tfrac{i}{2}\,\bar{\psi}\gamma^a e^\mu_{\ a}\partial_\mu\psi - \tfrac{i}{2}\,e^\mu_{\ a}\partial_\mu\bar{\psi}\gamma^a\psi \;-\; m\,\bar{\psi}\psi \;+\mathcal{L}_{int}\;, \tag{2.4.20}$$

and, to study the interaction term, let us now start from the explicit expression

$$\mathcal{L}_{int} = \tfrac{1}{4}\,\bar{\psi}\;\epsilon^c_{abd}\,\gamma_5\,\gamma^d\,A^{ab}_c\;\psi\;. \tag{2.4.21}$$

Here $a = \{0, \alpha\}$ and we consider the role of the gauge fields by analyzing its components $A_0^{0\alpha}$, $A_0^{\alpha\beta}$, $A_\gamma^{0\alpha}$, $A_\gamma^{\alpha\beta}$. We now impose the *time-gauge* condition $A_0^{\alpha\beta} = 0$ associated to this picture and neglect the term $A_0^{0\alpha}$ since it sums over the completely anti-symmetric symbol $\epsilon^0_{0\alpha d} \equiv 0$. The interaction Lagrangian density rewrites now

$$\mathcal{L}_{int} = \;\psi^\dagger\,C_0\,\gamma^0\gamma_5\gamma^0\,\psi \;+\; \psi^\dagger\,C_\alpha\,\gamma^0\gamma_5\gamma^\alpha\,\psi\;, \tag{2.4.22}$$

with the following identifications

$$C_0 = \tfrac{1}{4}\,\epsilon^\gamma_{\alpha\beta 0}A_\gamma^{\alpha\beta}\;, \qquad\qquad C_\alpha = \tfrac{1}{4}\,\epsilon^\gamma_{0\beta\alpha}A_\gamma^{0\beta}\;. \tag{2.4.23}$$

Varying now the total action built up from the fermion Lagrangian density WRT $\psi^\dagger$, we get the Modified Dirac Equation

$$(i\,\gamma^0\gamma^0\partial_0 \;+\; C_\alpha\,\gamma^0\gamma_5\gamma^\alpha \;+\; i\,\gamma^0\gamma^\alpha\partial_\alpha \;+\; C_0\,\gamma^0\gamma_5\gamma^0)\,\psi \;=\; m\,\gamma^0\,\psi\;, \tag{2.4.24}$$

which governs the 4-spinor ψ interacting with the gauge fields described here by the C_0 and C_α.

Stationary solutions Let us now look for stationary solutions of the Dirac Equation expanded as

$$\psi(\mathbf{r},t) \to \psi(\mathbf{r})\,e^{-i\mathcal{E}t}, \qquad \psi = \begin{pmatrix}\chi\\ \phi\end{pmatrix}\;, \qquad \psi^\dagger = (\,\chi^\dagger\;,\;\phi^\dagger\,)\;,$$

where $\mathcal{E}$ denotes the spinor total energy and the 4-component spinor $\psi(\mathbf{r})$ is expressed in terms of the two 2-spinors $\chi(\mathbf{r})$ and $\phi(\mathbf{r})$ (here $\mathbf{r}$ denotes the radial vector and $r = \mid \mathbf{r} \mid$). Using now the standard representation of the Dirac matrices,

$$\gamma^\alpha = \begin{pmatrix} 0 & \sigma_\alpha \\ -\sigma_\alpha & 0 \end{pmatrix} \qquad \gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$$

where σ_α denote Pauli matrices, the Modified Dirac Equation (2.4.24) splits into two coupled equations (here we write explicitly the *3*-momentum $p^\alpha = i\partial^\alpha$):

$$(\mathcal{E} - \sigma_\alpha C^\alpha)\,\chi \;-\; (\sigma_\alpha p^\alpha + C_0)\,\phi \;=\; m\,\chi\,, \tag{2.4.25a}$$

$$(\mathcal{E} - \sigma_\alpha C^\alpha)\,\phi \;-\; (\sigma_\alpha p^\alpha + C_0)\,\chi \;=-\; m\,\phi\,. \tag{2.4.25b}$$

The low-energy limit Let us now investigate the non-relativistic limit by splitting the spinor energy in the form

$$\mathcal{E} = E + m\,. \tag{2.4.26}$$

Substituting this expression in the system (2.4.25), we note that both the $|E|$ and $|\,\sigma_\alpha C^\alpha|$ terms are small in comparison WRT the mass term m, in the low-energy limit. Then, eq. (2.4.25b) can be solved approximately as

$$\phi \;=\; \tfrac{1}{2m}\,(\sigma_\alpha p^\alpha + C_0)\,\chi\,. \tag{2.4.27}$$

It is immediate to see that ϕ is smaller than χ by a factor of order p/m (*i.e.*, v/c where v is the magnitude of the velocity): in this scheme, the 2-component spinors ϕ and χ form the so-called *small* and *large components*, respectively [119].

Substituting the small components (2.4.27) in eq. (2.4.25a), after standard manipulation we finally get

$$E\,\chi = \tfrac{1}{2m}\left[p^2 \;+\; C_0^2 \;+\; 2\,C_0\,(\sigma_\alpha p^\alpha) \;+\; \sigma_\alpha C^\alpha\right]\,\chi\,. \tag{2.4.28}$$

This equation exhibits strong analogies with the electro-magnetic case. In particular, it is interesting to investigate the analogue of the so-called Pauli Equation used in the spectral analysis of the energy levels as in the Zeeman effect [119]:

$$E\,\chi = \left[\tfrac{1}{2m}\,(p^2 + e^2\mathcal{A}^2 + 2e\mathcal{A}_\alpha p^\alpha) + \mu_B(\sigma_\alpha B^\alpha) \;-\, e\,\Phi^{(E)}\right]\chi\,, \tag{2.4.29}$$

where $\mu_B = e/2m$ is the Bohr magneton (here e denotes the electron charge) and $\mathcal{A}_\alpha$ are the vector-potential components, B^α being the components of the external magnetic field and $\Phi^{(E)}$ the electric potential.

Corrections for one-electron atoms Let us now neglect the second order term C_0^2 in eq. (2.4.28) and implement the symmetry

$$\partial_\mu \to \partial_\mu + \mathcal{A}_\mu^{U(1)} + A_\mu{}^{ab}\Sigma_{ab}\,, \tag{2.4.30}$$

with a vanishing electromagnetic vector potential, *i.e.*, $\mathcal{A}_\alpha = 0$. In such a way, we introduce a Coulomb central potential

$$V(r) = Ze^2/4\pi\varepsilon_o r \ , \tag{2.4.31}$$

where Z is the atomic number and ε_o is the vacuum dielectric constant. Substituting now $E \to E - V(r)$ in eq. (2.4.28), we can derive the total Hamiltonian of the system:

$$H_{tot} = H_0 + H' \ , \tag{2.4.32}$$

where

$$H_0 = \frac{p^2}{2m} - \frac{Ze^2}{(4\pi\epsilon_0)r} \ , \qquad H' = H_1 + H_2 \ , \tag{2.4.33}$$

$$H_1 = \ C_0 \, (\sigma_\alpha \, p^\alpha) \ / \ m \ , \qquad H_2 = \sigma_\alpha \, C^\alpha \ / \ 2m \ , \tag{2.4.34}$$

which characterize the electron dynamics in a one-electron atom. The solutions of the unperturbed Hamiltonian are the well-known modified two-component Schrödinger wave function

$$H_0 \, \psi_{n \, \ell \, m_\ell \, m_s} = E_n \ \psi_{n \, \ell \, m_\ell}(r) \ \chi_{1/2 \, , m_s} \ , \qquad E_n = -m \, (Z\alpha)^2 \ / \ 2n^2 \ , \tag{2.4.35}$$

using the unperturbed basis $|\, n; \, \ell \, m_\ell \, s \, m_s \rangle$.

Since H_1 and H_2 have to be treated like perturbations, the gauge fields can be considered as independent, in the low-energy (linearized) regime. The analysis of H_1 can be performed substituting the operator $\sigma_\alpha \, p^\alpha$ with $J_\alpha \, p^\alpha$, where J_α denote the components of the total angular-momentum operator (in fact, $L_\alpha \, p^\alpha = 0$). H_1 is diagonal in the basis $|\, n; \, \ell \, s \, j \, m_j \rangle$ and according to basic tensor analysis, we decompose the term $J_\alpha \, p^\alpha$ into spherical-harmonics components. In particular, the Cartesian tensor operator p_α can be factorized into three components $V^{(k)}_q$ where $q = 0, \pm 1$ ($k = 1$ for any vectorial operator) and, by the harmonics formalism, we can use the following identification $V^{(k)}_q = \mathcal{Y}^{m=q}_{l=k}$. This way, we can decompose the corrective matrix element $\langle H_1 \rangle$ into

$$\begin{aligned} \langle H_1 \rangle = & \frac{c_{\ell j} C_0}{m} \, m_j \, \langle n'; \, \ell' \, s' \, j' \, m'_j |\, V^{(1)}_0 \, |\, n; \, \ell \, s \, j \, m_j \rangle \, + \\ & + \frac{c_{\ell j} C_0}{m} \, \sqrt{(j \mp m_j)(j \pm m_j + 1)} \, \langle n'; \, \ell' \, s' \, j' \, m'_j |\, V^{(1)}_{\pm 1} \, |\, n; \, \ell \, s \, j \, m_j \pm 1 \rangle \ , \end{aligned} \tag{2.4.36}$$

where $c_{\ell j}$ are the Clebsch-Gordan coefficients to change the basis $|\, n; \, \ell \, m_\ell \, s \, m_s \rangle$ into $|\, n; \, \ell \, s \, j \, m_j \rangle$. The terms above can be evaluated using the Wigner-Eckart Theorem.

Wigner-Eckart Theorem: Such a fundamental theorem of the quantum mechanics [120] states that matrix elements of a generic spherical tensor operator on the basis of angular-momentum eigenstates can be expressed as the product of two factors:

$$\langle\alpha';\, j'\, m'_j|\, V^{(k)}_q\, |\alpha;\, j\, m_j\rangle \overset{WE}{\equiv} \langle j\, k;\, m_j\, q|\, j\, k;\, j'\, m'_j\rangle\, \frac{\langle\alpha'\, j'||\, V^{(k)}_q\, ||\,\alpha\, j\rangle}{\sqrt{2j+1}}\;, \tag{2.4.37}$$

The first one is is just the Clebsch-Gordan coefficient for adding j and k to get j', while the second one is independent of angular momentum orientation and we indicate with the double bar a matrix element not depending on m_j and m'_j and on the geometry of the system. By other words, the Wigner-Eckart Theorem says that operating with a spherical tensor operator of rank k on an angular-momentum basis is like adding a state with angular momentum k to the state. The matrix element one finds for the spherical tensor operator is proportional to a Clebsch-Gordan coefficient, which arises when considering adding two angular momenta. The selection rules for the tensor operator matrix elements (2.4.37) can be now easily derived using the angular-momentum sums. In fact, in order to have non-vanishing Clebsch-Gordan coefficients, we obtain the triangular relation $|\, j-k\,| \leq j' < j+k$ and the constraint $m'_j = m_j + q$.

Let us now apply the Wigner-Eckart formula to our elements (2.4.36). For each harmonics components, we get

$$\begin{aligned}
\langle V^{(1)}_0\rangle &\sim \langle j\,1;\, m_j\, 0\,|\, j\,1;\, j'\, m'_j\rangle\;,\\
\langle V^{(1)}_{+1}\rangle &\sim \langle j\,1;\, m_j\,(+1)\,|\, j\,1;\, j'\,(m'_j+1)\rangle\;,\\
\langle V^{(1)}_{-1}\rangle &\sim \langle j\,1;\, m_j\,(-1)\,|\, j\,1;\, j'\,(m'_j-1)\rangle\;,
\end{aligned}$$

obtaining the following selection rules

$$j' = j+1\;, \qquad\qquad m'_j = m_j\;. \tag{2.4.38}$$

These conditions correspond to have the same parity $P = (-1)^{m_j}$ for the in- and out-state. Anyhow, since $J_\alpha\, p^\alpha$ is a pseudo-scalar operator and it connects states of opposite parity, no transition is eventually allowed.

The analysis of H_2 requires a different approach. We assume that the gauge fields are directed along the z direction. This way, only the component C_3 is considered and, for the sake of simplicity, we impose that only one between A_1^{02} and A_2^{01} contributes, in order to recast the correct DOF . The effect of C_3 corresponds to that of an external "magnetic"-like field generated by the fields $A_\gamma^{0\beta}$, which can be considered the vector bosons (spin-1 and massless particles) of such an interaction. H_2 is now diagonal in the unperturbed basis $|n;\, \ell\, m_\ell\, s\, m_s\rangle$ and produce an energy-level split of the order

$$\Delta E = \frac{C_3}{m}\, m_s \,, \tag{2.4.39}$$

where $m_s = \pm 1/2$. Nevertheless, since we are dealing with spin-1 and massless gauge bosons, the usual electric-dipole selection rules [119] can be used. This way, we have to impose $\Delta m_s = 0$ and no correction to the well-known transitions results to be detectable.

Collecting all the results together, we conclude that no new spectral line arises. Because of this properties of the Hamiltonian, it is not possible to evaluate an upper bound for the coupling constant of the interaction.

2.4.3 Curved space-time and the role of torsion

The considerations developed for a flat space-time are assumed here to be directly generalized in curved space-time. This way we postulate the presence of the general connections $\bar{\omega}_\mu{}^{ab} = \omega_\mu{}^{ab} + A_\mu{}^{ab}$, where spin connections $\omega_\mu{}^{ab}$ allow one to recover the proper Dirac algebra for Dirac matrices.

In what follows, within the framework of curved space-time, the relation between the gauge fields $A_\mu{}^{ab}$ and the geometrical properties of metric-compatible space-times will be investigated. In particular, in the Second-Order Approach, the possibility of identifying the contortion field with the new connections will be investigated. While, in the First-Order Approach, the geometrical hypotheses for the introduction of torsion as a gauge field will be addressed. The two approaches will be compared in the linearized regime.

First-Order Approach Within the framework of First-Order Approach [121], considering a space-time in presence of torsion field $\mathcal{T}^{\rho}_{\cdot\mu\nu}$, the II Cartan Structure Equation (2.2.61) rewrites

$$\partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a - \tilde{\omega}_\mu{}^{ab} e_{\nu b} + \tilde{\omega}_\nu{}^{ab} e_{\mu b} = e_\rho{}^a \tilde{\Gamma}^\rho_{[\mu\nu]} = \mathcal{T}_{\mu\nu}{}^a \,. \tag{2.4.40}$$

The total connections $\tilde{\omega}_{\mu}{}^{ab}$, solution of this equation, are

$$\tilde{\omega}_{\mu}{}^{ab} = \omega_{\mu}{}^{ab} + \mathcal{K}_{\mu}{}^{ab} \,, \tag{2.4.41}$$

where $\mathcal{K}_{\mu}{}^{ab}$ is the projected contortion field, derived by the standard relation

$$\mathcal{K}^{\rho}_{\cdot\mu\nu} = \tfrac{1}{2}\left[\,\mathcal{T}^{\rho}_{\cdot\mu\nu} - \mathcal{T}^{\rho}_{\mu\cdot\nu} - \mathcal{T}^{\rho}_{\nu\cdot\mu}\,\right] , \tag{2.4.42}$$

while the $\omega_{\mu}{}^{ab}$'s are the standard spin connections of eq. (2.4.1). As a result, new torsion dependent connections $\tilde{\omega}_{\mu}{}^{ab}$ enters the dynamics. In GR, nevertheless, such connections do not describe any physical field: after substituting the solutions (2.4.41) into the EH Action [6], one finds that connections $\mathcal{K}_{\mu}{}^{ab}$ appear only in a non-dynamical term, unless spinors are taken into account. In this case, such connections become proportional to the spin density of the matter, thus giving rise to the ECT, where the already discussed spin-spin contact term arises.

To establish a suitable geometrical interpretation of the gauge fields $A_{\mu}{}^{ab}$, let us now introduce general connections $\bar{\omega}_{\mu}{}^{ab}$ for our model and postulate the following interaction term

$$\mathcal{S}_{conn} = 2\!\int\! \det(e)\, d^4x \;\; e^{\mu}{}_{a} e^{\nu}{}_{b}\, \bar{\omega}_{\mu c}^{[a}\, A_{\nu}{}^{bc]} \,. \tag{2.4.43}$$

In such an approach, the action describing the dynamics of the fields $A_{\mu}{}^{ab}$ is derived form the gauge Lagrangian (2.4.13), *i.e.*,

$$\mathcal{S}_{A} = -\tfrac{1}{4}\!\int\! \det(e)\, d^4x \; F_{\mu\nu}{}^{ab} F^{\mu\nu}{}_{ab} \,, \tag{2.4.44}$$

while the action that accounts for the generalized connections can be taken as the gravitational action S_G (2.2.60), but now the projected Riemann tensor (2.2.59) is constructed by the general connections $\bar{\omega}_{\mu}{}^{ab}$. Such a new fundamental Lorentz invariant can be denoted by $\bar{R}_{\mu\nu}{}^{ab}$, and it reads

$$\bar{R}_{\mu\nu}{}^{ab} = \partial_{\nu}\bar{\omega}_{\mu}{}^{ab} - \partial_{\mu}\bar{\omega}_{\nu}{}^{ab} + \mathcal{F}^{ab}_{cdef}\bar{\omega}_{\mu}{}^{cd}\bar{\omega}_{\nu}{}^{ef} \,, \tag{2.4.45}$$

[6]Let $\mathcal{S}\left(q_\alpha, Q_\beta\right)$ be an action depending on two sets of dynamical variables, q_α and Q_β. The solutions of the dynamical equations are extrema of the action WRT both the two sets of variables: if the dynamical equations $\partial\mathcal{S}/\partial q_\alpha = 0$ have a unique solution, $q^{(0)}_\alpha\left(Q_\beta\right)$ for each choice of Q_β, then the extrema of the pullback $\mathcal{S}\left(q_\alpha\left(Q_\beta\right), Q_\beta\right)$ of the action to the set of solution are precisely the extrema of the total total action $\mathcal{S}\left(q_\alpha, Q_\beta\right)$. For an application of this theorem, see, for example [122].

yielding

$$\bar{\mathcal{S}}_G(e,\bar{\omega}) = -\tfrac{1}{2}\int \det(e)\, d^4x \;\; e^{\mu}{}_{a} e^{\nu}{}_{b} \bar{R}_{\mu\nu}{}^{ab}\ . \tag{2.4.46}$$

Collecting all terms together, *i.e.*, $\mathcal{S}_{tot} = \bar{\mathcal{S}}_G + \mathcal{S}_A + \mathcal{S}_{conn}$, one can get the total action for the model. Two cases can now be distinguished according to the absence or presence of spinors.

(1.) If fermion matter is absent, variation of the total action WRT connections $\bar{\omega}_{\mu}{}^{ab}$ gives the Generalized II Cartan Structure Equation

$$\partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a - \bar{\omega}_\mu{}^{ab} e_{\nu b} + \bar{\omega}_\nu{}^{ab} e_{\mu b} = A_\mu{}^{ab} e_{\nu b} - A_\nu{}^{ab} e_{\mu b}\,, \tag{2.4.47}$$

which admits, of course, the solutions

$$\bar{\omega}_\mu{}^{ab} = \omega_\mu{}^{ab} + A_\mu{}^{ab}\ , \tag{2.4.48}$$

As a result, confronting the expression above with the solution (2.4.41), the new gauge fields $A_\mu{}^{ab}$ mimic the dynamics of the contortion field $\mathcal{K}_\mu{}^{ab}$, once field equations are considered. Since solution (2.4.48) is unique, the total action $\mathcal{S}_{tot}$ can be pulled back to the given solutions to obtain the reduced action for the system.

(2.) If the fermion matter contribution is taken into account in the total action (*i.e.*, we add to $\mathcal{S}_{tot}$ the fermion action derived by (2.4.4)) variation WRT total generalized connections leads to additional terms in the RHS of eq. (2.4.47), *i.e.*,

$$\begin{aligned} &\partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a - \bar{\omega}_\mu{}^{ab} e_{\nu b} + \bar{\omega}_\nu{}^{ab} e_{\mu b} = \\ &\qquad = A_\mu{}^{ab} e_{\nu b} - A_\nu{}^{ab} e_{\mu b} - \tfrac{1}{4}\,\epsilon^{ab}_{cd}\, e_\mu{}^c e_{b\nu}\, j^d_{(ax)} + \tfrac{1}{4}\,\epsilon^{ab}_{cd}\, e_\nu{}^c e_{b\mu}\, j^d_{(ax)}\ , \end{aligned} \tag{2.4.49}$$

being $j^d_{(ax)} = \bar{\psi}\gamma_5\gamma^d\psi$ the spin axial current introduced above. The presence of spinors prevents one to identify connections $A_\mu{}^{ab}$ as the only torsion-like components, since all the terms in the RHS of the II Cartan Structure Equation (which, in this picture, is generalized by eq. (2.4.49)) have to be interpreted as torsion. This way, both the gauge fields and the spin axial current contribute to the torsion of space-time. It is worth noting that, if the fields $A_\mu{}^{ab}$ vanishes, we obtain the usual result of PGT [54, 55], *i.e.*, the ECT, in which torsion is directly connected with the density of spin and does not propagate [103]. In our scheme, we obtain the the unique solution for eq. (2.4.49):

$$\bar{\omega}_\mu{}^{ab} = \omega_\mu{}^{ab} + A_\mu{}^{ab} + \tfrac{1}{4}\,\epsilon^{ab}_{cd}\, e_\mu{}^c\, j^d_{(ax)}\ . \tag{2.4.50}$$

Substituting such an expression in the total action, variations leads to dynamical equations where the spin density of the fermion matter is present in the source term of the connections, and the Einstein Equations contain in the RHS not only the EMT of the matter, but also a four-fermion interaction term. The dynamical equations of spinors are formally the same as those ones of the ECT with the addition of the interaction with the connections $A_\mu{}^{ab}$.

As a result, the EC contact interaction is recovered in the limit of vanishing $A_\mu{}^{ab}$, which modifies profoundly the dynamics of the gravitational field both in absence and in presence of fermion matter. In particular, in the first case, the connections $A_\mu{}^{ab}$ are in strict relation with the torsion tensor modifying the Riemannian structure of ordinary space-time, while, in the second case, the presence of fermions already modifies the structure of space-time and the new connections contribute to the torsion tensor with a boson term. Moreover, the bosonic and fermionic parts of torsion interact, the latter being a source for the boson part of torsion, and the former the mediator of the interaction between two-fermion torsion terms. In the most general metric structure, curvature, torsion and non-metricity are present (see for example [123] for the relation between Riemannian curvature and generalized curvature). In [124], the most general parity-conserving quadratic Lagrangian has been established for this metric structure, in terms of the irreducible pieces of non-metricity, torsion and curvature, and a cosmological term is also included.

Second-Order Approach Let us now consider the space-time as a curved manifold, in presence of torsion, in which the tetrad basis is formed by dynamical fields, which describe pure gravity.

The Ricci rotation coefficients write usually as (2.2.54) and we remind that the symbol $\tilde{\nabla}_\mu$ denotes the covariant derivatives implemented with torsion-dependent affine connections. In curved space-time, the validity of the Dirac Equation is ensured as far as the Dirac algebra is valid in the non-Minkowskian metric, *i.e.*,

$$[\,\gamma^a, \gamma^b\,]_+ = 2I\eta^{ab}\ . \tag{2.4.51}$$

The affine connection coefficients are written of the following form,

$$\tilde{\Gamma}^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} - \mathcal{K}^\mu_{\cdot\nu\rho}\ , \tag{2.4.52}$$

where $\Gamma^{\mu}_{\nu\rho}$ are the usual Christoffel symbols. As in the torsionless case of eq. (2.2.57), we now aim to look for a geometrical covariant-derivative operator $\tilde{D}^{(S)}_{\mu}$ which guarantees the relation

$$\tilde{D}^{(S)}_{\mu}\gamma_{\nu} = 0 \ . \tag{2.4.53}$$

If we deal with a generic geometrical object, such an operator is found to be

$$\tilde{D}^{(S)}_{\mu}A = \tilde{\nabla}_{\mu}A - [\,\tilde{\Gamma}^{(S)}_{\mu}, A\,] \ . \tag{2.4.54}$$

This way, we obtain the relation

$$\tilde{D}^{(S)}_{\mu}\psi = \partial_{\mu}\psi - \tilde{\Gamma}^{(S)}_{\mu}\psi \ , \tag{2.4.55}$$

for spinor fields, which yields the following matter lagrangian density

$$\mathcal{L}_F = \tfrac{i}{2}\,\bar{\psi}\gamma^a e^{\mu}{}_a\tilde{D}^{(S)}_{\mu}\psi - \tfrac{i}{2}\,e^{\mu}{}_a\tilde{D}^{(S)}_{\mu}\bar{\psi}\gamma^a\psi \ . \tag{2.4.56}$$

Substituting the affine connections (2.4.52) in (2.4.54) with $A = \gamma^{\nu}$, after standard manipulation, one finds

$$\tilde{\Gamma}^{(S)}_{\mu} = \Gamma^{(S)}_{\mu} + \Gamma^{(\mathcal{K})}_{\mu} = \tfrac{1}{2}\,\omega_{\mu}{}^{ab}\,\Sigma_{ab} \ + \ \tfrac{1}{2}\,\mathcal{K}_{\mu}{}^{ab}\,\Sigma_{ab} \ , \tag{2.4.57}$$

where $\omega_{\mu}{}^{ab}$ are the usual spin connections $\omega_{\mu}{}^{ab} = e_{\mu}{}^{c}\gamma^{ba}{}_{c}$. The connections $\tilde{\Gamma}^{(S)}_{\mu}$ defined by (2.4.54) split up into two different terms: the connections $\Gamma^{(S)}_{\mu}$, which restore the Dirac algebra in the physical space-time (as in the standard tetrad approach to gravity, see eq. (2.2.55)) and torsion dependent connections $\Gamma^{(\mathcal{K})}_{\mu}$, respectively.

In flat space, we have $R = 0$ and we can choose the $\omega_{\mu}{}^{ab}$'s to vanish. Such a scenario matches the result of eq. (2.4.9), *i.e.*, $\tilde{D}^{(S)}_{\mu} = D^{(A)}_{\mu}$, so that torsion-dependent connections $\Gamma^{(K)}_{\mu}$ can be interpreted as the gauge fields

$$\mathcal{K}_{\mu}{}^{ab} = A_{\mu}{}^{ab} \ , \tag{2.4.58}$$

because they are non-vanishing quantities even in flat space-time, as requested for any gauge field.

Since gauge connections are primitive objects, the total action $\mathcal{S}_{tot}$ must depend on the independent fields ψ, $e_{\mu}{}^{a}$, and $A_{\mu}{}^{ab}$, such as

$$\mathcal{S}_{tot} = \int \det(e)d^4x\,\big(-\tfrac{1}{2}\,e^{\mu}{}_a e^{\nu}{}_b R_{\mu\nu}{}^{ab} + \mathcal{L}_F - \tfrac{1}{4}F_{\mu\nu}{}^{ab}F^{\mu\nu}{}_{ab}\big) \ , \tag{2.4.59}$$

which is the generalization on curved space-time of eq. (2.4.18) and $\mathcal{L}_F$ is now defined by eq. (2.4.56). Variation WRT bein vectors, leads to the bein projection of the Einstein Equations, with a Yang-Mills tensor $T_{\mu\nu}$ as source

$$e^{\nu}{}_{a}R_{\mu\nu} - \tfrac{1}{2}\,e^{\nu}{}_{a}g_{\mu\nu}R = e^{\nu}{}_{a}T_{\mu\nu}\ , \tag{2.4.60}$$

while variation WRT connections $A_{\mu}{}^{ab}$ brings Yang-Mills equations, with the spinor current density as a source: eq. (2.4.19). Finally, the Dirac Equation in curved space-time is derived by variations WRT spinors.

This picture allows one to obtain the expression for conserved quantities. Since the current density defined in (2.4.19), admits the conservation law

$$D_{\mu}J^{\mu}{}_{ab} = 0\ , \tag{2.4.61}$$

a conserved (gauge) charge can be defined

$$Q_{ab} = \textstyle\int d^3x J^0{}_{ab} = const.\ , \tag{2.4.62}$$

where this quantity is a conserved one if one assumes that the fluxes through the boundaries of the space integration vanish.

Remarks Since, in the First-Order Approach, the gravitational field plays the role of source for torsion, it should be compared with the "current" term of the Second-Order Formalism. We will restrict our analysis to the linearized regime in the transverse-traceless (TT) gauge.

For small perturbations $h_{\mu\nu}$ of a flat-Minkowskian metric $\eta_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\ , \tag{2.4.63}$$

the tetrad fields rewrite as the sum of the Minkowskian bein projection $\delta_{\mu}{}^{a}$ and the infinitesimal perturbation $\xi_{\mu}{}^{a}$, $e^a_{\mu} = \delta_{\mu}{}^{a} + \xi_{\mu}{}^{a}$ and the following first-order identifications hold

$$\eta_{\mu\nu} = \delta_{\mu}{}^{a}\delta_{\nu a}\ , \qquad\qquad h_{\mu\nu} = \delta_{\mu a}\xi_{\nu}{}^{a} + \delta_{\nu a}\xi_{\mu}{}^{a}\ . \tag{2.4.64}$$

The linearized spin connections $\omega_{\mu}{}^{ab} = e^{\nu a}\nabla_{\mu}e_{\nu}{}^{b}$ rewrite

$$\omega_{\mu}{}^{ab} = \delta^{\nu b}\big(\partial_{\nu}\,\xi_{\nu}{}^{a} - \Gamma(\xi)^{\rho}_{\mu\nu}\,\delta_{\rho}{}^{b}\big), \tag{2.4.65}$$

where $\Gamma(\xi)^{\rho}_{\mu\nu}$ are the linearized Christoffel symbols

$$\Gamma(\xi)^{\rho}_{\mu\nu} = \tfrac{1}{2}\,\delta^{\rho\sigma}(\xi_{\sigma\mu,\,\nu} + \xi_{\sigma\nu,\,\mu} - \xi_{\mu\nu,\,\sigma})\ . \tag{2.4.66}$$

Because of the interaction term (2.4.43) postulated in the First-Order Approach, it is possible to solve the Generalized II Cartan Structure Equation and to express connections as a sum of pure gravitational connections plus other contributions, both in absence and in presence of spinor matter. From the Einstein Lagrangian density in the TT gauge,

$$\mathcal{L}_G = \left(\partial_\rho h_{\mu\nu}\right)\left(\partial^\rho h^{\mu\nu}\right) \ , \tag{2.4.67}$$

the spin-current density associated with the spin angular momentum operator $M^\tau_{\rho\sigma}$ can be evaluated for a Lorentz transformation of the metric. In fact, if we consider the transformation

$$g_{\mu\nu} \rightarrow \partial_\mu x^{\rho'}\, \partial_\nu x^{\sigma'}\, g_{\rho'\sigma'} \ , \tag{2.4.68}$$

where $x'^\rho = x^\rho + \epsilon^\rho_\tau x^\tau$, then the current reads

$$M^\tau_{\rho\sigma} = \tfrac{\partial\mathcal{L}_G}{\partial h_{\mu\nu,\tau}} \Sigma^{\varepsilon\rho\sigma\upsilon}_{\mu\nu} h_{\varepsilon\upsilon} = \left(\eta^{\mu c}\xi^{\nu,\tau}_{\ \ c} + \eta^{\nu c}\xi^{\mu,\tau}_{\ \ c}\right)\Sigma^{\varepsilon\rho\sigma\upsilon}_{\mu\nu}\left(\eta_{\varepsilon f}\xi_\upsilon^{\ f} + \eta_{\upsilon f}\xi_\varepsilon^{\ f}\right) \ , \tag{2.4.69}$$

where

$$\Sigma^{\varepsilon\rho\sigma\upsilon}_{\mu\nu} = \eta^{\gamma[\rho}\left(\delta^\varepsilon_\gamma\delta^{\sigma]}_\mu\delta^\upsilon_\nu + \delta^\varepsilon_\mu\delta^\upsilon_\gamma\delta^{\sigma]}_\nu\right) \ . \tag{2.4.70}$$

The two quantities (2.4.65) and (2.4.69) do not coincide: in fact, (2.4.65) is linear in the $\xi_\mu^{\ a}$ terms, because the interaction term (2.4.43) is linear itself, while (2.4.69) is second order in $\xi_\mu^{\ a}$ by construction. As suggested by the comparison with gauge theories, and with eq. (2.4.69) in particular, the interaction term should be quadratic. In this case, however, it would be very difficult to split up the solution of the II Cartan Structure Equation as the sum of the pure gravitational connections plus other contributions.

2.5 Concluding remarks

This Chapter is aimed at investigating the possibility to describe torsion as a propagating field, from both a macroscopic and microscopic point of view after having described, in some details, several approach to torsion gravity in Section 2.2.

— In Section 2.3, we have exposed the formulation of a macroscopic geometrical theory, which is able to predict propagating torsion. Starting from the static ECT, we have introduced two torsion potentials. To determine the equation of motion of a test particle in presence of this new geometric quantity, we have established a principle of minimal substitution which implies that Autoparallels are the right trajectories. Finally, we have analyzed the analogue of the Geodesic Deviation for Autoparallels and studied the non-relativistic limit of this deviation. Within this scheme, Autoparallel deviation illustrates that the torsion scalar potential enters the dynamics just the same way as the gravitational field, thus letting us envisage an arduous experimental detection.

— In Section 2.4, we have developed a gauge theory of the group $SO(3,1)$, in flat space-time, by choosing vanishing spin connections. In treating spinor fields, a covariant derivative that accounts for the new gauge fields has been formulated. The analysis, in flat space, has been addressed considering the non-relativistic limit of the interaction between spin-$1/2$ fields and the gauge ones. This way, a generalization of the so-called Pauli Equation has been formulated and applied to a one-electron atom in presence of a Coulomb central potential. Energy-level modifications are present but selection rules do not allow for new detectable spectral lines. Then, we directly generalize this picture in curved space-time and a mathematical relation between the new connections and the contortion field has been found from the II Cartan Structure Equation if a (unique) interaction term between the gauge fields and generalized internal connections is introduced. Moreover, the predictions of First- and Second-Order Approaches have been compared in the linearized regime. The two results did not match, in this approximation, thus suggesting one to introduce a quadratic interaction term. Despite many formal differences from PGT, a pure contact interaction for spinor fields has been recovered for vanishing Lorentz connections, for which the II Cartan Structure Equation provides non-zero torsion even when gauge bosons are absent. From this point of view, PGT can be qualitatively interpreted as the First-Order approximation of our scheme.

Outlooks

During this work, we have exposed some peculiar features of space-time and matter source. The latter has been extended by including viscous processes into the dynamics and the effects on the gravitational instability have been analyzed. Concerning the space-time symmetries, we have focused the attention on the non-symmetric properties of the affine connection, *i.e.*, on the torsion filed. In this respect, we have analyzed the macroscopic effects on a test-particle motion and whether such a field affects the gravitational potential in the non-relativistic limit. From a microscopic point of view, we have identified *a posteriori* the torsion with new gauge internal connections and the effects induced on a one-electron atom spectral lines are discussed in flat space-time.

In what follows, we want to briefly discuss some interesting developments of the models here proposed.

First of all, the analysis dealing with the presence of the torsion field during early Universe evolution should be linked with the study of the effects induced by dissipative processes in the primordial eras. Furthermore, a very intriguing problem is how induce propagating torsion including such a quantity directly in the gravitational Lagrangian. In this scheme, further analyses will be performed in order to relate torsion with the gauge filed of Lorentz Group.

The dissipative-cosmology analysis, can be improved by studying the generalization of the Lemaître-Tolman-Bondi (LTB) dust solution. An extension of the model can be analyzed by keeping the usual inhomogeneous spherically symmetric LTB line element and replacing the dust matter source with an imperfect fluid with a generic Equation of State. The Gravitational Equations, in the co-moving reference frame, can be reduced to a set of differential equations: such a system involves the state parameters of the fluid, the scale factor and the curvature, which enter as function of time and of the radial coordinate. In particular, the new matter source requires an additional equation (namely, the 2-2 component of the Einstein Equations) with respect to the standard LTB analysis with dust. The presence of inhomogeneities implies a very complex and intriguing scenario, whose analysis requires both analytical an numerical techniques. Another complication due to the inhomogeneity is the presence of the so-called shear viscosity. In fact, this kind of dissipative effect is related to the matter friction generated during the fluid evolution. The form of the EMT is then extended to include into the dynamics this effect, which generates an additional traceless term.

Another natural development, within the context of dissipative cosmology, consists of pursuing the characterization of the EMT matter source of Gravitational Equations in the so-called Knudsen Regime of non-interacting particles. In fact, at temperature $T > \mathcal{O}(10^{16} GeV)$, the particle mean free-path overcomes the causal horizon scale acquiring a divergent behavior. Following the hydrodynamic procedure, concerning kinetic approach, the structure of the EMT can be derived starting from the relativistic Boltzmann Equation. In this scheme, the non-equilibrium transport phenomena of the gas flow can be described by a set of generalized hydrodynamic equations, where the well-known Navier-Stokes and Fourier laws are replaced by a new set of constitutive equations, which incorporate the non-local stress relation phenomena in addition to the dissipative effects. Starting from these equations, the matter term can be constructed using the standard thermodynamical laws. A well-grounded source term of the Gravitational Equations is very important for the study of the very early Universe and for many interesting applications to the perturbative dynamics of gravitational collapses.

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Attachments

Attachment 1

Modern Physics Letters A

FERMION DYNAMICS BY INTERNAL AND SPACE-TIME SYMMETRIES

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This manuscript is devoted to introduce a gauge theory of the Lorentz Group based on the ambiguity emerging in dealing with isometric diffeomorphism-induced Lorentz transformations. The behaviors under local transformations of fermion fields and spin connections (assumed to be ordinary world vectors) are analyzed in flat space-time and the role of the torsion field, within the generalization to curved space-time, is briefly discussed. The fermion dynamics is then analyzed including the new gauge fields and assuming time-gauge. Stationary solutions of the problem are also studied in the non-relativistic limit, to study the spinor structure of an hydrogen-like atom.

Keywords: Gauge Theory; Lorentz Symmetry; Pauli Equation.

02.04.-k; 11.30.Cp;

Attachment 2

International Journal of Modern Physics D

JEANS INSTABILITY IN PRESENCE OF VISCOUS EFFECTS

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An analysis of the gravitational instability in presence of dissipative effects is addressed. In particular, the standard Jeans Mechanism and the generalization in treating the Universe expansion are both analyzed when bulk viscosity affects the first-order Newtonian dynamics. As results, the perturbation evolution is founded to be damped by dissipative processes and the top-down mechanism of structure fragmentation is suppressed. In such a scheme, the Jeans Mass remain unchanged also in presence of viscosity.

Keywords: Jeans Model; Dissipative Cosmology; Bulk Viscosity.

International Journal of Modern Physics A
Vol. 23, No. 8 (2008) 1282–1285

LORENTZ GAUGE THEORY AND SPINOR INTERACTION

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A gauge theory of the Lorentz group, based on the different behavior of spinors and vectors under local transformations, is formulated in a flat space-time and the role of the torsion field within the generalization to curved space-time is briefly discussed.

The spinor interaction with the new gauge field is then analyzed assuming the *time gauge* and stationary solutions, in the non-relativistic limit, are treated to generalize the Pauli equation.

Keywords: Lorentz Gauge Theory; Spinors.

PACS numbers: 02.40.-k, 04.20.Fy, 04.50.+h, 11.15.-q

Attachment 4

International Journal of Modern Physics A
Vol. 23, No. 8 (2008) 1248–1252

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ON THE ROLE OF VISCOSITY IN EARLY COSMOLOGY

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We present a discussion of the effects induced by *bulk viscosity* on the very early Universe stability. The viscosity coefficient is assumed to be related to the energy density ρ via a power-law of the form $\zeta = \zeta_0 \rho^s$ (where ζ_0, $s = const.$) and the behavior of the *density contrast* in analyzed. In particular, we study both Einstein and hydrodynamic equations up to first and second order in time in the so-called quasi-isotropic collapsing picture near the cosmological singularity. As a result, we get a power-law solution existing only in correspondence to a restricted domain of ζ_0. The particular case of pure isotropic FRW dynamics is then analyzed and we show how the asymptotic approach to the initial singularity admits an unstable collapsing picture.

Keywords: Early Universe; Viscosity.

PACS numbers: 98.80.-k, 95.30.Wi, 51.20.+d

Attachment 5

Macroscopic and Microscopic Paradigms for the Torsion Field: from the Test-Particle Motion to a Lorentz Gauge Theory

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Abstract: Torsion represents the most natural extension of General Relativity and it attracted interest over the years in view of its link with fundamental properties of particle motion. The bulk of the approaches concerning the torsion dynamics focus their attention on their geometrical nature and they are naturally led to formulate a non-propagating theory.

Here we review two different paradigms to describe the role of the torsion field, as far as a propagating feature of the resulting dynamics is concerned. However, these two proposals deal with different pictures, *i.e.*, a macroscopic approach, based on the construction of suitable potentials for the torsion field, and a microscopic approach, which relies on the identification of torsion with the gauge field associated with the local Lorentz symmetry. We analyze in some detail both points of view and their implications on the coupling between torsion and matter will be investigated. In particular, in the macroscopic case, we analyze the test-particle motion to fix the physical trajectory, while, in the microscopic approach, a natural coupling between torsion and the spin momentum of matter fields arises.

P.A.C.S.: 02.40.-k; 04.20.Fy; 04.50.+h; 11.15.-q

Attachment 6

International Journal of Modern Physics D
Vol. 17, No. 6 (2008) 881 896

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STUDY OF THE QUASI-ISOTROPIC SOLUTION NEAR THE COSMOLOGICAL SINGULARITY IN THE PRESENCE OF BULK VISCOSITY

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Received 21 September 2007
Revised 7 November 2007
Communicated by V. G. Gurzadyan

We analyze the dynamical behavior of a quasi-isotropic universe in the presence of a cosmological uid endowed with *bulk viscosity*. We express the viscosity coefficient as a power law of the uid energy density: $\zeta = \zeta_0\,\epsilon^s$. Then we x $s = 1/2$ as the only case in which viscosity plays a signi cant role in the singularity physics but does not dominate the universe dynamics (as required by its microscopic perturbative origin). The parameter ζ_0 is left free to de ne the intensity of the viscous effects.

In spirit of the work by Lifshitz and Khalatnikov on the quasi-isotropic solution, we analyze both Einstein and hydrodynamic equations up to rst and second order in time. As a result, we get a power law solution existing only in correspondence to a restricted domain of ζ_0.

Keywords: Early cosmology; viscosity; perturbation theory.

Attachment 7

Gravitational Stability and Bulk Cosmology

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Abstract. We present a discussion of the effects induced by *bulk viscosity* either on the very early Universe stability and on the dynamics associated to the extreme gravitational collapse of a gas cloud. In both cases the viscosity coefficient is related to the energy density ρ via a power-law of the form $\zeta = \zeta_0 \rho^s$ (where ζ_0, $s = const.$) and the behavior of the *density contrast* in analyzed.

In the first case, matter filling the isotropic and homogeneous background is described by an ultra-relativistic equation of state. The analytic expression of the density contrast shows that its growth is suppressed forward in time as soon as ζ_0 overcomes a critical value. On the other hand, in such a regime, the asymptotic approach to the initial singularity admits an unstable collapsing picture.

In the second case, we investigate the *top-down* fragmentation process of an uniform and spherically symmetric gas cloud within the framework of a Newtonian approach, including the negative pressure contribution associated to the bulk viscous phenomenology. In the extreme regime toward the singularity, we show that the density contrast associated to an adiabatic-like behavior of the gas (which is identified by a particular range of the politropic index) acquire, for sufficiently large viscous contributions, a vanishing behavior which prevents the formation of sub-structures. Such a feature is not present in the isothermal-like collapse. We also emphasize that in the adiabatic-like case bulk viscosity is also responsible for the appearance of a threshold scale (equivalent to a Jeans length) beyond which perturbations begin to increase.

Keywords: Bulk viscosity, Cosmology, General relativity
PACS: 98.80.-k, 95.30.Wi, 51.20.+d

CP966, *Relative Astrophysics - 4th Italian-Sino Workshop*, edited by C. L. Bianco and S.-S. Xue