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EXISTENCE OF SOLUTIONS OF SOME QUADRATIC INTEGRAL EQUATIONS

Giuseppe Anichini,
Facoltà di Ingegneria, Università di Firenze,
giuseppe.anichini@unifi.it

Giuseppe Conti,
Facoltà di Architettura, Università di Firenze,
gconti@unifi.it

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Abstract

In this paper we study the existence of continuous solutions of a quadratic integral equations. The theory of quadratic integral equations has many useful applications in mathematical physics, economics, biology, as well as in describing real world problems. The main tool used in our investigations is a fixed point result for the multivalued solution's map with acyclic values.

Keywords: Fixed point property; measure of noncompactness, compact mappings; acyclic valued; quadratic integral equation.

1 Introduction and Notations

In this paper, we are going to study, in an abstract setting, the solvability of a nonlinear quadratic integral equation of the type

$$x(t) = h(t) + (Tx)(t)\int_0^t k(t,s)u(t,s,x(s))ds.$$

We will look for solutions of that equation in the Banach space of real functions being defined and continuous on a bounded and closed interval. The main tools used in our investigations are the measure of noncompactness and a fixed point result for acyclic (multivalued) maps. Let us mention that the theory of integral equations has many useful applications in describing numerous events and problems of the real world. It is caused by the fact that this theory is frequently applicable in other branches of mathematics and in mathematical physics, economics, biology as well as in describing real world problems.

Moreover, in the theory in question, several types of integral operators, both of linear and nonlinear types are investigated. Let us mention, for instance, the classical linear integral operators of Fredholm or Volterra type and the nonlinear operators of Hammerstein or Urysohn type. These operators and integral equations associated with them were considered in numerous papers and monographs (see [?], [?], [?], [?], [?], etc).

The oldest research in this field includes (see, for istance, [?], [?]) the study of the existence of continuous solutions of the following nonlinear Urysohn integral equation,

$$x(t) = h(t) + \int_0^t g(t,s,x(s))ds, \quad t \in \mathbb{R} \tag{1}$$

or of a Volterra integral equations like:

$$\begin{cases} x(t) &= h(t) + \int_0^t k(t,s)g(s,x(s))ds, t \in \mathbb{R}, \\ x(0) &= x_0, \end{cases} \tag{2}$$

Equations of the type we are dealing with are widely considered for instance in [?], [?], [?], [?]. In [?], the nonlinear integral equation of the type

$$x(t) = 1 + tx(t)\int_0^1 \frac{\psi(t)x(t)}{x+t}dt$$

where $\psi(t)$ is a given function on $[0,1]$ and $x(\cdot)$ is the unknown function, is studied. Such equations arise in theories of radiative transfer, neutron transport and in the kinetic theory of gases.

In this paper, following certain generalizations of the previous integral equation, we will consider (instead of the term $tx(t)$) a general operator T as a continuous operator from suitable Banach spaces.

In [?] the following equation is considered

$$u\frac{dI(t,u\phi)}{dt} = I(t,u,\phi)-$$

$$\frac{1}{4\pi}\frac{\sigma_s}{\sigma_s+\sigma_a}\int_0^{2\pi}\int_{-1}^{1} p(u',\phi';u\phi)I(t,u',\phi')du',d\phi',$$

where $I(t,u,\phi)$ is the unknown intensity (of radiative transfer) at optical depth t considered as non-interacting beams of radiation in all directions. The coefficients σ_s and σ_a denote the scattering and absorption coefficients of the medium; the phase function p specifies the probability distribution of scattering from incident direction (u',ϕ'), to direction (u,ϕ), where u is the cosine of polar angle and ϕ is azimuthal angle.

In the engineering setting, a quadratic integral equation of this kind can for istance arise in the design of bandlimited signals for binary communication using simple memoryless correlation, detection when the signals are disturbed by additive white Gaussian noise. It is shown that a bandlimited signal can be designed which eliminates intersymbol interference for signalling at Nyquist rate: this signal is a solution to a quadratic integral equation (for some suitable references, see [?], [?], [?], [?], [?], [?]).

Equations of this type and some of their generalizations were considered in several papers (see [?], [?], [?]). In those papers the authors proved existence results for more general equations. For instance, in [?] the classic Hammerstein equation is considered (i.e. $T(x(t)) = 1$) and the existence of a solution in the space of all bounded and continuous functions over $\mathbb{R}^+$ is achieved. More recently, in [?] a quadratic integral equation with linear modification of the argument is studied by means of a technique associated with measures of noncompactness, in order to prove the existence of non-decreasing solutions in the space $C[0,1]$. The conditions imposed in these papers are a bit heavier than ours even if, in the proof of the result, the same tool (i.e., the measure of noncompactness) is used. Some other properties, such as uniqueness, location of solutions, and convergence of successive approximations, were also studied in the papers mentioned .

2 Preliminaries

In what follows $B_r(x_0)$ will denote an $r-$ ball (in a metric space ($\mathcal{N}$,d)) i.e., the set $\{x \in \mathcal{N} : d(x, x_0) < r\}$, centered at x_0, where x_0 is any point in $\mathcal{N}$.

We will denote by $\mathcal{B}(I, \mathbb{R})$ the Banach space of all continuous functions (from I to $\mathbb{R}$) equipped with the sup-norm, where $I = [0, L]$

Definition 1 Let X be a topological space and let $\breve{H}^p(X, Z)$ denote the reduced Čech cohomology group of X in dimension p with coefficients in Z. Then by an *acyclic* we mean a (non empty) topological space X such that $\breve{H}^p(X, Z) = 0$ for every $p \geq 0$.

Remark Cohomology can be viewed as a method of assigning algebraic invariants to a topological space. The reduced Čech cohomology is a minor modification made to homology theory designed to make a point have all its homology groups zero.

In the 1930s, Brouwer proved that any non constant selfmapping from a connected and acyclic polyhedron has a fixed point: since then, many generalizations in the field of fixed point theory have been made (see, for instance , Proposition 1 below).

Contractible (and hence convex) sets are examples of acyclics. As for topological properties of acyclics it is known that they are connected and even simply connected. Moreover it is known (see, for istance [?], [?], [?]) that an $R_\delta-$set is an acyclic set (in the Čech homology). There are many papers dealing with the research of conditions in such a way that the lack of convexity can be overcome. There are also cases in which the set of solutions is not convex neither acyclic as, for istance, in the case in which a finite number of solutions arises.

The following results will be very useful in achieving our purpose:

Proposition 1: (see [?]) *Let $\mathcal{B}$ be a Banach space and let $S : B_r(0) \subset \mathcal{B} \longrightarrow B_r(0)$ be a multivalued upper semicontinuous compact operator with acyclic values. Then S admits a fixed point.*

Proposition 2:(see [?] and related references). *Let $(t, x, y) \longrightarrow g(t, x, y)$ be be a continuous function defined on a compact subset D of the space $\mathbb{B} = [t_1, t_2] \times B_R \times \mathbb{R}^n$ into $\mathbb{R}^n$, for some $R > 0$; then there exists a sequence of Lipschitzean functions $\{g_k\}$, leading from B into $\mathbb{R}^n$, such that*

$$\lim_{k\to\infty} g_k(w) = g(w), \quad \text{for all} \quad w = (t, x, y) \in D,$$

$$||g_k|| = \sup\{|g_k(w)|,\, w \in D\} \leq ||g|| = \sup\{|g(w)|,\, w \in D\}.$$

Proposition 3:(see [?] and related references). *Let E be a Banach space, $V \subset E$ be a (suitable nonempty) open set and S be a non linear compact continuous operator from the closure of V into E. Then, if there is a sufficiently ϵ small such that there exists a compact and continuous operator S_ϵ (from the closure of V into E) satisfying $||S(x) - S_\epsilon(x)|| < \epsilon$, $\forall x$ and such that the equation $x - S_\epsilon(x) = b$, has at most one solution if $||b|| < \epsilon$, then the set of fixed points of it is an acyclic set.*

The well known *Gronwall Lemma*, from the standard theory of Ordinary Differential Equations, will also be used:

If, for $t_0 \leq t \leq t_1$, $\phi(t) \geq 0$ and $\psi(t) \geq 0$ are continuous functions such that the inequality $\phi(t) \leq K + M \int_{t_0}^{t} \psi(s)\phi(s)ds$ holds on $t_0 \leq t \leq t_1$, with K and M positive constants, then $\phi(t) \leq K \exp\left(M \int_{t_0}^{t} \psi(s)ds\right)$ on $t_0 \leq t \leq t_1$ (see, e.g., [?]).

3 Main result

Here we want to deal with an integral equation as above in order to establish an existence result for the initial value problem:

$$x(t) = h(t) + (Tx)(t) \int_0^t u(t, s, x(s))ds, \quad t \in I = [0, L] \tag{3}$$

The following theorem holds:

Theorem 1 *: Let us assume that:*

1. *$u : I \times I \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function such that $|u(t, s, x)| \leq \alpha + \beta|x|$, for every $(t, s, x) \in I \times I \times \mathbb{R}$, $\alpha, \beta, \in \mathbb{R}^+ = (, +\infty)$.*
2. *$h : I \longrightarrow \mathbb{R}$ is a continuous function;*

3. *T is a continuous operator from the Banach space $\mathcal{B}(I, \mathbb{R})$ into itself such that there exists $a > 0$ with $|(Tx)(t)| \leq a|x(t)|$ for every $t \in I$;*

4. *$a\,L\,\alpha < 1$.*

Then integral equation (??) has at least one solution in the space $\mathcal{B}(I, \mathbb{R})$

.

Proof: Let M be a suitable ball $B_\rho(0)$ in the space $\mathcal{B}(I, \mathbb{R})$ and let us consider, $\forall q \in M$, the map $\mathbb{U} : M \subset \mathcal{B}(I, \mathbb{R}) \longrightarrow \mathcal{B}(I, \mathbb{R})$ defined as follows: $x = \mathbb{U}(q)$ if and only if $x(t) = h(t) + (Tq)(t) \int_0^t u(t, s, x(s))ds$.

Any possible fixed point of the (as usual multivalued) function $\mathbb{U}$ will be a solution of the integral problem (??).

In order to prove the theorem, the following steps in the proof have to be established:

i): $\mathbb{U}$ *is a (relatively) compact operator.*

To obtain such a result, we prove (by using Ascoli's theorem) that $\mathbb{U}$ is an equicontinuous and equibounded operator.

We take a function $t \to q(t) \in M \subset \mathcal{B}(I, \mathbb{R})$; so $||q|| \leq \rho$; now, using the Gronwall Lemma, we obtain:

$$|x(t)| \leq |h(t)| + a||q|| \int_0^t (\alpha + \beta|x(s)|\, ds \leq (||h|| + a\rho\alpha L) \exp(a\rho\beta L).$$

Thus we can say that there is some constant ρ_0 such that $\mathbb{U}(q) \subset B_{\rho_0}(0)$ for every $q \in M$. So the set $\mathbb{U}(M)$ is equibounded.

ii): $\mathbb{U}$ *is equicontinuous.*

Since $||q|| \leq \rho$, then $||Tq|| \leq a\rho$. Let $x \in \mathbb{U}(q)$ and let us assume that $t_1, t_2 \in [0, L]$ are such that $|t_2 - t_1| < \delta$, for a given positive constant δ.

Thus:

$$|x(t_2) - x(t_1)| \leq |(Tq)(t_2) \int_0^{t_2} u(t_2, s, x(s))ds - (Tq)(t_1) \int_0^{t_1} u(t_1, s, x(s))ds| \leq$$

$$\leq |(Tq)(t_2)|| \int_0^{t_2} u(t_2, s, x(s))ds - (Tq)(t_2) \int_0^{t_2} u(t_1, s, x(s))ds +$$

$$+(Tq)(t_2)\int_0^{t_2} u(t_1,s,x(s))ds-(Tq)(t_2)\int_0^{t_1} u(t_1,s,x(s))ds+$$

$$+(Tq)(t_2)\int_0^{t_1} u(t_1,s,x(s))ds-(Tq)(t_1)\int_0^{t_1} u(t_1,s,x(s))ds|\leq$$

$$|(Tq)(t_2)|\int_0^{t_2}|u(t_2,s,x(s))-u(t_1,s,x(s))|ds+$$

$$|(Tq)(t_2)|\int_{t_1}^{t_2}|u(t_1,s,x(s))|ds+|(Tq)(t_2)-(Tq)(t_1)|\int_0^{t_1}|u(t_1,s,x(s))|ds\leq$$

$$a\rho L\epsilon_1+a\rho(\alpha+\beta\rho)|t_2-t_1|+L(\alpha+\beta\rho)\epsilon_2\leq\epsilon$$

whenever $t_1,t_2\in I$ are such that $|t_2-t_1|<\delta$.

So $\mathbb{U}$ is an equicontinuous operator.

ii): $\mathbb{U}$ *is an upper semicontinuous operator.*

Indeed, let $q_n\longrightarrow q_0$ and let $x_n\in\mathbb{U}(q_n)$, $x_n\longrightarrow x_0$. We need to show that $x_0\in\mathbb{U}(q_0)$.

From $x_n(t)=h(t)+(Tq_n)(t)\int_0^t u(t,s,x_n(s))ds$, from the continuity of the operator T and the function u (w.r.t. x), there follows $lim_{n\to+\infty}(Tq_n)(t)=(Tq_0)(t)$ and $lim_{n\to+\infty}\int_0^t u(t,s,x_n(s))ds=\int_0^t lim_{n\to+\infty}u(t,s,x_n(s))ds$, from condition $i)$ and the Dominated Lebesgue Convergence Theorem. So we get

$$lim_{n\to+\infty}x_n(t)=h(t)+(Tq_0)(t)\int_0^t u(t,s,x_0(s))ds$$

The latter means that $lim_{n\to+\infty}x_n(t)=x_0(t)$, or, equivalently, that $x_0\in\mathbb{U}(q_0)$.

iii): $\mathbb{U}$ (q) *is acyclic for every* $q\in M$.

To get that we want to apply Proposition 3. So let $u_n(t,s,\circ)\longrightarrow u(t,s,\circ)$ be a sequence of Lipschitzean functions (w.t.r. the third variable) such that (see Proposition 2) $||u_n||\leq||u||$, $\forall t\in I$, $x\in M$ Now, let us put $\mathbb{U}_n:M\longrightarrow\mathcal{B}(I,\mathbb{R})$ and, as before, let $y\in\mathbb{U}_n(q)$ if $y(t)=h(t)+(Tq)(t)\int_0^t u_n(t,s,y(s))ds$, $\forall t\in I$.

The operators $\mathbb{U}_n$ are compact and continuous (by using the same argument as before).

So, in order to apply Proposition 3, we need to verify that the equation $x = \mathbb{U}_n(q)$ admits at most one solution. To this end, let x and y be two solutions such that $x = \mathbb{U}_n(q)$ and $y = \mathbb{U}_n(q)$.

There hods, simultaneously,

$$x(t) = h(t) + (Tq)(t)) \int_0^t u_n(t, s, x(s))ds,$$

and

$$y(t) = h(t) + (Tq)(t) \int_0^t u_n(t, s, y(s))ds.$$

But

$$|x(t) - y(t)| \leq |(T_q)(t) \int_0^t u_n(t, s, x(s))ds - (T_q)(t) \int_0^t u_n(t, s, y(s))ds| \leq$$

$$|(T_q)(t)| \int_0^t k_0|x(s) - y(s)|ds$$

where k_0 is the Lipschitz constant of the sequence of the functions u_n.

So, $\forall t \in I$, there is

$$|x(t) - y(t)| \leq a\rho \int_0^t k_0|x(s) - y(s)|ds$$

.

This means that $|x(t) - y(t)| = 0$ for every $t \in I$.

Moreover, in order to prove that $||\mathbb{U}_n - \mathbb{U}|| \leq \epsilon$, it will be enough to observe that if $y(t)$ and $z(t)$ are solutions of $\mathbb{U}_n(q)$ and $\mathbb{U}(q)$, respectively, then

$|y(t) - z(t)| \leq |(T_q)(t)| \int_0^t |u_n(t, s, x(s)) - u(t, s, x(s))|ds \leq a\rho L\epsilon.$

iv): There is a ball $B_R(0)$ such that $\mathbb{U}(B_R(0)) \subset B_R(0)$.

If $||q|| \leq R$ we get:

$||x|| \leq a||q|| \, |\int_0^t(\alpha+\beta|x(s)|)ds| \leq a\,||q||L(\alpha+\beta||x||) \leq \alpha\, R\, a\, L + a\, R\, \beta\, L||x||.$

Thus

$||x||(1 - a\, R\, \beta\, L) \leq \alpha\, a\, R\, L$; hence we obtain $||x|| \leq \frac{\alpha\, a\, R\, L}{1 - a\, R\, \beta\, L}$.

The last condition of the Theorem allows us to take $R \leq \frac{1 - a\,\alpha\,L}{a\,\beta}$ and so $||x|| \leq R$ and, consequently, $\mathbb{U}(B_R(0)) \subset B_R(0)$.

Remark: The method used here for the solution set allows us to extend, under a weaker set of conditions, some previous results regarding the subject (see e.g. [?], [?]).

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