

## FLORE

Repository istituzionale dell'Università degli Studi di Firenze

## Some properties of the solution set for integral differential equations

Questa è la Versione finale referata (Post print/Accepted manuscript) della seguente pubblicazione:
Original Citation:
Some properties of the solution set for integral differential equations / G.Anichini; G.Conti. - In: FAR EAST JOURNAL OF MATHEMATICAL SCIENCES: FJMS. - ISSN 0972-0871. - STAMPA. - 24:(2007), pp. 415-423.

Availability:
This version is available at: $2158 / 403257$ since:

Terms of use:
Open Access
La pubblicazione è resa disponibile sotto le norme e i termini della licenza di deposito, secondo quanto stabilito dalla Policy per l'accesso aperto dell'Università degli Studi di Firenze
(https://www.sba.unifi.it/upload/policy-oa-2016-1.pdf)

Publisher copyright claim:

## 1 Introduction and Notations

In this paper we are concerned with the solution sets for Volterra integral equation and integrodifferential equations like:

$$
\left\{\begin{align*}
x(t) & =h(t)+\int_{0}^{t} k(t, s) g(s, x(s) d s  \tag{1}\\
x(0) & =x_{0},
\end{align*}\right.
$$

or

$$
\left\{\begin{align*}
x(t) & =f\left(t, x(t), \int_{0}^{t} k(t, s) g(s, x(s)) d s\right)  \tag{2}\\
x(0) & =x_{0}
\end{align*}\right.
$$

where $h: I=[0, T) \longrightarrow \mathbb{R}^{n}, k: I \times \mathbb{R}^{n} \longrightarrow \mathbb{R}^{n}$ are continuous functions, $x_{0}$ is a given vector of $\mathbb{R}^{n}, I$ a (possible unbounded) interval of $\mathbb{R}$.

In the following $B\left(x_{0}, r\right)$ will denote an $r$ - ball (in the metric space $(X, d))$ i.e. the set $\left\{x \in X: d\left(x, x_{0}\right)<r\right\}$ where $x_{0}$ is any point in $X ; B(0, r)$ wil denote the closed ball centered in $x_{0}=0$.

Let now consider the (Hilbert) space $L^{2}\left(I, \mathbb{R}^{n}\right)$ normed, as usually, by $\|x\|_{2}=\left(\int_{I} x^{2}(t) d t\right)^{\frac{1}{2}}$ and its (affine) subspace $E=\left\{x \in L^{2}\left(I, \mathbb{R}^{n}\right): x(0)=\right.$ $\left.x_{0}\right\}$. Let $X$ be some some Banach space; $\mathrm{f} V \subset X$ is some subset then $\left.\overline{( } V\right)$ will denote its (topological) closure and $V^{c}$ will denote the complement of $V$. Finally $\mathcal{B}(\mathcal{X})$ will denote the set of all nonempty and bounded subsets of $X$.

Definition 1 : Let $X$ be a Banach space and $A \subset$ a subset. A measure $\mu: B_{d}(X) \longrightarrow \mathbb{R}^{+}$defined by $\mu(V)=\inf \{\epsilon>0: V \in \mathcal{B}(\mathcal{X})$ admits a finite cover by sets of diameter $\leq \epsilon\}$ where diameter of $V$ is the $\sup \{\|x-y\|: x \in$ $V, y \in V\}$, is called the (Kuratowski) measure of noncompactness.

A measure like $\mu$ has interesting properties, some of which are listed in the sequel:
a) $\mu(V)=0$ if and only if $\bar{V}$ is compact;
b) $\quad \mu(V)=\mu(\bar{V}) ; \quad \mu(\operatorname{conv}(V))=\mu(V) ;(\operatorname{conv}(V)=$ convex hull of $V)$;
c) $\mu\left(\alpha\left(V_{1}\right)+(1-\alpha) V_{2}\right) \leq \alpha \mu\left(V_{1}\right)+(1-\alpha) \mu\left(V_{2}\right), \quad \alpha \in[0,1]$;
d) if $V_{1} \subset V_{2}$ then $\mu\left(V_{1}\right) \leq \mu\left(V_{2}\right)$;
e) if $\left\{V_{n}\right\} \quad$ is a nested sequence of closed sets of $B_{d}(X)$

$$
\text { and if } \lim _{n \rightarrow+\infty} \mu\left(V_{n}\right)=0 \quad \text { then } \quad \cap_{n=1}^{\infty} V_{n} \neq \emptyset
$$

The analogous measure of noncompactness for an operator is defined by $\mu(F(V))=\inf \{k>0: \mu(F(V)) \leq k \mu(V)\}$ for all bounded subsets $V \subset X$.

When $X$ is a complete metric space and $f: X \longrightarrow X$ is a continuous mapping $f$ is called an
$m u$-set contraction if there exists $k \in[0,1)$ such that, for all bounded noncompact subsets $V$ of $X$, the following relation holds: $\mu(f(V)) \leq k \alpha(V)$ ( [?], pag 160).

A continuous operator $F: X \longrightarrow X$ such that $\mu(F(V))<\mu(V)$, for any bounded $V \subset X$, is called condensing or densifying.
(The concept of measure of noncompactness is considerably dealed with in the references [?], [?] or [?].)

Let $S$ and $S_{1}$ be topological spaces and let $f: S \longrightarrow S_{1}$. Then $f$ is said to be proper if, whenever $K_{1}$ is a compact subset of $S_{1}, f^{-1}\left(K_{1}\right)$ is a compact set in $S$. It is also known ( [?], pag 160) that if $X$ is a Banach space and $f: X \longrightarrow X$ is a continuous $k$-set contraction, then $I-f$ is a proper mapping.

The following result, due to R.K. Juberg ([?]), will be useful in the proof of our main result:

Proposition 1 : Let ( $a, b$ ) be any real (possible unbounded) interval and let $L^{p}(a, c), 1 \leq p \leq+\infty$ be the Lebesgue's space of (the power $p$ ) summable
functions over ( $a, c$ ) for every $c \in(a, b)$. For $u \in L^{p}(c, b), \quad v \in L^{q}(a, c)$, where $\frac{1}{p}+\frac{1}{q}=1$, we set

$$
\begin{aligned}
\rho & =\lim _{\epsilon \rightarrow 0} \sup \left\{\left[\int_{x}^{a+\epsilon}|u(y)|^{p} d y\right]^{\frac{1}{p}}\left[\int_{a}^{x}|v(y)|^{q} d y\right]^{\frac{1}{q}}, a<x \leq a+\epsilon\right\}+ \\
& +\lim _{\delta \rightarrow 0} \sup \left\{\left[\int_{x}^{b}|u(y)|^{p} d y\right]^{\frac{1}{p}}\left[\int_{b-\delta}^{x}|v(y)|^{q} d y\right]^{\frac{1}{q}}, b-\delta \leq x<b\right\} .
\end{aligned}
$$

Let $D$ be the linear operator defined by: $D(f(y))(x)=\int_{0}^{x} u(x) v(y) f(y) d y$; in the sequel wh shall assume that $D$ is a bounded operator in the space $L^{p}(0, T)$. We want to recall that the operator $D$ is bounded (in the $L^{p}(a, b)$ space) if and only if the function
$\psi(x)=\left[\int_{x}^{b}|u(y)|^{p} d y\right]^{\frac{1}{p}}\left[\int_{a}^{x}|v(y)|^{q} d y\right]^{\frac{1}{q}}$ is bounded on $(a, b)$. This operator is not necessarily a compact operator; as matter of fact it is well known (see [?], for istance), that $D$ is a compact operator if the functions $u(\cdot)$ e $v(\cdot)$ belongs to $L^{2}(a, b)$.

Furthermore the measure of noncompactness of $D$, i.e. $\mu(D)$ satisfies $\left(\frac{1}{2}\right)^{1+\frac{1}{p}} \leq \mu(D) \leq p^{\frac{1}{q}} q^{\frac{1}{p}} \rho$; in the special case when $p=q=2$, i.e. when the (Lebesgue) space $L^{p}$ is a Hilbert space $L^{2}$, we obtain $\rho \sqrt{\frac{1}{8}} \leq \mu(D) \leq 2 \rho$.

Definition 2:An $R_{\delta}-$ set is the intersection of a decreasing sequnce $\left\{A_{n}\right\}$ of compact $A R$ (metric absolute retracts; see [?] or [?], for a reference.) Moreover it is known (see [?] for istance) that an $R_{\delta}-$ set is an acyclic set in the Cȩch homology.

The following result also will be crucially used in teh sequel:
Proposition 2 : ( $[?]$, pag 159). Let $X$ be a space and let $Y,\|\cdot\|$ be a Banach space and $f: X \longrightarrow X$ be a proper mapping. Assume further that for each $\epsilon_{n}>0, n>0 \in \mathbb{N}$ a proper mapping $f_{n}: X \longrightarrow X$ is given and the couple of conditions is satisfied:

- $\left\|f_{n}(x)-f(x)\right\|<\epsilon_{n}, \forall x \in X ;$
- for any $\epsilon_{n}>0$ and $y \in E$ such that $\|y\| \leq \epsilon_{n}$, the equation $f_{\epsilon_{n}}(x)=y$ has exactly one solution.

Then the set $S=f^{-1}(0)$ is an $R_{\delta}-$ set.
Remark: a sequence $f_{\epsilon_{n}}$ is called an $\epsilon_{n}$ approximation (of the function $f$ ).
Proposition 3 : ([?], pag ???). Let $F, F_{n}: \bar{B}(0, r) \longrightarrow Y$ be condensing operators such that

- $\delta_{n}=\sup \{$
$\left.F_{n}(x)-F(x) \|, x \in \bar{B}(0, r)\right\} \rightarrow 0$, as $n \rightarrow+\infty$;
- the equation $x=F_{n}(x)+y$ has at most one solution if $\|y\| \leq \delta_{n}$.

Then the set of fixed points of $F$ is an $R_{\delta}-$ set.
Main result

We are ready to establish out (main) existence result for the (initial value problems for) integral equations of the type here introduced.

First of all let $F: B(0, r) \rightarrow E$ be defined as follows:
$F(y)=h(t)+\int_{0}^{t} k(t, s) g(s, y(s) d s$
where $r$ is a real number (suitably defined below) and put $m_{0}=\|F(0)\|_{2}$.
Theorem 1 : Let $\rho$ the number defined in Proposition 1; then we assume that:

1. i) there are functions $\alpha, \phi,: I \rightarrow \mathbb{R}^{n}$ belonging to $L^{2}(I)$ such that $k(t, s)=\alpha \phi(s)$ for every $(t, s) \in I \times I$; moreover we assume that $\|k\|_{2}<2 \rho ;$
2. ii) $\|g(t, x)\| \leq \frac{1}{2 \rho}\|x\|+b(t)$, for $(t, x) \in I \times \mathbb{R}^{n}, b \in L^{2}(I), b(t) \geq 0$;
3. iii) there is a ball $B(0, r)$ such that $r>\frac{2 m_{0} \rho}{2 \rho-\|k\|_{2}}$.

Then the set of solution of the integral problem (??) is an $R_{\delta}-$ set.
Remark: The first part of the assumption $i$ ) is satisfied in many cases: for istance when $k(t, s)$ is a Green function; see, for istance, [?] for similar cases.

Proof: Clearly the above operator $F$ is a single value mapping and a possible fixed point of $F$ is a solution of the integral problem (??).

In order to prove the theorem the following steps in the proof have to be established:
a ) $F$ has a closed graph;
b) $F$ is a condensing mapping;
c) The set of fixed point of $F$ is $R_{\delta}-$ set.

Proof of Step a): in fact, let $y_{n} \rightarrow y_{0}$ and put $G(y)(t)=g(t, y(t))$. Now, from assumption ii), it follows that the superposition operator $G$ mapping the space $L^{2}$ into $L^{2}$ is condensing (see [?]); thus we have $\lim _{n}\left\|G\left(y_{n}\right)-G\left(y_{0}\right)\right\|_{2}=$ 0 . By using the Holder inequality, we get:
$\left\|F\left(y_{n}\right)-F\left(y_{0}\right)\right\|_{2}=\left[\int_{I}\left|F\left(y_{n}\right)(s)-F\left(y_{0}\right)(s)\right|^{2} d s\right]^{\frac{1}{2}}=$
$=\left[\int_{I}\left[\int_{)}^{t}\left(k(t, s) g\left(s, y_{n}(s)\right)-k(t, s) g\left(s, y_{0}(s)\right) d s\right]^{2} d t\right]^{\frac{1}{2}} \leq\|k\|_{2}\| \| G\left(y_{n}\right)-\right.$ $G\left(y_{0}\right) \|_{2}$
and this quantity is gioing to zero whenever $n \rightarrow+\infty$.
Proof of Step b): Always working from $B(0, r)$ into $E$, we have $F(y)=$ $(H \circ G)(y))$, where
$H(y)(t)=\int_{0}^{t} \phi(s) \alpha(t) y(s) d s+h(t)$.
Now, by assumptions $i$ ) and $i i$ ), we have (see [?]) $\mu(G(V)) \leq \frac{1}{2 \rho} \mu(V)$, for any bounded set $V \subset L^{2}\left(I \times \mathbb{R}^{n}\right)$ and also $\mu(H)<2 \rho$; so (see [?]) $\mu(F)=\mu(H \circ G)(y)) \leq \mu(H) \mu(G)<1$.

Proof of Step c): Finally we have to prove that the set of fixed points of the operator $F$ is an $R_{\delta}$-set (in the sequel we assume that $(a, b)=(0, T)$.)

Let us consider the mappings $F_{n}: L^{2}(0, T) \rightarrow L^{2}(0, T)$ defined as:

$$
F_{n}(x)(t)=\left\{\begin{array}{ll}
h(t) & =\text { if } 0 \leq t \leq \frac{T}{n}  \tag{3}\\
h(t)+\int_{0}^{t-\frac{T}{n}} \phi(s) \alpha(s) g(s, y(s)) d s & =
\end{array} \quad \text { if } \frac{T}{n} \leq t \leq T .\right.
$$

The mappings $F_{n}$ are continuous mappings; by assumption $i$ ) and $\left.i i\right)$ we have that they are also condensing. The intervals $\left[0, \frac{T}{n}\right],\left[\frac{T}{n}, \frac{2 T}{n}\right], \cdots\left[\frac{k T}{n}, \frac{(k+1) T}{n}\right], \cdots\left[\frac{(n-1) T}{n}, T\right]$ are now coming in one after the other: each time the mappings $F_{n}$ are bijective and their inverses $F_{n}^{-1}$ are continuous. Moreover we have $\left\|F_{n}-F\right\|_{2} \rightarrow 0$ as $n \rightarrow+\infty$. The latter fact allows us to say that the mappings $I-F_{n}$ and $I-F$ are proper maps. Finally we can conclude that the set of fixed points of $F$ is an $R_{\delta}-$ set.

## Riferimenti bibliografici

[1] G. Anichini - G. Conti Existence of Solutions of a Boundary Value Problem through the solution mapping of a linearized type problem, Rendiconti del Seminario Mate. Univ. Torino, Fascicolo speciale dedicato a Mathematical theory of dynamical systems and ordinary differential equations, 1990, vol 48 (2), p. 149 - 160,
[2] G. Anichini - G. Conti - P. Zecca Using solution sets for solving boundary value problems for ordinary differential equations, Nonlinear Analysis Theory Meth.\& Appl., 1991, vol 5, p. 465-474,
[3] G. Anichini - G. Conti A direct approach to the existence of solutions of a Boundary Value Problem for a second order differential system, Differential Equations and Dynamical Systems, 1995, vol 3 (1), p. 23 34,
[4] G. Anichini - G. Conti About the Existence of Solutions of a Boundary Value Problem for a Carathéodory Differential System, Zeitschrift für Analysis und ihre Anwendungen, 1997, vol 16 (3), p. 621 - 630,
[5] G. Anichini - G. Conti Boundary Value Problem for Implicit ODE's in a singular case, Differential Equations and Dynamical Systems, 1999, vol 7 (4), p. 437 - 459,
[6] G. Anichini - G. Conti How to make use of the solutions set to solve Boundary Value Problems, Progress in Nonlinear Differential Equations and their Applications, Springer Verlag (Basel), 2000, vol 40,
[7] G. Anichini - G. Conti Boundary value problems for perturbed differential systems on an unbounded interval, International Mathematical Journal, 2002, vol 2 (3), p. 221 - 234 (?),
[8] J. Banas - K. Goebel Measures of noncompactness in Banach spaces, Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, New York, 1980,
[9] F.E. Browder - C.P. Gupta Topological Degree and Nonlinear Mappings of Analytic Type in Banach spaces, Journal of Mathematical Analysis and Applications, 1969, vol 26 (4), p. $390-402$ ?),
[10] G. Conti - J. Pejsachowicz Fixed point theorems for multivalued maps, Annali Matem. Pura Appl., 1980, vol 126 (4), p. 319 - 341
[11] G. Darbo Punti uniti in trasformazioni a codominio non compatto, Rend. Sem. Matem. Univ. Padova, 1955, vol 24, p. $84-92$
[12] A. Deimling Nonlinear Functional Analysis, Springer Verlag, Berlin, 1984
bibitem13 L. Gorniewicz, Topological Approach to differential inclusions, NATO-ASI Series, A.Granas - M. Frigon editors, Kluwer, 1990, vol 472, p. 129-190,
[13] H. Hochstadt Integral Equations, Pure and Applied Matheamtics, Wiley, New York, 1973,
[14] V.I. Istrăţescu Fixed point theory, D. Reidel Publishing Company, Dordrecht, 1981,
[15] R.K. Juberg The measure of noncompactness in $L^{p}$ for a Class of Integral Operators, Indiana Math. Journal, 1973/74, vol 23, p. 925 936,
[16] M.A. Krasnoselkii - P.P. Zabreiko Geometrical methods of nonlinear analysis, Springer Verlag, Berlin, 1984
[17] J. Lasry - R.Robert Analyse nonlineare multivoque, U.E.R. Math de la Décision, 1979, vol 249, Paris Dauphine,
[18] W.V. Petryshyn Solvability of various boundary value problems for the equation $x^{\prime \prime}=f\left(t, x, x^{\prime}, x^{\prime \prime}\right)-y$, Pacific Journal of Math. 1986, vol. 122, p. 169-195
[19] E.Spanier Algebraic Topology, McGraw Hill, New York, 1966

