Harnack type estimates and Hölder continuity for non-negative solutions to certain sub-critically singular parabolic partial

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Abstract. A two-parameter family of Harnack type inequalities for non-negative solutions of a class of singular, quasilinear, homogeneous parabolic equations is established, and it is shown that such estimates imply the Hölder continuity of solutions. These classes of singular equations include p-Laplacean type equations in the sub-critical range $1 < p \le \frac{2N}{N+1}$ and equations of the porous medium type in the sub-critical range $0 < m \le \frac{(N-2)_+}{N}$.

1. Introduction and main results

Let E be an open set in $\mathbb{R}^N$ and for $T > 0$ let E_T denote the cylindrical domain $E \times (0, T]$. Consider quasi-linear, parabolic differential equations of the form

$$\begin{gathered} u \in C_{\rm loc}\left(0, T; L^2_{\rm loc}(E)\right) \cap L^p_{\rm loc}\left(0, T; W^{1,p}_{\rm loc}(E)\right) \\ u_t - \operatorname{div} \mathbf{A}(x, t, u, Du) = 0 \quad \text{weakly in } E_T \end{gathered} \tag{1.1}$$

where the function $\mathbf{A} : E_T \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ is only assumed to be measurable and subject to the structure conditions

$$\begin{cases} \mathbf{A}(x, t, u, Du) \cdot Du \ge C_o |Du|^p \\ |\mathbf{A}(x, t, u, Du)| \le C_1 |Du|^{p-1} \end{cases} \quad \text{a.e. } (x, t) \in E_T \tag{1.2}$$

where C_o and C_1 are given positive constants, and p is in the sub-critical range

$$1 < p \le p_* \stackrel{\text{def}}{=} \frac{2N}{N+1}. \tag{1.3}$$

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The homogeneous prototype of such a class of parabolic equations is

$$u_t - \operatorname{div} |Du|^{p-2} Du = 0 \quad \text{weakly in } E_T. \tag{$(1.1)_o$}$$

The parameters $\{N, p, C_o, C_1\}$ are the data, and we say that a generic constant $\gamma = \gamma(N, p, C_o, C_1)$ depends upon the data, if it can be quantitatively determined a priori only in terms of the indicated parameters. For $\rho > 0$ let B_ρ be the ball of center the origin on $\mathbb{R}^N$ and radius ρ and for $y \in \mathbb{R}^N$ let $B_\rho(y)$ denote the homothetic ball centered at y. For $\tau > 0$ and for $(y, s) \in \mathbb{R}^N \times \mathbb{R}$ set also

$$Q_\rho(\tau) = B_\rho \times (-\tau, 0], \qquad (y, s) + Q_\rho(\tau) = B_\rho(y) \times (s - \tau, s].$$

Let u be a non-negative weak solution of (1.1–1.3). Having fixed $(x_o, t_o) \in E_T$, and $B_{4\rho}(x_o) \subset E$, introduce the quantities

$$\fint_{B_\rho(x_o)} u^q(x, t_o)dx, \qquad \delta \stackrel{\text{def}}{=} \left[\varepsilon \left(\fint_{B_\rho(x_o)} u^q(\cdot, t_o)dx \right)^{\frac{1}{q}} \right]^{2-p} \rho^p \tag{1.4}$$

where $\varepsilon \in (0, 1)$ is to be chosen, and $q \geq 1$ is arbitrary. If $\delta > 0$, set also

$$\eta \stackrel{\text{def}}{=} \left[\frac{\left(\fint_{B_\rho(x_o)} u^q(\cdot, t_o)dx \right)^{\frac{1}{q}}}{\left(\fint_{B_{4\rho}(x_o)} u^r(\cdot, t_o - \delta)dx \right)^{\frac{1}{r}}} \right]^{\frac{rp}{\lambda_r}} \tag{1.5}$$

where $r \geq 1$ is any number such that

$$\lambda_r \stackrel{\text{def}}{=} N(p-2) + rp > 0. \tag{1.6}$$

Theorem 1.1. *Let u be a non-negative, locally bounded, local, weak solution of* (1.1–1.3). *Introduce δ as in* (1.4) *and assume that $\delta > 0$. There exist constants $\varepsilon \in (0, 1)$, and $\gamma > 1$, depending only on the data and the parameters q, r, and a constant $\beta > 1$, depending only upon the data and independent of q, r, such that*

$$\inf_{(x_o,t_o)+Q_\rho(\frac{1}{2}\delta)} u \geq \gamma \left[\frac{\left(\fint_{B_\rho(x_o)} u^q(\cdot, t_o)dx \right)^{\frac{1}{q}}}{\left(\fint_{B_{4\rho}(x_o)} u^r(\cdot, t_o - \delta)dx \right)^{\frac{1}{r}}} \right]^{\beta \frac{rp}{\lambda_r}} \sup_{(x_o,t_o)+Q_\rho(\delta)} u \tag{1.7}$$

provided $q \geq 1$ and $r \geq 1$ satisfies (1.6) *and $(x_o, t_o) + Q_{8\rho}(\delta) \subset E_T$. The constants $\varepsilon \to 0$, and $\gamma \to \infty$ as either $\lambda_r \to 0$ or $\lambda_r \to \infty$.*

Remark 1.1. The estimate is vacuous if $\delta = 0$. This does occur for certain solutions of (1.1) for t_o larger than the extinction time ([8]).

Remark 1.2. Inequality (1.7) is not a Harnack inequality per se, since η depends upon the solution itself. It would reduce to a Harnack inequality if $\eta \geq \eta_o > 0$ for some absolute constant η_o depending only upon the data. This however cannot occur since a Harnack inequality for solutions of (1.1–1.3) does not hold, as shown by the counterexamples of [7]. Further comments in this direction are in Remark 4.1.

Remark 1.3. An estimate similar to (1.7) has been derived in [3] for non-negative solutions of the prototype equation $(1.1)_o$, by means of maximum and comparison principles, and some asymptotic estimates of [8]. However the Harnack inequality is a structural property of a parabolic equation, unrelated to comparison and maximum principles. This emerges from the pioneering work of Moser [9,10], and the results of [1,4,6,11]. Theorem 1.1 is in this direction.

Remark 1.4. Inequality (1.7) actually holds for non-negative solutions of (1.1–1.2) for all $1 < p < 2$, provided $r \geq 1$ satisfies (1.6). For super-critical values of $p > \frac{2N}{N+1}$ one has $\lambda = \lambda_1 > 0$, and (1.6) can be realized for $r = 1$. However, for $\lambda > 0$ the strong form of a Harnack estimate holds ([7]). Therefore (1.7), while true for all $1 < p < 2$, holds significance only for critical and sub-critical values $1 < p \leq \frac{2N}{N+1}$. In this sense (1.7) can be regarded as a "weak" form of a Harnack estimate. Neverthless (1.7) is sufficient to establish the local Hölder continuity of locally bounded, weak solutions of (1.1–1.2), irrespective of their sign, as we show in Sect. 4.

2. Components of the proof of Theorem 1.1

2.1. L^r_{loc}-L^∞_{loc} Estimates For $r \geq 1$ Such That $\lambda_r > 0$

Proposition 2.1. *Let u be a non-negative, locally bounded, local, weak solution to (1.1–1.3), and assume that $u \in L^r_{loc}(E_T)$ for some $r \geq 1$, satisfying (1.6). There exists a positive constant γ_r depending only upon the data, and r, such that*

$$\sup_{B_\rho(y)\times[s,t]} u \leq \frac{\gamma_r}{(t-s)^{\frac{N}{\lambda_r}}}\left(\int_{B_{2\rho}(y)} u^r(x, 2s-t)dx\right)^{\frac{p}{\lambda_r}} + \gamma_r\left(\frac{t-s}{\rho^p}\right)^{\frac{1}{2-p}} \tag{2.1}$$

for all cylinders

$$B_{2\rho}(y) \times [s-(t-s), s+(t-s)] \subset E_T. \tag{2.2}$$

The constant $\gamma_r \to \infty$ if either $\lambda_r \to 0$ or $\lambda_r \to \infty$.

Remark 2.1. The values of u in the upper part of the cylinder (2.2) are estimated by the integral on the lower base of the cylinder.

Remark 2.2. The local boundedness of a weak solution is insured by the integrability $u \in L^r_{\text{loc}}(E_T)$ for some $r \geq 1$ satisfying (1.6). For $\frac{2N}{N+2} < p \leq \frac{2N}{N+1}$, such an integrability condition is a consequence of the notion of weak solution. Indeed by parabolic embedding (see [5], Chapter I, Proposition 3.1), $u \in L^m_{\mathrm{loc}}(E_T)$ with $m = \frac{N+2}{N}p$, and $\lambda_m > 0$. For $1 < p \leq \frac{2N}{N+2}$ this is no longer the case, and the integrability requirement is an extra assumption imposed on the notion of weak solution, to insure its local boundedness. Indeed for $1 < p \leq \frac{2N}{N+2}$ there exist unbounded, local, weak solutions to (1.1) ([5]).

The proof of Proposition 2.1 follows arguments similar to those in of [5] Chap. V, with minor modifications outlined in Appendix A.

2.2. *Expansion of positivity*

Proposition 2.2. *Let u be a non-negative, local, weak solution to (1.1–1.2), for $1 < p < 2$, satisfying*

$$\left|[u(\cdot, t) > M] \cap B_\rho(y)\right| > \alpha |B_\rho| \tag{2.3}$$

for all times

$$s - \epsilon M^{2-p}\rho^p \leq t \leq s \tag{2.4}$$

for some $M > 0$, and some $\alpha, \epsilon \in (0, 1)$. Assume moreover that

$$B_{8\rho} \times (s - \epsilon M^{2-p}\rho^p, s) \subset E_T.$$

There exists $\sigma \in (0, 1)$ that can be determined a priori, quantitatively only in terms of the data, and the numbers α and ϵ, independent of M, such that

$$u(x, t) \geq \sigma M \quad \textit{for all } \ x \in B_{2\rho}(y) \tag{2.5}$$

for all times

$$s - \tfrac{1}{2}\epsilon M^{2-p}\rho^p < t \leq s. \tag{2.6}$$

Remark 2.3. Thus measure-theoretical information on the measure of the “positivity set” in $B_\rho(y)$ for all times in (2.4) implies that such a positivity set actually expands to $B_{2\rho}(y)$ for comparable times. This is the main underlying structural fact of a Harnack inequality.

Remark 2.4. The proof, given in [5], Chap. IV, and in [7], shows that the functional dependence of σ on ϵ and α is of the form

$$\sigma(\epsilon, \alpha) \approx a^{1/\epsilon^b \alpha^c} \tag{2.7}$$

for constants $a \in (0, 1)$ and $b, c > 1$ depending only upon the data.

Remark 2.5. Proposition 2.2 holds for all $1 < p < 2$, irrespective of p belonging to the sub-critical or super-critical range.

2.3. L^r_{loc} estimates backward in time

Proposition 2.3. *Let u be a non-negative, local, weak solution to (1.1–1.2), for $1 < p < 2$, and assume that $u \in L^r_{loc}(E_T)$ for some $r \geq 1$. There exists a constant $\bar{\gamma}_r$ depending only upon the data and r, such that for all cylinders $B_{2\rho}(y) \times [\tau, t] \subset E_T$*

$$\sup_{\tau \leq s \leq t} \int_{B_\rho(y)} u^r(x, s)dx \leq \bar{\gamma}_r \int_{B_{2\rho}(y)} u^r(x, \tau)dx + \bar{\gamma}_r \left[\frac{(t-\tau)^r}{\rho^{\lambda_r}} \right]^{\frac{1}{2-p}} \tag{2.8}$$

where λ_r is defined in (1.6), but it is not required to be positive.

The proof is in Appendix A. If $r = 1$ this estimate can be given the form of a Harnack inequality in the L^1_{loc} topology.

Proposition 2.4. *Let u be a non-negative, local, weak solution to (1.1–1.2), for $1 < p < 2$. There exists a positive constant $\bar{\gamma}$ depending only upon the data, such that for all cylinders $B_{2\rho}(y) \times [\tau, t] \subset E_T$*

$$\sup_{\tau \leq s \leq t} \int_{B_\rho(y)} u(x, \tau)dx \leq \bar{\gamma} \inf_{\tau \leq s \leq t} \int_{B_{2\rho}(y)} u(x, \tau)dx + \bar{\gamma} \left(\frac{t-\tau}{\rho^\lambda} \right)^{\frac{1}{2-p}} \tag{2.9}$$

where $\lambda = \lambda_1$ is defined in (1.6), but it is not required to be positive.

If $p_* < p < 2$ then $\lambda > 0$, whereas $1 < p \leq p_*$ implies $\lambda \leq 0$. However (2.9) holds true for all $1 < p < 2$ and accordingly, λ could be of either sign. The constant $\bar{\gamma} = \bar{\gamma}(p) \to \infty$ as either $p \to 2$ or $p \to 1$. The proof is in [7].

3. Estimating the positivity set of the solutions

Having fixed $(x_o, t_o) \in E_T$, assume it coincides with the origin, write $B_\rho(0) = B_\rho$ and introduce the quantity δ as in (1.4), which is assumed to be positive. From (2.8) and the definition of δ

$$\int_{B_\rho} u^q(\cdot, 0)dx \leq \bar{\gamma}_q \int_{B_{2\rho}} u^q(\cdot, \tau)dx + \bar{\gamma}_q \, \varepsilon^q \int_{B_\rho} u^q(\cdot, 0)dx$$

for all $q \geq 1$ and for all $\tau \in (-\delta, 0]$. Choosing $\bar{\gamma}_q \, \varepsilon^q \leq \frac{1}{2}$ yields

$$\int_{B_{2\rho}} u^q(\cdot, \tau)dx \geq \frac{1}{2\bar{\gamma}_q} \int_{B_\rho} u^q(\cdot, 0)dx \quad \text{for all } \tau \in (-\delta, 0]. \tag{3.1}$$

Next apply the sup-estimate (2.1) over the cylinder $B_{2\rho}\times(-\frac{1}{2}\delta,0]$ with $r\geq 1$ such that $\lambda_r>0$, to get

$$\begin{aligned}\sup_{B_{2\rho}\times(-\frac{1}{2}\delta,0]} u &\leq \frac{\gamma_r[\omega_N(4\rho)^N]^{\frac{p}{\lambda_r}}}{\delta^{\frac{N}{\lambda_r}}}\left(\fint_{B_{4\rho}} u^r(\cdot,-\delta)dx\right)^{\frac{1}{r}\frac{rp}{\lambda_r}}+\gamma_r\left(\frac{\delta}{\rho^p}\right)^{\frac{1}{2-p}}\\ &\leq \frac{\gamma_r'}{\varepsilon^{\frac{N(2-p)}{\lambda_r}}}\frac{1}{\eta}\left(\fint_{B_\rho} u^q(\cdot,0)dx\right)^{\frac{1}{q}}+\gamma_r'\varepsilon\left(\fint_{B_\rho} u^q(\cdot,0)dx\right)^{\frac{1}{q}}\\ &=\gamma_r'\varepsilon\left(1+\frac{1}{\eta\,\varepsilon^{\frac{rp}{\lambda_r}}}\right)\left(\fint_{B_\rho} u^q(\cdot,0)dx\right)^{\frac{1}{q}}\end{aligned}$$

for a constant γ_r' depending only upon the data and r. One verifies that $\gamma_r'\to\infty$ as either $\lambda_r\to 0$ or $\lambda_r\to\infty$.

Assume momentarily that $0<\eta<1$ so that in the round brackets containing η, the second term dominates the first. In such a case

$$\sup_{B_{2\rho}\times(-\frac{1}{2}\delta,0]} u\leq M\overset{\text{def}}{=}\frac{1}{\varepsilon'\eta}\left(\fint_{B_\rho} u^q(\cdot,0)dx\right)^{\frac{1}{q}}\quad\text{where }\ \varepsilon'=\frac{\varepsilon^{\frac{N(2-p)}{\lambda_r}}}{2\gamma_r'}. \tag{3.2}$$

and therefore

$$\varepsilon'\eta M=\left(\fint_{B_\rho} u^q(\cdot,0)dx\right)^{\frac{1}{q}}. \tag{3.3}$$

Let $\nu\in(0,1)$ to be chosen. Combining (3.3) with (3.1) gives

$$\begin{aligned}(\varepsilon'\eta M)^q &\leq 2^{N+1}\bar{\gamma}_q\fint_{B_{2\rho}} u^q(\cdot,\tau)dx\\ &\leq 2^{N+1}\bar{\gamma}_q\nu^q(\eta M)^q+2^{N+1}\bar{\gamma}_q M^q\frac{|[u(\cdot,\tau)>\nu\eta M]\cap B_{2\rho}|}{|B_{2\rho}|}\end{aligned}$$

for all $\tau\in(-\frac{1}{2}\delta,0]$. From this

$$|[u(\cdot,\tau)>\nu\eta M]\cap B_{2\rho}|\geq\alpha\eta^q|B_{2\rho}|\quad\text{where }\ \alpha=\frac{\varepsilon'^q-\nu^q\,2^{N+1}\bar{\gamma}_q}{2^{N+1}\bar{\gamma}_q} \tag{3.4}$$

for all $\tau\in(-\frac{1}{2}\delta,0]$. By choosing $\nu\in(0,1)$ sufficiently small, only dependent on the data and $\bar{\gamma}_q$, we can insure that $\alpha\in(0,1)$ depends only upon the data and q, and is independent of η.

Proposition 3.1. *Let u be a non-negative, locally bounded, local, weak solution of (1.1–1.2) for $1<p<2$. Fix $(x_o,t_o)\in E_T$, let $B_{4\rho}(x_o)\subset E$ and let δ and η be defined by (1.4–1.6) for some $\varepsilon\in(0,1)$. For every $r\geq 1$ satisfying (1.6) and every $q\geq 1$, there exist constants $\varepsilon,\nu,\alpha\in(0,1)$, depending only upon the data and q and r, such that*

$$|[u(\cdot,t)>\nu\eta M]\cap B_{2\rho}(x_o)|\geq\alpha\eta^q|B_{2\rho}|\quad\textit{for all }\ t\in(t_o-\tfrac{1}{2}\delta\,,\,t_o]. \tag{3.5}$$

3.1. A first form of the Harnack inequality

The definition of (1.4) of δ and the parameters in (3.2–3.4), imply that

$$\tfrac{1}{2}\delta = \epsilon(\nu\eta M)^{2-p}\rho^{p} \qquad \text{where } \epsilon = \tfrac{1}{2}\left(\frac{\varepsilon\varepsilon'}{\nu}\right)^{2-p}. \tag{3.6}$$

Therefore by Proposition 2.2 with M replaced by $\nu\eta M$ and α replaced by $\alpha\eta^q$

$$u(\cdot,t) > \sigma(\alpha\eta^q,\epsilon)\nu\eta M \quad \text{in } B_{4\rho}(x_o), \quad \text{for all } t \in (t_o - \tfrac{1}{4}\delta\,,\ t_o].$$

Proposition 3.2. *Let u be a non-negative, locally bounded, local, weak solution of (1.1–1.3). Fix $(x_o,t_o) \in E_T$, let $B_{4\rho}(x_o) \subset E$ and let δ and η be defined by (1.4–1.6) for some $\varepsilon \in (0,1)$. For every $r \geq 1$ satisfying (1.6) and every $q \geq 1$, there exist a constant ε, depending only upon the data and q and r, and a continuous, increasing function $\eta \to f(\eta)$ defined in $\mathbb{R}^+$ and vanishing at $\eta = 0$, that can be quantitatively determined a priori only in terms of the data, such that*

$$\inf_{B_{4\rho}(x_o)} u(\cdot,t) \geq f(\eta) \sup_{(x_o,t_o)+Q_{2\rho}(\frac{1}{4}\delta)} u, \quad \textit{for all } t \in (t_o - \tfrac{1}{4}\delta\,,\ t_o]. \tag{3.7}$$

provided $(x_o,t_o) + Q_{8\rho}(\delta) \subset E_T$.

Remark 3.1. In view of (2.7) the function $f(\cdot)$ can be taken of the form

$$f(\eta) \approx \eta\, B^{-\frac{1}{\eta^d}}$$

for constants $B, d > 1$ depending only upon the data and q and r.

Remark 3.2. The function $f(\cdot)$ depends on δ only through the parameter ε in the definition (1.4) of δ.

Remark 3.3. The inequality (3.7) is a Harnack type estimate of the same form as that established in [7], where however the constant $f(\eta)$ depends on the solution itself, through η defined in (1.5), as a proper quotient of the $L^q_{\rm loc}$ and $L^r_{\rm loc}$ averages of u, respectively at time $t = t_o$ on ball $B_\rho(x_o)$, and at time $t = t_o - \delta$ on ball $B_{4\rho}(x_o)$.

Remark 3.4. The inequality (3.7) has been derived by assuming that $0 < \eta < 1$. If $\eta \geq 1$ the same proof gives (3.7) where $f(\eta) \geq f(1)$, thereby establishing a strong form of the Harnack estimate for these solutions. As shown in [7] such a strong form is false for p in the sub-critical range (1.3).

It turns out that (3.7) is actually sufficient to establish that any locally bounded, possibly of variable sign, local, weak solutions of (1.1–1.2) for $1 < p < 2$, is locally Hölder continuous in E_T. In turn, such a Hölder continuity permits one to improve the lower bound in (3.7) by estimating $f(\cdot)$ to a power of its argument, as indicated in (1.7).

4. The first form of the Harnack inequality implies the Hölder continuity of u

Let u be a locally bounded, possibly of variable sign, local, weak solution of (1.1–1.3) in E_T. It is shown in [5] (Chap. IV, Proposition 2.1 and Lemma 2.1), that u is locally Hölder continuous in E_T if there exist constants $\theta \in (0, 1)$ and $C, A > 1$, depending only upon the data and independent of u, such that, for every $(x_o, t_o) \in E_T$, constructing the sequences

$$R_o = R, \quad R_n = \frac{R}{C^n}; \qquad \omega_o = \omega, \quad \omega_{n+1} = \theta\omega_n \quad \text{for } n = 0, 1, 2, \dots$$

for positive R and ω, and the cylinders

$$Q_n = B_{R_n}(x_o) \times \left(t_o - \left(\frac{\omega_n}{A}\right)^{2-p} R_n^p, t_o \right] \quad \text{for } n = 1, 2, \dots$$

there holds

$$Q_{n+1} \subset Q_n \subset Q_o \subset E_T \quad \text{and} \quad \operatorname*{ess\,osc}_{Q_n} u \le \omega_n.$$

We will show that (3.7) permits one to construct such sequences for an arbitrary $(x_o, t_o) \in E_T$. Having fixed $(x_o, t_o) \in E_T$ assume it coincides with the origin of $\mathbb{R}^{N+1}$ and for $\rho > 0$ set

$$R_o = 4\rho \quad \text{and} \quad Q = B_{4\rho} \times (-(4\rho)^p, 0] \tag{4.1}$$

where ρ is so small that $Q \subset E_T$. Set also

$$\mu_o^+ = \operatorname*{ess\,sup}_Q u, \quad \mu_o^- = \operatorname*{ess\,inf}_Q u, \quad \omega_o = \mu_o^+ - \mu_o^- = \operatorname*{ess\,osc}_Q u.$$

Since u is locally bounded in E_T, without loss of generality we may assume that $\omega_o \le 1$ so that

$$Q_o \overset{\text{def}}{=} B_{4\rho} \times \left(-\left(\frac{\omega_o}{A}\right)^{2-p} (4\rho)^p, 0 \right] \subset Q \subset E_T \quad \text{and} \quad \operatorname*{ess\,osc}_{Q_o} u \le \omega_o$$

for a number $A \ge 1$ to be chosen. Now set

$$\mu^+ = \operatorname*{ess\,sup}_{Q_o} u, \qquad \mu^- = \operatorname*{ess\,inf}_{Q_o} u, \qquad \bar{\omega} = \operatorname*{ess\,osc}_{Q_o} u$$

and introduce the two functions defined in Q_o

$$v_+ = \mu^+ - u, \qquad v_- = u - \mu^-.$$

Without loss of generality may assume that

$$\mu^+ - \tfrac{1}{4}\omega_o \ge \mu^- + \tfrac{1}{4}\omega_o. \tag{4.2}$$

Indeed otherwise $\bar{\omega} \le \frac{1}{2}\omega_o$ and thus passing from Q to any smaller cylinder the essential oscillation of u is reduced by a factor $\frac{1}{2}$, and there is nothing to prove. Then either

$$\begin{aligned} &\big|\big[v_-(\cdot,0) \ge \tfrac{1}{4}\omega_o\big] \cap B_\rho\big| \ge \tfrac{1}{2}|B_\rho| \quad \text{or} \\ &\big|\big[v_+(\cdot,0) \ge \tfrac{1}{4}\omega_o\big] \cap B_\rho\big| > \tfrac{1}{2}|B_\rho|. \end{aligned} \tag{4.3}$$

Indeed by virtue of (4.2)

$$\big[u \le \mu^+ - \tfrac{1}{4}\omega_o\big] \cap B_\rho \supset \big[u \le \mu^- + \tfrac{1}{4}\omega_o\big] \cap B_\rho.$$

Therefore if the first of (4.3) is violated, then

$$\big|\big[u \le \mu^+ - \tfrac{1}{4}\omega_o\big] \cap B_\rho\big| > \tfrac{1}{2}|B_\rho|.$$

Compute and estimate the values $\delta_\pm$, as defined by (1.4), relative to the functions $v_\pm$, over B_ρ at the time level $t = 0$. Assuming the first of (4.3) holds

$$\begin{aligned} \omega_o^q &\ge \tfrac{1}{|B_\rho|} \int_{B_\rho} \big(u(\cdot,0) - \mu^-\big)^q\, dx \\ &\ge \tfrac{1}{|B_\rho|} \int_{B_\rho \cap [v_- > \frac{1}{4}\omega_o]} [u(\cdot,0) - \mu^-]^q dx \ge \frac{1}{2}\left(\frac{\omega_o}{4}\right)^q. \end{aligned}$$

Therefore if the first of (4.3) holds

$$\frac{1}{2^{\frac{2-p}{q}}}\left(\frac{\omega_o}{4A_o}\right)^{2-p} \rho^p \le \delta_- \le \left(\frac{\omega_o}{A_o}\right)^{2-p} \rho^p \qquad \text{for } A_o^{-1} = \varepsilon \tag{4.4}$$

and there holds the inclusion

$$B_{4\rho} \times (-\delta_-, 0] \subset B_{4\rho} \times \left(-\left(\frac{\omega_o}{A_o}\right)^{2-p} \rho^p, 0\right].$$

Similar estimates hold for δ_+ if the second of (4.3) is in force. By the structure conditions (1.2) both $v_\pm$ are solutions of (1.1–1.6) for the same constants C_o and C_1 and hence the Harnack-type inequality (3.7) holds for either v_- or v_+, i.e.,

$$\inf_{Q_{4\rho}(\frac{1}{4}\delta_\pm)} v_\pm \ge f(\eta_\pm) \sup_{Q_{2\rho}(\frac{1}{2}\delta_\pm)} v_\pm. \tag{4.5}$$

where $\eta_\pm$ are defined as in (1.5) for $v_\pm$. By virtue of (4.4), which holds for either δ_- or δ_+, and Remark 3.2, the function $f(\cdot)$ can be taken to be the same. Assume now that the first of (4.3) holds true. Then as shown before

$$\fint_{B_\rho} v_-^q(\cdot,0)dx \ge \frac{1}{|B_\rho|} \int_{B_\rho \cap [v \ge \frac{1}{4}\omega_o]} v_-^q(x,0)dx \ge \frac{1}{2}\left(\tfrac{\omega_o}{4}\right)^q.$$

On the other hand

$$\fint_{B_{4\rho}} v_-^r(x,-\delta_-)dx \le \omega_o^r$$

and therefore recalling the definition (1.5) of η_-

$$f(\eta_-) \geq f\left(\left(\frac{1}{2^{1/q}4}\right)^{\frac{pr}{\lambda r}}\right) \stackrel{\text{def}}{=} 1-\theta$$

for $\theta \in (0,1)$ depending only on the data and q and r. This and (4.5) imply

$$\inf_{B_{4\rho}\times(-\frac{1}{4}\delta,0]} v_- \geq (1-\theta) \sup_{B_{2\rho}\times(-\frac{1}{2}\delta,0]} v_- \tag{4.6}$$

from which

$$\operatorname*{ess\,osc}_{Q_1} u \leq \omega_1 \stackrel{\text{def}}{=} \theta\omega_o$$

where

$$Q_1 = B_\rho \times \left(-\left(\frac{\omega_o}{A}\right)^{2-p}\rho^p, 0\right] \quad \text{and} \quad A = 2^{1/q}\, 4^{1+\frac{1}{2-p}} A_o.$$

This and (4.4) determine A depending only upon the data and q, r. Taking into account (4.1) the cylinder Q_1 is determined from Q_o by the indicated choice of A and for $C = 4$. A similar argument holds if the second of (4.3) is in force. This process can now be iterated and continued to yield:

Proposition 4.1. *Let u be a locally bounded, local, weak solution of* (1.1–1.2) *for* $1 < p < 2$*, in* E_T*. There exist constants* $\bar{\gamma} > 1$ *and* $\epsilon_o \in (0,1)$*, depending only upon the data and r and q, such that for all* $(x_o, t_o) \in E_T$*, setting*

$$M = \operatorname*{ess\,sup}_{(x_o,t_o)+Q_R(R^p)} u \quad \textit{for} \;\; (x_o,t_o) + Q_R(R^p) \subset E_T, \tag{4.7}$$

there holds

$$\operatorname*{ess\,osc}_{(x_o,t_o)+Q_\rho(\delta_M)} u \leq \bar{\gamma} M \left(\frac{\rho}{R}\right)^{\epsilon_o} \quad \textit{where} \;\; \delta_M = \left(\frac{M}{A}\right)^{2-p}\rho^p \tag{4.8}$$

for all $0 < \rho \leq R$ *and all cylinders*

$$(x_o,t_o) + Q_\rho(\delta_M) \subset (x_o,t_o) + Q_R(R^p) \subset E_T.$$

Remark 4.1. Returning to Remark 1.2, the previous arguments show that either η_+ or η_- are bounded below by an absolute, positive constant η_o. Thus (4.5) implies that either $\mu^+ - u$ or $u - \mu^-$ satisfy a strong form of the Harnack Inequality. By the results of [7] a strong form of the Harnack estimate need not hold simultaneously for $\mu^+ - u$ and $u - \mu^-$.

5. Proof of Theorem 1.1 concluded

Assume (x_o, t_o) coincides with the origin of $\mathbb{R}^{N+1}$. Returning to (3.3) observe that by (3.2) and the same argument leading to (3.4)

$$|[u(\cdot, 0) > \nu\eta M] \cap B_\rho| \geq \alpha\eta^q |B_\rho| \quad \text{and} \quad \sup_{B_{2\rho}\times(-\frac{1}{2}\delta,0]} u \leq M$$

for the same values of ν and α and with δ given by (3.7). Since u is locally Hölder continuous, there exists $x_1 \in B_\rho$ such that

$$u(x_1, 0) = \nu\eta M.$$

Using the parameter A claimed by Proposition 4.1, construct the cylinder with "vertex" at $(x_1, 0)$

$$(x_1, 0) + Q_{2r}\left[\left(\frac{\nu\eta M}{A}\right)^{2-p} r^p\right] \subset B_{2\rho} \times \left(-\tfrac{1}{4}\delta, 0\right].$$

In view of (3.7) and the choice (4.4–4.6) of the parameter A, such an inclusion can be realized by possibly increasing A by a fixed quantitative factor depending only on the data, and by choosing r sufficiently small. Assuming the choice of r has been made, by Proposition 4.1

$$|u(x, t) - u(x_1, 0)| \leq \bar{\gamma} M \left(\frac{r}{\rho}\right)^{\epsilon_o}$$

for all

$$(x, t) \in \tilde{Q}_1 \stackrel{\text{def}}{=} (x_1, 0) + Q_r\left[\left(\frac{\nu\eta M}{A}\right)^{2-p} r^p\right].$$

From this

$$u(x, t) \geq \tfrac{1}{2}\nu\eta M \quad \text{for all} \;\; (x, t) \in \tilde{Q}_1$$

provided r is chosen to be so small that

$$\frac{\bar{\gamma}}{\nu\eta}\left(\frac{r}{\rho}\right)^{\epsilon_o} = \frac{1}{2} \quad \text{that is} \quad r = \varepsilon_1 \eta^{\frac{1}{\epsilon_o}} \rho \quad \text{where} \;\; \varepsilon_1 = \left(\frac{\nu}{2\bar{\gamma}}\right)^{\frac{1}{\epsilon_o}} \tag{5.1}$$

Therefore by Proposition 2.2

$$u \geq \sigma[\nu\eta M] \quad \text{in} \quad (x_1, 0) + Q_{2r}\left[\left(\frac{\sigma[\nu\eta M]}{A}\right)^{2-p} (2r)^p\right]$$

for $\sigma \in (0, 1)$ depending only on A and p. This process can now be iterated to give

$$u \geq \sigma^n[\nu\eta M] \quad \text{in} \quad (x_1, 0) + Q_{2^n r}\left[\left(\frac{\sigma^n[\nu\eta M]}{A}\right)^{2-p} (2^n r)^p\right]$$

for all $n \in \mathbb{N}$. Choose n as the smallest integer for which

$$2^n r \geq 4\rho \quad \text{that is} \quad n \geq \log_2 \left(\frac{4}{\varepsilon_1 \eta^{\frac{1}{\epsilon_o}}} \right).$$

For such a choice

$$u \geq \gamma \eta^\beta M \quad \text{in } Q_{2\rho} \left[\left(\frac{\gamma \eta^\beta M}{A} \right)^{2-p} \rho^p \right]$$

for some $\beta = \beta(\text{data})$.

6. Equations of porous medium type

The techniques apply, by minor variants, to non-negative solutions of the class of quasi-linear, singular, parabolic equations of the porous-medium type. Precisely

$$\begin{gathered} u \in C_{\mathrm{loc}} \left(0, T; L^2_{\mathrm{loc}}(E)\right) \text{ such that } |u|^{\frac{m+1}{2}} \in L^2_{\mathrm{loc}} \left(0, T; W^{1,2}_{\mathrm{loc}}(E)\right) \\ u_t - \operatorname{div} \mathbf{A}(x, t, u, Du) = 0 \quad \text{weakly in } E_T. \end{gathered} \tag{6.1}$$

The functions $\mathbf{A} : E_T \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ are only assumed to be measurable and subject to the structure conditions

$$\begin{cases} \mathbf{A}(x, t, u, Du) \cdot Du \geq C_o |u|^{1-m} \big| D|u|^m \big|^2 \\ |\mathbf{A}(x, t, u, Du)| \leq C_1 \big| D|u|^m \big| \end{cases} \quad \text{a.e. in } E_T \tag{6.2}$$

where C_o and C_1 are given positive constants, and m is in the critical and sub-critical range

$$0 < m \leq \frac{(N-2)_+}{N}. \tag{6.3}$$

The homogeneous prototype of such a class is

$$u_t - \Delta |u|^{m-1} u = 0 \quad \text{weakly in } E_T. \tag{6.1$_o$}$$

For $\tau > 0$ and for $(y, s) \in \mathbb{R}^N \times \mathbb{R}$ set

$$Q_\rho(\tau) = B_\rho \times (-\tau, 0], \qquad (y, s) + Q_\rho(\tau) = B_\rho(y) \times (s - \tau, s].$$

Let u be a non-negative weak solution of (6.1–6.3). Having fixed $(x_o, t_o) \in E_T$, and $B_{4\rho}(x_o) \subset E$, introduce the quantities

$$\fint_{B_\rho(x_o)} u^q(x, t_o) dx, \qquad \delta \stackrel{\text{def}}{=} \left[\varepsilon \left(\fint_{B_\rho(x_o)} u^q(\cdot, t_o) dx \right)^{\frac{1}{q}} \right]^{1-m} \rho^2 \tag{6.4}$$

where $\varepsilon \in (0,1)$ is to be chosen, and $q \geq 1$ is arbitrary. If $\delta > 0$, set also

$$\eta \stackrel{\text{def}}{=} \left[\frac{\left(\fint_{B_\rho(x_o)} u^q(\cdot, t_o) dx \right)^{\frac{1}{q}}}{\left(\fint_{B_{4\rho}(x_o)} u^r(\cdot, t_o - \delta) dx \right)^{\frac{1}{r}}} \right]^{\frac{2r}{\lambda_r}} \tag{6.5}$$

where $r \geq 1$ is any number such that

$$\lambda_r \stackrel{\text{def}}{=} N(m-1) + 2r > 0. \tag{6.6}$$

Theorem 6.1. *Let u be a non-negative, locally bounded, local, weak solution of (6.1–6.3). Introduce δ as in (6.4) and assume that $\delta > 0$. There exist constants $\varepsilon \in (0,1)$, and $\gamma > 1$, depending only on the data and the parameters q, r, and a constant $\beta > 1$, depending only upon the data and independent of q, r, such that*

$$\inf_{(x_o,t_o)+Q_\rho(\frac{1}{2}\delta)} u \geq \gamma \left[\frac{\left(\fint_{B_\rho(x_o)} u^q(\cdot, t_o) dx \right)^{\frac{1}{q}}}{\left(\fint_{B_{4\rho}(x_o)} u^r(\cdot, t_o - \delta) dx \right)^{\frac{1}{r}}} \right]^{\beta \frac{2r}{\lambda_r}} \sup_{(x_o,t_o)+Q_\rho(\delta)} u \tag{6.7}$$

provided $q \geq 1$ and $r \geq 1$ satisfies (6.6) and $(x_o, t_o) + Q_{8\rho}(\delta) \subset E_T$. The constants $\varepsilon \to 0$, and $\gamma \to \infty$ as either $\lambda_r \to 0$ or as $\lambda_r \to \infty$.

Remark 6.1. An estimate similar to (6.7) has been derived in [2] for non-negative solutions of the prototype equation $(6.1)_o$, by means of maximum and comparison principles. The arguments for the classes $(1.1)_o$ and $(6.1)_o$ are conceptually and technically similar.

Remark 6.2. Inequality (6.7) actually holds for non-negative solutions of (6.1–6.2) for all $0 < m < 1$, provided $r \geq 1$ satisfies (6.6). For super-critical values of $m > \frac{(N-2)_+}{N}$ one has $\lambda = \lambda_1 > 0$, and (6.6) can be realized for $r = 1$. However, for $\lambda > 0$ the strong form of a Harnack estimate holds ([7]). Therefore (6.7), while true for all $0 < m < 1$, holds significance only for critical and sub-critical values $0 < m \leq \frac{(N-2)_+}{N}$. In this sense (6.7) can be regarded as a "weak" form of a Harnack estimate. Neverthless it can be shown (6.7) is sufficient to establish the local Hölder continuity of locally bounded, weak solutions of (6.1–6.2), irrespective of their sign.

Appendix

A. Proof of Propositions 2.1 and 2.3

Proposition A.1. *Let u be a non-negative, locally bounded, local, weak solution of (1.1–1.2) for $1 < p < 2$. For every $r \geq 1$ satisfying (1.6), there exists a positive constant $\bar{\gamma}_r$, depending only upon the data and r, such that for all $B_{2\rho}(y) \times [2s-t, t] \subset E_T$, for $s < t$*

$$\sup_{B_\rho(y)\times[s,t]} u \leq \tilde{\gamma}_r \left(\frac{\rho^p}{t-s}\right)^{\frac{N}{\lambda_r}} \left(\frac{1}{\rho^N(t-s)} \int_{2s-t}^{t} \int_{B_{2\rho}(y)} u^r \, dx d\tau\right)^{\frac{p}{\lambda_r}} + \tilde{\gamma}_r \left(\frac{t-s}{\rho^p}\right)^{\frac{1}{2-p}}. \tag{A.1}$$

The proof is in [5] Chap. V.

A.1. Proof of Proposition 2.3

If $r = 1$ this follows from Proposition 2.4. Assume $r > 1$, take $(y, t) = (0, 0)$, fix $\sigma \in (0, 1]$ and let ζ be a non-negative piecewise smooth cutoff function in $\mathbb{R}^N$ vanishing outside $B_{(1+\sigma)\rho}$ and satisfying

$$0 \leq \zeta \leq 1 \text{ in } B_{(1+\sigma)\rho}; \quad \zeta = 1 \text{ in } B_\rho; \quad |D\zeta| \leq \frac{C}{\sigma\rho} \text{ in } B_{(1+\sigma)\rho}.$$

In the weak formulation of (1.1–1.2), take the testing function $u^{r-1}\zeta^p$, modulo a standard Steklov time averaging process, and integrate over the cylinder $Q = B_{(1+\sigma)\rho} \times (0, s]$. This gives

$$\frac{1}{r} \iint_Q \zeta^p u^r_\tau dx d\tau + (r-1) \iint_Q u^{r-2} \zeta^p \mathbf{A}(x, t, u, Du) \cdot Du dx d\tau$$
$$+ p \iint_Q \zeta^{p-1} u^{r-1} \mathbf{A}(x, t, u, Du) \cdot D\zeta dx d\tau = \frac{1}{r} T_1 + (r-1) T_2 + T_3.$$

Compute

$$T_1 = \int_{B_{(1+\sigma)\rho}} u^r(x, s) \zeta^p dx - \int_{B_{(1+\sigma)\rho}} u^r(x, 0) \zeta^p dx$$

and estimate

$$\iint_Q u^{r-2} \zeta^p \mathbf{A}(x, t, u, Du) \cdot Du \, dx d\tau \geq C_o \iint_Q u^{r-2} \zeta^p |Du|^p dx d\tau$$

$$|T_3| \leq C_o(r-1) \iint_Q \zeta^p u^{r-2} |Du|^p dx d\tau + \frac{C}{(\sigma\rho)^p} \iint_Q u^{p-2+r} dx d\tau$$

for a constant C depending only upon the data and r and such that $C \to \infty$ if either $\lambda_r \to 0$ or $\lambda_r \to \infty$. Combining these estimate yields

$$\sup_{0\le s\le t} \int_{B_\rho} u^r(x,s)dx \le \int_{B_{(1+\sigma)\rho}} u^r(x,0)dx + \frac{C}{(\sigma\rho)^p} \iint_Q u^{p-2+r} dx d\tau$$
$$\le \int_{B_{(1+\sigma)\rho}} u^r(x,0)dx + \frac{C}{\sigma^p} \left(\frac{t^r}{\rho^{\lambda_r}}\right)^{\frac{1}{r}} \left(\sup_{0\le s\le t} \int_{B_{(1+\sigma)\rho}} u^r dx\right)^{\frac{p-2+r}{r}}.$$

The proof is concluded by a standard interpolation argument as in Lemma 4.3 of Chap. I of [5].

A.2. *Proof of Proposition 2.1*

The proof of Proposition 2.1 follows by combining (A.1) and Proposition 2.3.

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