Modelling a slow smoldering combustion process

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We present a mathematical model for the slow and partial combustion process of a sheet of paper ignited on one side and in the presence of a flow of air confined in a narrow gap above the paper. The model includes mass and thermal balance for the various components. After having introduced some simplifications, a suitable rescaling is performed and some limit cases are examined. Two classes of travelling wave solutions are analyzed, corresponding to the opposite cases of a sufficiently large or of a moderate air flow. Copyright © 2009 John Wiley & Sons, Ltd.

Keywords: Combustion; Travelling waves.

1. Introduction

We model the process of slow and partial combustion (smoldering) of a sheet of paper exposed to a flow of air confined in a narrow gap above the paper.

We consider a particularly simple geometry in which the sheet is ignited on one side rising the local temperature by means of a heat pulse, and the combustion front proceeds keeping the form of a straight line $x = s(t)$ travelling from the ignition front $x = 0$ to the opposite side $x = L$. The air flow is parallel to the paper sheet, orthogonal to the combustion front and in the opposite direction. The experimental apparatus is shown in Fig. 1.

Experiments show that if the speed of the gas is large enough, then the combustion front is flat and the problem is one-dimensional with very good approximation.

Unlikely the expanding growth in normal gravity, it is observed in microgravity that smolder spreading over a thin filter paper exhibits a complex and unexpected fingering growth pattern [1]. In order to study more qualitatively, E. Moses and his group made the 2D experimental device of Hele-Shaw geometry where the fuel is filter paper. A rectangular sheet of the paper is put between two plates. Here the oxidizing gas is supplied in a uniform flow, opposite to the direction of the front propagation as shown in Fig. 1 (see [2], [3] for more precise explanation). Taking the oxygen flow velocity V_{o_2} as a control parameter, diverse char growth patterns are observed. For V_{o_2} = 11.4 cm/s, the growing front is even (Fig. 2a). As the value of V_{o_2} is decreasing, for V_{o_2} = 10.2 cm/s, the front becomes uneven (Fig. 2b) and for V_{o_2} = 9.2 cm/s, the front is periodic (Fig. 2c). While for V_{o_2} = 1.3 cm/s, the pattern exhibits fingers with tip splitting (Fig. 2d). Finally for V_{o_2} = 0.1 cm/s, it exhibits fingers without tip splitting (Fig. 2e) [2]. Consequently, we found that fingering instability in smoldering combustion.

The purpose of this paper is to model smoldering combustion introduced in the above. Particularly, for simplicity only, we consider the 1 dimensional problem which corresponds to the case where the gas flow velocity is large enough, as shown in Fig. 2a. We consider the situation where the combustion front proceeds from the ignition point $x = 0$ to the opposite side $x = L$.

The combustion reaction takes place between cellulose and oxygen as e.g.

$$C_6H_{10}O_5 + 6O_2 \longrightarrow 6CO_2 + 5H_2O, \tag{1}$$

the reaction rate being a function of temperature.

For our purposes, the state of the paper is described by the quantities

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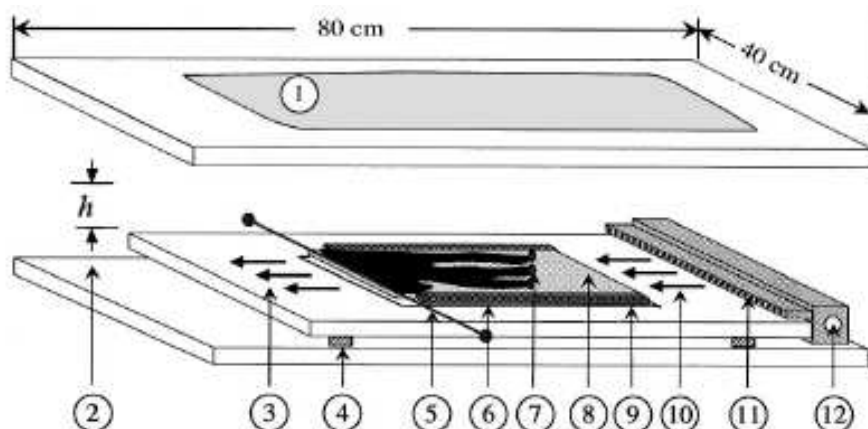
Figure 1. Schematic representation of the setup. 1, glass top; 2, variable gap between top and bottom plates h; 3, outflow of combustion products; 4, spacers to control h; 5, ignition wire; 6, heat conducting boundaries; 7, flame front; 8, fuel; 9, interchangeable bottom plate; 10, uniform flow of O_2/N_2; 11, gas diffuser; 12, gas inlet. (from [2]).

$T(x, t)$, paper temperature [K]

$\mu(x, t)$, linear molar density of cellulose [mole/cm].

The flowing gas has variable composition, due to reaction (1) and we distinguish the following components: oxygen (O_2), reaction production (CO_2 and H_2O in scheme (1)), gases not entering the reaction (N_2 etc., including e.g. CO_2 injected with fresh air). We define

$T_G(x, t)$, gas temperature [K]

$\mu_0(x, t)$, $\mu_1(x, t)$, $\mu_2(x, t)$ linear molar densities of oxygen, reaction products and inert gases, respectively [mole/cm].

2. Mass balance in the gas

The mass balance equations for the three classes of gaseous components can be written

$$\frac{\partial \mu_i}{\partial t} + \frac{\partial q_i}{\partial x} = (-1)^i \alpha_i \frac{\partial \mu}{\partial t}, \quad i = 0, 1, 2, \tag{2}$$

where $\frac{\partial \mu}{\partial t} \leq 0$ is the cellulose consumption rate [mole/(cm sec)] and α_0, α_1 are positive constants involving the stoichiometric ratios in the reaction, while $\alpha_2 = 0$. Functions $q_i(x, t)$ are the molar mass fluxes of the components

$$q_i(x, t) = -V \mu_i(x, t) - D_i \frac{\partial \mu_i}{\partial x}, \tag{3}$$

where $V(x, t)$ is the speed of the gas (flowing from $x = L$ to $x = 0$) and D_i are the diffusivities.

For each component we can introduce the linear density [g/cm] as $\rho_i = M_i \mu_i$. Here M_i is the molar weight when dealing with a single species, otherwise it represents suitable combination of molar weights. Therefore, introducing mass fluxes $Q_i = M_i q_i$ we write

$$\frac{\partial \rho_i}{\partial t} + \frac{\partial Q_i}{\partial x} = (-1)^i \beta_i \frac{\partial \rho}{\partial t}, \tag{4}$$

where $\beta_0, \beta_1 > 0$, $\beta_2 = 0$ and $\frac{\partial \rho}{\partial t}$ is the linear mass consumption rate of cellulose [g/(cm sec)].

So far we have disregarded the presence of moisture in the paper and its possible evaporation due to the air flow and the increase of temperature due to combustion.

3. Thermal balance in the gas

Let c_0, c_1, c_2 be the specific heats of the gaseous components which we assume to be constant and let K_G be the heat conductivity of the gas.

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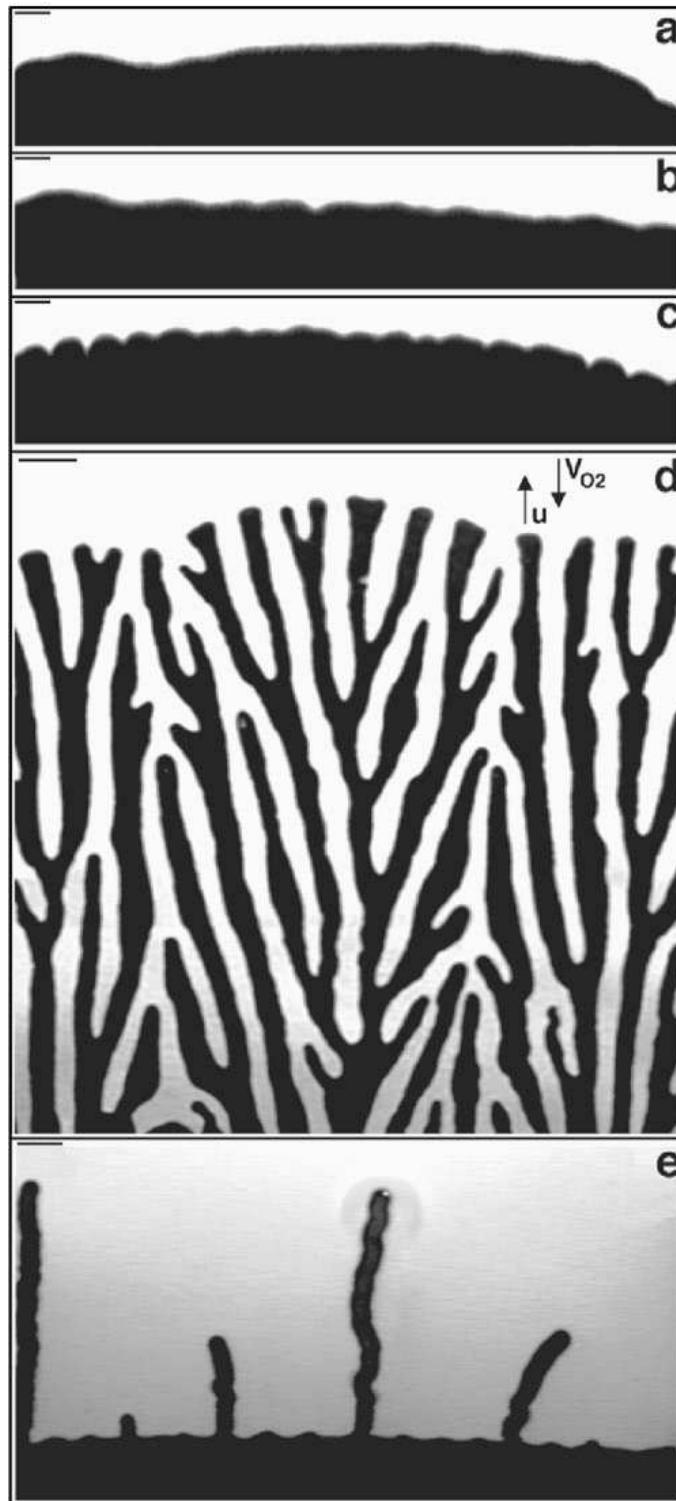Math. Meth. Appl. Sci. **2009**, 00 1–11

Figure 2. Various patterns of burned paper which appear in the experiment ([2])

The overall thermal balance in the gas can be written

$$\frac{\partial}{\partial t}\left(\sum_{i=0}^{2} c_i \rho_i T_G\right) + \frac{\partial}{\partial x}\left(\sum c_i Q_i T_G - K_G \frac{\partial T_G}{\partial x}\right) = S_E + c_0 \beta_0 T_G \frac{\partial \rho}{\partial t} - c_1 \beta_1 T \frac{\partial \rho}{\partial t}. \tag{5}$$

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At the r.h.s. of (5) S_E describes the heat exchange rate (per unit length) with the paper and the surroundings and will be specified soon. The second and the third term take into account the fact that oxygen is taken from the gas at the temperature T_G, while the reaction products are delivered to the gas at the temperature T of the paper. From the point of view of physical dimensions it is important to remember that we are considering heat fluxes through the whole paper or gap cross section. Therefore heat conductivities are such that

$$\left[K \frac{\partial T}{\partial x} \right] = \text{cal sec}^{-1},$$

i.e.

$$[K] = \text{cal cm sec}^{-1} \text{ K}^{-1}.$$

Defining

$$\rho_G c_G = \sum_{i=0}^{2} c_i \rho_i \tag{6}$$

and using (4) we get

$$\rho_G c_G \frac{\partial T_G}{\partial t} - \rho_G c_G V \frac{\partial T_G}{\partial x} - K_G \frac{\partial^2 T_G}{\partial x^2} - \sum c_i D_i \frac{\partial \rho_i}{\partial x} \frac{\partial T_G}{\partial x} = S_E - c_1 (T - T_G) \beta_1 \frac{\partial \rho}{\partial t}, \tag{7}$$

which emphasises the contribution of the of hot gases produced in the reaction to the increase of the gas temperature.

Passing to S_E, we split it into two parts: the rate of heat exchange by radiation with the surroundings of the apparatus (through the plate 1 of fig. 1), namely

$$S_A = \gamma_1 (T_A - T_G), \tag{8}$$

(T_A is the ambient temperature), and the rate of heat exchange with the paper,

$$S_P = \gamma_2 (T - T_G). \tag{9}$$

We are not going to investigate the full fluid dynamics of the problem. The flow in the narrow gap could be considered of Darcy type (i.e. driven by pressure gradient). This requires the introduction of one more unknown (gas pressure), for which the boundary conditions are fairly obvious. The real complication however is represented by the thermal expansion of the gas. Even though the densities ρ_i could be taken independent of pressure (the flow is largerly subsonic) they should be treated as $\rho_i(x, t, T_G(x, t))$, making the determination of V quite difficult.

Since the attempts of using an over-simplified model to compute approximate solutions proved to be rather successful, see [4], there is a strong evidence that it makes sense to avoid the intricacy of the fluid dynamical problem, just assuming that V *is a given constant*. The key point justifying this approximation is that in the regime we are considering the mass flow rate is much larger than the mass exchange rate due to combustion, so that V is practically not much perturbed by the combustion.

In the same spirit *diffusive and conductive terms in (7) will be neglected*.

Note that, even if $\rho(x, t)$ and $T(x, t)$ were known, equation (7) would not be sufficient for the determination of T_G, also in the simplified scheme in which V is a given constant and the third and fourth term on the l.h.s. are neglected.

Indeed, $\rho_G = \rho_G(x, t)$ should also be considered among the unknowns of the problem and equation (4) should be used for its determination. Since neglecting the variation of the composition of the gas is clearly a less drastic approximation than neglecting thermal expansivity, *we will take $\rho_G =$ constant from now on.*

4. Thermal balance in the paper

In (9) we implicitly assumed that K is a constant, disregarding not only its dependence on temperature but also the possible influence of moisture. In the sequel we will also assume that the heat conduction through a cross section of the paper along the direction x can be simply expressed as

$$q_{TH}(x, t) = -K T_x(x, t), \tag{10}$$

with K constant, neglecting the fact that it should be taken dependent on ρ, since the amount of heat transported in the paper through a cross section depends on the local value of the residual paper volume fraction (and on its composition). As a matter of fact, we will always consider processes in which the combustion affects just a small portion of the paper (possibly also because of components different from cellulose that do not take part in the combustion process).

In the same spirit we take the heat capacity independent of ρ and we write

$$\bar{\rho} \bar{c} \frac{\partial T}{\partial t} - K \frac{\partial^2 T}{\partial x^2} = S_A' - S_P + \mathcal{R} \tag{11}$$

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where $\bar{\rho}\bar{c}$ is an average heat capacity of the paper and

$$S'_A = \sigma(T_A^4 - T^4). \tag{12}$$

The term

$$\mathcal{R} = -R(T)\frac{\partial\rho}{\partial t} \tag{13}$$

encompasses all the thermal effect of the combustion also accounting for the discrepancy between the capacitive thermal energy of the reactants and the one of the products of combustion. In principle function R in (13) should depend on T_G as well, so to account for the energy needed to heat the oxygen from temperature T_G to temperature T, but the latter is of course negligible w.r.t. the heat of reaction.

5. Mass balance of the cellulose

Let us go back to formula (13) defining the rate of heat release. The basic quantity entering $\mathcal{R}$ (and driving the whole process) is the mass consumption of cellulose. This can be expressed as

$$\frac{\partial\rho}{\partial t} = -\lambda A(T)\rho\rho_0, \tag{14}$$

where $A(T)$ is a non-dimensional Arrhenius type factor, vanishing below a critical temperature T_I and that, to be specific, we will take as

$$A(T) = \begin{cases} 0, & T < T_I \\ A, & T > T_I, \end{cases} \tag{15}$$

with A positive constant.

In (14) λ [sec^{-1}] measures the speed of the reaction.

6. Governing equations and initial/boundary conditions

As we said, we assume that the thermal balance in the gas is governed by the following version of (7)

$$\rho_G c_G \left(\frac{\partial T_G}{\partial t} - V\frac{\partial T_G}{\partial x}\right) = \gamma_1(T_A - T_G) + \gamma_2(T - T_G) + c_1(T - T_G)\beta_1\lambda A(T)\rho\rho_0, \tag{16}$$

where (14) has been used and ρ_G, c_G, V, σ, K, δ, c_1 and β_1 are given positive constants.

We write the mass balance of oxygen (see (4)) in the form

$$\frac{\partial\rho_0}{\partial t} - V\frac{\partial\rho_0}{\partial x} - D\frac{\partial^2\rho_0}{\partial x^2} = -\beta_0\lambda A(T)\rho\rho_0, \tag{17}$$

where D is the oxygen diffusivity in air and β_0 is the dimensionless constant appearing in (4).

The heat balance in the paper is

$$\bar{\rho}\bar{c}\frac{\partial T}{\partial t} - K\frac{\partial^2 T}{\partial x^2} = \sigma(T_A^4 - T^4) - \gamma_2(T - T_G) + \lambda R(T)A(T)\rho\rho_0, \tag{18}$$

where, as pointed out above, $\bar{\rho}$, $\bar{c}$ can be taken as given positive constants, at least in the assumption that the paper contains many inert components and/or the combustion involves just a small fraction of it.

The problem is completed by equation (14).

The initial conditions for $0 < x < L$ are:

$$\rho_o = \tilde{\rho}_o, \qquad \rho = \tilde{\rho},$$
$$\tag{19}$$
$$T_G = T_A, \qquad T = T_A,$$

where $\tilde{\rho}_o$ is the density of oxygen in air in the experiment room and $\tilde{\rho}$ is the linear concentration of cellulose in the paper sample at the beginning of the experiment.

The boundary conditions are

$$\rho_o(L, t) = \tilde{\rho}_o \text{ (fresh air injection)}, \tag{20}$$

$$\frac{\partial\rho_o}{\partial x}(0, t) = 0 \text{ (purely convective outflow)}, \tag{21}$$

$$T_G(L, t) = T_A, \tag{22}$$

$$\frac{\partial T_G}{\partial x}(0, t) = 0, \tag{23}$$

$$T(L, t) = T_A \text{ (can be different but not very influential)}, \tag{24}$$

$$T(0, t) = T^* H(\epsilon - t), \tag{25}$$

where $T^* > T_I$, ϵ is some sufficiently small time and $H(\cdot)$ is the Heaviside function. Condition (25) mimics the impulsive onset of combustion.

7. Rescaling

We rescale densities ρ_o, ρ by their initial values and set:

$$w = \frac{\rho_o}{\tilde{\rho}_0}, \qquad p = \frac{\rho}{\tilde{\rho}}, \tag{26}$$

and the temperatures by the temperature T_A

$$u = \frac{T}{T_A}, \qquad z = \frac{T_G}{T_A}. \tag{27}$$

The independent variables x, t will be rescaled by L and

$$t_D = \frac{L^2}{D}, \tag{28}$$

respectively.

Still using the same symbols x, t for the rescaled variables and setting $\hat{A}(u) = A(T_A u)$, we obtain from (16):

$$\frac{\partial z}{\partial t} - \frac{VL}{D} \frac{\partial z}{\partial x} = \frac{\gamma_2 L^2}{D \rho_G c_G}(u - z) + \frac{c_1 \beta_1 \lambda \hat{A}(u) L^2}{D \rho_G c_G} \tilde{\rho} \tilde{\rho}_0 (u - z) pw + \frac{\gamma_1}{D \rho_G c_G} L^2 (1 - z^4). \tag{29}$$

Thus it is natural to introduce the following characteristic times;

$$t_G = (\lambda \tilde{\rho})^{-1}, \quad t_R = \left(\frac{\gamma_1}{\tilde{\rho} \tilde{c}}\right)^{-1}, \quad t_A = \left(\frac{\gamma_1}{\rho_G c_G}\right)^{-1}, \quad t_P = \left(\frac{\gamma_2}{\rho_G c_G}\right)^{-1}, \tag{30}$$

associated to the respective processes: gas heating by combustion, heat radiation, heat exchange to the paper and the surroundings. So we write

$$\frac{\partial z}{\partial t} - Pe \frac{\partial z}{\partial x} = \frac{t_D}{t_P}(u - z) + \frac{t_D}{t_G} \beta_1 \frac{\tilde{\rho}_0 c_1}{\rho_G c_G} \hat{A}(u - z) pw + \frac{t_D}{t_A}(1 - z), \tag{31}$$

where $Pe = VL/D$ is the Péclet number.

Similarly, the non-dimensional form of (17) and (18) are

$$\frac{\partial w}{\partial t} - Pe \frac{\partial w}{\partial x} - \frac{\partial^2 w}{\partial x^2} = -\beta_0 \frac{t_D}{t_G} pw \hat{A}(u) \tag{32}$$

$$\frac{\partial u}{\partial t} - Le \frac{\partial^2 u}{\partial x^2} = \frac{t_D}{t_R}(1 - u) + \frac{\tilde{\rho}_0}{\hat{\rho}} \frac{t_D}{t_G} \hat{A} \hat{R} pw - \frac{t_D}{t_P} \frac{\rho_G c_G}{\tilde{\rho} \tilde{c}}(u - z), \tag{33}$$

where we set

$$\hat{R}(u) = \frac{R(T_A u)}{\bar{c} T_A}, \text{ and}$$

$$Le = \frac{K}{\tilde{\rho} \tilde{c} D}$$

is the Lewis number (the ratio between tha equivalent heat diffusivity of paper and oxygen diffusivity).

Finally, (14) becomes

$$\frac{\partial \rho}{\partial t} = -\hat{A} \frac{\tilde{\rho}_0}{\tilde{\rho}} \frac{t_D}{t_G} pw. \tag{34}$$

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The initial and boundary conditions are now

$$z(x, 0) = z(1, t) = 1 \tag{35}$$

$$w(x, 0) = w(1, t) = 1, \qquad w_x(0, t) = 0 \tag{36}$$

$$u(x, 0) = u(1, t) = 1, \quad u(0, t) = \frac{T^*}{T_A} H\left(\frac{\epsilon}{t_D} - t\right). \tag{37}$$

8. Special cases

We consider two particular limit cases of large speed V.

Assume that $Pe >> 1$, while the remaining coefficients in the equations (31)-(34) remain $O(1)$. Then, (32) implies $w \approx 1$. Since, in addition, γ_2 tends to infinity as $V \to \infty$, we will have also $u \approx z \approx 1$. This means that the temperature of the system stays close to T_A, then extinguishing the reaction, as expected.

Another important particular case is when V is such that the thermal contact between gas and paper may be considered perfect, but the convective term $Pe\frac{\partial w}{\partial x}$ is not so important to remove all the heat produced by the reaction and is of the same order of magnitude as the r.h.s of (32). In that case we may reformulate the entire problem defining a cumulative linear density of enthalpy

$$E = (\bar{\rho}\bar{c}d_P + \rho_G c_G d_G)T, \tag{38}$$

where d_P and d_G measure the relative widths of the paper sheet and the gas channel ($d_P + d_G = 1$) and the enthalpy flux is defined as

$$q_{TH} = -(k d_P + K_G d_G)T_x(x, t) - V\rho_G c_G d_G T(x, t). \tag{39}$$

Consequently, we will have a single equation expressing the energy balance in the form

$$\hat{\rho}\hat{c}\frac{\partial T}{\partial t} - \hat{K}\frac{\partial^2 T}{\partial x^2} - \rho_G c_G d_G V \frac{\partial T}{\partial x} = \lambda R(T)A(T)\rho\rho_0 + \hat{Q}_A, \tag{40}$$

where

$$\hat{\rho}\hat{c} = \bar{\rho}\bar{c}d_P + \rho_G c_G d_G,$$
$$\hat{K} = K d_P + K_G d_G,$$

and $\hat{Q}_A$ is the heat exchange rate with the surroundings, which from now on we linearize to $\mathcal{D}(1 - u)$ while $\lambda R(T)A(T)\rho\rho_0$ is the rate of heat release by combustion. To (40) we must couple equations (14), (17).

Equation (40) can be rescaled to

$$\frac{\partial u}{\partial t} - \hat{L}e\frac{\partial^2 u}{\partial x^2} - Pe\frac{\rho_G c_G}{\hat{\rho}\hat{c}}d_G\frac{\partial u}{\partial x} = \frac{t_D}{t_G}\frac{\tilde{\rho}_0}{\hat{\rho}}\hat{R}(u)\hat{A}(u)pw + \frac{t_D}{t_A}\frac{\rho_G c_G}{\hat{\rho}\hat{c}}(1 - u^4), \tag{41}$$

where $\hat{L}e$ is defined using the global diffusivity $\frac{\hat{K}}{\hat{\rho}\hat{c}}$ and $\hat{R}(u)$ is defined replacing $\bar{c}$ by $\hat{c}$.

Note that the presence of the residual convection term still allows to have flame extinction in the limit $Pe \to \infty$.

9. Travelling wave solution

In this section we consider two classes of travelling waves in the situation described by (41). The first class deals with sufficiently large air flux, so that in equation (41) we keep the convective term and correspondingly we neglect diffusion in the oxygen transport. In the second class we deal with the converse situation of moderate gas flow, hence neglecting convection in (41) while keeping the whole second order equation for oxygen transport.

9.1. Sufficiently large flow

We recall that the product $\hat{R}(u)\hat{A}(u)$ is a step function vanishing for $u < u_l = \frac{T_l}{T_A}$. Instead of u we take $v = u - 1$, $v_l = u_l - 1$. We set $\xi = x - ct$ and we look for a solution of the form $v = v(\xi)$, $w = w(\xi)$, $p = p(\xi)$ (for simplicity we keep the same symbols) satisfying

$$\hat{L}ev'' + \gamma Pev' + cv' = -H(v - v_l)\Gamma_1 pw + \mathcal{D}v, \tag{42}$$

$$Pew' + cw' = H(v - v_l)\Gamma_2 pw, \tag{43}$$

$$cp' = H(v - v_l)\Gamma_3 pw, \tag{44}$$

where the constants Γ_1, Γ_2, Γ_3 are easily deducible from the original system, and H, is the Heaviside function. In addition we impose

$$v \longrightarrow 0, \text{ as } \xi \longrightarrow \pm\infty \tag{45}$$

$$p \longrightarrow 1, \text{ as } \xi \longrightarrow +\infty \tag{46}$$

$$w \longrightarrow 1, \text{ as } \xi \longrightarrow +\infty \tag{47}$$

With no loss of generality we take $\xi = 0$ as the rightmost point of the closure of $supp\{H(v - v_l)\}$.

We start by solving for $\xi > 0$, where $H(v - v_l) \equiv 0$. Let us define $\omega(c) = \frac{c+\gamma Pe}{Le}$. Then we immediately have

$$w \equiv 1, \quad p \equiv 1, \quad \xi > 0 \tag{48}$$

and the equation for v is

$$v'' + \omega v' - rv = 0, \quad \xi > 0 \tag{49}$$

with $r = \frac{\mathcal{D}}{Le}$. Hence, setting $\omega_\pm = \frac{-\omega \pm \sqrt{\omega^2 + 4r}}{2}$

$$v(\xi) = v_l e^{\omega_- \xi}, \quad \xi > 0 \tag{50}$$

which implies $v'(0) = \omega_- v_l$.

Now we solve for $\xi < \xi_0$ where $\xi_0 < 0$ is the still unknown $\inf\{supp\{H(v - v_l)\}\}$. Here w and p will take some unknown constant values w^*, p^*, while the solution of (49) has to be

$$v(\xi) = v_l e^{\omega_+ (\xi - \xi_0)}, \quad \xi < \xi_0. \tag{51}$$

Let us consider the interval $\xi_0 < \xi < 0$.

We rewrite (43) in the form

$$w' = \Omega(c)p', \quad \Omega(c) = \frac{c}{Pe + c} \frac{\Gamma_2}{\Gamma_3}$$

whence $w = 1 - \Omega(1 - p)$, which, put back into (44), provides

$$p' = \frac{\Gamma_3}{c} p[1 - \Omega + \Omega p],$$

to be integrated in $(\xi_0, 0)$ with the condition $p(0) = 1$.

Supposing $\Omega \neq 1$, we have

$$p(\xi) = (1 - \Omega) \left\{ \exp\left[-\frac{\Gamma_3}{c}(1 - \Omega)\xi \right] - \Omega \right\}^{-1}, \tag{52}$$

whence

$$w(\xi) = (1 - \Omega) \left\{ 1 - \Omega \exp\left[\frac{\Gamma_3}{c}(1 - \Omega)\xi \right] \right\}^{-1} \tag{53}$$

and finally

$$pw = \exp\left[\frac{\Gamma_3}{c}(1 - \Omega)\xi \right] \left[\frac{1 - \Omega}{1 - \Omega \exp\left[\frac{\Gamma_3}{c}(1 - \Omega)\xi \right]} \right]^2 \equiv F(\xi). \tag{54}$$

When $\Omega = 1$, we have $p = w = \left(1 - \frac{\Gamma_3}{c}\xi\right)^{-1}$, so that $F(\xi)$ in (54) becomes $\left(1 - \frac{\Gamma_3}{c}\xi\right)^{-2}$.

We have to solve

$$v'' + \omega v' - rv = -\hat{\Gamma}_1 F(\xi) \text{ for } \xi \in (\xi_0, 0) \tag{55}$$

where $\hat{\Gamma}_1 = \Gamma_1 / \hat{Le}$. We note that $F(0) = 1$, and F is bounded for negative values of ξ.

We write

$$v(\xi) = \alpha_+ e^{\omega_+ \xi} + \alpha_- e^{\omega_- \xi} + \bar{v}(\xi) \tag{56}$$

where α_+ and α_- are unknown constants and

$$\bar{v}(\xi) = C_+ e^{\omega_+ \xi} + C_- e^{\omega_- \xi}, \tag{57}$$

and

$$C_\pm(\xi) = \pm \frac{\hat{\Gamma}_1}{\sqrt{\omega^2 + 4r}} \int_\xi^0 F(\eta) e^{-\omega_\pm \eta} d\eta \tag{58}$$

and we note that

$$\bar{v}(0) = \bar{v}'(0) = 0. \tag{59}$$

Thus imposing $v(0) = v_I$ and $v'(0) = \omega_- v_I$ according to (49), we find

$$v(\xi) = v_I e^{\omega_- \xi} + \bar{v}(\xi), \quad \xi \in (\xi_0, 0) \tag{60}$$

and, imposing $v(\xi_0) = v_I$, ξ_0 is found (as a function of c) as the root of

$$\bar{v}(\xi_0) = v_I(1 - e^{\omega_- \xi_0}). \tag{61}$$

To find the speed c we impose the continuity of v' across ξ_0, namely

$$\omega_+ v_I = \omega_-(v_I + C_-(\xi_0))e^{\omega_- \xi_0} + \omega_+ C_+(\xi_0)e^{\omega_+ \xi_0} + \frac{\hat{\Gamma}_1}{\sqrt{\omega^2 + 4r}}F(\xi_0)\{e^{-\omega_- \xi_0} - e^{-\omega_+ \xi_0}\},$$

which reduces to

$$v_I = C_+(\xi_0)e^{\omega_+ \xi_0} + \frac{\hat{\Gamma}_1}{\omega^2 + 4r}F(\xi_0)\{e^{-\omega_- \xi_0} - e^{-\omega_+ \xi_0}\}. \tag{62}$$

9.2. Moderate flow

In stead of (42)-(44), we have

$$\hat{L}e v'' + cv' = -H(v - v_I)\Gamma_1 pw + Dv \tag{63}$$
$$w'' + Pe w' + cw' = H(v - v_I)\Gamma_2 pw \tag{64}$$
$$cp' = H(v - v_I)\Gamma_3 pw \tag{65}$$

with the boundary conditions (45)-(47).
We divide the line $\xi \in (-\infty, +\infty)$ into three parts $(-\infty, 0)$, $[0, Z]$ and $(Z, +\infty)$.
Setting

$$\rho_\pm = \frac{-c \pm \sqrt{c^2 + 4D\hat{L}e}}{2\hat{L}e}, \tag{66}$$

we have

$$v(\xi) = v_I e^{\rho_+ \xi}, \quad p = p^*, \quad w = w^*, \quad \xi \in (-\infty, 0), \tag{67}$$

where p^*, w^* are constants to be determined. Moreover,

$$\begin{cases} v(\xi) = v_I e^{\rho_-(\xi - Z)}, \quad p = 1 \\ w(\xi) = w(Z) + \frac{w'(Z)}{Pe+c} - \frac{w'(Z)}{Pe+c}e^{-(Pe+c)(\xi-Z)}, \quad \xi \in (z, +\infty) \end{cases} \tag{68}$$

in (68), $w(Z)$ and $w'(Z)$ are such that

$$w(Z) + \frac{w'(Z)}{Pe + c} = 1. \tag{69}$$

Introducing

$$V = v', \quad W = w', \tag{70}$$

we have to look for a solution of

$$\begin{cases} v' = V \\ V' = \frac{-cV + Dv - \Gamma_1 pw}{\hat{L}e} \\ p' = \frac{\Gamma_3 pw}{c} \qquad\qquad \xi \in [0, Z] \\ w' = W \\ W' = -(Pe + c)W + \Gamma_2 pw \end{cases} \tag{71}$$

that connects the point

$$(v_I, \rho_+ v_I, p^*, w^*, 0) \text{ for } \xi = 0$$

with the point

$$(v_I, \rho_- v_I, 1, w(Z), W(Z)) \text{ for } \xi = Z,$$

where the constants p^*, w^*, c and Z are to be determined along with $w(Z)$ and $W(Z)$ satisfying (69), i.e.

$$w(Z) + \frac{W(Z)}{Pe + c} = 1. \tag{72}$$

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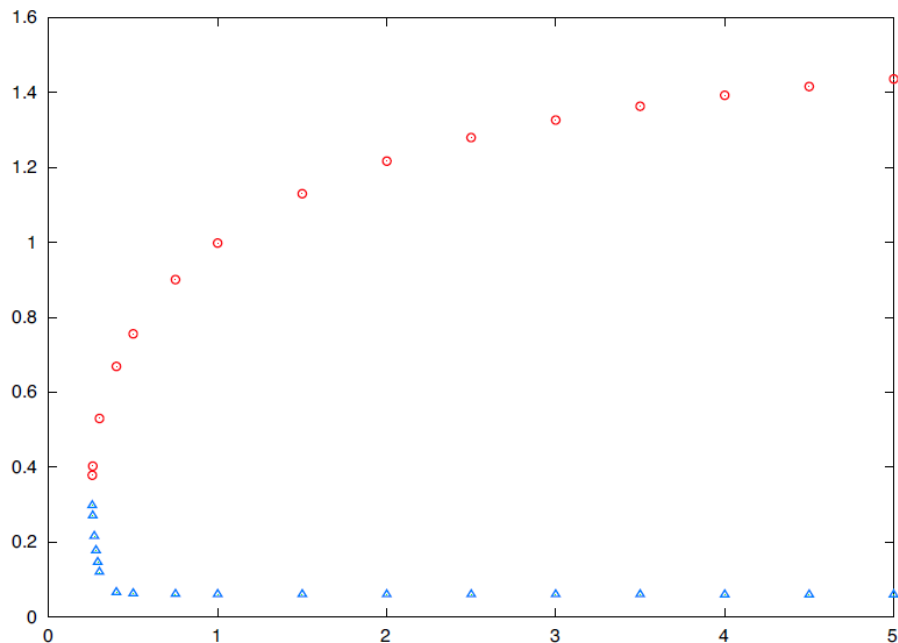Copyright © 2009 John Wiley & Sons, Ltd.   9

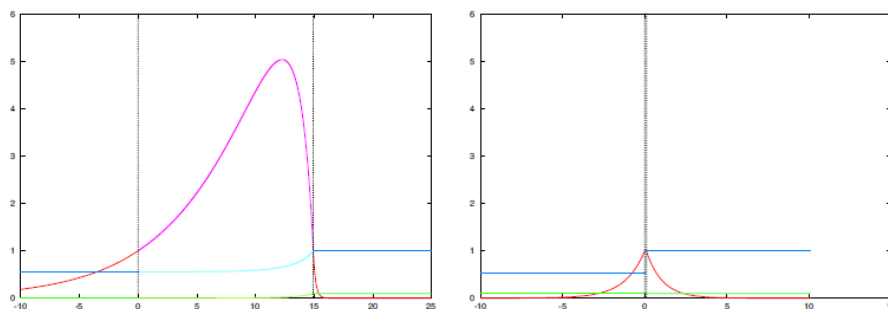Figure 3. Horizontal axis Pe, vertical axis c.

Figure 4. Profiles of two solutions at $Pe = 5.0$ with $c = 1.436$ (figure to the left) and $c = 0.060$ (figure on the right).

In conclusion, we have six equations for six unknowns. Solving this system is by no means trivial. In a forthcoming paper we will describe a numerical shooting method and will analyze how the solution depends on the parameters.

Here we just anticipate one of the most interesting results: for each Pe beyond some lower threshold there exist two solutions: a fast wave and a slow wave. The corresponding bifuration diagram is shown in Fig. 3 . The two waves have peculiar qualitative differences (see Fig. 4). In particular the low waves are unstable and the fast waves are stable. The fast waves tend to stabilize for large values of Pe.

Acknowledgements

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