STEADY WATER WAVES WITH MULTIPLE CRITICAL LAYERS: INTERIOR DYNAMICS

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Abstract. We study small-amplitude steady water waves with multiple critical layers. Those are rotational two-dimensional gravity-waves propagating over a perfect fluid of finite depth. It is found that arbitrarily many critical layers with cat's-eye vortices are possible, with different structure at different levels within the fluid. The corresponding vorticity depends linearly on the stream function.

1. Introduction

Steady two-dimensional gravity-waves constitute one of the most common types of water waves at sea. Dispersion separates waves of different wavelength and wave trains of essentially periodic, two-dimensional waves with permanent shape and velocity appear. The study thereof can be traced all the way back to Euler, but most famous are probably the *Stokes waves*. Named after Sir George Gabriel Stokes those have a surface profile which rises and falls once in every minimal period, being symmetric around their crest [23] (for Stokes' work on water waves we refer to [6]).

In the mathematical theory of such waves it is common, and physically realistic, to consider water as inviscid and of constant density [14, 16]. Historically, most research has concentrated on irrotational waves [7, 12]. This is in part for technical reasons, and in part because of Kelvins circulation theorem: waves emanating from a region with irrotational flow will remain irrotational [15]. There are, however, many situations when vorticity plays an important role [5, 20], and the recent study of exact steady water waves with a general vorticity distribution has cast new light on rotational waves (see [4] and the research following it).

Perhaps somewhat surprisingly, several studies indicate major similarities between rotational and irrotational Stokes waves, e.g. [2, 3, 24]. A notable exception from this rule arises when one examines flows with internal stagnation, i.e. points where the velocity of an individual fluid particle coincides with that of the entire wave. Even when the vorticity is only constant, *critical layers* of closed streamlines with cat's-eye vortices arise [10]. Very recently, the existence of such waves with one critical layer as solutions of the full Euler equations with exact nonlinear boundary conditions was established [25], thereby connecting the research on exact Stokes waves to the study of waves with critical layers.

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The ways to introduce critical layers are often with the aid of viscous layers, discontinuities in the vorticity, density stratification, or some combination of those. Much research has been devoted to such phenomena, not least in conjunction with the current interest for internal gravity waves. A few examples are [1, 11, 13, 18, 22]. The monograph [21] contains a general overview of vortex dynamics.

We present here small-amplitude waves with arbitrary many critical layers. Those layers, we emphasize, are not due to viscosity, density stratification, non-continuous vorticity, or any other auxiliary effect; they are a generic feature of certain rotational background flows. The results describe—in a sense precisely stated below—a family of exact gravity-waves found in [8] via bifurcation from a simple eigenvalue (the same article also contains a proof for the existence of exact bichromatic waves with vorticity, but those are not considered here). This allows for a precise description of the vortex pattern, streamlines and the possible stagnation points which goes beyond a sole phenomenological understanding. At the moment there is no exact large-amplitude theory for those waves, and, as far as we know, no numerical investigation of how the waves evolve along the bifurcation continuum. We have also not been able to find a laboratory experiment that captures the setting with multiple critical layers. All those questions are certainly very interesting.

The disposition is as follows. Section 2 describes the governing equations, with Section 3 narrowing in on laminar flows and the first-order perturbations thereof. In Section 4 we describe four solution classes of our problem (the additional fifth being the case of constant vorticity), and detail at which levels stagnation can occur in the different cases. Finally, Section 5 presents the main structure of the interesting solution class with multiple critical layers, and some examples are given. For a quick glance of the waves, see the last section.

2. Preliminaries

Let (x, y) be Cartesian position coordinates, and $(u, v) = (\dot{x}, \dot{y})$ the corresponding velocity field. Here

$$u := u(t, x, y), \qquad v := v(t, x, y),$$

are 2π-periodic in the x-variable and the vertical coordinate y ranges from the flat bed at $y = 0$ to the (normalized) free water surface at $y = 1 + \eta(t, x)$. Let $p := p(t, x, y)$ denote the *pressure*, and g the *gravitational constant of acceleration*. The Euler equations

$$\begin{aligned} u_t + uu_x + vu_y &= -p_x, \\ v_t + uv_x + vv_y &= -p_y - g, \end{aligned} \tag{2.1a}$$

then models the motion within the fluid. The equations

$$u_x + v_y = 0 \qquad \text{and} \qquad v_x - u_y = \omega \tag{2.1b}$$

additionally describes incompressibility and the *vorticity* ω, respectively. At the surface the conditions

$$p = p_0 \qquad \text{and} \qquad v = \eta_t + u\eta_x \tag{2.1c}$$

separate the air from the water, p_0 being the *atmospheric pressure*. Note that the second condition in (2.1c) states that $y(t) - \eta(t, x(t))$ is constant over time, so that the same particles constitute the interface at all times. Similarly, no water penetrates the flat bed, whence we have

$$v = 0 \qquad \text{at} \qquad y = 0. \tag{2.1d}$$

The equations (2.1) govern the motion of two-dimensional gravitational water waves on finite depth.

An important class of waves are *travelling waves*, propagating with constant shape and speed. Mathematically, such waves are solutions of (2.1) with an $(x - ct)$-dependence, where $c > 0$ ist the constant wavespeed, and we have restricted attention to waves travelling rightward with respect to the fixed Cartesian frame. Since $\mathrm{D}_t(x - ct) = u - c$, it is natural to introduce *steady variables*,

$$X := x - ct, \qquad U := u - c.$$

We shall also write Y for y to indicate when we are in the travelling frame. In the steady variables the fluid occupies

$$\Omega_\eta := \left\{(X, Y) \in \mathbb{R}^2 \colon 0 < Y < 1 + \eta(X)\right\}.$$

Define the *relative pressure* P through

$$p =: p_0 + g(1 + P - Y).$$

Since the term $-gY$ measures the hydrostatic pressure distribution, the relative pressure is a measure of the pressure perturbation induced by a passing wave. All together we obtain the governing equations

$$\left.\begin{aligned} UU_X + VU_Y &= -gP_X, \\ UV_X + VV_Y &= -gP_Y, \\ U_X + V_Y &= 0, \\ V_X - U_Y &= \omega, \end{aligned}\right\} \quad \text{in} \quad \Omega_\eta \tag{2.2a}$$

with boundary conditions

$$\begin{aligned} P &= \eta, \\ V &= U\eta_X, \end{aligned} \qquad \text{on} \quad Y = 1 + \eta(X), \tag{2.2b}$$

and

$$V = 0, \qquad \text{on} \quad Y = 0. \tag{2.2c}$$

The problem of finding (U, V, P, η) such that (2.2) is satisfied is the *steady water-wave problem*. Since η is an *a priori* unknown, (2.2) is a free-boundary problem.

The α-problem. When $\eta \in C^3(\mathbb{R})$, and $u, v \in C^2(\overline{\Omega}_\eta)$, one can use the fact that the velocity field is divergence-free (cf. (2.1b)) to introduce a *stream function* $\psi \in C^3(\overline{\Omega})$ with

$$\psi_x := -V \qquad \text{and} \qquad \psi_y := U. \tag{2.3}$$

Define $\{f, g\} := f_X g_Y - f_Y g_X$.

Proposition 2.1 (Stream-function formulation). *The water-wave problem* (2.2) *is equivalent to that*

$$\begin{aligned} \Delta\psi &= -\omega, && \text{in} && \Omega_\eta, \\ \{\psi, \Delta\psi\} &= 0, && \text{in} && \Omega_\eta, \\ |\nabla\psi|^2 + 2gy &= C, && \text{on} && Y = 1+\eta(X), \\ \psi &= m_0, && \text{on} && Y = 1+\eta(X), \\ \psi &= m_1, && \text{on} && Y = 0, \end{aligned} \tag{2.4}$$

for some constants m_0, m_1, *and* C.

Proof. Identify ψ with U and V through (2.3). Given the regularity assumptions and that Ω_η is simply connected, we see that $U_X + V_Y = 0$ is equivalent to the existence of ψ. The relations $P = \eta$ in (2.2b) and $V = 0$ in (2.2c) mean that ψ is constant on the surface and on the flat bed, just as $V_X - U_Y = -\omega$ means that $\Delta\psi = -\omega$. It remains to show how the equations of motion relate to the Bernoulli surface condition and the bracket condition.

Given (2.2) one can eliminate the relative pressure by taking the curl of the Euler equations. That yields

$$U\Delta V - V\Delta U = 0. \tag{2.5a}$$

Moreover, by differentiating the relation $P = \eta$ along the surface, and using (2.2a), we find that

$$U^2 + V^2 + 2gY = C, \qquad Y = 1 + \eta(X). \tag{2.5b}$$

Hence (2.4) holds. Contrariwise, if (U, V) fulfil (2.5a) and (2.5b), one can define P up to a constant through (2.2a), and with the right choice of constant P satisfies (2.2b). □

Consider now the case when ψ_y may vanish, but $\Delta\psi_y/\psi_y$ can be extended to a continuous function on $\overline{\Omega}_\eta$, i.e.

$$\alpha := \frac{\Delta\psi_Y}{\psi_Y} \in C^0(\overline{\Omega}_\eta) \tag{2.6}$$

One can then exchange the bracket condition $\{\psi, \Delta\psi\} = 0$ in (2.4) for

$$(\Delta - \alpha)\,\nabla\psi = 0. \tag{2.7}$$

When α is a constant there exists an affine *vorticity function* γ with $\gamma' = -\alpha$, meaning that

$$\Delta\psi = -\gamma(\psi) = \alpha\psi + \beta, \qquad \beta \in \mathbb{R}.$$

Observe that this does not rule out the existence of *stagnation points* $\nabla\psi = 0$. Without loss of generality we may take β to be zero; changing it corresponds to changing m_0 and m_1. The choice $\alpha = 0$ models constant vorticity and was investigated in [10, 25]. The next natural step is a constant but non-vanishing α. That is the setting of this investigation.

3. Laminar flows and their first-order perturbations

Laminar flows are solutions of the steady water-wave problem (2.2) with $\eta(X) = 0$. Those are the *running streams* for which

$$U(X,Y) = U_0(Y) \quad \text{and} \quad V = P = \eta = 0.$$

We shall require that $U_0 \in C^2([0,1],\mathbb{R})$. The function U_0 is the (rotational) *background current*, upon which we will impose a small disturbance: the system (2.2) will be linearized at a point $(U,V,P,\eta) = (U_0,0,0,0)$, and the solutions of the constructed linear problem analyzed. We thus assume that U, V, P and η allow for expansions of the form

$$f = f_0 + \varepsilon f_1 + \mathcal{O}\left(\varepsilon^2\right), \quad \text{as} \quad \varepsilon \to 0. \tag{3.1}$$

Here U_0 is a background current as described above, and

$$V_0 = P_0 = \eta_0 = 0.$$

By inserting these expansions into (2.2), and retaining only first-order terms in ε, we obtain the linearized system

$$\left.\begin{aligned} \partial_X U_1 + \partial_Y V_1 &= 0,\\ U_0\,\partial_X U_1 + V_1\,\partial_Y U_0 &= -\partial_X P_1,\\ U_0\,\partial_X V_1 &= -\partial_Y P_1, \end{aligned}\right\} \quad \text{in} \quad \mathbb{R}\times(0,1) \tag{3.2a}$$

with boundary conditions

$$\left.\begin{aligned} V_1 &= U_0\,\partial_X \eta_1,\\ P_1 &= \eta_1, \end{aligned}\right\} \quad \text{on} \quad Y = 1, \tag{3.2b}$$

as well as

$$V_1 = 0 \qquad \text{on} \qquad Y = 0. \tag{3.2c}$$

The following result allows us to eliminate the relative pressure from (3.2).

Proposition 3.1. *Let the background current U_0 be given. Under the condition that*

$$\int_{-\pi}^{\pi} \eta_1(X)\,dX = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} U_1(X,Y)\,dX = 0, \quad Y \in [0,1], \tag{3.3}$$

the solutions (U_1,V_1,P_1,η_1) of (3.2) are in one-to-one-correspondence with the solutions V_1 of

$$\begin{aligned} U_0 \Delta V_1 &= U_0'' V_1, && 0 < Y < 1,\\ (1 + U_0 U_0')V_1 &= U_0^2\,\partial_Y V_1, && Y = 1,\\ V_1 &= 0, && Y = 0. \end{aligned} \tag{3.4}$$

Proof. Taking the curl of the linearized Euler equations, and differentiating $p = \eta$ along the linearized surface $Y = 1$ yields (3.4). If (U_1,V_1,P_1,η_1) is a solution of (3.2) then V_1 fulfills (3.4), and if V_1 is a solution of (3.4), then one can find (U_1,P_1,η_1) such that (3.2) holds. One defines U_1 through the first equation in (3.2a), and then P_1 through the two last equations in (3.2a). The linear surface η_1 can be determined by (3.2b), and the boundary condition at $Y = 1$ in (3.4) guarantees that (3.2b) is consistent with (3.2a). Notice, however, that for a given V_1, a solution U_1 is only determined modulo functions $f(y)$, and η_1 up to a constant. We shall therefore require that the periodic mean of the first-order solution equals that of the running stream, meaning that (3.3) holds. In particular, this implies that the solution (U_1, V_1, P_1, η_1) of (3.2) is unique with respect to the solution V_1 of (3.4). □

Laminar vorticity. Now, suppose that U_0''/U_0 can be extended to a continuous function on $[0,1]$, and introduce the *laminar vorticity*

$$\alpha_0 := \frac{U_0''}{U_0} \in C([0,1], \mathbb{R}). \tag{3.5}$$

Let (cf. (3.4))

$$\mu_1 := 1 + U_0(1)U_0'(1) \quad \text{and} \quad \mu_2 := U_0^2(1).$$

We may then consider the system

$$\begin{aligned} \Delta V_1 &= \alpha_0 V_1, && 0 < Y < 1,\\ \mu_1 V_1 &= \mu_2\, \partial_Y V_1, && Y = 1,\\ V_1 &= 0, && Y = 0. \end{aligned} \tag{3.6}$$

In our case $\alpha = \alpha_0 \in \mathbb{R}$, and constant vorticity is captured by $\alpha_0 = 0$.

Relation to exact nonlinear solutions. In what comes we will find and investigate four solution classes of (3.6) and thus of (3.2). Any exact solution of the steady water-wave problem (2.4) with $\Delta\psi = \alpha\psi$ that allows for an expansion as in (3.1) and adheres to the normalization (3.3) will satisfy the velocity fields here investigated up to an error of order ε^2 in the appropriate space. Solution class 1 corresponds to a class of solutions found in [8] by linearizing around a running stream with background current $U_0(Y) = a\sin(\theta_0(Y-1)+\lambda)$. Those solutions do not necessarily satisfy the normalization (3.3); while $\int_{-\pi}^{\pi} \eta_1\, dX = 0$ the strength of the first-order background current may change with ε. This is the reason why a depends on ε in the following proposition, which is a consequence of the results from [8].

Proposition 3.2. *Let $\varepsilon \mapsto (\psi, \eta) \in C^2(\overline{\Omega}_\eta) \times C^2(\mathbb{R})$ be a solution curve found in [8] by bifurcation from a one-dimensional kernel of minimal period 2π. Pick $0 < \delta << 1$. For any ε small enough, there exists $a(\varepsilon)$ such that each component of the velocity field $(U, V) = (\psi_Y, -\psi_X)$ satisfies that of solution class 1 with $U_0(Y) = a(\varepsilon)\sin(\theta_0(Y-1)+\lambda)$ up to addition of terms $\mathcal{O}(\varepsilon^2)$ in $C^2(\overline{\Omega}_{-\delta})$. The map $\varepsilon \mapsto a(\varepsilon)$ is smooth and $a(0)$ fulfils the bifurcation condition (4.3).*

4. Solution classes

Even when α_0 is a constant, the linear system (3.6) contains a rich variety of solutions, including asymmetric ones (cf. [9]). We shall see that restricting attention to the first Fourier mode of V_1 still produces a wide range of linear waves. We thus search for a solution of the form

$$V_1 = \sin(X) f(Y), \qquad f \in C^2([0,1], \mathbb{R}). \tag{4.1}$$

The *Ansatz* (4.1) reduces the system (3.6) to a (trivial) Sturm–Liouville problem:

$$\begin{aligned} -f'' + (\alpha_0 + 1)f &= 0, \\ \mu_1 f(1) - \mu_2 f'(1) &= 0, \\ f(0) &= 0, \end{aligned} \tag{4.2}$$

with $\alpha_0 \in \mathbb{R}$, $\mu_1^2 + \mu_2^2 > 0$, and $\mu_2 \geq 0$. Since the case $\alpha_0 = 0$ has already been treated in [10] we restrict our attention to $\alpha_0 \neq 0$, corresponding to non-constant vorticity. Define

$$\theta_0 := \sqrt{|\alpha_0|} \quad \text{and} \quad \theta_1 := \sqrt{|\alpha_0 + 1|}.$$

The solutions belong to one of the following four classes:

Solution class 1 (Laminar vorticity $\alpha_0 < -1$). The solutions of (4.2) are generated by $f(Y) = \sin(\theta_1 Y)$ with

$$\begin{aligned} U_0(Y) &= a\sin(\theta_0(Y-1)+\lambda), \\ a^{-2} &= \sin^2(\lambda)\left(\theta_1\cot(\theta_1) - \theta_0\cot(\lambda)\right), \\ \lambda &\in \left(\operatorname{arccot}\left(\tfrac{\theta_1\cot(\theta_1)}{\theta_0}\right), \pi\right). \end{aligned} \tag{4.3}$$

Up to the first order in ε,

$$\begin{aligned} U(X,Y) &= U_0(Y) + \varepsilon\theta_1\cos X\cos(\theta_1 Y), \\ V(X,Y) &= \varepsilon\sin X\sin(\theta_1 Y). \end{aligned} \tag{4.4}$$

Solution class 2 (Laminar vorticity $\alpha_0 = -1$). The solutions of (4.2) are generated by $f(Y) = Y$ with

$$\begin{aligned} U_0(Y) &= a\sin(Y - 1 + \lambda), \\ a^{-2} &= \sin^2(\lambda)\left(1 - \cot(\lambda)\right), \qquad \lambda \in \left(\tfrac{\pi}{4}, \pi\right). \end{aligned} \tag{4.5}$$

Up to the first order in ε,

$$\begin{aligned} U(X,Y) &= U_0(Y) + \varepsilon\cos(X), \\ V(X,Y) &= \varepsilon Y\sin(X). \end{aligned} \tag{4.6}$$

Solution class 3 (Laminar vorticity $-1 < \alpha_0 < 0$). The solutions of (4.2) are generated by $f(Y) = \sinh(\theta_1 Y)$ with

$$\begin{aligned} U_0(Y) &= a\sin(\theta_0(Y-1)+\lambda), \\ a^{-2} &= \sin^2(\lambda)\left(\theta_1\coth(\theta_1) - \theta_0\cot(\lambda)\right), \\ \lambda &\in \left(\operatorname{arccot}\left(\tfrac{\theta_1\coth(\theta_1)}{\theta_0}\right), \pi\right). \end{aligned} \tag{4.7}$$

Up to the first order in ε,

$$\begin{aligned} U(X,Y) &= U_0(Y) + \varepsilon\theta_1\cos X\cosh(\theta_1 Y), \\ V(X,Y) &= \varepsilon\sin X\sinh(\theta_1 Y). \end{aligned} \tag{4.8}$$

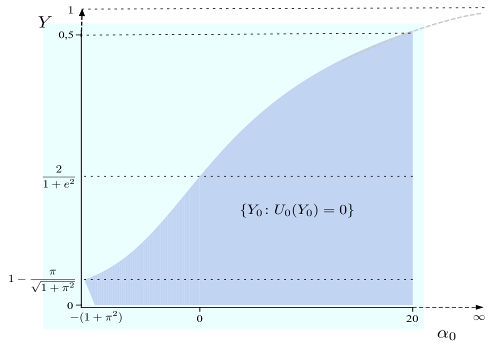

FIGURE 1. The possible zeros Y_0 of the background current U_0 as a (multivalued) function of the laminar vorticity α_0; those are the levels of the critical layers in the limit $\varepsilon \to 0$. For each pair (α_0, Y_0) in the shaded region, and only for those, there exists λ such that the bifurcation condition is fulfilled and U_0 has precisely one zero at Y_0. As $\alpha_0 \to \infty$ we have $\max Y_0 \to 1$, but for any given α_0 the stagnation points are bounded away from the surface as $\varepsilon \to 0$. In contrast, stagnation at the flat bed is possible whenever $\alpha_0 \geq -\pi^2$. For $\alpha_0 < -1-\pi^2$ more zeros appear and the situation is not as transparent.

Solution class 4 (Laminar vorticity $\alpha_0 > 0$). The solutions of (4.2) are generated by $f(Y) = \sinh(\theta_1 Y)$ with

$$\begin{aligned} U_0(Y) &= a\sinh(\theta_0(Y-1)) + \lambda\cosh(\theta_0(Y-1)),\\ a &= \frac{\lambda^2\theta_1\coth(\theta_1)-1}{\lambda\theta_0}, \qquad \lambda \neq 0. \end{aligned} \tag{4.9}$$

Up to the first order in ε,

$$\begin{aligned} U(X,Y) &= U_0(Y) + \varepsilon\theta_1\cos X\cosh(\theta_1 Y),\\ V(X,Y) &= \varepsilon\sin X\sinh(\theta_1 Y). \end{aligned} \tag{4.10}$$

Stagnation. The explicit solutions allow us to determine the possible levels of stagnation. From the following proposition one obtains Figure 1.

Proposition 4.1 (Stagnation). *The following hold for the background current U_0 for the solution classes 1–4:*

- SC 1. *For any $Y_0 \in [0,1)$ there exists $\alpha_0 < -1$ and λ such that $U_0(Y_0) = 0$, and the number of zeros of U_0 in $[0,1]$ can be made arbitrarily large. In the interval $-1-\pi^2 < \alpha_0 < -1$ U_0 has one zero at $Y_0 = 1-\lambda/\theta_0$ for $\operatorname{arccot}(\theta_1\cot(\theta_1)/\theta_0) < \lambda \leq \theta_0$, and none for $\lambda \in (\theta_0, \pi)$.*
- SC 2. *U_0 has one zero at $Y_0 = 1-\lambda$ for $\lambda \in (\pi/4, 1]$ and none for $\lambda \in (1,\pi)$.*
- SC 3. *U_0 has one zero at $Y_0 = 1-\lambda/\theta_0$ for $\operatorname{arccot}(\theta_1\coth(\theta_1)/\theta_0) < \lambda \leq \theta_0$ and none for $\lambda \in (\theta_0,\pi)$.*
- SC 4. *U_0 has one zero at $Y_0 = 1-\theta_0^{-1}\operatorname{arctanh}\left(\frac{\lambda^2\theta_0}{\lambda^2\theta_1\coth(\theta_1)-1}\right)$ for $\lambda^2 \geq (\theta_1\coth(\theta_1)-\theta_0\coth(\theta_0))^{-1}$ and none for other $\lambda \neq 0$.*

Proof. The analysis is carried out separately for each solution class .

SC 1. The function

$$\alpha_0 \mapsto \frac{\theta_1 \cot(\theta_1)}{\theta_0},$$

spans the real numbers (it blows up at $\alpha_0 = -1 - n^2\pi^2$, $n \in \mathbb{N}$). We thus see from (4.3) that the set $(\operatorname{arccot}(\theta_1 \cot(\theta_1)/\theta_0), \pi)$ may be empty. But for any $\varepsilon > 0$ and $n \in \mathbb{N}$, there exists $\delta > 0$ such that if $\alpha_0 < -1 - n^2\pi^2 < \alpha_0 + \delta$, then (4.3) is solvable for all $\lambda \in (\varepsilon, \pi)$. The number of zeros is at least as large as $\lfloor \theta_0/\pi \rfloor \to \infty$ as $\alpha_0 \to -\infty$. The last proposition then follows by checking that U_0 can have at most one zero Y_0 for $-1 - \pi^2 < \alpha < -1$.

SC 2. Consider (4.5). The function

$$\lambda \mapsto \sin(\lambda)\,(\sin(\lambda) - \cos(\lambda)) > 0 \quad \text{exactly when} \quad \frac{\pi}{4} < \lambda < \pi,$$

and since it is bounded, the amplitude a is bounded away from 0. There thus exist λ and $Y_0 \in [0,1]$ such that $U_0(Y_0) = 0$ if and only if $Y_0 = 1 - \lambda \in [0, 1 - \frac{\pi}{4}]$.

SC 3. The right-hand side of (4.7) is positive when

$$\arctan\left(\frac{\theta_0}{\theta_1 \coth(\theta_1)}\right) < \lambda < \pi. \tag{4.11}$$

To have $Y_0 \in [0,1]$ with $U_0(Y_0) = 0$ necessarily $\lambda \in [0, \theta_0]$. The assertion follows from that θ_0 is strictly larger than the lower bound in (4.11).

SC 4. The background current has at most one zero, and to see what zeros there are in $[0,1]$ we consider

$$\tanh(\theta_0(Y-1)) = -\frac{\lambda}{a} = \frac{\lambda^2\theta_0}{1 - \lambda^2\theta_1\coth(\theta_1)}.$$

For $Y_0 \in [0,1)$ the left-hand side is negative, so that we must at least have $\lambda^2 > (\theta_1\coth(\theta_1))^{-1}$, and a closer look yields that $\lambda^2 \geq (\theta_1\coth(\theta_1) - \theta_0\coth(\theta_0))^{-1}$ is required to match $Y \geq 0$. The right-hand side is then an increasing function of λ^2, and

$$-\tanh(\theta_0) < \frac{\lambda^2\theta_0}{1 - \lambda^2\theta_1\coth(\theta_1)} \leq \frac{-\theta_0}{\theta_1\coth(\theta_1)}.$$

Since also arctanh is an increasing function, the question reduces to whether

$$0 \leq Y_0 < 1 - \frac{1}{\theta_0}\operatorname{arctanh}\left(\frac{\theta_0}{\theta_1\coth(\theta_1)}\right).$$

The right-hand side is positive, strictly increasing in α_0, and tends to 1 as $\alpha_0 \to \infty$. □

5. Hamiltonian formulation and phase-portrait analysis

In the analysis to come the region of interest is

$$0 \leq Y \leq 1 \pm \mathcal{O}(\varepsilon)\cos(X),$$

where the sign indicates that for each of the solution classes 1–4 one finds that $X = 0$ may be either a crest or a trough, depending on the signs of a and λ. Recall that $(u,v) = (\dot{x}, \dot{y})$. In view of that $X = x - ct$ and $U = u - c$, one similarly obtains

$$(\dot{X}, \dot{Y}) = (U, V).$$

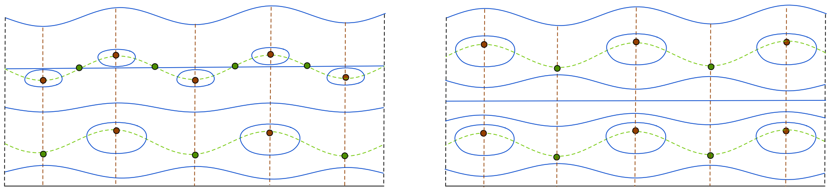

FIGURE 2. Two scenarios for solution class 1. Blue lines are streamlines, dotted lines isoclines [∞-isocline green, 0-isocline red], and fat dots critical points [centers red, saddle points green]. Left: an uppermost critical layer with horizontal streamlines cutting through it as in Prop. 5.2 ii.b), and a lower critical layer as in Prop. 5.2 ii.a). Right: two critical layers as in Prop. 5.2 ii.a) separated by a horizontal streamline. Note that the rotational flow near the bottom evolves under a "rigid lid".

The paths $(X(t), Y(t))$ describe the particle trajectories in the steady variables, and any such solution is entirely contained in one streamline.

Proposition 5.1 (Hamiltonian formulation). *The solution classes 1–4 all admit a Hamiltonian*

$$H(X, Y) := \varepsilon \cos(X) G(Y) + \int_0^Y U_0(s)\, ds, \tag{5.1}$$

with

$$G(Y) := \begin{cases} \sin(\theta_1 Y), & \text{for solution class } 1, \\ Y, & \text{for solution class } 2, \\ \sinh(\theta_1 Y), & \text{for solution classes } 3\text{–}4. \end{cases} \tag{5.2}$$

Classes 2–4 can be dealt with as the class $\alpha = 0$ (constant vorticity) in [10, 25], and do not yield any qualitatively new results. Indeed, their appearance and the analysis thereof is captured within that of the interesting class 1.

Proposition 5.2 (Main proposition). *The following hold for small-amplitude waves of solution class 1 (ε infinitesimally small).*

i. *The fluid motion is divided into vertical layers, each separated from the others by flat sets of streamlines $\{(X, Y_*) \colon \sin(\theta_1 Y_*) = 0\}$.*

ii. *For each Y_* with $U_0(Y_*) = 0$ there is a smooth connected part of the ∞-isocline passing through all points $(\pi/2 + n\pi, Y_*)$, $n \in \mathbb{Z}$, along which centers (cats-eye vortices) and saddle points alternate in one of the following ways:*
   a) *when Y_* is not a common zero of U_0 and $\sin(\theta_1 \cdot)$ centers appear at every other $X = n\pi$ and saddle points at every other $(n+1)\pi$;*
   b) *when Y_* is a common zero of U_0 and $\sin(\theta_1 \cdot)$ centers appear at $X = n\pi$ and saddle points at $\pi/2 + n\pi$.*

Remark 5.3. The situation in ii.b) is not exceptional; it might occur whenever ε is not infinitesimally small. Starting with the situation in ii.b) one might fix ε and α, and then vary the zero of the background flow. The saddle point

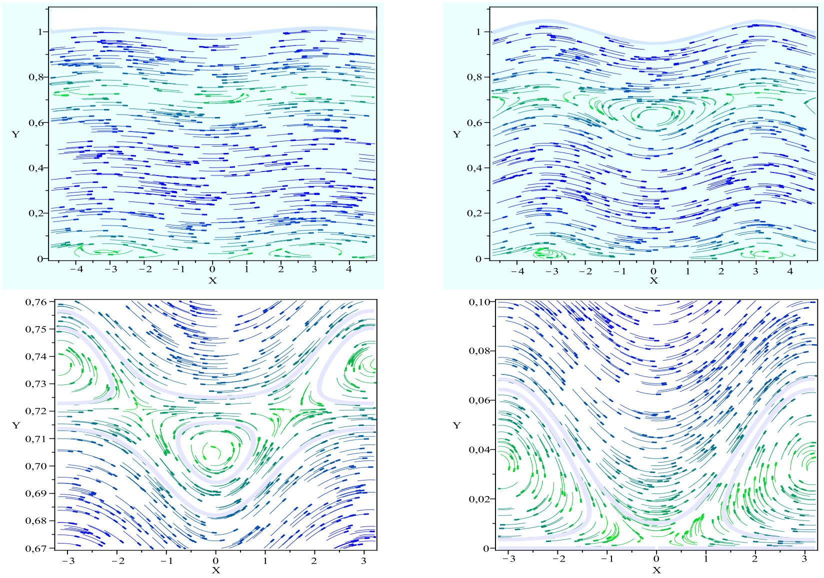

FIGURE 3. Numerical plots of solution class 1. Colours indicate the strength of the velocity field. Top left [$\alpha = -20$, $\lambda = 4.39$, $\varepsilon = 0.05$]: the case of a common zero as in Prop. 5.2 ii.b). The upper critical layer is magnified in the plot bottom left, where the three centers, the two saddle points and the 0-isocline cutting through them are clearly visible. The bottom right plot shows the same flow near the bottom with only two centers as in Prop. 5.2 ii.a). The situation is similar to that in Figure 2, left, with the difference that the ∞-isocline actually crosses the flat bed in this numerical plot. Right [$\alpha = -20$, $\lambda = 4.60$, $\varepsilon = 0.05$]: as the zero of the background current shifts, the horizontal 0-isocline climbs above the critical points in the upper critical layer and the saddle nodes merge with one center each, turning the situation in Prop. 5.2 ii.b) to the one in Prop. 5.2 ii.b).

at $X = \pi/2$ then continuously and monotonically approaches the center at either $X = 0$ or $X = \pi$, eventually merging with it and wiping it out.

Proof. It follows from (5.1) that the fluid motion is 2π-periodic and symmetric around the vertical $X = 0$ axis. It therefore suffices to investigate the strip $0 \leq X \leq \pi$.

i). The velocity field is

$$\dot{X} = U_0(Y) + \varepsilon\theta_1 \cos(X)\cos(\theta_1 Y),$$
$$\dot{Y} = \varepsilon \sin(X)\sin(\theta_1 Y),$$

whence the 0-*isocline*—defined as the set where $\dot{Y} = 0$—is given by the vertical axes $X = 0 \bmod \pi$ and the horizontal lines where $\sin(\theta_1 Y) = 0$.

ii). Let Y_* be a zero of U_0. Then $(\pi/2, Y_*)$ belongs to the set $\{(X,Y)\colon \dot{X} = 0\}$, called the ∞-*isocline*. Since $U_0'(Y_*) = a\theta_0 \neq 0$, we have that

$$D_Y\dot{X} = U_0'(Y) - \varepsilon\theta_1^2\cos(X)\sin(\theta_1 Y)$$

is nonzero at $(\pi/2, Y_*)$. The implicit function theorem allows us to locally parametrize the ∞-isocline as the graph of a smooth function $Y_\infty(X)$ with slope

$$\mathrm{D}_X\, Y_\infty = \frac{\varepsilon\theta_1 \sin(X)\cos(\theta_1 Y)}{U_0'(Y) - \varepsilon\theta_1^2 \cos(X)\sin(\theta_1 Y)}. \tag{5.3}$$

A continuity argument yields that for ε small enough Y_∞ extends to a 2π periodic function on $\mathbb{R}$, strictly rising and falling between the zeros of $\sin(X)$, and with $|Y_\infty - Y_*| = \mathcal{O}(\varepsilon)$. Hence $U_0'(Y_\infty) \neq 0$. At $X = n\pi$, $n \in \mathbb{Z}$, the graph of Y_∞ intersects the 0-isocline. At those *critical points*, the Hessian of the Hamiltonian is given by

$$\begin{aligned}&\mathrm{D}^2\, H(n\pi, Y_\infty|_{X=n\pi})\\ &= \begin{bmatrix} (-1)^{n+1}\varepsilon\sin(\theta_1 Y_\infty) & 0 \\ 0 & (-1)^{n+1}\varepsilon\theta_1^2\sin(\theta_1 Y_\infty) + U_0'(Y_\infty)\end{bmatrix}\Bigg|_{X=n\pi},\end{aligned} \tag{5.4}$$

There are now two cases.

a) When $\sin(\theta_1\cdot)$ and U_0 have no common zero. From $|Y_\infty - Y_*| = \mathcal{O}(\varepsilon)$ wee find that $\sin(\theta_1 Y_\infty)$ is non-vanishing and thus of constant sign. For ε small enough, the Hessian (5.4) thus has one negative and one positive eigenvalue at every other $X = n\pi$, and two of the same sign at every other $X = (n+1)\pi$ in between. The assertion ii.a) then follows from the Morse lemma [17].

b) When $\sin(\theta_1\cdot)$ and U_0 have a common zero. In this case there are additional critical points at $X = \pi/2 + n\pi$, $n \in \mathbb{Z}$, all similar to the one at $X = \pi/2$. There

$$\mathrm{D}^2\, H(\pi/2, Y_*) = \begin{bmatrix} 0 & -\varepsilon\theta_1\cos(\theta_1 Y_*) \\ -\varepsilon\theta_1\cos(\theta_1 Y_*) & U_0'(Y_*)\end{bmatrix},$$

with one strictly positive and one strictly negative eigenvalue. Hence, the critical point $(\pi/2, Y_*)$ is always a saddle point.

We want to show that the Hessian (5.4) has two eigenvalues of the same sign at all critical points $(n\pi, Y_\infty|_{X=n\pi})$. Since the slope of Y_∞ changes direction exactly at $X = n\pi$ it follows that also in the case when $U_0(Y_*) = \sin(\theta_1 Y_*) = 0$ we have $\sin(\theta_1 Y_\infty|_{x=n\pi}) \neq 0$, but with

$$\operatorname{sgn}\sin(\theta_1 Y_\infty|_{X=n\pi}) = -\operatorname{sgn}\sin(\theta_1 Y_\infty|_{X=n\pi}),$$

all given that ε is small enough. We now claim that $-\varepsilon\sin(\theta_1 Y_\infty|_{X=0})$ and $U_0'(Y_\infty|_{X=0})$ have the same sign (cf. (5.4)). The slope of $Y_\infty|_{X\in(0,\pi)}$ is determined by the sign of $\cos(\theta_1 Y_\infty)/U_0'(Y_\infty)$ in the same interval. We have $\cos(\theta_1\cdot) > 0$ when $\sin(\theta_1\cdot)$ is increasing, and contrariwise. The assertion now follows from that $\sin(\theta_1 Y_*) = 0$. □

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