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# A BOUNDARY VALUE PROBLEM ON A HALF-LINE FOR DIFFERENTIAL EQUATIONS WITH INDEFINITE WEIGHT

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*Dedicated to Professor J.R.L. Webb on the occasion of his retirement*

**ABSTRACT.** The boundary value problem on the half-line for the second order differential equation with general Φ-Laplacian

$$\big(a(t)\Phi(x')\big)' = b(t)F(x), \quad t \ge 0,$$
$$x(0) = c > 0,\ 0 < \lim_{t\to\infty} x(t) < \infty,\ x(t) > 0,\ \lim_{t\to\infty} x'(t) = 0,$$

is considered, where $a, b$ are continuous functions on $[0,\infty)$, $a$ is positive and $b$ can change its sign. The cases of regular variation, slow variation, and rapid variation of the inverse function $\Phi^*$ of $\Phi$ are considered. Some applications of the main results complete the paper.

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## 1. INTRODUCTION

Consider the second order nonlinear differential equation

$$\big(a(t)\Phi(x')\big)' = b(t)F(x), \quad t \ge 0, \tag{1.1}$$

where $\Phi$ is an increasing odd homeomorphism defined on an open interval $(-\rho,\rho), 0 < \rho \le \infty$, and Im $\Phi = (-\sigma,\sigma), 0 < \sigma \le \infty$, $F$ is a real continuous nondecreasing function on $\mathbb{R}$ such that $F(u)u > 0$ for $u \ne 0$ and $a, b$ are continuous functions for $t \ge 0$ such that

$$a(t) > 0, \quad \liminf_{t\to\infty} a(t) > 0.$$

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We are interested in solving the boundary value problem on the whole half-line associated to (1.1)

$$x(0) = c > 0,\ x(t) > 0,\ 0 < \lim_{t\to\infty} x(t) < \infty,\ \lim_{t\to\infty} x'(t) = 0, \tag{1.2}$$

especially when the weight $b$ changes its sign, that is, if there exist $t_1, t_2 \geq 0$ satisfying $b(t_1)b(t_2) < 0$.

Let $b_+, b_-$ be respectively the positive and the negative part of $b$, i.e.,

$$b_+(t) = \max\{b(t), 0\}, \quad b_-(t) = -\min\{b(t), 0\}.$$

Clearly $b(t) = b_+(t) - b_-(t)$.

Many papers deal with the existence of eventually positive solutions of (1.1) when the weight $b$ does not change sign. We refer, for instance, to [7, 8] and references therein for the case $b_- \equiv 0$. Similar results for the opposite case $b_+ \equiv 0$ can be found in [5, 19] and references therein. Some results can be obtained from papers dealing with coupled first order differential systems, see, e.g., [3, 4, 16]. Finally, other contributions can be found in the monographs [1, 13, 18].

When $\Phi$ is the classical $\Phi$-Laplacian, i.e.,

$$\Phi(u) = \Phi_\alpha(u) = |u|^\alpha \operatorname{sgn} u, \quad \alpha > 0, \tag{1.3}$$

and $b(t) \geq 0$ on $[0, \infty)$, it is well-known that (1.1) has nonnegative nonincreasing solutions $x$ such that $x(0) = c$ for any $c > 0$, see, e.g., [13, 18]. These solutions are called *Kneser solutions* and are widely studied in the literature, see, e.g., [3] and references therein. Thus, the existence of these solutions, tending to non-zero constants, is a special case of the boundary value problem (1.1), (1.2).

When $b$ takes negative values, the existence of globally positive solutions of (1.1), that is solutions which are positive for any $t \geq 0$, is a difficult problem and very few is known. As far as our knowledge, the only result on this topic is [6, Theorem 3.2], which deals with the particular case

$$\Phi(u) = \Phi_C(u) = \frac{u}{\sqrt{1 + |u|^2}}, \tag{1.4}$$

which arises in studying radially symmetric solutions of partial differential equations with the mean curvature operator.

Motivated by the previous quoted papers, our aim here is to study the solvability of the boundary value problem (1.1), (1.2), in particular when the weight $b$ changes its sign. Our results give also the global positiveness of certain solutions of (1.1) and, in this sense, generalize the previous quoted papers. The paper is completed by some applications concerning the particular cases of the classical $\Phi$-Laplacian (1.3), of the map $\Phi_C$ and its inverse

$$\Phi_R(u) = \frac{u}{\sqrt{1 - |u|^2}}, \tag{1.5}$$

which, similarly to $\Phi_C$, arises in searching radial solutions of partial differential equations with the relativity operator.

Our approach is based on the Tychonov fixed point theorem and, in order to focus the behavior of $\Phi$ near zero, on the notion of Karamata functions [11, 12]. These functions are widely employed in many fields, like, for instance, probability theory, number theory, complex analysis. Recently, in [17] the role of the Karamata functions in studying asymptotic qualitative problems associated to second order Emden-Fowler type differential equations has been pointed out. Other contributions concerning ordinary differential equations, in which the Karamata functions have been used to describe the growth of nonoscillatory solutions at infinity, can be found in [10, 14, 15] and references therein. The definition and the main properties of these functions are recalled in Section 2. The regular variation case is treated in Section 3, and in Section 4 the solvability of (1.1), (1.2) is considered in case of rapid variation or slow variation. Some applications complete the paper.

## 2. PRELIMINARIES

In this section we give the definition and the main properties of Karamata functions in the form that is useful for our aim, even if this theory has been formulated in a more general context.

Let $g$ be a positive continuous function, defined on the right neighborhood $(0, \delta)$ of zero. Following [11, 12], see also [2], the function $g$ is called *regularly varying at* $u = 0$ *of index* $p > 0$ if, for any $\lambda > 0$,

$$\lim_{u \to 0^+} \frac{g(\lambda u)}{g(u)} = \lambda^p.$$

If

$$\lim_{u \to 0^+} \frac{g(\lambda u)}{g(u)} = 1,$$

then $g$ is called *slowly varying at* $u = 0$. Finally, $g$ is said to be *rapidly varying at* $u = 0$ if

$$\lim_{u \to 0^+} \frac{g(\lambda u)}{g(u)} = \begin{cases} 0 & \text{for } 0 < \lambda < 1 \\ \infty & \text{for } \lambda > 1. \end{cases}$$

For example, the functions $g_1(u) = u^p$, $g_2(u) = u^p / |\log u|$, are regularly varying at $u = 0$ of index $p$. The functions $g_3(u) = |\log u|$, $g_4(u) = |\log u|^{-1}$, are slowly varying at $u = 0$, and $g_5(u) = e^{-1/u}$, is rapidly varying at $u = 0$. We refer to [2, 17] for more sophisticated examples.

The following properties of the Karamata functions will be useful in our later considerations.

**Lemma 2.1.** $i_1$) *If* $L$ *is a slowly varying function at* $u = 0$, *then* $\lim_{u \to 0^+} u^{\varepsilon} L(u) = 0$, $\lim_{u \to 0^+} u^{-\varepsilon} L(u) = \infty$ *for every* $\varepsilon > 0$.

$i_2$) *If* $g$ *is a regularly varying function at* $u = 0$ *with index* $p > 0$, *then* $g$ *can be represented in the form*

$$g(u) = u^p L(u), \tag{2.1}$$

*where $L$ is a slowly varying function at $u = 0$. Moreover, for any $\varepsilon, T$ with $0 < \varepsilon < p$ and $0 < T < \delta$, there exists $M = M_{\varepsilon,T}$ such that*

$$g(u) \le M u^{p-\varepsilon} \quad \textit{on } (0, T].$$

*$i_3$) If $R$ is a rapidly varying function at $u = 0$ and increasing in a right neighborhood of zero, then $\lim_{u\to 0^+} u^{-\varepsilon} R(u) = 0$ for every $\varepsilon > 0$.*

*Proof. Claim $i_1$).* It follows from [2, Proposition 1.3.6.], with minor changes.

*Claim $i_2$).* Since $g$ is regularly varying with index $p$, the function $g(u)/u^p$ is slowly varying, and the representation formula (2.1) follows. In virtue of Claim $i_1$) we have $\lim_{u\to 0^+} u^{\varepsilon} L(u) = 0$ for every $\varepsilon > 0$. Thus, the function $u^{\varepsilon} L(u)$ is bounded on $(0, T]$, i.e., there exists a positive constant $M$, depending on $\varepsilon$ and $T$, such that $L(u) \le M u^{-\varepsilon}$, and the assertion follows.

*Claim $i_3$).* Since the inverse of $R$ is slowly varying [2, Theorem 2.4.7], the assertion follows from Claim $i_1$). □

Let $g$ be regularly varying at $u = 0$ with index $p > 0$, and set

$$G(\lambda, u) = \frac{g(\lambda u)}{\lambda^p g(u)}, \quad \lambda \in (0, 1], u \in (0, \delta). \tag{2.2}$$

Then $G$ can be unbounded in the square $Q = (0, 1] \times (0, \delta)$, as the following example shows.

**Example 2.2.** Consider the function

$$g(u) = u^3 |\log u|, \quad u \in (0, 1),$$

which is regularly varying at $u = 0$ with index 3. Since

$$G(\lambda, u) = \frac{g(\lambda u)}{\lambda^3 g(u)} = \left| \frac{\log \lambda}{\log u} + 1 \right|,$$

choosing $0 < \lambda < 1$ and

$$u = e^{-\sqrt{|\log \lambda|}}$$

we obtain $G(\lambda, e^{-\sqrt{|\log \lambda|}}) = \sqrt{|\log \lambda|} + 1$, showing that $G$ is unbounded on $Q$.

Since the boundedness of the function $G$ given by (2.2) plays a role in the sequel, we close this section with sufficient conditions assuring this boundedness.

**Lemma 2.3.** *Let $g$ be regularly varying at $u = 0$ with index $p > 0$. Assume that the function*

$$L(u) = \frac{g(u)}{u^p} \tag{2.3}$$

*satisfies on $(0, T], 0 < T < \delta$, any of the following assumptions:*

*$i_1$) There exist $m_1, m_2 > 0$ such that $m_1 \le L(u) \le m_2$ ;*

*$i_2$) The function $L$ is nondecreasing.*

*Then there exists $M = M_T > 0$ such that*

$$g(\lambda u) \le M \lambda^p g(u) \quad \textit{for } u \in (0, T], \lambda \in (0, 1]. \tag{2.4}$$

*Proof.* In view of (2.3) we have

$$g(\lambda u) = \lambda^p g(u) \frac{L(\lambda u)}{L(u)}. \tag{2.5}$$

If the assumption $i_1$) holds, then the inequality (2.4) immediately follows. If the assumption $i_2$) holds, then $L(\lambda u) \leq L(u)$. Thus from (2.5) we obtain (2.4) with $M = 1$. □

The following example illustrates Lemma 2.3.

**Example 2.4.** The functions $g_6(u) = ue^{-u}, u > 0$, and $g_7(u) = u/(\log u)^2, 0 < u < 1$, are regularly varying at $u = 0$ with index $p = 1$. Clearly, $g_6(u)/u$ and $g_7(u)/u$ satisfy the assumptions $i_1$) and $i_2$) of Lemma 2.3, respectively. Then (2.4) holds for both the functions $g_6$ and $g_7$.

## 3. THE REGULAR VARIATION CASE

Denote by $\Phi^*$ the inverse map of $\Phi$. We start by considering the existence of solutions of (1.1), (1.2), when $\Phi^*$ is regularly varying at $u = 0$ of index $p > 0$. The following holds.

**Theorem 3.1.** *Let $\Phi^*$ be regularly varying at $u = 0$ with index $p > 0$, and assume that there exists $q$, $0 < q < p$, such that*

$$\lim_{u \to 0^+} \frac{F(u)}{u^{1/q}} = 0, \tag{3.1}$$

*and for some $\varepsilon$, $0 < \varepsilon < p - q$*

$$\begin{aligned} I_+ &= \int_0^\infty \left( \frac{1}{a(s)} \int_s^\infty b_+(r)dr \right)^{p-\varepsilon} ds < \infty \\ I_- &= \int_0^\infty \left( \frac{1}{a(s)} \int_s^\infty b_-(r)dr \right)^{p-\varepsilon} ds < \infty. \end{aligned} \tag{3.2}$$

*Then the boundary value problem* (1.1), (1.2) *is solvable for any small positive $c$. Moreover, every solution is of bounded variation on* $[0, \infty)$.

*Proof.* Choose $\mu > 0$ such that

$$T_\mu = \mu \max \left\{ \max_{t \geq 0} \left( \frac{1}{a(t)} \int_t^\infty b_+(s)ds \right), \max_{t \geq 0} \left( \frac{1}{a(t)} \int_t^\infty b_-(s)ds \right) \right\} < \sigma. \tag{3.3}$$

Since $\Phi^*$ is regularly varying with index $p > 0$ at $x = 0$, from Lemma 2.1, fixed $\varepsilon < p - q$, a positive constant $M$ exists such that

$$\Phi^*(u) \leq M u^{p-\varepsilon} \tag{3.4}$$

for $0 < u \leq T_\mu$. Choose $c > 0$ sufficiently small such that

$$F(2c) < \mu \tag{3.5}$$

and consider the Fréchet space $C[0,\infty)$ of all continuous functions on $[0,\infty)$, endowed with the topology of uniform convergence on compact subintervals of $[0,\infty)$. Let $\Omega$ be the subset of $C[0,\infty)$ given by

$$\Omega = \left\{ u \in C[0,\infty) : \frac{c}{2} \le u(t) \le 2c \right\} \tag{3.6}$$

and define in $\Omega$ the operator $\mathcal{T}$ as follows

$$\mathcal{T}(u)(t) = c + \int_0^t \Phi^* \left( \frac{1}{a(s)} \left( \int_s^\infty b_-(r) F(u(r)) dr - \int_s^\infty b_+(r) F(u(r)) dr \right) \right) ds. \tag{3.7}$$

In view of (3.3) and (3.5), we have

$$\frac{1}{a(s)} \int_s^\infty b_-(r) F(u(r)) dr \le \mu \frac{1}{a(s)} \int_s^\infty b_-(r) dr \le T_\mu < \sigma, \tag{3.8}$$

and

$$\frac{1}{a(s)} \int_s^\infty b_+(r) F(u(r)) dr \le T_\mu < \sigma. \tag{3.9}$$

So, the operator $\mathcal{T}$ is well defined. Hence

$$\begin{aligned} \mathcal{T}(u)(t) &\le c + \int_0^\infty \Phi^* \left( F(2c) \frac{1}{a(s)} \int_s^\infty b_-(r) dr \right) ds \\ \mathcal{T}(u)(t) &\ge c - \int_0^\infty \Phi^* \left( F(2c) \frac{1}{a(s)} \int_s^\infty b_+(r) dr \right) ds. \end{aligned} \tag{3.10}$$

In view of (3.8) and (3.9), from (3.4), we obtain

$$\begin{aligned} \Phi^* \left( F(2c) \frac{1}{a(s)} \int_s^\infty b_-(r) dr \right) &\le M \left( F(2c) \right)^{p-\varepsilon} \left( \frac{1}{a(s)} \int_s^\infty b_-(r) dr \right)^{p-\varepsilon} \\ \Phi^* \left( F(2c) \frac{1}{a(s)} \int_s^\infty b_+(r) dr \right) &\le M \left( F(2c) \right)^{p-\varepsilon} \left( \frac{1}{a(s)} \int_s^\infty b_+(r) dr \right)^{p-\varepsilon}. \end{aligned} \tag{3.11}$$

Thus, from (3.10), we get

$$c - M \left( F(2c) \right)^{p-\varepsilon} I_+ \le \mathcal{T}(u)(t) \le c + M \left( F(2c) \right)^{p-\varepsilon} I_-$$

Since $p - \varepsilon > q$, in view of (3.1) we have

$$\lim_{u \to 0^+} \frac{(F(u))^{p-\varepsilon}}{u} = \lim_{u \to 0^+} \left\{ \left[ \frac{(F(u)}{(u)^{1/q}} \right]^q F(u)^{p-q-\varepsilon} \right\} = 0,$$

and a sufficiently small $c > 0$ exists, such that

$$\frac{(F(2c))^{p-\varepsilon}}{2c} \le \min \left\{ \frac{1}{2MI_-}, \frac{1}{4MI_+} \right\}.$$

Then

$$\frac{c}{2} \le \mathcal{T}(u)(t) \le 2c,$$

i.e., $\mathcal{T}(\Omega) \subset \Omega$.

Let us show that $\mathcal{T}(\Omega)$ is relatively compact, i.e. $\mathcal{T}(\Omega)$ consists of functions equibounded and equicontinuous on every compact interval of $[0,\infty)$. Since $\mathcal{T}(\Omega) \subset \Omega$, the equiboundedness follows. Moreover, for any $u \in \Omega$ we have

$$-\Phi^*\left(F(2c)\frac{1}{a(t)}\int_t^\infty b_+(r)dr\right) \le \frac{d}{dt}\mathcal{T}(u)(t) \le \Phi^*\left(F(2c)\frac{1}{a(t)}\int_t^\infty b_-(r)dr\right)$$

which proves the equicontinuity of the elements of $\mathcal{T}(\Omega)$.

Now we prove the continuity of $\mathcal{T}$ in $\Omega$. Let $\{u_n\}$, $n \in \mathbb{N}$, be a sequence in $\Omega$ which uniformly converges on every compact interval of $[0,\infty)$ to $\bar{u} \in \Omega$. Since $\mathcal{T}(\Omega)$ is relatively compact, the sequence $\{\mathcal{T}(u_n)\}$ admits a subsequence $\{\mathcal{T}(u_{n_j})\}$ converging, in the topology of $C[0,\infty)$, to $\bar{z} \in \overline{\mathcal{T}(\Omega)}$. In addition, since $\Phi^*$ is odd, we have

$$\left|\Phi^*\left(\frac{1}{a(t)}\Big(\int_t^\infty b(\tau)F(u_n(\tau))d\tau\Big)\right)\right| \le \Phi^*\left(F(2c)\frac{1}{a(t)}\Big(\int_t^\infty |b(\tau)|d\tau\Big)\right)$$

and, in view of (3.2) and (3.11), the function

$$\Phi^*\left(F(2c)\frac{1}{a(t)}\Big(\int_t^\infty |b(\tau)|d\tau\Big)\right)$$

belongs to $L^1[0,\infty)$. Thus, from the Lebesgue dominated convergence theorem, the sequence $\{\mathcal{T}(u_{n_j})(t)\}$ pointwise converges to $\mathcal{T}(\bar{u})(t)$. In view of the uniqueness of the limit, $\mathcal{T}(\bar{u})$ is the only cluster point of the compact sequence $\{\mathcal{T}(u_n)\}$, and so $\mathcal{T}(\bar{u}) = \bar{z}$. The continuity of $\mathcal{T}$ in the topology of $C[0,\infty)$ is proved.

Applying the Tychonov fixed point theorem, we get the existence of $x \in \Omega$ such that $x(t) = \mathcal{T}(x)(t)$, i.e. $x$ is a solution of (1.1). Clearly,

$$x'(t) = \Phi^*\left(\frac{1}{a(t)}\left(\int_t^\infty b_-(r)F(x(r))dr - \int_t^\infty b_+(r)F(x(r))dr\right)\right)$$

and so

$$-\Phi^*\left(\frac{F(2c)}{a(t)}\int_t^\infty b_+(r)dr\right) \le x'(t) \le \Phi^*\left(\frac{F(2c)}{a(t)}\int_t^\infty b_-(r)dr\right).$$

Since $1/a(t)$ is bounded as $t \to \infty$, we have $\lim_{t\to\infty} x'(t) = 0$ and, from (3.2), $x' \in L^1[0,\infty)$. Thus $x$ is of bounded variation on $[0,\infty)$ and the limit

$$\lim_{t\to\infty} x(t)$$

is finite. Since $x$ belongs to the set $\Omega$, the assertion follows. □

**Remark 3.2.** Condition (3.1) is satisfied, for instance, if $F$ is regularly varying at $u = 0$ with index $\beta > 1/q$.

If $\Phi^*$ satisfies the assumptions on $g$ in Lemma 2.3, then the following holds.

**Theorem 3.3.** *Let $\Phi^*$ be regularly varying at $u = 0$ with index $p > 0$, such that the function*

$$L(u) = \Phi^*(u)/u^p$$

*satisfies any of the conditions* $i_1$), $i_2$) *in Lemma* 2.3. *Assume that*

$$\lim_{t\to 0^+} \frac{F(u)}{u^{1/p}} = 0, \tag{3.12}$$

*and that a constant* $k > 0$ *exists, such that*

$$\begin{aligned} J_+^k &= \int_0^\infty \Phi^*\left(\frac{k}{a(s)}\int_s^\infty b_+(r)\,dr\right) ds < \infty, \\ J_-^k &= \int_0^\infty \Phi^*\left(\frac{k}{a(s)}\int_s^\infty b_-(r)\,dr\right) ds < \infty. \end{aligned} \tag{3.13}$$

*Then the boundary value problem* (1.1), (1.2) *is solvable for any small positive c. Moreover, every solution is of bounded variation on* $[0,\infty)$.

*Proof.* The argument is similar to the one given in the proof of Theorem 3.1. From Lemma 2.3, a positive constant $M$ exists, such that

$$\Phi^*(\lambda u) \le M\lambda^p \Phi^*(u), \tag{3.14}$$

for $0 < u \le T$, $0 < \lambda \le 1$. Since $\Phi^*$ is monotone increasing, without loss of generality we can assume that $k$ satisfies

$$k \max\left\{\max_{t\ge 0}\left(\frac{1}{a(t)}\int_t^\infty b_+(s)ds\right), \max_{t\ge 0}\left(\frac{1}{a(t)}\int_t^\infty b_-(s)ds\right)\right\} \le T.$$

Let $c > 0$ sufficiently small, such that $F(2c) \le k$. Consider in $C[0,\infty)$ the subset $\Omega$ given by (3.6), and let $\mathcal{T}$ be the operator given by (3.7). In view of (3.14) the estimate

$$\begin{aligned} \Phi^*\left(F(2c)\frac{1}{a(s)}\int_s^\infty b_-(r)dr\right) &= \Phi^*\left(\frac{F(2c)}{k}\frac{k}{a(s)}\int_s^\infty b_-(r)dr\right) \\ &\le M\left(\frac{F(2c)}{k}\right)^p \Phi^*\left(\frac{k}{a(s)}\int_s^\infty b_-(r)dr\right) \end{aligned}$$

holds. Clearly, also the corresponding estimate involving $b_+$ instead of $b_-$, is valid. Thus, from (3.10), we obtain

$$c - M\left(\frac{F(2c)}{k}\right)^p J_+^k \le \mathcal{T}(u)(t) \le c + M\left(\frac{F(2c)}{k}\right)^p J_-^k.$$

From (3.12), a sufficiently small $c > 0$ exists, such that

$$\frac{(F(2c))^p}{2c} \le \min\left\{\frac{k^p}{2MJ_-^k}, \frac{k^p}{4MJ_+^k}\right\}.$$

Then $\frac{c}{2} \le \mathcal{T}(u)(t) \le 2c$, i.e., $\mathcal{T}(\Omega) \subset \Omega$. To conclude, it is sufficient to use the same argument to the one given in the proof of Theorem 3.1. □

**Remark 3.4.** Let $\Phi^*$ be regularly varying at $u = 0$ with index $p > 0$. Since the function

$$f(t) = \frac{1}{a(t)}\int_t^\infty b_+(s)ds$$

tends to zero as $t \to \infty$, we have $J_+^k < \infty$ for some $k > 0$ if and only if $J_+^m < \infty$ for any $m > 0$ such that $\max_{t\ge o} mf(t) < \sigma$. A similar statement holds for $J_-^k$.

A comparison between Theorem 3.1 and Theorem 3.3 is given by the following examples.

**Example 3.5.** Consider equation (1.1) with

$$\Phi^*(u) = u|\log u|,\ 0 < u < e^{-1}, \quad F(u) = u|u|,$$

and

$$b_+(t) \equiv 0, \quad \frac{1}{a(t)} \int_t^\infty b_-(t)\, dt = \frac{1}{(t+1)^2}.$$

Thus Theorem 3.1 can be applied, but not Theorem 3.3, since $\Phi^*$ does not satisfy either $i_1$) or $i_2$) of Lemma 2.3.

**Example 3.6.** Consider equation (1.1) with

$$\Phi^*(u) = \frac{u}{(\log u)^2},\ 0 < u < 1, \quad F(u) = u|u|,$$

and

$$b_+(t) \equiv 0, \quad \frac{1}{a(t)} \int_t^\infty b_-(t)\, dt = \frac{1}{t+2}.$$

Then Theorem 3.3 can be applied, since $\Phi^*$ satisfies $i_2$) of Lemma 2.3, and

$$\int_0^\infty \Phi^*(1/(t+2))\, dt < \infty.$$

Nevertheless, Theorem 3.1 cannot be applied because

$$\int_0^\infty 1/(t+2)^{1-\epsilon}\, dt = \infty, \qquad \text{for any } \varepsilon > 0.$$

**Remark 3.7.** In both Theorems 3.1 and 3.3, the assumptions on the behaviour of $F$ in a neighborhood of zero, i.e., assumptions (3.1) and (3.12), respectively, play a fundamental role in assuring the existence of a globally positive solution on $[0, \infty)$, as it appears from the proofs. Without these assumptions, the same argument of the proofs leads to the existence of an eventually positive solution of (1.1), with a positive limit as $t \to \infty$, and such that its derivative has zero limit as $t \to \infty$.

## 4. THE RAPID AND SLOW VARIATION CASE

If $\Phi^*$ is rapidly varying at $u = 0$, an analogous result to Theorem 3.1 holds.

**Theorem 4.1.** *Let $\Phi^*$ be rapidly varying at $u = 0$, and assume that there exists $q > 0$, such that* (3.1) *is verified, and*

$$\int_0^\infty \left( \frac{1}{a(s)} \int_s^\infty b_+(r) dr \right)^q ds < \infty, \quad \int_0^\infty \left( \frac{1}{a(s)} \int_s^\infty b_-(r) dr \right)^q ds < \infty. \tag{4.1}$$

*Then the boundary value problem* (1.1), (1.2) *is solvable for any small positive c. Moreover, every solution is of bounded variation on* $[0, \infty)$.

*Proof.* If $\Phi^*$ is rapidly varying at $u = 0$, then by Lemma 2.1 we have

$$\lim_{u \to 0^+} \frac{\Phi^*(u)}{u^q} = 0.$$

Therefore for $T > 0$ fixed, a positive constant $M$ exists, such that

$$\Phi^*(u) \le M u^q, \quad \text{for } u \in [0, T].$$

The argument is therefore analogous to the one given in the proof of Theorem 3.1. □

The case of slow variation of $\Phi^*$ at $u = 0$ is more delicate, since in this case, in general, is not possible to have a good upper bound for the function $\Phi^*$ in a neighborhood of zero. A general result for the solvability of (1.1), (1.2) is the following. Clearly, this result makes sense in case $\Phi^*$ is slowly varying at $u = 0$.

**Theorem 4.2.** *Assume $k > 0$ exists, such that* (3.13) *holds, and*

$$\max\left\{F(3J_+^k), F(2J_-^k)\right\} < k. \tag{4.2}$$

*Then the boundary value problem* (1.1), (1.2) *has solution for infinitely many $c > 0$. Moreover, every solution is of bounded variation on* $[0, \infty)$.

*Proof.* In virtue of (4.2), taking into account that $F$ is nondecreasing, there exist infinitely many positive constants $\eta$ such that

$$\eta \ge \max\{3J_+^k, 2J_-^k\}, \quad F(\eta) \le k. \tag{4.3}$$

Fixed $c$ such that $\eta = 2c$ satisfies (4.3), consider in $C[0, \infty)$ the set $\Omega$ given by

$$\Omega = \{u \in C[0, \infty) : 0 \le u(t) \le 2c\}\ .$$

Define in $\Omega$ the operator $\mathcal{T}$ given by (3.7). Hence (3.10) holds and, since $F(2c) \le k$, we obtain

$$c - J_+^k \le \mathcal{T}(u)(t) \le c + J_-^k.$$

Taking into account that $J_+^k \le 2c/3$, and $J_-^k \le c$, we have

$$\frac{1}{3} c \le \mathcal{T}(u)(t) \le 2c,$$

i.e., $\mathcal{T}(\Omega) \subset \Omega$ and $\mathcal{T}(u)$ is bounded from below away from zero for any $u \in \Omega$. Using a similar argument to the one given in the final part of the proof of Theorem 3.1, the assertion follows. □

## 5. APPLICATIONS

A prototype of $\Phi$ is the classical $\Phi$-Laplacian given by (1.3). In this case $\Phi_\alpha^*$ is regularly varying at $u = 0$ of index $1/\alpha$. So, for the generalized Emden-Fowler type equation ($\alpha > 0, \beta > 0$)

$$\big(a(t)\Phi_\alpha(x')\big)' = b(t)\Phi_\beta(x), \tag{5.1}$$

Theorem 3.3 reads as follows

**Corollary 5.1.** *If* $\alpha < \beta$ *and*

$$\int_0^\infty \left( \frac{1}{a(s)} \int_s^\infty b_-(r)dr \right)^{1/\alpha} ds < \infty, \tag{5.2}$$

$$\int_0^\infty \left( \frac{1}{a(s)} \int_s^\infty b_+(r)dr \right)^{1/\alpha} ds < \infty, \tag{5.3}$$

*then the boundary value problem* (5.1), (1.2) *is solvable for any small positive* $c$ *and the solutions are of bounded variation on* $[0, \infty)$.

**Remark 5.2.** When $b_+ \equiv 0$, the condition (5.2) is necessary and sufficient for the existence of eventually increasing solutions of (5.1) tending to a positive constant, the so-called *subdominant solutions,* see, e.g., [9]. The assumption $\alpha < \beta$ in Corollary 5.1 plays the role of ensuring the global positivity of these solutions. Similarly, when $b_- \equiv 0$, the condition (5.3) is necessary and sufficient for the existence of solutions of (5.1) tending to a positive constant, see, e.g., [7, Proposition 1, Proposition 2] with minor changes.

Other prototypes of $\Phi$ are the maps $\Phi_C$ and $\Phi_R$, given by (1.4) and (1.5), respectively. Consider the equations

$$\big(a(t)\Phi_C(x')\big)' = b(t)F(x), \tag{5.4}$$

$$\big(a(t)\Phi_R(x')\big)' = b(t)F(x). \tag{5.5}$$

We have $\Phi_C^* = \Phi_R$ and $\Phi_R^* = \Phi_C$, and both the functions $\Phi_C^*, \Phi_R^*$ are regularly varying at $u = 0$ of index 1. Since for $u \in [0, 1/2]$ we have

$$u \le \Phi_C^*(u) \le \frac{2}{\sqrt{3}}u, \quad \frac{2}{\sqrt{5}}u \le \Phi_R^*(u) \le u,$$

in view of condition $i_1$) in Lemma 2.3 and Theorem 3.3, we obtain the following result.

**Corollary 5.3.** *If*

$$\int_0^\infty \frac{1}{a(s)} \int_s^\infty b_-(r)drds < \infty, \quad \int_0^\infty \frac{1}{a(s)} \int_s^\infty b_+(r)drds < \infty,$$

*and*

$$\lim_{u \to 0^+} \frac{F(u)}{u} = 0,$$

*then both the boundary value problems* (5.4), (1.2), *and* (5.5), (1.2), *are solvable for any small positive* $c$. *Moreover, every solution is of bounded variation on* $[0, \infty)$.

**Remark 5.4.** The problem (5.4), (1.2), together with its uniqueness, has been investigated in [6, Theorem 3.2] in case $b_+(t) \equiv 0$ and $\liminf_{t\to\infty} a(t) = 0$.

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