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THE SUPERCOOLED STEFAN PROBLEM IN RADIAL SYMMETRY

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ABSTRACT: We prove existence of solutions to a supercooled Stefan problem in radial symmetry. Our analysis includes the critical case in which at the starting point for the free boundary, the initial temperature approaches the threshold value under which no solution can exist. We also give some new a priori estimates of the speed of the free boundary.

1. INTRODUCTION

Let $T > 0$, $b > r_0 > 0$ be positive given constants and let $f : [0,T] \to \mathbf{R}$, $h : [r_0,b] \to \mathbb{R}$ be piecewise continuous given functions. Consider the following

Problem (P) Find a triple (z,s,T_0) with $0 < T_0 \le T$ and

(i) $s \in C[0,T_0] \cap C^1(0,T_0)$, $s(0) = b$, $s(t) > r_0$, $0 \le t < T_0$;

(ii) $z \in C^{2,1}(D_{T_0}) \cap C^*(\bar{D}_{T_0})$ [1] where

$$D_{T_0} = \{(r,t) : r_0 < r \le s(t),\ 0 < t < T_0\} ;$$

moreover z_r is required to be continuous up to $r = s(t)$, $t > 0$;

(iii)
$$z_t - z_{rr} - \frac{1}{r} z_r = 0 , \quad \text{in } D_{T_0} , \tag{1}$$
$$z(r_0,t) = f(t) , \quad 0 < t < T_0 , \tag{2}$$
$$z(r,0) = h(r) , \quad r_0 < r < b , \tag{3}$$
$$z(s(t),t) = 0 , \quad 0 < t < T_0 , \tag{4}$$

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[1] $C^*(\bar{D}_{T_0})$ denotes the set of the functions which are continuous and bounded in D_{T_0} and continuous up to the boundary at points of continuity of the chain of the data.

$$z_r(s(t),t) = -\dot{s}(t), \quad 0 < t < T_0. \tag{5}$$

Remark 1.1 Equation (1) is the heat equation $u_t - \Delta u = 0$ in cylindrical symmetry. Writing the equation in the form

$$z_t - z_{rr} - \frac{n-1}{r} z_r = 0, \qquad n = 1, 2, 3,$$

encompasses the three cases of planar, cylindrical and spherical symmetry respectively. In what follows we will consider $n = 2$ with no loss of generality.

Remark 1.2 The problem with Neumann data

$$z_r(r_0,t) = g(t), \quad 0 < t < T_0, \tag{2}'$$

can be treated by the methods developed here for problem (P). In the following we restrict ourselves to the case of Dirichlet data, the extension to the case (2)′ being straightforward.

When $f, h \geq 0$, (P) is a classical one-phase Stefan problem. On the other hand, when no sign restrictions are imposed on the data, we can not attach to (P) the meaning of a model of change of phase. Here we note that problem (P) has been considered in [3] in connection with the motion of the liquid in a Hele-Shaw cell [4), 5)]. The cell is formed by two parallel plates and a liquid can be injected or extracted from the space between them through a horifice which we assume to be a circle of radius $r_0 > 0$. Here z is the pressure in the liquid. Then the zone $z \equiv 0$ (the zone to the right of the free boundary $r = s(t)$) is the "dry" zone, while when $z \geq 0$ the liquid is being injected, and when $z \leq 0$ it is being extracted.

Clearly, under appropriate assumptions on the state of the system at the initial time, the problem is radially symmetric, accounting for the form taken by the equation of conservation of mass (for the liquid) in (1).

In [1] the qualitative properties of solutions to (P) and to similar schemes have been considered.

Here we consider the problem of existence of (local in time) solutions

to (P), when $h(b) \neq 0$, and especially in the case $h(b) < 0$. **This subject is not trivial; it is known that, in the case of planar symmetry a solution exists,** roughly speaking, if and only if $h(r) > -1$ in some left neighbourhood of $r = b$ [7]. As far as we know, solutions to problem (P) are known to exist if $h(b) = 0$, and h is Hölder continuous near $r = b$ [2]. Moreover, in [1] it is proved that if $h(r) \leq -1$ in $(b-\epsilon, b)$ for some $\epsilon > 0$, then no solution to (P) may exist.

Our main purpose here is to prove the following more general existence result.

Theorem 1.3 *Let*

$$h(r) > -1, \quad b - \epsilon < r < b, \tag{6}$$

for some $\epsilon \in (0, b - r_0]$. *Then a solution to* (P) *exists.*

Remark 1.4 We remark explicitly that $h(b) = -1$ is allowed in Theorem 1.3 as long as (6) is fullified. This case is of some interest in the framework of the general theory of free boundary problems; this fact also illustrates the interest of Theorem 1.3.

Remark 1.5 Uniqueness of solutions to (P), under assumption (6) and for given T_0, can be proved as in [7]; we omit the details.

The proof of Theorem 1.3, given in Section 3 below, relies on two main ingredients. The first one is a monotone approximation technique following the ideas of [7]. The second one is a new suitable a priori estimate of $|\dot{s}|$, independent of the boundary data, that is proved in Section 2 in a more general setting.

2. A PRIORI ESTIMATES

In this section we establish some estimates of $|\dot{s}(t)|$ not depending on the behaviour of either the initial or the boundary data. Under this respect our estimates differ from those given in [6), 1)].

Let $s \in C^1[0,T]$, $T>0$, and let $d>0$ be a given constant such that $s(t) - r_0 \geq d$ for all $0 \leq t \leq T$. Define

$$E = \{(r,t) \mid s(t) - d < r < s(t),\ 0<t<T\}. \tag{7}$$

Consider a function $z \in C^{2,1}(E) \cap C^{1,0}(\bar{E})$, satisfying

$$Lz := z_t - \alpha(r,t) z_{rr} + \beta(r,t) z_r + \gamma(r,t) z = 0 \text{ in } E, \tag{8}$$

$$z(s(t),t) = 0, \quad z_r(s(t),t) = -\dot{s}(t), \qquad 0<t<T, \tag{9}$$

where

$$\alpha, \beta, \gamma \in C^0(\bar{E}), \tag{10}$$

$$\alpha(r,t) \geq \alpha_0 > 0, \quad |\beta(r,t)| \leq \beta_0, \quad \gamma(r,t) \geq 0, \tag{11}$$

with $\alpha_0, \beta_0 > 0$ given constants.

In the following we denote $x_+ = \max(x,0)$, $x_- = \max(-x,0)$, $x \in \mathbf{R}$.

We prove the following

Theorem 2.1 *Let* z, s *be as above, and assume*

$$z(r,t) \geq -z_0, \qquad (r,t) \in \bar{E}, \tag{12}$$

for some $0 < z_0 < \alpha_0$. *Then there exists a positive constant* K, *depending only on* z_0, α_0, β_0 *such that*

$$-\dot{s}(t) \leq K\Big(1 + \frac{1}{d} + \frac{d}{t}\Big), \qquad t \in (0,T). \tag{13}$$

Proof We consider the function

$$w(r,t) = -\frac{z_0}{1-e^{-da}}\,(1 - \exp A(r,t)), \qquad (r,t) \in \bar{E}_0,$$

where

$$A(r,t) = a\,(r - s(t) - \epsilon(\bar{t} - t)_+^2), \quad E_0 = E \cap \{t > t_0\}.$$

Here $\bar{t} \in (0,T)$ is given, and $a, \epsilon > 0$, $t_0 \in (0,\bar{t})$ are to be chosen later.

Standard calculations show that

$$Lw = -\frac{z_0\, a}{1-e^{-da}} \exp A \Big\{ \dot{s}(t) - 2\epsilon\,(\bar{t}-t)_+ + \alpha a - \beta - \gamma\, \frac{1-\exp(-A)}{a} \Big\}$$

for $(r,t) \in E_0$. We define

$$\sigma_0 := \max_{t_0 \leq t \leq T} [\dot{s}(t)]_-, \qquad a := (\sigma_0 + 2\epsilon(\bar{t} - t_0) + \beta_0)\,\alpha_0^{-1}. \tag{14}$$

It is easily checked that, with this choice of a we have

$$Lw(r,t) \leq 0, \qquad (r,t) \in E_0. \tag{15}$$

Moreover for $t_0 \leq t \leq T$

$$w(s(t),t) = -\frac{z_0}{1-e^{-da}}\Big(1 - e^{-a\epsilon(\bar{t}-t)_+^2}\Big) \leq z(s(t),t) = 0, \tag{16}$$

$$w(s(t)-d,t) = -\frac{z_0}{1-e^{-ad}}\Big(1 - e^{-ad}\, e^{-a\epsilon(\bar{t}-t)_+^2}\Big) \leq -z_0. \tag{17}$$

Also, for $s(t_0) - d \leq r \leq s(t_0)$, we have

$$w(r,t_0) \leq -\frac{z_0}{1-e^{-ad}}\Big(1 - e^{-a\epsilon\,(\bar{t}-t_0)^2}\Big) \leq -z_0, \tag{18}$$

provided we select $\epsilon = d(\bar{t} - t_0)^{-2} > 0$, so that, owing to (14),

$$a = (\sigma_0 + 2d(\bar{t} - t_0)^{-1} + \beta_0)\,\alpha_0^{-1}. \tag{19}$$

Thus we invoke the maximum principle to infer from (15)-(18) that $w \leq z$ in $E \cap \{t \geq t_0\}$, if $a, \epsilon > 0$ are chosen as above. Using this lower estimate for z, and the equality

$$w(s(t),t) = 0 = z(s(t),t), \qquad \bar{t} \leq t \leq T,$$

we get

$$[\dot{s}(t)]_- = [z_r(s(t),t)]_+ \leq w_r(s(t),t) = \frac{a z_0}{1-e^{-ad}}, \quad \bar{t} \leq t \leq T. \tag{20}$$

Next, for $\theta \in (0,1)$ to be chosen, we set $t_0 = \bar{t}\theta$, and we define for all $0<t<T$

$$\sigma_t^+ = \max\{[\dot{s}(\tau)]_- \mid t \leq \tau \leq T\},$$

$$\sigma_t = \max\{[\dot{s}(\tau)]_- \mid t\theta \leq \tau \leq t\}.$$

Let us assume for the moment

$$\sigma_t^+ \geq \theta\,\sigma_t\,; \tag{21}$$

then $\sigma_0 \le \sigma_t^+ \theta^{-1}$, and (20) implies, setting $\xi = \sigma_t^+$,

$$\xi \le \frac{a\, z_0}{(1 - e^{[illegible]})} \le \frac{z_0}{\alpha_0} \, \frac{\delta + \theta^{-1}\xi}{1 - \exp\{-\alpha_0^{-1} d(\delta+\xi)\}} . \tag{22}$$

where we denote

$$[illegible] = 2d\,(\bar{t} - \theta t)^{-1} + \beta_0 > 0 .$$

We restate (22) as

$$[illegible]\Big(1 - e^{-\frac{d}{\alpha_0}(\xi+\delta)} - \frac{z_0}{\alpha_0 \theta}\Big) \le \frac{\delta\, z_0}{\alpha_0} . \tag{23}$$

We fix now $z_0 \alpha_0^{-1} < \theta < 1$ (this is possible by virtue of (12)), and distinguish between the two cases

i) $1 - [illegible]\alpha_0\theta)^{-1} \le [illegible]\, e^{-\frac{d}{\alpha_0}(\xi+\delta)}$,

ii) $1 - [illegible]\alpha_0\theta)^{-1} [illegible]\, 2 e^{-\frac{d}{\alpha_0}(\xi+\delta)}$.

If the former holds, [illegible] directly implies

$$[illegible] \le -\frac{\alpha_0}{d} \ln \tfrac{1}{2}\Big(1 - \frac{z_0}{\alpha_0 \theta}\Big) - \delta . \tag{24}$$

If the inequality (ii) is satisfied, we combine it with (23) to get

$$[illegible] \le 2\delta z_0 [illegible] \Big(1 - \frac{z_0}{\alpha_0 \theta}\Big)^{-1} . \tag{25}$$

Collecting (24) and (25), we find

$$[illegible] = \xi \le K\Big(1 + \frac{1}{d} + \frac{d}{t}\Big) \tag{26}$$

where K is a positive constant depending on z_0, α_0, β_0 and θ chosen as above, but not on [illegible].

Let us recall that estimate (26) has been proved under assumption (21).

If (21) is not fulfilled, i.e. if

$$[illegible] < \theta\, \sigma_{[illegible]} \tag{27}$$

let us define

$$a_n^+ = \sigma^+_{\bar{t}\theta^n} , \qquad a_n^- = \sigma^-_{\bar{t}\theta^n} ,$$

Since $\dot{s}$ is continuous over $[0,T]$, we have

$$\lim_n a_n^+ = \max_{0 \le t \le T} [\dot{s}(t)]_- \ge [\dot{s}(0)]_- = \lim_n a_n^- .$$

Thus there exists a unique positive integer $m \ge 1$ such that $a_m^+ \ge \theta\, a_m^-$, and $a_n^+ < \theta\, a_n$ for all $0 \le n < m$. Then

$$\sigma^+_{\bar{t}\theta^m} \ge \theta\, \sigma^-_{\bar{t}\theta^m} ,$$

and the proof above guarantees that

$$\sigma^+_{\bar{t}\theta^m} \le K\Big(1 + \frac{1}{d} + \frac{d}{t\,\theta^m}\Big) \tag{28}$$

Finally from (21), (28) and the choice of m,

$$\sigma_t^+ < \theta\, \sigma_t^- \le \theta\, \sigma^+_{t\theta} < \theta^2 \sigma_{t\theta} \le \cdots \le \theta^m \sigma^+_{t\theta^m} \le K\Big(1 + \frac{1}{d} + \frac{d}{t}\Big)$$

The proof of (13) is completed.

□

Our next result provides us with an estimate of $\dot{s}_+$.

<u>Theorem 2.2</u> *Let [illegible] fulfil (7)-(11) and assume*

$$[illegible](r,t) \le M , \qquad (r,t) \in E , \tag{29}$$

where $M > 0$ *is given. Then there exists a positive constant* $K > 0$ *such that*

$$[illegible](t) \le K\Big(1 + \frac{1}{d} + \frac{d}{t} + \sigma\Big) , \qquad 0 < t < T , \tag{30}$$

where

$$[illegible] := \max_{0 \le t \le T} |\dot{s}(t)| .$$

The constant K *depends on* M, α_0, β_0, *but not on* t *or* s, *the dependence on* M *being linear.*

<u>Proof</u> We introduce the function, for $t \in (0,T)$ given,

$$w(r,t) = \frac{M}{1-e^{-ad}}\,(1-\exp A(r,t)), \qquad (r,t) \in \bar{E}_0,$$

$$A(r,t) = a\,(r - s(t) - \epsilon\,(\bar{t}-t)_+^2),$$

where $E_0 = E \cap \{t > \frac{\bar{t}}{2}\}$, and a, ϵ are given by

$$\epsilon = 4d\,\bar{t}^{-2}, \qquad a = \frac{1}{d} + (\sigma + 4d\bar{t}^{-1} + \beta_0)\,\alpha_0^{-1}.$$

Reasoning as above, we see that

$$Lw \geq 0, \qquad (r,t) \in E_0, \tag{31}$$

$$w(s(t),t) \geq 0 = z(s(t),t), \qquad \tfrac{\bar{t}}{2} < t < T, \tag{32}$$

$$w(s(t)-d,t) \geq M \geq z(s(t)-d,t), \qquad \tfrac{\bar{t}}{2} < t < T, \tag{33}$$

$$w(r,\tfrac{\bar{t}}{2}) \geq M \geq z(r,\tfrac{\bar{t}}{2}), \qquad s(\tfrac{\bar{t}}{2}) - d \leq r \leq s(\tfrac{\bar{t}}{2}). \tag{34}$$

Therefore $w \geq z$ in $\bar{E}_0$. Again we note that

$$w(s(\bar{t}),\bar{t}) = 0 = z(s(\bar{t}),\bar{t}), \qquad \bar{t} \leq t < T.$$

Hence

$$\dot{s}(t) = -z_r(s(t),t) \leq -w_r(s(t),t) = \frac{M\,a}{1-e^{-ad}}, \qquad \bar{t} \leq t < T.$$

Thus, owing to the definition of a,

$$\dot{s}(\bar{t}) \leq M\,a\,(1-e^{-1})^{-1} \leq K\left(1 + \frac{1}{d} + \frac{d}{\bar{t}} + \sigma\right),$$

and the proof is completed.

□

We conclude this section stating the estimates for $|\dot{s}|$ in a form more precise and more suitable to our purposes.

<u>Theorem</u> <u>2.3</u> *Let z satisfy the assumptions of Theorem 2.1: then*

$$-\dot{s}(t) \leq K_1\left(1 + \frac{1}{\sqrt{t}}\right). \tag{35}$$

with $K_1 = K_1(M,\alpha_0,\beta_0,d)$.

If moreover $z \leq M$ *in* E, *then*

$$|\dot{s}(t)| \leq K_2\left(1 + \frac{1}{\sqrt{t}}\right), \tag{36}$$

with $K_2 = K_2(M,z_0,\alpha_0,\beta_0,d)$.

<u>Proof</u> It is obvious that, for $0 < t \leq d^2$, we may take $d = \sqrt{t}$ in (13), so proving (35) in this case. Indeed it suffices to note that for $\bar{t} \leq d^2$,

$$E' = \{(r,t) \mid s(t) - \sqrt{\bar{t}} < r < s(t),\ 0 < t < T\} \subset E.$$

When $T > t > d$, (13) trivially implies (35).

If $z \leq M$ in E, (36) follows from combining (35) with Theorem 2.2 and the reasoning above.

□

<u>Remark</u> <u>2.4</u> We note that estimate (36) is sharp in the sense that explicit solutions to Stefan-like problems are known with $\dot{s}(t) = c\,t^{-1/2}$, $t > 0$, with either $c > 0$ or $c < 0$.

Moreover, assumption (12) is necessary to get a bound below for $\dot{s}$ [6]. We also remark that, even in planar symmetry, the case $z_0 = \alpha_0$ is known to produce an asymptotic behaviour (as $t \downarrow 0$) which is worse than a parabola.

3. PROOF OF THE EXISTENCE THEOREM

We look first at the case

$$\liminf_{r \to b-} h(r) > -1; \tag{37}$$

the case when equality holds in (37) will be considered later. Due to (37) we may assume

$$h(r) \geq -k > -1, \quad r_1 < r < b, \tag{38}$$

for a suitable $0 < k < 1$, $r_0 < r_1 < b$. Next we fix a constant k' such that $k < k' < 1$, and we consider the solution Z to

$$Z_t - (Z_{rr} + \tfrac{1}{r} Z_r) = 0, \quad r_0 < r < +\infty, \quad 0 < t < T, \tag{39}$$

$$Z(r,0) = \begin{cases} \min(-k, h(r)), & r_0 < r < b, \\ 0, & r \geq b, \end{cases} \tag{40}$$

$$Z(r_0,t) = -[f(t)]_- , \qquad 0<t<T . \tag{41}$$

Clearly $Z \le 0$, **and, redefining** r_1 **if necessary, we may find** $0<t_1<T$ **such that**

$$Z(r,t) \ge -k' > -1 , \quad r_1 < r < +\infty , \quad 0<t<t_1 . \tag{42}$$

Next we **introduce** a suitable sequence of approximating problem. Let (z_n,s_n,T_n) be, for $n \ge 1$, the solution to

$$z_{nt} - (z_{nrr} + \tfrac{1}{r} z_{nr}) = 0 , \qquad r_0 < r < s_n(t), \; 0<t<T_n , \tag{43}$$

$$z_n(r_0,t) = f(t) , \qquad 0<t<T_n , \tag{44}$$

$$z_n(r,0) = h_n(r) , \qquad r_0 < r < b_n , \tag{45}$$

$$z_n(s_n(t),t) = 0 , \qquad 0<t<T_n , \tag{46}$$

$$z_{nr}(s_n(t),t) = -\dot{s}_n(t) , \qquad 0<t<T_n , \tag{47}$$

$$s_n(t)(0) = b_n ; \tag{48}$$

here special care has to be taken in the choice of h_n, b_n. Namely, we choose two sequences $\{a_n\}$, $\{b_n\}$ such that $a_n, b_n \to b^-$, $n \to \infty$ and

$$r_1 < a_n < a_{n+1} < b_n < b_{n+1} , \quad (b_{n+1}^2 - b_n^2) \ge k\,(a_{n+1}^2 - a_n^2) \quad \forall\, n \ge 0 \tag{49}$$

(such sequences can be constructed easily). Then we define

$$h_n(r) = \begin{cases} 0 , & a_n \le r \le b_n , \\ -k , & a_{n-1} \le r \le a_n , \\ h(r) , & r_0 \le r \le a_{n-1} , \end{cases} \tag{50}$$

for all $n \ge 1$; (41)-(50) are used in the proof of Lemma 3.1 below.

Since $h_n(b_n) = 0$, **and** h_n **is smooth near** $r = b_n$, **existence and uniqueness of a solution to (43)-(48) follow from the techniques of** [2]. As a consequence of **the maximum principle we have**

$$z_n(r,t) \ge Z(r,t) , \quad r_0 \le r \le s(t) , \; 0 \le t \le T_n ;$$

thus, if we set

$$T'_n = \sup\{t \in (0,T_n) \mid t \le t_1 , \; s_n(\tau) > r_1 , \; 0 < \tau < t\} \le t_1 ,$$

it follows

$$z_n(r,t) \ge -k' > -1 , \quad r_1 \le r \le s_n(t) , \quad 0<t<T'_n . \tag{51}$$

We give below a monotonicity result **which will be instrumental in proving** the convergence of the approximating sequence $\{s_n\}$.

Lemma 3.1 *For all* $n \ge 1$, *we have* $s_{n+1}(t) > s_n(t)$, $0 \le t \le \theta_n$, *where* $\theta_n = \min(T'_n, T'_{n+1})$.

Proof Define for $j = n, n+1$, $r_0 \le r \le s_j(t)$, $0 \le t \le \theta_n$,

$$u_j(r,t) = \int_r^{s_j(t)} \xi^{-1}\, d\xi \int_\xi^{s_j(t)} y(z_j(y,t)+1)\, dy .$$

Then

$$u_{jt} - (u_{jrr} + \tfrac{1}{r} u_{jr}) = -1 , \qquad r \in (r_0, s_j(t)) , \; t \in (0,\theta_n) ,$$

$$u_j(s_j(t),t) = 0 , \quad u_{jr}(s_j(t),t) = 0 , \quad t \in (0,\theta_n) ,$$

$$u_j(r_0,t) = u_j(r_0,0) + \int_0^t f(\tau)\, d\tau , \qquad t \in (0,\theta_n) ,$$

$$u_j(r,t) \ge 0 , \qquad r \in (r_1, s_j(t)) , \; t \in (0,\theta_n) .$$

Define $w = u_{n+1} - u_n$. Thus, setting $\tilde{s}(t) = \min(s_n(t), s_{n+1}(t))$,

$$w_t - (w_{rr} + \tfrac{1}{r} w_r) = 0 , \qquad r_0 < r < \tilde{s}(t), \; 0<t<\theta_n ,$$

$$w(r_0,t) = u_{n+1}(r_0,0) - u_n(r_0,0) \ge 0 , \qquad 0<t<\theta_n ,$$

$$w(r,0) = u_{n+1}(r,0) - u_n(r,0) \ge 0 , \qquad r_0 \le r \le b_n ,$$

$$w(s_n(t),t) = u_{n+1}(s_n(t),t) \ge 0 , \qquad 0<t<t^* ,$$

where

$$t^* = \sup\{t \in (0,\theta_n) \mid s_{n+1}(\tau) > s_n(\tau) , \; 0<\tau<t\} ;$$

indeed $u_{n+1}(r,0) \ge u_n(r,0)$ follows from a direct calculation employing (49)-(50) (see the Appendix).

If $s_n(t^*) = s_{n+1}(t^*)$, we get, by the boundary point principle,

$$w_r(s_n(t^*),t^*) < 0 ,$$

contradicting

$$w_r(s_n(t^*),t^*) = u_{n+1\,r}(s_{n+1}(t^*),t^*) - u_{n\,r}(s_n(t^*),t^*) = 0 .$$

Therefore $t^* = \theta_n$, and $s_{n+1}(t) > s_n(t)$, $0 \le t \le \theta_n$.

□

Next we prove a bound below for T'_n, uniform on n.

Lemma 3.2 *For all* $n \ge 1$, $T'_n \ge T'_1$.

Proof Owing to Lemma 3.1, and to the results of [1], to show $T'_{n+1} \ge T'_n$ we only have to rule out

$$\liminf_{t \to T'_{n+1}} \dot{s}_{n+1}(t) = -\infty . \tag{52}$$

But, estimate (51) and Theorem 2.3 guarantee that (52) can not hold. Hence $T'_{n+1} \ge T'_n$ $\forall\, n \ge 1$ and the claim follows.

□

Define $T_0 = T'_1/2$.

Lemma 3.3 *For* $n \to \infty$, $s_n(t) \uparrow s(t)$, $t \in [0,T_0]$, $s \in C^0([0,T_0])$, $s(0) = b$, $s \in Lip_{loc}((0,T_0])$.

Proof Theorem 2.3, together with (51) imply $\forall\, n \ge 1$

$$|\dot{s}_n(t)| \le c_0 + c_1\, t^{-1/2}, \quad 0 < t \le T_0 . \tag{53}$$

where c_0, $c_1 > 0$ do not depend on n. Hence the limit s of $\{s_n\}$ is locally Lipschitz continuous in $(0,T_0]$.

Moreover, (53) implies that the s_n are Hölder continuous over $[0,T_0]$, uniformly with respect to n. Thus the limit s is continuous even at $t = 0$, and the claim follows.

□

Finally, we remark that existence of a solution to (P) follows from standard arguments by virtue of the regularity of the free boundary s.

To complete the proof of Theorem 1.3, we have to remove assumption (37), i.e., we let

$$\liminf_{r \to b-} h(r) = -1 , \quad h(r) > -1 , \quad b - \epsilon < r < b ,$$

(we may assume that $(b-\epsilon, b)$ is an interval of continuity of $h(r)$).

We start considering the family of approximating problems given by (43)-(44), (46)-(48), coupled with

$$z_n(r,0) = h(r) , \quad r_0 < r < b_n ; \tag{45'}$$

existence of a solution (z_n, s_n, T_n) follows from the first part of the proof.

Consider also the solution Z to (39)-(41), where we now let $k = 0$.

Though (42) can not hold any more, we have

$$z^*(t_0) = \inf\{Z(r,t) \mid r_2 \le r < +\infty,\ t_0 < t < t_2\} > -1 ,$$

for all $0 < t_0 < t_2$, where $r_2 \in (r_0, b)$, $t_2 \in (0,T)$ are suitably chosen. Then for all $0 < t_0 < t_2$,

$$z_n(r,t) \ge z^*(t_0) > -1 , \quad r_2 \le r \le s_n(t),\ t_0 \le t \le T'_n , \tag{54}$$

where

$$T'_n = \sup\{t \in (0,T_n) \mid t \le t_2 ,\ s_n(\tau) > r_2 ,\ 0 < \tau < t\} .$$

Lemma 3.1 still holds; the proof, which is similar to the one given above, relies on the inequality

$$u_{n+1}(r,0) = \int_r^{b_{n+1}} \rho^{-1}\, d\rho \int_\rho^{b_{n+1}} y\,(h(y)+1)\, dy \ge$$
$$\ge \int_r^{b_n} \rho^{-1}\, d\rho \int_\rho^{b_n} y\,(h(y)+1)\, dy = u_n(r,0) , \tag{55}$$

$r_0 \le r \le b_n$ (using the notation of Lemma 3.1). Indeed $h(r) + 1 \ge 0$ when $b - \epsilon < b_n < r < b_{n+1}$.

The proof of Lemma 3.2 can be reproduced without changes, with the help of (54); thus $T'_{n+1} \geq T'_n$.

A different approach is needed in the proof of Lemma 3.3. In fact, estimate (53) is now replaced by

$$|\dot{s}_n(t)| \leq c(t_0), \qquad t_0 < t \leq T_0 = T'_1/2, \tag{56}$$

which is a consequence of (54) and Theorem 2.3. Note that $c(t_0)$ becomes unbounded as $t_0 \to 0$.

The bound (56) yields $s \in Lip_{loc}((0,T_0])$, but we have to use an additional argument to show that s is continuous at $t = 0$.

Let

$$\omega = \liminf_{t \to 0+} s(t).$$

If $\omega < b$, there exists a sequence $t_m \downarrow 0$, $m \geq 1$, such that

$$s_n(t_m) \leq s(t_m) < (b+\omega)/2 < b, \qquad \forall\, n, m \geq 1.$$

On letting $m \to \infty$, we have $s_n(0) \leq (b+\omega)/2$, a contradiction to $s_n(0) = b_n \to b$ as $n \to \infty$. Therefore, to prove

$$\liminf_{t \to 0+} s(t) = \limsup_{t \to 0+} s(t) = b,$$

it will be enough to prove $s_n(t) \leq S(t)$ for all $n \geq 1$, $t \in [0,T_0]$, where $S \in C^0[0,T_0]$, $S(0) = b$.

We choose S as the free boundary of a Stefan problem posed for the heat equation in plane symmetry, with constant, positive (and sufficiently large) boundary data.

If the data are suitably chosen, inequality $S(t) \geq s_n(t)$, $0 < t < T_0$, follows from a standard application of the maximum principle.

The proof of Theorem 1.3 is completed.

□

Remark 3.4 A few comments on the second part of the proof are perhaps in order. Clearly (55) (and even (54), under additional assumptions on the data) still holds if $h(r) \equiv -1$ in a left neighbourhood of $r = b$. Nevertheless, we know [1] that in such a case, no solution to problem (P) exists. Indeed, the proof given above fails, since in it we exploit the strict inequality $h(b_n) > -1$ to apply the existence result proved earlier. On the other hand, if we approximate h near $r = b_n$ in order to guarantee local existence, the ordering $u_{n+1}(r,0) \geq u_n(r,0)$ is no longer granted. Then we cannot infer the monotonicity of $\{s_n\}$, which was essential in proving the continuity of the limit function s at $t = 0$. In this connection, it is perhaps of some interest mentioning that two sequences $\{a_n\}$, $\{b_n\}$ obeying the requirements in (49) *can not* be found if $k = 1$.

Remark 3.5 In order to clarify the remarks above, let us consider the case where $f \equiv 0$, $0 > h(r) > -1$ in $(r_0, \bar{b})$, $h(r) = -1$ in $(\bar{b}, b)$, with $r_0 < \bar{b} < b$. Let $\{h_n\}$ be any approximating sequence satisfying $h_n(r) \to h(r)$ in (r_0, b) $n \to \infty$, $h \leq h_n \leq 0$, $h_n(b) > -1$, h_n smooth near $r = b$. Let (z_n, s_n, T_n) be the solution to the corresponding approximating problem, and let $(\bar{z}, \bar{s}, \bar{T})$ be the solution to the problem obtained from (P) by substituting b with $\bar{b}$ (i.e., $\bar{s}(0) = \bar{b}$); we denote this problem $(\bar{P})$.

Then, reasoning as in Lemma 3.1 (see also (55) above), we get

$$s_n(t) > \bar{s}(t), \qquad 0 < t < \bar{T}_n := \min(T_n, \bar{T}).$$

Owing to Theorem 2.3 and to suitable estimates below for z_n, proved as above, we find

$$|\dot{s}_n(t)| \leq C(t_0), \qquad 0 < t_0 \leq t \leq \bar{T}_n - t_0, \quad n \geq 1. \tag{57}$$

As a first consequence of (57) (and of the results of [1]), note that s_n and $\bar{s}$ can be continuated until they hit the boundary $r = r_0$ (so that $T_n \geq \bar{T}$); also note that $\dot{s}_n, \dot{\bar{s}} < 0$, $t > 0$.

Moreover, by virtue of (57), there exists a subsequence $\{s_{n_k}\}$ converging to s^*, $s^* \in Lip_{loc}((0,\bar{T}))$, s^* decreasing and

$$\bar{b} \leq \lim_{t \to 0+} s^*(t) =: \sigma^* \leq b.$$

The inequality $\sigma^* > \bar{b}$ would lead us to an inconsistency, by virtue of the non existence result mentioned in Remark 3.4. Hence $\sigma^* = \bar{b}$. Then s^* is the free boundary of a solution to $(\bar{P})$. By uniqueness, it follows $s^* \equiv \bar{s}$

in $(0,\bar{T})$. Thus, the whole sequence $\{s_n\}$ converges to $\bar{s}$ in $(0,\bar{T})$, though $s_n(0) = b \ \ \forall n \geq 1$.

4. APPENDIX

We outline below the proof of the ordering

$$u_{n+1}(r,0) = \int_r^{b_{n+1}} \rho^{-1}\, d\rho \int_\rho^{b_{n+1}} y\,(h_{n+1}(y)+1)\, dy \geq$$

$$\geq \int_r^{b_n} \rho^{-1}\, d\rho \int_\rho^{b_n} y\,(h_n(y)+1)\, dy = u_n(r,0)\,, \tag{58}$$

$r_0 \leq r \leq b_n$, that is essential in proving Lemma 3.1.

Actually, the calculation are cumbersome, and we do not reproduce them here entirely. Let us note again that (49)-(50) play a basic role in proving (58).

By using $h_{n+1}(r) \geq h_n(r)$ if $r_0 \leq r \leq a_n$, one can show that

$$u_{n+1}(r,0) - u_n(r,0) \geq I_1(r) + I_2(r)\,, \quad r_0 \leq r \leq b_n\,, \tag{59}$$

where

$$I_1(r) = \int_r^{b_n} \xi^{-1}\, d\xi \left\{ \int_{a_n}^{a_{n+1}} y\, h_{n+1}(y)\, dy + \int_{b_n}^{b_{n+1}} y\, dy \right\}$$

$$I_2(r) = \int_{b_n}^{b_{n+1}} y \ln \frac{y}{b_n}\, dy - \int_{a_n}^{a_{n+1}} y\, h_{n+1}(y) \ln \frac{b_n}{y}\, dy\,.$$

Since $h_{n+1}(y) = -k < 0$ if $y \in (a_n, a_{n+1})$, we have $I_2(r) \geq 0$ in (59). Moreover, $I_1(r) \geq 0$ if (49)-(50) hold; indeed

$$\frac{1}{2}(b_{n+1}^2 - b_n^2) = \int_{b_n}^{b_{n+1}} y\, dy \geq -\int_{a_n}^{a_{n+1}} y\, h_{n+1}(y)\, dy = \frac{k}{2}(a_{n+1}^2 - a_n^2)\,.$$

Therefore (58) is proved.

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