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SYMMETRIES IN AN OVERDETERMINED PROBLEM FOR THE GREEN'S FUNCTION

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Abstract. We consider in the plane the problem of reconstructing a domain from the normal derivative of its Green's function with pole at a fixed point in the domain. By means of the theory of conformal mappings, we obtain existence, uniqueness, (non-spherical) symmetry results, and a formula relating the curvature of the boundary of the domain to the normal derivative of its Green's function.

1. **Introduction.** Overdetermined boundary value problems in partial differential equations have connections to various fields in mathematics; they emerge in the study of isoperimetric inequalities, optimal design and ill-posed and free boundary problems, to name a few. In many such problems one's interest is focused on a specific feature: the shape of the domain considered; mainly, its (spherical) symmetry, as in Serrin's landmark paper [13] and its many offsprings (see [14], [1], [4], [8], [10], and the references therein).

With the present paper, we want to start a more detailed analysis of overdetermined problems in the plane, by exploiting the full power of the theory of analytic functions. As a case study, we shall analyse what appears to be the simplest situation: in a planar bounded simply connected domain Ω with boundary $\partial\Omega$ of class $C^{1,\alpha}$, we shall consider the problem

$$-\Delta U = \delta_{\zeta_c} \qquad \text{in } \Omega, \tag{1}$$

$$U = 0 \qquad \text{on } \partial\Omega, \tag{2}$$

$$\frac{\partial U}{\partial \nu} = \varphi \qquad \text{on } \partial\Omega. \tag{3}$$

where ν is the *interior* normal direction to $\partial\Omega$, δ_{ζ_c} is the Dirac delta centered at a given point $\zeta_c \in \Omega$ and $\varphi : \partial\Omega \to \mathbb{R}$ is a positive given function of arclength, measured counterclockwise from a reference point on $\partial\Omega$.

Problem (1)-(3) can be interpreted as a free-boundary problem: find a domain Ω whose Green's function U with pole at ζ_c has gradient with values on the boundary

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that fit those of the given function φ. This formulation serves as a basis to model, for example, the Hele-Shaw flow, as done in [6] and [12].

By means of the Riemann Mapping Theorem, the solution of (1)-(2) can be explicitly written in terms of a conformal mapping f from the unit disk D to Ω, which is uniquely determined if it satisfies some suitable normalizing conditions. Since it turns out that the normal derivative of U on $\partial\Omega$ is proportional to the modulus of the inverse of f, then by (3) and classical results on holomorphic functions, we can derive an explicit formula for f in terms of φ (see section §2 for details). With the help of such a formula, we obtain the following results:

(i) existence and uniqueness theorems for a domain Ω satisfying (1)-(3) (Theorems 2.2 and 2.3);
(ii) symmetry results relating the invariance of φ under certain groups of transformations to that of Ω (Theorems 3.1 and 3.2); of course, when φ in constant, we obtain that Ω is a disk — a well-known result (see [10], [8] [1]);
(iii) a formula relating the interior normal derivative of the Green's function to the curvature of $\partial\Omega$.

2. **Construction of a forward operator and its inverse.**

In what follows, D will always be the open unit disk in $\mathbb{C}$ centered at 0.

Let us recall some basic facts of harmonic and complex analysis. We refer the reader to [5] and [9] for more details. If $\Omega \subseteq \mathbb{C}$ is a simply connected domain bounded by a Jordan curve and $\zeta_c \in \Omega$, then, by the Riemann Mapping Theorem, Ω is the image of an analytic function $f : D \to \Omega$ which induces a homeomorphism between the closures $\overline{D}$ and $\overline{\Omega}$, has non-zero derivative f' in D and is such that $f(0) = \zeta_c$. Moreover, if Ω is of class $C^{1,\alpha}$, $0 < \alpha < 1$, that is its boundary $\partial\Omega$ is locally the graph of a function of class $C^{1,\alpha}$, then, by Kellogg's theorem, we can infer that $f \in C^{1,\alpha}(\overline{D})$ (see [5]).

In the following elementary lemma, which will be useful in the sequel, we relate f' to the so called *outer function* (see [2]).

Lemma 2.1. *Let Ω be a bounded simply connected domain in $\mathbb{C}$ and $f : D \to \Omega$ be one-to-one and analytic with $f \in C^1(\overline{D})$. Then there exists $\gamma \in \mathbb{R}$ such that*

$$f'(z) = e^{i\gamma} \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log|f'(e^{it})|dt\right\} \tag{4}$$

for every $z \in D$.

Proof. The function

$$f'(z) \exp\left\{-\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log|f'(e^{it})|dt\right\}, \qquad z \in D,$$

is analytic, never zero in D and has unitary modulus on ∂D; hence it equals the number $e^{i\gamma}$ for some $\gamma \in \mathbb{R}$. □

With these premises, given two distinct numbers ζ_c and $\zeta_b \in \mathbb{C}$, we consider

the set $\mathscr{O}$ of all $C^{1,\alpha}$, $0 < \alpha < 1$, simply connected
bounded domains such that $\zeta_c \in \Omega$ and $\zeta_b \in \partial\Omega$.

We can put $\mathscr{O}$ in one-to-one correspondence with

the class $\mathscr{F}$ of all one-to-one analytic mappings
$f \in C^{1,\alpha}(\overline{D})$ such that $f(0) = \zeta_c$ and $f(1) = \zeta_b$.

In fact, the arbitrary parameter γ in (4) can be determined by observing that

$$\zeta_b - \zeta_c = \int_0^1 f'(t)dt. \tag{5}$$

We now construct our forward operator $\mathcal{T}$ as the one that associates to each Ω in $\mathscr{O}$ the interior normal derivative $\frac{\partial U}{\partial \nu}$ — as function of the arclength, measured counterclockwise on $\partial\Omega$, starting from ζ_b — of the solution of (1)-(2). With our identification of $\mathscr{O}$ with $\mathscr{F}$ in mind, for $f \in \mathscr{F}$, $\mathcal{T}(f)$ is a function of arclength $s \in [0, |\partial\Omega|]$ and it is defined by the following remarks.

First, notice that, by Gauss-Green's formula, if U satisfies (1)-(2), then

$$v(\zeta_c) = \int_{\partial\Omega} v(\zeta) \frac{\partial U}{\partial \nu}(\zeta) ds(\zeta)$$

for every function $v \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ which is harmonic in Ω.

Secondly, recall that any such function v satisfies the well-known Poisson integral formula

$$v(\zeta) = \frac{1}{2\pi} \int_{\partial\Omega} v(\zeta') \frac{1 - |f^{-1}(\zeta)|^2}{|f^{-1}(\zeta) - f^{-1}(\zeta')|} \frac{ds(\zeta')}{|f'(f^{-1}(\zeta'))|}, \qquad \zeta \in \Omega,$$

if $\partial\Omega$ is rectifiable (see [9]). By comparing the last two formulas (with $\zeta = \zeta_c = f(0)$), we obtain that

$$\frac{\partial U}{\partial \nu}(\zeta) = \frac{1}{2\pi |f'(f^{-1}(\zeta))|}, \qquad \zeta \in \partial\Omega.$$

Thirdly, since the arclength on $\partial\Omega$ is related to f by the formula

$$s(\theta) = \int_0^\theta |f'(e^{it})| dt, \qquad \theta \in [0, 2\pi], \tag{6}$$

the values $\mathcal{T}(f)(s)$, $s \in [0, |\partial\Omega|]$, can be defined parametrically by

$$s = \int_0^\theta |f'(e^{it})| dt, \quad \mathcal{T}(f) = \frac{1}{2\pi |f'(e^{i\theta})|}, \quad \theta \in [0, 2\pi]. \tag{7}$$

It is clear that $\mathcal{T}(f) \in C^{0,\alpha}([0, |\partial\Omega|])$ and also that

$$\int_0^{|\partial\Omega|} \mathcal{T}(f)(s) ds = 1, \qquad \mathcal{T}(f) > 0 \text{ on } [0, |\partial\Omega|],$$

for all $f \in \mathscr{F}$.

We shall now prove that $\mathcal{T}$ is injective by showing that each φ in the range of $\mathcal{T}$ determines only one $f \in \mathscr{F}$. In fact, for $\varphi \in C^{0,\alpha}([0, |\partial\Omega|])$ in the range of $\mathcal{T}$, by formulas (7) it turns out that

$$2\pi \varphi(s(\theta)) s'(\theta) = 1, \qquad \theta \in [0, 2\pi]. \tag{8}$$

This last formula, once integrated between 0 and θ, gives

$$s(\theta) = \Phi^{-1}(\theta), \qquad \theta \in [0, 2\pi], \tag{9}$$

where Φ^{-1} is the inverse of $\Phi : [0, |\partial\Omega|] \to [0, 2\pi]$ defined by

$$\Phi(s) = 2\pi \int_0^s \varphi(\sigma) d\sigma, \qquad s \in [0, |\partial\Omega|]. \tag{10}$$

By the same formulas (7), we then obtain that

$$|f'(e^{i\theta})| = \frac{1}{2\pi \varphi(\Phi^{-1}(\theta))}, \qquad \theta \in [0, 2\pi], \tag{11}$$

and hence (4) gives

$$f'(z) = e^{i\gamma} \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log\frac{1}{2\pi\varphi(\Phi^{-1}(t))}dt\right\}, \qquad z \in D, \tag{12}$$

where γ is determined by (5). Therefore, for any φ in the range of $\mathcal{T}$, a unique $f \in \mathscr{F}$ such that $\mathcal{T}(f) = \varphi$ is determined by

$$f(z) = \zeta_c + \int_0^1 f'(tz)z\,dt, \qquad z \in D,$$

with f' given by (12).

We collect these remarks in the following theorem.

Theorem 2.2. *Given $\Omega \in \mathscr{O}$, let ζ_b be a reference point on $\partial\Omega$ from which the arclength on $\partial\Omega$ is measured counterclockwise.*

Let φ be in the range of $\mathcal{T}$, that is φ is the interior normal derivative of the Green's function on $\partial\Omega$ (as function of the arclength).

Then a function $f \in \mathscr{F}$ is uniquely determined such that $\mathcal{T}(f) = \varphi$ and its derivative is given by

$$f'(z) = e^{i\gamma} \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log\frac{1}{2\pi\varphi(s(t))}dt\right\}, \quad z \in D, \tag{13}$$

where s and Φ are defined by (9) and (10), respectively.

Moreover, the constant γ is determined by

$$e^{i\gamma}\int_0^1 \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\tau}+t}{e^{i\tau}-t} \log\frac{1}{2\pi\varphi(s(\tau))}d\tau\right\}dt = \zeta_b - \zeta_c. \tag{14}$$

Theorem 2.2 tells us that the operator $\mathcal{T}$ is injective. A discussion about its surjectivity is beyond the aims of this paper. Far from being complete, we want here to suggest the following criterion.

Referring to [3], let us introduce the so called *boundary rotation* of a function f defined in D:

$$\rho = \lim_{r\to 1^-}\int_0^{2\pi} \left|\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| d\theta, \quad z = re^{i\theta} \in D.$$

We consider the class $\mathscr{V}$ of all normalized functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$$

which are analytic, locally univalent and with $\rho < +\infty$. The proof of the surjectivity of $\mathcal{T}$ relies on the problem of finding an analytic and univalent function f from the disk to $f(D) = \Omega$. The following theorem is based on a sufficient condition, due to Paatero, that says that any function in the class $\mathscr{V}$ with $\rho \le 4\pi$ is univalent (see [3]).

Theorem 2.3. *Let $\varphi \in C^1(\mathbb{R})$ be L-periodic, strictly positive and satisfying the compatibility condition $\int_0^L \varphi(s)ds = 1$. If, moreover, φ satisfies the condition*

$$\max_{[0,L]}\left|\frac{\varphi'(s)}{\varphi^2(s)}\right| \le 2\pi,$$

then there exists $\Omega \in \mathscr{O}$ with perimeter L and a solution of the overdetermined boundary value problem (1)-(3); thus, $\mathcal{T}$ is surjective.

Proof. By Theorem 2.2, we know that a function $f \in \mathscr{F}$ such that $\mathcal{T}(f) = \varphi$ must satisfy (13). Thus, we have to check Paatero's condition on (13). From that expression we deduce that

$$\frac{f''(z)}{f'(z)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{2e^{it}}{(e^{it}-z)^2}\log\frac{1}{2\pi\varphi(s(t))}dt,$$

being s defined as in (9) and (10). Now, by observing that

$$\frac{d}{dt}\left(\frac{e^{it}+z}{e^{it}-z}\right) = \frac{-2ize^{it}}{(e^{it}-z)^2},$$

we can integrate by parts and obtain that

$$\frac{-izf''(z)}{f'(z)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\frac{\varphi'(s(t))s'(t)}{\varphi(s(t))}dt.$$

By the maximum modulus principle, we can estimate, for $z \in D$,

$$\begin{aligned}\left|\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}\right| &\le 1+\left|\frac{-izf''(z)}{f'(z)}\right| \\ &\le 1+\max_{[0,2\pi]}\left|\frac{\varphi'(s(t))s'(t)}{\varphi(s(t))}\right|,\end{aligned}$$

and, from (8), we have that $\varphi'(s)s'/\varphi(s) = \varphi'(s)/2\pi\varphi^2(s)$. Therefore, we can estimate the boundary rotation of f in the following way:

$$\rho \le \int_0^{2\pi}\left(1+\max_{[0,2\pi]}\left|\frac{\varphi'(s(t))}{2\pi\varphi^2(s(t))}\right|\right)d\theta = 2\pi\left(1+\max_{[0,L]}\left|\frac{\varphi'(s)}{2\pi\varphi^2(s)}\right|\right).$$

By our assumptions, it follows that $\rho \le 4\pi$ and hence, from Paatero's criterion for univalence, f is a homeomorphism from the disk onto $f(D)$. □

3. **Symmetries.**

Remark 1. Theorem 2.2 allows us to rediscover a result already proved in [10] and also in [8] and [1]: if φ is constant, then Ω is a disk. More precisely, given $\Omega \in \mathscr{O}$ with perimeter L, let φ be constantly equal to $C > 0$. From (13), we obtain that

$$f'(z) = e^{i\gamma}\exp\left\{\frac{1}{2\pi}\log\frac{1}{2\pi C}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}dt\right\} = \frac{e^{i\gamma}}{2\pi C},$$

since $\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}dt = 2\pi$. Therefore, we get that

$$f(z) = \zeta_c + \frac{e^{i\gamma}}{2\pi C}z, \qquad z \in D,$$

that is Ω is the disk centered at ζ_c with radius $\frac{1}{2\pi C}$.

Now we want to show how some other symmetry properties of Ω can be derived from some invariance properties of φ and viceversa.

In what follows, for $\Omega \in \mathscr{O}$, let $L = |\partial\Omega|$ and let φ denote the values of the interior normal derivative on $\partial\Omega$ (as function of arclength) of the Green's function of Ω.

In the next theorem, we will identify φ with its L-periodic extension to $\mathbb{R}$ and $\mathcal{R}_{\zeta,\beta}$ will denote the clockwise rotation of an angle β around a point ζ.

Theorem 3.1. *Let $\Omega \in \mathscr{O}$ and $n \in \{2,3,4,...\}$. Then:*

$$\mathcal{R}_{\zeta_c,\frac{2\pi}{n}}(\Omega) = \Omega \textit{ if and only if } \varphi \textit{ is } \tfrac{L}{n}\textit{-periodic.}$$

Proof. Let us fix n and suppose φ measured counterclockwise from $\zeta_b \in \partial\Omega$. Let $f \in \mathscr{F}$ be the unique analytic function from D to Ω such that $f(0) = \zeta_c$ and $f(1) = \zeta_b$.

(i) If Ω is invariant by rotations of angle $\frac{2\pi}{n}$ around ζ_c, then f satisfies

$$f(ze^{i\frac{2\pi}{n}}) = \zeta_c + [f(z) - \zeta_c]e^{i\frac{2\pi}{n}}, \qquad z \in D.$$

By differentiating this expression, we obtain $f'(ze^{i\frac{2\pi}{n}}) = f'(z)$, from which

$$s\left(\theta + \frac{2\pi}{n}\right) = \int_0^{\theta+\frac{2\pi}{n}} |f'(e^{it})|dt = s(\theta) + \int_\theta^{\theta+\frac{2\pi}{n}} |f'(e^{it})|dt,$$

and hence

$$s\left(\theta + \frac{2\pi}{n}\right) = s(\theta) + s\left(\frac{2\pi}{n}\right), \quad \theta \in \mathbb{R}. \tag{15}$$

Since

$$L = s(2\pi) = s\left(\frac{n-1}{n}2\pi\right) + s\left(\frac{2\pi}{n}\right) = ... = ns\left(\frac{2\pi}{n}\right),$$

we have that $s\left(\theta + \frac{2\pi}{n}\right) = s(\theta) + \frac{L}{n}$. Thus, (15) and (8)-(11) imply that

$$\varphi\left(s(\theta) + \frac{L}{n}\right) = \varphi\left(s\left(\theta + \frac{2\pi}{n}\right)\right) = \frac{1}{2\pi|f'(e^{i(\theta+\frac{2\pi}{n})})|} = \frac{1}{2\pi|f'(e^{i\theta})|} = \varphi(s(\theta)),$$

and hence, for every $s \in \mathbb{R}$,

$$\varphi\left(s + \frac{L}{n}\right) = \varphi(s).$$

(ii) If now φ is $\frac{L}{n}$-periodic, from (10) we write

$$\Phi\left(s + \frac{L}{n}\right) = 2\pi \int_0^{s+\frac{L}{n}} \varphi(\sigma)d\sigma = \Phi(s) + \Phi\left(\frac{L}{n}\right). \tag{16}$$

Since (9) holds, it follows that

$$2\pi = \Phi(s(2\pi)) = \Phi(L) = \Phi\left(\frac{n-1}{n}L\right) + \Phi\left(\frac{L}{n}\right) = ... = n\Phi\left(\frac{L}{n}\right),$$

and hence

$$\Phi\left(\frac{L}{n}\right) = \frac{2\pi}{n} = \theta + \frac{2\pi}{n} - \theta = \Phi\left(s\left(\theta + \frac{2\pi}{n}\right)\right) - \Phi(s(\theta)).$$

From this and (16), we infer that

$$\Phi\left(s\left(\theta + \frac{2\pi}{n}\right)\right) = \Phi(s(\theta)) + \Phi\left(\frac{L}{n}\right) = \Phi\left(s(\theta) + \frac{L}{n}\right),$$

and, thanks to the invertibility of Φ, we obtain

$$s\left(\theta + \frac{2\pi}{n}\right) = s(\theta) + \frac{L}{n}, \qquad \theta \in \mathbb{R}.$$

By this formula, (13) and the periodicity of φ, it follows that

$$\begin{aligned} f'(z) &= e^{i\gamma}\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log\frac{1}{2\pi\varphi(s(t))}dt\right\} \\ &= e^{i\gamma}\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log\frac{1}{2\pi\varphi\left(s\left(\frac{2\pi}{n}+t\right)-\frac{L}{n}\right)}dt\right\} \\ &= e^{i\gamma}\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log\frac{1}{2\pi\varphi\left(s\left(\frac{2\pi}{n}+t\right)\right)}dt\right\}. \end{aligned}$$

By a change of variables, we thus get

$$\begin{aligned} f'(z) &= e^{i\gamma}\exp\left\{\frac{1}{2\pi}\int_{\frac{2\pi}{n}}^{2\pi+\frac{2\pi}{n}}\frac{e^{i(t-\frac{2\pi}{n})}+z}{e^{i(t-\frac{2\pi}{n})}-z}\log\frac{1}{2\pi\varphi(s(t))}dt\right\} \\ &= e^{i\gamma}\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+ze^{i\frac{2\pi}{n}}}{e^{it}-ze^{i\frac{2\pi}{n}}}\log\frac{1}{2\pi\varphi(s(t))}dt\right\} \\ &= f'(ze^{i\frac{2\pi}{n}}). \end{aligned}$$

Finally we find

$$f(z)-\zeta_c=\int_0^1 f'(tz)zdt=\int_0^1 f'(tze^{i\frac{2\pi}{n}})zdt=[f(ze^{i\frac{2\pi}{n}})-\zeta_c]e^{-i\frac{2\pi}{n}},$$

and hence $\mathcal{R}_{\zeta_c,\frac{2\pi}{n}}\Omega=\Omega$. □

In what follows, $\mathcal{M}$ will denote mirror-reflection with respect to a given axis.

Theorem 3.2. *A domain $\Omega \in \mathscr{O}$ is symmetric with respect to a generic axis passing through ζ_c if and only if $\varphi(s)=\varphi(L-s)$ for all $s\in[0,L]$.*

Here φ is measured counterclockwise starting from an intersection point of the axis with $\partial\Omega$.

Proof. (i) Suppose Ω symmetric with respect to a given axis passing through ζ_c, that is $\mathcal{M}(\Omega)=\Omega$. Short of rotations and translations, we can assume the symmetry axis to coincide with the real axis, so that $\mathcal{M}z$ is the conjugate $\overline{z}$ of z.

Let $f\in\mathscr{F}$ be the unique mapping from D to Ω such that $f(0)=\zeta_c$ and $f(1)=\zeta_b$, where ζ_b is supposed to be one of the intersection point of $\partial\Omega$ with the symmetry axis. We keep in mind that arclength on $\partial\Omega$ is measured counterclockwise from ζ_b.

It is clear that $\zeta_c-\zeta_b\in\mathbb{R}$ and

$$\overline{f(z)}=f(\overline{z}); \tag{17}$$

thus,

$$\overline{f(e^{i\theta})}=f(e^{i(2\pi-\theta)}),\quad \theta\in[0,2\pi].$$

Differentiating the latter formula with respect to θ and taking the modulus, yields

$$|f'(e^{i\theta})|=|f'(e^{i(2\pi-\theta)})|,\quad \theta\in[0,2\pi]; \tag{18}$$

thus, from (6), we have that

$$s(2\pi-\theta)=L-s(\theta),\qquad \theta\in\mathbb{R}.$$

From this formula and (11), we obtain:

$$\varphi(L-s(\theta))=\varphi(s(2\pi-\theta))=\frac{1}{2\pi|f'(e^{i(2\pi-\theta)})|},\qquad \theta\in\mathbb{R}.$$

Finally, from (18), it follows that

$$\varphi(s) = \varphi(L-s), \qquad s \in [0, L].$$

(ii) Suppose now $\varphi(s) = \varphi(L-s)$ for all $s \in \mathbb{R}$. From (10) we write

$$\Phi(L-s) = 2\pi \int_0^{L-s} \varphi(\sigma)d\sigma = 2\pi \int_s^L \varphi(L-\sigma)d\sigma = 2\pi - \Phi(s), \quad s \in [0, L].$$

This property of Φ and (9) imply that

$$\Phi(s(2\pi-\theta)) = 2\pi - \theta = 2\pi - \Phi(s(\theta)) = \Phi(L - s(\theta)),$$

and hence

$$s(2\pi-\theta) = L - s(\theta), \qquad \theta \in \mathbb{R},$$

by the invertibility of Φ. Then, by differentiating, we have that

$$|f'(e^{i(2\pi-\theta)})| = s'(2\pi-\theta) = s'(\theta) = |f'(e^{i\theta})|$$

for every $\theta \in \mathbb{R}$. Thus, by a change of variable and by simple properties of the complex conjugate, we can write that, for $z \in D$,

$$\begin{aligned}
\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log|f'(e^{it})|dt &= \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log|f'(e^{i(2\pi-t)})|dt \\
&= \int_0^{2\pi} \frac{e^{i(2\pi-t)}+z}{e^{i(2\pi-t)}-z} \log|f'(e^{it})|dt \\
&= \overline{\left(\int_0^{2\pi} \frac{e^{it}+\overline{z}}{e^{it}-\overline{z}} \log|f'(e^{it})|dt\right)}.
\end{aligned}$$

Therefore, modulo a rotation, we have obtained that

$$f'(z) = \overline{f'(\overline{z})}, \quad z \in D,$$

and hence

$$f(z) = \overline{f(\overline{z})}, \quad z \in D,$$

modulo a translation. Thus, $\mathcal{M}(\Omega) = \Omega$ for some reflection $\mathcal{M}$. □

4. **A formula involving curvature.** Recall that the curvature (with sign) κ of a planar curve can be defined by the formula

$$\kappa = \frac{d\psi}{ds}, \tag{19}$$

where ψ is the angle between the positive real axis and the tangent (unit) vector.

By using the conformal map $f : D \to \Omega$ already introduced and the Hilbert transform, we can express the curvature κ of $\partial\Omega$ in terms of the interior normal derivative φ of the Green's function of Ω.

Theorem 4.1. *Let $\Omega \in \mathscr{O}$ and φ be defined as usual. Then φ and the curvature κ of $\partial\Omega$ are related by the formula:*

$$\kappa(s) = 2\pi\varphi(s)\left[1 - \frac{1}{2\pi}\int_0^{|\partial\Omega|} \cot\left(\frac{\Phi(s)-\Phi(\sigma)}{2}\right)\frac{d}{d\sigma}(\log\varphi)(\sigma)d\sigma\right], \tag{20}$$

for $s \in [0, |\partial\Omega|]$, where Φ is defined as in (10).

Proof. Let $f : D \to \Omega$ be as usual. Now we compute κ in terms of f. Define

$$\omega(\theta) = \arg(f'(e^{i\theta}))$$

for $\theta \in [0, 2\pi]$; the angle ψ in (19) is given by

$$\psi(\theta) = \arg\left(\frac{d}{d\theta} f(e^{i\theta})\right) = \omega(\theta) + \frac{\pi}{2} + \theta.$$

From (19) and (8), we have that

$$\kappa(s) = \frac{d\psi}{d\theta}\frac{d\theta}{ds} = 2\pi\varphi(s)[1 + \omega'(\theta)], \qquad s \in [0, \partial\Omega]. \tag{21}$$

As is well-known (see [7] and [11]), since $\log|f'|$ and $\arg f'$ are the real and the imaginary part of the analytic function $\log f'$, we have that

$$\arg f'(e^{i\theta}) = \mathcal{H}(\log s')(\theta), \tag{22}$$

being $s'(\theta) = |f'(e^{i\theta})|$. Here, $\mathcal{H}$ is the Hilbert transformation on the unit circle, namely,

$$\mathcal{H}(\log s')(\theta) = \frac{1}{2\pi}\int_0^{2\pi} \cot\left(\frac{\theta - t}{2}\right)\log(s'(t))dt.$$

In our notations, (22) can be rewritten as

$$\omega = \mathcal{H}(\log s');$$

thus,

$$\omega' = \mathcal{H}(s''/s'),$$

since $\mathcal{H}$ and $\dfrac{d}{d\theta}$ commute. From (21), we infer that

$$\kappa(s(\theta)) = 2\pi\varphi\left[1 + \mathcal{H}\left(\frac{s''}{s'}\right)(\theta)\right], \quad \theta \in [0, 2\pi],$$

and hence

$$\kappa(s(\theta)) = 2\pi\varphi(s(\theta))\left[1 - \frac{1}{2\pi}\int_0^{2\pi} \cot\left(\frac{\theta - t}{2}\right)\frac{\varphi'(s(t))}{2\pi\varphi^2(s(t))}dt\right], \quad \theta \in [0, 2\pi],$$

from (8). Finally, we obtain (20) by operating the change of variable $\sigma = s(t)$ and by using (9). □

Remark 2. Let $\mathcal{D}2\mathcal{N}$ denote the Dirichlet-to-Neumann operator, that is $\mathcal{D}2\mathcal{N}$ maps the values on $\partial\Omega$ of any harmonic function in Ω to the values of its (interior) normal derivative on $\partial\Omega$. Then, formula (20) can be rewritten as

$$\kappa = 2\pi\varphi[1 + \mathcal{D}2\mathcal{N}(\log(\varphi))].$$

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REFERENCES

[1] G. Alessandrini and E. Rosset, *Symmetry of singular solutions of degenerate quasilinear elliptic equations*, Rend. Sem. Mat. Univ. Trieste, **39** (2007), 1–8.
[2] P. L. Duren, "Theory of H^p Spaces," Academic Press, New York, 1970.
[3] P. L. Duren, "Univalent Functions," Springer-Verlag, New York, 1983.
[4] L. E. Fraenkel, "An Introduction to Maximum Principles and Symmetry in Elliptic Problems," Cambridge University Press, Cambridge, 2000.
[5] G. M. Goluzin, "Geometric Theory of Functions of a Complex Variable," American Mathematical Society, Providence, 1969.
[6] B. Gustafsson and A. Vasil'ev, "Conformal and Potential Analysis in Hele-Shaw Cells," Birkhäuser Verlag, Basel, 2006.
[7] P. Koosis, "Introduction to H_p Spaces," Cambridge University Press, Cambridge, 1998.
[8] J. L. Lewis and A. Vogel, *On some almost everywhere symmetry theorems*, in "Progr. Nonlinear Differential Equations Appl.," **7**, Birkhäuser Boston, Massachusetts, (1992), 347–374.
[9] A. I. Markushevich, "Theory of Functions of a Complex Variable," Prentice-Hall, Englewood Cliffs, 1965.
[10] L. E. Payne and P. W. Schaefer, *Duality theorems in some overdetermined boundary value problems*, Math. Meth. Appl. Sci., **11** (1989), 805–819.
[11] J. Privalov, *Sur les fonctions conjuguées*, Bulletin de la S. M. F., **44** (1916), 100–103.
[12] M. Sakai, "Quadrature Domains," Springer-Verlag, Berlin, 1982.
[13] J. Serrin, *A symmetry problem in potential theory*, Arch. Rational Mech. Anal., **43** (1971), 304–318.
[14] H. F. Weinberger, *Remark on the preceding paper of Serrin*, Arch. Rational Mech. Anal., **43** (1971), 319–320.

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