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ON MODULI OF INSTANTON BUNDLES ON \mathbb{P}^{2n+1}

VINCENZO ANCONA AND G. OTTAVIANI

Let $MI_{\mathbb{P}^{2n+1}}(k)$ be the moduli space of stable instanton bundles on \mathbb{P}^{2n+1} with $c_2 = k$. We prove that $MI_{\mathbb{P}^{2n+1}}(2)$ is smooth, irreducible, unirational and has zero Euler-Poincaré characteristic, as it happens for \mathbb{P}^3 . We find instead that $MI_{\mathbb{P}^5}(3)$ and $MI_{\mathbb{P}^5}(4)$ are singular.

1. Definition and preliminaries.

Instanton bundles on a projective space $\mathbb{P}^{2n+1}(\mathbb{C})$ were introduced in **[OS]** and **[ST]**. In **[AO]** we studied their stability, proving in particular that special symplectic instanton bundles on \mathbb{P}^{2n+1} are stable, and that on \mathbb{P}^5 every instanton bundle is stable.

In this paper we study some moduli spaces $MI_{\mathbb{P}^{2n+1}}(k)$ of stable instanton bundles on \mathbb{P}^{2n+1} with $c_2 = k$. For k = 2 we prove that $MI_{\mathbb{P}^{2n+1}}(2)$ is smooth, irreducible, unirational and has zero Euler-Poincaré characteristic (Theor. 3.2), just as in the case of \mathbb{P}^3 [Har].

We find instead that $MI_{\mathbb{P}^5}(k)$ is singular for k = 3, 4 (theor. 3.3), which is not analogous with the case of \mathbb{P}^3 [**ES**], [**P**]. To be more precise, all points corresponding to symplectic instanton bundles are singular. Theor. 3.3 gives, to the best of our knowledge, the first example of a singular moduli space of stable bundles on a projective space. The proof of Theorem 3.3 needs help from a personal computer in order to calculate the dimensions of some cohomology group [**BaS**].

We recall from [OS], [ST] and [AO] the definition of instanton bundle on $\mathbb{P}^{2n+1}(\mathbb{C})$.

Definition 1.1. A vector bundle E of rank 2n on \mathbb{P}^{2n+1} is called an instanton bundle of quantum number k if

- (i) The Chern polynomial is $c_t(E) = (1 t^2)^{-k} = 1 + kt^2 + \binom{k+1}{2}t^2 + \dots$
- (ii) E(q) has natural cohomology in the range $-2n 1 \le q \le 0$ (that is $h^i(E(q)) \ne 0$ for at most one i = i(q))

(iii) $E|_r \simeq \mathcal{O}_r^{2n}$ for a general line r.

Every instanton bundle is simple [**AO**]. There is the following characterization:

Theorem 1.2 ([**ST**], [**AO**]). A vector bundle E of rank 2n on \mathbb{P}^{2n+1} satisfies the properties (i) and (ii) if and only if E is the cohomology of a monad

(1.1)
$$\mathcal{O}(-1)^k \xrightarrow{A} \mathcal{O}^{2n+2k} \xrightarrow{B} \mathcal{O}(1)^k.$$

With respect to a fixed system of homogeneous coordinates the morphism A (resp. B) of the monad can be identified with a $k \times (2n + 2k)$ (resp. $(2n + 2k) \times k$) matrix whose entries are homogeneous polynomials of degree 1. Then the conditions that (1.1) is a monad are equivalent to:

A, B have rank k at every point
$$x \in \mathbb{P}^{2n+1}$$
, $A \cdot B = 0$.

Definition 1.3. A bundle S appearing in an exact sequence:

(1.2)
$$0 \to S^* \to \mathcal{O}^d \xrightarrow{B} \mathcal{O}(1)^c \to 0$$

is called a Schwarzenberger type bundle (STB).

The kernel bundle Ker B in the monad (1.1) is the dual of a STB.

Definition 1.4. An instanton bundle is called special if it arises from a monad (1.1) where the morphism B is defined in some system of homogeneous coordinates $(x_0, \ldots, x_n, y_0, \ldots, y_n)$ on \mathbb{P}^{2n+1} by the matrix

$$B = \begin{bmatrix} x_{0} & & \\ \vdots & \ddots & \\ x_{n} & x_{0} & \\ & \ddots & \vdots \\ & & x_{n} \\ y_{0} & & \\ \vdots & \ddots & \\ y_{n} & y_{0} \\ & \ddots & \vdots \\ & & & y_{n} \end{bmatrix}$$

Example 1.5. Take

$$A = \begin{bmatrix} y_{n} \cdots y_{0} & -x_{n} \cdots -x_{0} \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ y_{n} \cdots y_{0} & -x_{n} \cdots -x_{0} \end{bmatrix} \qquad B = \begin{bmatrix} x_{0} \\ \vdots & \ddots \\ x_{n} & x_{0} \\ \vdots & \ddots \\ y_{0} \\ \vdots & \ddots \\ y_{n} & y_{0} \\ \vdots & \ddots \\ y_{n} & y_{0} \\ \vdots & \ddots \\ y_{n} & y_{0} \end{bmatrix}$$

E = Ker B / Im A is a special instanton bundle. Property (iii) of the definition 1.1 can be checked by the following:

Theorem 1.6 [OS]. Let E = Ker B/Im A as in (1.1). Let r be the line joining two distinct points $P, Q \in \mathbb{P}^{2n+1}$. Then

 $E|_r \simeq \mathcal{O}_r^{2n} \Leftrightarrow A(P) \cdot B(Q)$ is an invertible matrix.

Example 1.7. Consider the special instanton bundle E of the example 1.5. Let P = (1, 0, ...; 0, ..., 0), Q = (0, ..., ; 0, ..., 1). Then

$$A(P) = \begin{bmatrix} & & & -1 \\ & & \cdot \\ & & \cdot \\ & & -1 \end{bmatrix} \qquad B(Q) = \begin{bmatrix} 1 \\ & \cdot \\ & \cdot \\ & & 1 \end{bmatrix}$$

and $A(P) \cdot B(Q) = \begin{bmatrix} & -1 \\ & \cdot \\ & \cdot \\ & -1 \end{bmatrix}$ is invertible. Hence E is trivial on the line $\{x_1 = \ldots = x_n = y_0 = \ldots = y_{n-1} = 0\}.$

Proposition 1.8. Let E be an instanton bundle as in (1.1). Then

$$H^2(E\otimes E^*)=H^2[(\operatorname{Ker} B)\otimes (\operatorname{Ker} A^t)]$$

Proof. See [AO] Theorem 3.13 and Remark 2.22.

Remark 1.9. If $E \simeq E^*$, then

$$H^{2}(E \otimes E^{*}) = H^{2}[(\operatorname{Ker} A^{t}) \otimes (\operatorname{Ker} A^{t})] = H^{2}[(\operatorname{Ker} B) \otimes (\operatorname{Ker} B)].$$

Remark 1.10. The single complex associated with the double complex obtained by tensoring the two sequences

$$0 \to \operatorname{Ker} A^{t} \to \mathcal{O}^{2n+2k} \xrightarrow{A^{t}} \mathcal{O}(1)^{k} \to 0$$
$$0 \to \operatorname{Ker} B^{t} \to \mathcal{O}^{2n+2k} \xrightarrow{B^{t}} \mathcal{O}(1)^{k} \to 0$$

gives the resolution

$$0 \to (\operatorname{Ker} A^{t}) \otimes (\operatorname{Ker} B) \to \mathcal{O}^{2n+2k} \otimes \mathcal{O}^{2n+2k} \to \mathcal{O}^{2n+2k} \otimes \mathcal{O}(1)^{k} \oplus \mathcal{O}(1)^{k} \otimes \mathcal{O}^{2n+2k} \xrightarrow{\alpha} \mathcal{O}(1)^{k} \otimes \mathcal{O}(1)^{k} \to 0$$

where $\alpha = (A^t \otimes id, id \otimes B)$.

Hence

$$H^2(E\otimes E^*) = \operatorname{Coker} H^0(\alpha)$$

and its dimension can be computed using [BaS]. For the convenience of the reader we sketch the steps needed in the computations.

 A, B^t are given by $k \times (2n + 2k)$ matrices whose entries are linear homogeneous polynomials.

$$A \otimes \mathrm{Id}_k = (a_1, \ldots, a_{k(2n+2k)})$$

and

$$\mathrm{Id}_k\otimes B^t=(b_1,\ldots,b_{k(2n+2k)})$$

are both $k^2 \times (2n+2k)k$ matrices. Let

$$C = (a_1, \ldots, a_{k(2n+2k)}, b_1, \ldots, b_{k(2n+2k)}).$$

We will denote by $\operatorname{syz}_m C$ the dimension of the space of the syzygies of C of degree m. Then

$$\begin{aligned} h^2(E \otimes E^*) &= h^0(\mathcal{O}(2)^{k^2}) - (4n+4k)h^0(\mathcal{O}(1)^k) + \operatorname{syz}_1 C \\ &= k(n+1)[k(2n-5)-8n] + \operatorname{syz}_1 C \\ h^1(E \otimes E^*) &= h^2(E \otimes E^*) + 1 - k^2 + 8n^2k - 4n^2 + 3nk^2 - 2n^2k^2 \\ &= 1 - 6k^2 - 8kn - 4n^2 + \operatorname{syz}_1 C. \end{aligned}$$

Note also that $h^0(E(1)) = \operatorname{syz}_1 B^t - k$ and $h^0(E^*(1)) = \operatorname{syz}_1 A - k$. Remark 1.11. In the same way we obtain

$$h^1(E\otimes E^*(-1)) = \operatorname{syz}_0 C$$

 $h^2(E\otimes E^*(-1)) = 2k(nk-2n-k) + \operatorname{syz}_0 C.$

2. Example on \mathbb{P}^5 .

Let (a, b, c, d, e, f) be homogeneous coordinates in \mathbb{P}^5 . Example 2.1. (k = 3) Let

$$B^{t} = \begin{bmatrix} a \ b \ c & d \ e \ f \\ a \ b \ c & d \ e \ f \\ a \ b \ c & d \ e \ f \end{bmatrix}$$
$$A = \begin{bmatrix} f \ e \ d & -c \ -b \ -a \\ f \ e \ d & -c \ -b \ -a \end{bmatrix}$$

The corresponding monad gives a special symplectic instanton bundle on \mathbb{P}^5 with k = 3. With the notation of remark 1.10, using [**BaS**] we can compute $\operatorname{syz}_0 C = 14, \operatorname{syz}_1 C = 174$. Hence $h^2(E \otimes E^*) = 3$ from the formulas of Remark 1.10. Moreover $h^0(E(1)) = 4$.

Example 2.2. (k = 3) Let B^t as in the Example 2.1 and

$$A = egin{bmatrix} f \ e \ d & -c - b \ -a \ e \ d & 2f \ -b \ -a & -2c \ d & f \ e \ -a & -c \ -b \end{bmatrix}.$$

We have $\operatorname{syz}_0 C = 10$, $\operatorname{syz}_1 C = 171$. Hence $h^2(E \otimes E^*) = 0$. We can compute also the syzygies of B^t and A and we get $h^0(E(1)) = 4$, $h^0(E^*(1)) = 3$, hence E is not self-dual.

Example 2.3. (k = 4) Let

$$B^{t} = \begin{bmatrix} a \ b \ c & d \ e \ f \\ a \ b \ c & d \ e \ f \\ a \ b \ c & d \ e \ f \\ a \ b \ c & d \ e \ f \end{bmatrix}$$
$$A = \begin{bmatrix} f \ e \ d & -c \ -b \ -a \\ f \ e \ d & -c \ -b \ -a \\ f \ e \ d & -c \ -b \ -a \\ f \ e \ d & -c \ -b \ -a \end{bmatrix}$$

E is a special symplectic instanton bundle with k = 4. We compute

$$h^2(E\otimes E^*)=12.$$

Example 2.4. (k = 4) Let B^t as in the Example 2.3. Let

$$A = \begin{bmatrix} f e d & -c - b - a \\ e d & 2f - b - a & -2c \\ 3d & f e - 3a & -c - b \\ f e d & -c - b - a \end{bmatrix}$$

In this case $h^2(E \otimes E^*) = 6$, $h^0(E(1)) = 4$, $h^0(E^*(1)) = 3$. Example 2.5. (k = 4) Let B^t as in the Example 2.3. Let

$$A = \begin{bmatrix} f & e & d & -c & -b & -a \\ e & d & 2f & -b & -a & -2c \\ 3d & f & e & -3a & -c & -b \\ 5d & f & e & d + f & e & -5a & -c & -b & -a & -c & -b \end{bmatrix}.$$

Now $H^2(E \otimes E^*) = 0$, $h^0(E(1)) = 4$, $h^0(E^*(1)) = 2$.

3. On the singularities of moduli spaces.

The stable Schwarzenberger type bundles on \mathbb{P}^m (see (1.2)) form a Zariski open subset of the moduli space of stable bundles. Let $N_{\mathbb{P}^m}(k,q)$ be the moduli space of stable STB whose first Chern class is k and whose rank is q. The following proposition is easy and well known:

Proposition 3.1. The space $N_{\mathbb{P}^m}(k,q)$ is smooth, irreducible of dimension $1 - k^2 - (q+k)^2 + k(q+k)(m+1)$.

We denote by $MI_{\mathbb{P}^{2n+1}}(k)$ the moduli space of stable instanton bundles with quantum number k. It is an open subset of the moduli space of stable 2n-bundles on \mathbb{P}^{2n+1} with Chern polynomial $(1-t^2)^{-k}$.

On \mathbb{P}^5 (as on \mathbb{P}^3) all instanton bundles are stable by [**AO**], Theorem 3.6. $\mathrm{MI}_{\mathbb{P}^{2n+1}}(2)$ is smooth ([**AO**] Theorem 3.14), unirational of dimension $4n^2 + 12n - 3$ and has zero Euler-Poincaré characteristic ([**BE**], [**K**]).

Theorem 3.2. The space $MI_{\mathbb{P}^{2n+1}}(2)$ is irreducible.

Proof. The moduli space $N = N_{\mathbb{P}^{2n+1}}(2, n+2)$ of stable STB of rank 2n+2 and $c_1 = 2$ is irreducible of dimension $4n^2 + 8n - 3$ by Prop. 3.1

For a given instanton bundle E there is a STB S associated with E, which is stable ([**AO**], Theorem 2.8) and unique (ibid., Prop. 2.17). It is easy to prove that the map $\pi : M \to N$ defined by $\pi([E]) = [S]$ is algebraic, moreover π is dominant by [**ST**]. If $m = [E] \in M$, the fiber $\pi^{-1}(\pi(m))$ is a Zariski open subset of the grassmannian of planes in the vector space $H^0(\mathbb{P}^{2n+1}, S^*(1))$, where $\pi(m) = [S]$; by the Theorem 3.14 of [**AO**], $h^0(\mathbb{P}^{2n+1}, S^*(1)) = 2n+2$, hence dim $\pi^{-1}(\pi(m)) = 4n$.

In order to prove that M is irreducible, we suppose by contradiction that there are at least two irreducible components M_0 and M_1 of M. Then $M_0 \cap M_1 = \emptyset$ (M is smooth), $\pi(M_0)$ and $\pi(M_1)$ are constructible subset of N by Chevalley's theorem. Looking at the dimensions of M_0, M_1, N and the fibers of π we conclude that both $\pi(M_0)$ and $\pi(M_1)$ must contain an open subset of N, which implies $\pi(M_0) \cap \pi(M_1) \neq \emptyset$ by the irreducibility of N. This is a contradiction because the fibers of π are connected.

For $n \ge 2$ and $k \ge 3$, it is no longer true that $\mathrm{MI}_{\mathbb{P}^{2n+1}}(k)$ is smooth. In fact on \mathbb{P}^5 we have:

Theorem 3.3. The space $MI_{\mathbb{P}^5}(k)$ is singular for k = 3, 4. To be more precise, the irreducible component $M_0(k)$ of $MI_{\mathbb{P}^5}(k)$ containing the special instanton bundles is generically reduced of dimension 54(k = 3) or 65(k = 4), and $MI_{\mathbb{P}^5}(k)$ is singular at the points corresponding to special symplectic instanton bundles.

Proof. Let E_0 be the special instanton bundle on \mathbb{P}^5 of the Example 2.2(k = 3) or of the Example 2.5(k = 4). Then $h^2(E_0 \otimes E_0^*) = 0$ and $M_0(k)$ is smooth at the point corresponding to E_0 , of dimension $h^1(E_0 \otimes E_0^*) = 54(k = 3)$ or 65(k = 4). In particular, $M_0(k)$ is generically reduced. If E_1 is a special symplectic instanton bundle on \mathbb{P}^5 , the computations in 2.1 and 2.3 show that $h^2(E_1 \otimes E_1^*) = 3(k = 3)$ or 12(k = 4), and $h^1(E_1 \otimes E_1^*) = 57$ or 77 respectively. Hence $\mathrm{MI}_{\mathbb{P}^5}(k)$ is singular at E_1 for k = 3 and 4.

Remark 3.4. It is natural to conjecture that $MI_{\mathbb{P}^{2n+1}}(k)$ is singular for all $n \geq 2$ and $k \geq 3$.

Theorem 3.5. Let E be an instanton bundle on \mathbb{P}^{2n+1} with $c_2(E) = k$. Then

$$h^1(E(t)) = 0 \text{ for } t \le -2 \text{ and } k - 1 \le t.$$

Proof. The result is obvious for $t \leq -2$. It is sufficient to prove $h^1(S^*(t)) = 0$ for $t \geq k - 1$. We have

$$S^*(t) = \bigwedge^{2n+k-1} S(t-k).$$

Taking wedge products of (1.2) we have the exact sequence

$$0 \to \mathcal{O}(t+1-2n-2k)^{\alpha_0} \to \ldots \to \mathcal{O}(t-k-1)^{\alpha_{2n+k-2}}$$
$$\to \mathcal{O}(t-k)^{\alpha_{2n+k-1}} \to \bigwedge^{2n+k-1} S(t-k) \to 0$$

for suitable $\alpha_i \in \mathbb{N}$ and from this sequence we can conclude.

Ellia proves Theorem 3.5 in the case of \mathbb{P}^3 ([**E**], Prop. IV.1). He also remarks that the given bound is sharp. This holds on \mathbb{P}^{2n+1} as it is shown by the following theorem, which points out that the special symplectic instanton bundles are the "furthest" from having natural cohomology.

Theorem 3.6. Let E be a special symplectic instanton bundle on \mathbb{P}^{2n+1} with $c_2 = k$. Then

$$h^1(E(t)) \neq 0 \text{ for } -1 \leq t \leq k-2.$$

Proof. For n = 1 the thesis is immediate from the exact sequence

$$0 \to \mathcal{O}(t-1) \to E(t) \to \mathcal{J}_C(t+1) \to 0$$

where C is the union of k + 1 disjoint lines in a smooth quadric surface. Then the result follows by induction on n by considering the sequence

$$0 \to E(t-2) \to E(t-1)^2 \to E(t) \to E(t)|_{\mathbb{P}^{2n-1}} \to 0$$

and the fact that, for a particular choice of the subspace \mathbb{P}^{2n-1} , the restriction $E|_{\mathbb{P}^{2n-1}}$ splits as the direct sum of a rank-2 trivial bundle and a special symplectic instanton bundle on $\mathbb{P}^{2n-1}([\mathbf{ST}] 5.9)$.

Remark 3.7. In **[OT]** it is proved that if E_k is a special symplectic instanton bundle on \mathbb{P}^5 with $c_2 = k$ then $h^1(\text{End } E_k) = 20k - 3$.

In the following table we summarize what we know about the component $M_0(k) \subset \mathrm{MI}_{\mathbb{P}^5}(k)$ containing E_k .

	$h^1(E_k\otimes E_k^*)$) $h^2(E_k\otimes E_k^*)$	$\dim M_0(k)$	$\mathrm{MI}_{\mathbf{P}^5}(k)$
k = 1	14	0	14	open subset of \mathbb{P}^{14}
k = 2	37	0	37	smooth, irreduc., unirat.
k = 3	57	3	54	singular
k = 4	77	12	65	singular
$k \ge 2$	20k-3	$3(k-2)^2$?	?

Table 3.10

References

- [AO] V. Ancona and G. Ottaviani, On the stability of special instanton bundles on \mathbb{P}^{2n+1} , Trans. Amer. Math. Soc., **341** (1994), 677-693.
- [BE] J. Bertin and G. Elencwajg, Symétries des fibrés vectoriels sur \mathbb{P}^n et nombre d'Euler, Duke Math. J., **49** (1982), 807-831.
- [BaS] D. Bayer and M. Stillman, Macaulay, a computer algebra system for algebraic geometry.
- [BoS] G. Bohnhorst and H. Spindler, The stability of certain vector bundles on \mathbb{P}^n , Proc. Bayreuth Conference "Complex Algebraic Varieties", LNM 1507, Springer Berlin (1992), 39-50.
 - [E] Ph. Ellia, Some vanishings for the cohomology of stable rank two vector bundles on \mathbb{P}^3 , J. reine angew. Math., **451** (1994), 1-14.
- [ES] G. Ellingsrud and S.A. Stromme, Stable rank-2 bundles on \mathbb{P}^3 with $c_1 = 0$ and $c_2 = 3$, Math. Ann., 255 (1981), 123-137.
- [Har] R. Hartshorne, Stable vector bundles of rank 2 on \mathbb{P}^3 , Math. Ann., 238 (1978), 229-280.
 - [K] T. Kaneyama, Torus-equivariant vector bundles on projective spaces, Nagoya Math. J., 111 (1988), 25-40.
- [M] M. Maruyama, Moduli of stable sheaves, II, J. Math. Kyoto Univ., 18 (1978), 557-614.
- [OS] C. Okonek and H. Spindler, Mathematical instanton bundles on \mathbb{P}^{2n+1} , Journal reine angew. Math., **364** (1986), 35-50.
- [OT] G. Ottaviani and G. Trautmann, The tangent space at a special symplectic instanton bundle on \mathbb{P}^{2n+1} , Manuscr. Math., 85 (1994), 97-107.
 - [P] J. LePotier, Sur l'espace de modules des fibrés de Yang et Mills, in Mathématique et Physique, Sém. E.N.S. 1979-1982, Basel-Stuttgart-Boston 1983.
 - [S] R.L.E. Schwarzenberger, Vector bundles on the projective plane, Proc. London Math. Soc., 11 (1961), 623-640.
- [ST] H. Spindler and G. Trautmann, Special instanton bundles on \mathbb{P}^{2n+1} , their geometry and their moduli, Math. Ann., **286** (1990), 559-592.

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Added in proof. After this paper has been written we received a preprint of R. Miró-Roig and J. Orus-Lacort where they prove that the conjecture stated in the Remark 3.4 is true.