A NEW APPROACH TO THE EXPANSION OF POSITIVITY SET OF NON-NEGATIVE SOLUTIONS TO CERTAIN SINGULAR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

EMMANUELE DIBENEDETTO, UGO GIANAZZA, AND VINCENZO VESPRI

(Communicated by Matthew J. Gursky)

ABSTRACT. Let u be a non-negative solution to a singular parabolic equation of p-Laplacian type $(1 < p < 2)$ or porous-medium type $(0 < m < 1)$. If u is bounded below on a ball B_ρ by a positive number M, for times comparable to ρ and M, then it is bounded below by σM, for some $\sigma \in (0,1)$, on a larger ball, say $B_{2\rho}$, for comparable times. This fact, stated quantitatively in this paper, is referred to as the "spreading of positivity" of solutions of such singular equations and is at the heart of any form of Harnack inequality. The proof of such a "spreading of positivity" effect, first given in 1992, is rather involved and not intuitive. Here we give a new proof, which is more direct, being based on geometrical ideas.

1. INTRODUCTION AND MAIN RESULTS

Let E be an open set in $\mathbb{R}^N$ and for $T > 0$ let E_T denote the cylindrical domain $E \times (0, T]$. Consider quasi-linear, parabolic differential equations of the form

$$
(1.1) \qquad \begin{aligned} &u \in C_{\text{loc}}\big(0, T; L^2_{\text{loc}}(E)\big) \cap L^p_{\text{loc}}\big(0, T; W^{1,p}_{\text{loc}}(E)\big) \\ &u_t - \operatorname{div} \mathbf{A}(x, t, u, Du) = 0 \quad \text{weakly in } E_T \end{aligned} \qquad \text{for } 1 < p < 2,
$$

where the function $\mathbf{A} : E_T \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ is only assumed to be measurable and subject to the structure conditions

$$
(1.2) \qquad \begin{cases} \mathbf{A}(x, t, u, Du) \cdot Du \geq C_o |Du|^p \\ |\mathbf{A}(x, t, u, Du)| \leq C_1 |Du|^{p-1} \end{cases} \quad \text{a.e. } (x, t) \in E_T,
$$

where C_o and C_1 are given positive constants. The homogeneous prototype of such a class of parabolic equations is

$$
(1.1)_o \qquad u_t - \operatorname{div} |Du|^{p-2} Du = 0, \qquad 1 < p < 2, \quad \text{weakly in } E_T.
$$

These equations are termed singular since, for $1 < p < 2$, the modulus of ellipticity $|Du|^{p-2} \to \infty$ as $|Du| \to 0$.

The parameters $\{N, p, C_o, C_1\}$ are the data, and we say that a generic constant $\gamma = \gamma(N, p, C_o, C_1)$ depends upon the data if it can be quantitatively determined a priori only in terms of the indicated parameters. For $\rho > 0$ let B_ρ be the ball

of center the origin on $\mathbb{R}^N$ and radius ρ, and for $y \in \mathbb{R}^N$ let $B_\rho(y)$ denote the homothetic ball centered at y.

Theorem 1.1. *Let u be a non-negative, local, weak solution to (1.1)–(1.2), satisfying*

$$|[u(\cdot, t) > M] \cap B_\rho(y)| > \alpha|B_\rho| \tag{1.3}$$

for all times

$$s - \epsilon M^{2-p}\rho^p \le t \le s \tag{1.4}$$

for some $M > 0$, and some α and ϵ in $(0, 1)$, and assume that

$$B_{16\rho} \times [s - \epsilon M^{2-p}\rho^p, s] \subset E_T.$$

There exist $\sigma \in (0, 1)$ and $\varepsilon \in (0, \frac{1}{2}\epsilon]$, which can be determined a priori, quantitatively only in terms of the data, and the numbers α and ϵ, and independent of M, such that

$$u(x, t) \ge \sigma M \quad \text{for all } x \in B_{2\rho}(y) \tag{1.5}$$

for all times

$$s - \varepsilon M^{2-p}\rho^p < t \le s. \tag{1.6}$$

Thus measure-theoretical information on the measure of the "positivity set" in $B_\rho(y)$ for all times in (1.4) implies that such a positivity set actually "expands" to $B_{2\rho}(y)$ for comparable times. This fact is at the heart of any form of Harnack estimate, as evidenced in [3, 4]. Its proof, first given in [1] and reported in [2], Chapter IV, §5, is rather involved and not intuitive.

The goal of this paper is to give a new proof, which is more direct, being based on geometrical ideas.

2. PRELIMINARIES

For positive ρ and θ set

$$Q_\rho^-(\theta) = B_\rho \times (-\theta\rho^p, 0], \qquad Q_\rho^+(\theta) = B_\rho \times (0, \theta\rho^p]$$

and for $(y, s) \in \mathbb{R}^N \times \mathbb{R}$,

$$(y, s) + Q_\rho^-(\theta) = B_\rho(y) \times (s - \theta\rho^p, s],$$
$$(y, s) + Q_\rho^+(\theta) = B_\rho(y) \times (s, s + \theta\rho^p].$$

There exists a constant $\gamma = \gamma(\text{data})$ such that, for every cylinder $[(y, s) + Q_\rho^-(\theta)] \subset E_T$, every $k \in \mathbb{R}$ and every piecewise smooth, non-negative function ζ vanishing on $\partial B_\rho(y)$,

$$\begin{aligned}
\operatorname*{ess\,sup}_{s-\theta\rho^p < t < s} &\int_{B_\rho(y)} (u-k)_\pm^2 \zeta^p(x, t)dx - \int_{B_\rho(y)} (u-k)_\pm^2 \zeta^p(x, s - \theta\rho^p)dx \\
&+ C_o \iint_{(y,s)+Q_\rho^-(\theta)} |D(u-k)_\pm \zeta|^p dxd\tau \\
&\le \gamma \iint_{(y,s)+Q_\rho^-(\theta)} \left[(u-k)_\pm^p |D\zeta|^p + (u-k)_\pm^2 |\zeta_\tau| \right] dxd\tau,
\end{aligned} \tag{2.1}$$

where C_o is the constant appearing in the structure conditions. Similar energy estimates continue to hold for cylinders $[(y, s) + Q_\rho^+(\theta)] \subset E_T$.

Remark 2.1. The constant $\gamma(p) \to \infty$ as $p \to 1$, but it is stable as $p \to 2$. Thus (2.1) holds for all $1 < p \leq 2$, with the indicated behavior of the constant $\gamma(p)$.

2.1. **A DeGiorgi-type lemma.**

Lemma 2.2. *There exists a number ν depending upon the data θ and M, such that if*

$$\left| [u \leq M] \cap [(y,s) + Q_{2\rho}^-(\theta)] \right| \leq \nu \left| Q_{2\rho}^-(\theta) \right|,$$

then

$$(2.2) \qquad\qquad u \geq \tfrac{1}{2} M \quad a.e. \ in \ [(y,s) + Q_\rho^-(\theta)].$$

Proof. Without loss of generality, we may assume that $(y,s) = (0,0)$. For $n = 0, 1, 2, \ldots,$ set

$$\rho_n = \rho + \frac{\rho}{2^n}, \qquad B_n = B_{\rho_n}, \qquad Q_n = B_n \times (-\theta\rho_n^p, 0].$$

Apply (2.1) over B_n and Q_n to $(u - k_n)_-$, for the levels

$$k_n = \left(\frac{1}{2} + \frac{1}{2^n} \right) M.$$

The cutoff function ζ is taken to be of the form $\zeta(x,t) = \zeta_1(x)\zeta_2(t)$, where

$$\zeta_1 = \begin{cases} 1 & \text{in } B_{n+1}, \\ 0 & \text{in } \mathbb{R}^N - B_n, \end{cases} \qquad |D\zeta_1| \leq \frac{1}{\rho_n - \rho_{n+1}} = \frac{2^{n+1}}{\rho},$$

$$\zeta_2 = \begin{cases} 0 & \text{for } t < -\theta\rho_n^p, \\ 1 & \text{for } t \geq -\theta\rho_{n+1}^p, \end{cases} \qquad 0 \leq \zeta_{2,t} \leq \frac{1}{\theta(\rho_n^p - \rho_{n+1}^p)} \leq \frac{2^{p(n+1)}}{\theta\rho^p}.$$

The energy inequalities (2.1) with these stipulations yield

$$\operatorname*{ess\,sup}_{-\theta\rho_n^p < t < 0} \int_{B_n} (u - k_n)_-^2 \zeta^2(x,t) dx + \iint_{Q_n} |D(u - k_n)_- \zeta|^p dx d\tau$$

$$\leq \gamma \frac{2^{np}}{\rho^p} \left(\iint_{Q_n} (u - k_n)_-^p dx d\tau + \frac{1}{\theta} \iint_{Q_n} (u - k_n)_-^2 dx d\tau \right)$$

$$\leq \gamma \frac{2^{np}}{\rho^p} \left(M^p + \frac{M^2}{\theta} \right) |[u < k_n] \cap Q_n|.$$

By Proposition 3.1 of [2], Chapter I,

$$\iint_{Q_n} [(u - k_n)_- \zeta]^{p \frac{N+2}{N}} dx dt$$

$$\leq \iint_{Q_{n+1}} |D[(u - k_n)_- \zeta]|^p dx d\tau \left(\operatorname*{ess\,sup}_{-\theta\rho_n^p < t < 0} \int_{B_n(t)} [(u - k_n)_- \zeta]^2 dx \right)^{\frac{p}{N}}$$

$$\leq \gamma \left[\frac{2^{np}}{\rho^p} \left(M^p + \frac{M^2}{\theta} \right) \right]^{\frac{N+p}{N}} |[u < k_n] \cap Q_n|^{\frac{N+p}{N}}.$$

Estimate

$$\iint_{Q_n} [(u - k_n)_- \zeta]^{p \frac{N+2}{N}} dx dt \geq \left[2^{-(n+2)} M \right]^{p \frac{N+2}{N}} |[u < k_{n+1}] \cap Q_{n+1}|$$

and set

$$Y_n = \frac{|[u < k_n] \cap Q_n|}{|Q_n|}.$$

Then

$$Y_{n+1} \le \gamma b^n \left(\frac{\theta}{M^{2-p}} \right)^{\frac{p}{N}} \left(1 + \frac{M^{2-p}}{\theta} \right)^{\frac{N+p}{N}} Y_n^{1+\frac{p}{N}},$$

where

$$b = 2^{\frac{p}{N}[2(N+1)+p]} \quad \text{and} \quad \gamma = \gamma(\text{data}).$$

By Lemma 4.1 of [2], Chapter I, $\{Y_n\} \to 0$ as $n \to \infty$, provided that

$$Y_o \le \gamma_o \frac{\dfrac{M^{2-p}}{\theta}}{\left(1 + \dfrac{M^{2-p}}{\theta} \right)^{\frac{N+p}{p}}} \stackrel{\text{def}}{=} \nu, \qquad \text{where } \gamma_o = \frac{1}{\gamma^{\frac{N}{p}}} \frac{1}{b^{\left(\frac{N}{p}\right)^2}}.$$

Thus this choice of ν yields $Y_\infty = 0$, which is equivalent to (2.2). $\qquad\square$

2.2. A variant of Lemma 2.2.

Assume now that some information is available on the "initial data" relative to the cylinder $[(y,s) + Q_{2\rho}^+(\theta)]$, say, for example,

$$(2.3) \qquad\qquad u(x,s) \ge M \quad \text{for a.e. } x \in B_{2\rho}(y)$$

for some $M > 0$. Then, writing the energy inequalities (2.1) for $(u-k)_-$, for $k \le M$, over the cylinder $[(y,s) + Q_{2\rho}^+(\theta)]$, the integral extended over $B_{2\rho}$, at the time level $t = s$, vanishes in view of (2.3). Moreover, by taking cutoff functions $\zeta(x,t) = \zeta_1(x)$ independent of t, also the integral involving ζ_t on the right-hand side of (2.1) vanishes. We may now repeat the same arguments as in the previous proof for $(u - k_n)_-$, over the cylinders $Q_n^+ = B_n \times (0, \theta(2\rho)^p]$.

This leads to similar estimates as before, with Q_n replaced by Q_n^+, and with A_n replaced by $A_n^+ = [u < \xi_n M] \cap Q_n^+$. Proceeding as before gives

$$Y_{n+1}^+ \le \gamma b^n \left(\frac{\theta}{M^{2-p}} \right)^{\frac{p}{N}} (Y_n^+)^{1+\frac{p}{N}},$$

where $Y^+ = |A_n^+|/|Q_n^+|$ and $b = 2^{\frac{p}{N}(2N+2+p)}$. This in turn implies that $\{Y_n^+\} \to 0$ as $n \to \infty$, provided that

$$(2.4) \qquad\qquad Y_o^+ \le \delta \frac{M^{2-p}}{\theta}$$

for a constant $\delta \in (0,1)$ depending only upon the data, and independent of M, ρ, and θ. We summarise.

Lemma 2.3. *Let M be a positive number such that both* (2.3) *and* (2.4) *hold. Then*

$$u \ge \tfrac{1}{2} M \quad \text{a.e. in } B_\rho(y) \times (s, s + \theta(2\rho)^p].$$

Remark 2.4. Both Lemmas 2.2 and 2.3 are based on the energy estimates (2.1) and Proposition 3.1 of [2], Chapter I, which continue to hold in a stable manner for $p \to 2$. These results are therefore valid for all $1 < p \le 2$.

3. Transforming the variables and the PDE

Fix $(y,s) \in E_T$, and let $\rho > 0$ be so that the cylindrical domain

$$(3.1) \qquad\qquad B_{16\rho} \times \left(s - \epsilon M^{2-p} \rho^p , s \right) \subset E_T.$$

Introduce the change of variables and the new unknown function

$$(3.2) \qquad z = \frac{x-y}{\rho}, \qquad -e^{-\tau} = \frac{t-s}{\epsilon M^{2-p} \rho^p}, \qquad v(z,\tau) = \frac{1}{M} u(x,t) e^{\frac{\tau}{2-p}}.$$

This maps the cylinder in (3.1) into $B_{16} \times (0, \infty)$ and transforms (1.1) into

$$(3.3) \qquad v_\tau - \operatorname{div} \bar{\mathbf{A}}(z, \tau, v, D_z v) = \frac{1}{2-p} v \quad \text{weakly in } B_{16} \times (0, \infty),$$

where $\bar{\mathbf{A}}$ is a measurable vector-valued function of its arguments, satisfying the structure condition

$$(3.4) \qquad \begin{cases} \bar{\mathbf{A}}(z, \tau, v, D_z v) \cdot D_z v \geq \epsilon C_o |D_z v|^p \\ |\bar{\mathbf{A}}(z, \tau, v, D_z v)| \leq \epsilon C_1 |D_z v|^{p-1} \end{cases} \quad \text{a.e. in } B_{16} \times (0, \infty),$$

where C_o and C_1 are the constants in the structure conditions (1.2) and ϵ is the number appearing in (1.4). The assumption (1.3) of Theorem 1.1 becomes

$$(3.5) \qquad \left| [v(\cdot, \tau) \geq e^{\frac{\tau}{2-p}}] \cap B_1 \right| \geq \alpha |B_1| \quad \text{for all } \tau \in (0, +\infty).$$

Let $\tau_o > 0$ be chosen and set

$$k = e^{\frac{\tau_o}{2-p}}, \qquad \text{and } k_j = \frac{1}{2^j} k \quad \text{for } j = 0, 1, \dots.$$

With this symbolism (3.5) implies that

$$(3.6) \qquad \left| [v(\cdot, \tau) \geq k] \cap B_8 \right| \geq \alpha 8^{-N} |B_8| \quad \text{for all } \tau \in (\tau_o, +\infty).$$

Introduce the cylinders

$$Q_{\tau_o} = B_8 \times \left(\tau_o + k^{2-p}, \tau_o + 2k^{2-p} \right),$$
$$Q'_{\tau_o} = B_{16} \times \left(\tau_o, \tau_o + 2k^{2-p} \right)$$

and a piecewise smooth cutoff function on Q'_{τ_o} of the form $\zeta(z, \tau) = \zeta_1(z)\zeta_2(\tau)$, where

$$\zeta_1 = \begin{cases} 1 & \text{in } B_8 \\ 0 & \text{in } \mathbb{R}^N - B_{16} \end{cases} \qquad |D\zeta_1| \leq \frac{1}{8},$$

$$\zeta_2 = \begin{cases} 0 & \text{for } \tau < \tau_o \\ 1 & \text{for } \tau \geq \tau_o + k^{2-p} \end{cases} \qquad 0 \leq \zeta_{2,\tau} \leq \frac{1}{k^{2-p}}.$$

Energy estimates, over Q'_{τ_o}, are derived by taking $-(v - k_j)_- \zeta^p$ as a testing function in the weak formulation of (3.3). Discarding the non-positive contribution of the right-hand side, standard calculations give

$$\int_{Q'_{\tau_o}} |D(v - k_j)_- \zeta|^p dz d\tau \leq \gamma \int_{Q'_{\tau_o}} \left[(v - k_j)_-^p |D\zeta|^p + (v - k_j)_-^2 |\zeta_\tau| \right] dz d\tau,$$

where the constant γ depends only upon N, p, C_o, C_1 and the parameter ϵ appearing in the assumption (1.4) of Theorem 1.1 and the transformed structure conditions (3.4). From this

$$(3.7) \qquad \int_{Q_{\tau_o}} |D(v - k_j)_-|^p dz d\tau \leq 2\gamma k_j^p |Q_{\tau_o}|.$$

3.1. Estimating the measure of the set $[v < k_j]$ within Q_{τ_o}. Set

$$A_j(\tau) = [v(\cdot, \tau) < k_j] \cap B_8, \qquad A_j = [v < k_j] \cap Q_{\tau_o}.$$

By the measure-theoretical Lemma 2.2 of [2], Chapter I, and (3.6),

$$(k_j - k_{j+1})|A_{j+1}(\tau)| \leq \frac{\gamma(N)}{|B_8 - A_j(\tau)|} \int_{[k_{j+1} < v(\cdot, \tau) < k_j]} |Dv| dz$$

$$\leq \gamma(N, \alpha) \int_{[k_{j+1} < v(\cdot, \tau) < k_j]} |Dv| dz$$

for all $\tau \geq \tau_o$. Integrate this in $d\tau$ over $(\tau_o + k^{2-p}, \tau_o + 2k^{2-p})$, majorize the resulting integral on the right-hand side by the Hölder inequality, and use (3.7) to get

$$\frac{k_j}{2}|A_{j+1}| \leq \gamma(\text{data}, \alpha) \int_{[k_{j+1} < v < k_j]} |Dv| dz d\tau$$

$$\leq \gamma(\text{data}, \alpha) \left(\int_{[k_{j+1} < v < k_j]} |Dv|^p dz d\tau \right)^{\frac{1}{p}} |A_j - A_{j+1}|^{\frac{p-1}{p}}$$

$$\leq \gamma(\text{data}, \alpha, \epsilon) \left(\int_{Q_{\tau_o}} |D(v - k_j)_-|^p dz d\tau \right)^{\frac{1}{p}} |A_j - A_{j+1}|^{\frac{p-1}{p}}$$

$$\leq \gamma(\text{data}, \alpha, \epsilon) k_j |Q_{\tau_o}|^{\frac{1}{p}} |A_j - A_{j+1}|^{\frac{p-1}{p}}.$$

Taking the $p/(p-1)$-power yields the recursive inequalities

$$|A_{j+1}|^{\frac{p}{p-1}} \leq \gamma(\text{data}, \alpha, \epsilon) |Q|^{\frac{1}{p-1}} |A_j - A_{j+1}|.$$

Add these inequalities for $j = 0, 1, \ldots, j_* - 1$, where j_* is an integer to be chosen, and majorize the sum on the right-hand side by the corresponding telescopic series. This gives

$$(j_* - 1)|A_{j_*}|^{\frac{p}{p-1}} \leq \gamma(\text{data}, \alpha, \epsilon)|Q|^{\frac{p}{p-1}}.$$

Equivalently

$$(3.8) \qquad |[v < k_{j_*}] \cap Q_{\tau_o}| \leq \nu |Q_{\tau_o}|, \quad \text{where} \quad \nu = \left(\frac{\gamma(\text{data}, \alpha, \epsilon)}{j_*} \right)^{\frac{p-1}{p}}.$$

Remark 3.1. This estimation technically requires $p > 1$. The restriction however is not only technical in view of the geometrical significance of $(1.1)_o$ for $p = 1$ even in the elliptic case ([5]).

3.2. Segmenting Q_{τ_o}.
Assume momentarily that j_* and hence ν has been determined. By possibly increasing j_* to be not necessarily integer, we may assume, without loss of generality, that $(2^{j_*})^{2-p}$ is an integer. Then subdivide Q_{τ_o} into $(2^{j_*})^{2-p}$ cylinders, each of length $k_{j_*}^{2-p}$, by setting

$$Q_n = B_8 \times \left(\tau_o + k^{2-p} + n k_{j_*}^{2-p}, \tau_o + k^{2-p} + (n+1)k_{j_*}^{2-p} \right)$$

$$\text{for} \quad n = 0, 1, \ldots, (2^{j_*})^{2-p} - 1.$$

For at least one of these, say Q_n, we must have

$$|[v < k_{j_*}] \cap Q_n| \leq \nu |Q_n|.$$

Apply Lemma 2.2 to v over Q_n with $M = k_{j_*}$ and $\theta = k_{j_*}^{2-p}$. This gives

$$v\left(z, \tau_o + k^{2-p} + (n+1)k_{j_*}^{2-p}\right) \geq \tfrac{1}{2}k_{j_*} \quad \text{a.e. in } B_4$$

provided that

$$\frac{|[v < k_{j_*}] \cap Q_n|}{|Q_n|} \le 2^{-\frac{N+p}{p}} \bar{\gamma}_o(\text{data}, \epsilon) = \nu.$$

Now choose j_*, and hence ν, from this and (3.8). Summarising, for such a choice of j_*, and hence ν, there exists a time level τ_1 in the range

$$(3.9) \qquad\qquad \tau_o + k^{2-p} < \tau_1 < \tau_o + 2k^{2-p}$$

such that

$$v(z, \tau_1) \ge \sigma_o e^{\frac{\tau_o}{2-p}}, \qquad \text{where} \qquad \sigma_o = 2^{-(j_*+1)}.$$

Remark 3.2. Notice that j_* and hence ν are determined only in terms of the data and are independent of the parameter τ_o, which is still to be chosen.

3.3. **Returning to the original coordinates.** In terms of the original coordinates and the original function $u(x, t)$, this implies that

$$u(\cdot, t_1) \ge \sigma_o M e^{-\frac{\tau_1 - \tau_o}{2-p}} \overset{\text{def}}{=} M_o \qquad \text{in } B_{4\rho}(y),$$

where the time t_1 corresponding to τ_1 is computed from (3.2) and (3.9). Now apply Lemma 2.3 with M replaced by M_o over the cylinder

$$(t_1, y) + Q_{4\rho}^+ = B_{4\rho}(y) \times \left(t_1, t_1 + \theta(4\rho)^p\right].$$

By choosing

$$\theta = \delta M_o^{2-p}, \qquad \text{where} \qquad \delta = \delta(\text{data}),$$

the assumption (2.4) is satisfied, and Lemma 2.3 yields

$$(3.10) \qquad \begin{aligned} u(\cdot, t) &\ge \tfrac{1}{2} M_o = \tfrac{1}{2} \sigma_o M e^{-\frac{\tau_1 - \tau_o}{2-p}} \\ &\ge \tfrac{1}{2} \sigma_o e^{-\frac{2}{2-p} e^{\tau_o}} M \end{aligned} \qquad \text{in} \quad B_{2\rho}(y)$$

for all times

$$(3.11) \qquad\qquad t_1 \le t \le t_1 + \delta M_o^{2-p} (2\rho)^p.$$

If the right-hand side equals s, then (3.10) holds for all times in

$$(3.12) \qquad \left(s - \varepsilon M^{2-p}(2\rho)^p, s\right], \qquad \text{where} \quad \varepsilon = \delta \sigma_o^{2-p} e^{-e^{\tau_o}}.$$

Thus the conclusion of Theorem 1.1 holds, provided that the upper time level in (3.11) equals s. The transformed τ_o level is still undetermined, and it will be so chosen as to verify such a requirement. Precisely, taking into account (3.2),

$$\epsilon M^{2-p} \rho^p e^{-\tau_1} = -(t_1 - s) = \delta \sigma_o M^{2-p}(2\rho)^p e^{-(\tau_1 - \tau_o)} \quad \implies \quad e^{\tau_o} = \frac{\epsilon}{2^p \delta \sigma_o}.$$

This determines quantitatively $\tau_o = \tau_o(\text{data})$. The proof of Theorem 1.1 is now completed by inserting such a τ_o on the right-hand side of (3.10) and (3.12).

4. Stability for $p \approx 2$

The change of variables (3.2) and the subsequent arguments yield constants that deteriorate as $p \to 2$. The conclusion of Theorem 1.1 however continues to hold with constants that are "stable" as $p \to 2$. We indicate here the modifications of the arguments needed to achieve such a constant stability.

For $p \approx 2$, instead of (3.2) introduce the change of variables

$$(4.1) \qquad z = \frac{x-y}{\rho}, \qquad \tau = \frac{t-s}{\epsilon M^{2-p}\rho^p}, \qquad v(z,\tau) = \frac{1}{M}u(x,t).$$

This maps the cylinder in (3.1) into $B_{16} \times (-1,0)$ and transforms (1.1) into

$$(4.2) \qquad v_\tau - \operatorname{div}\bar{\mathbf{A}}(z,\tau,v,D_z v) = 0 \qquad \text{weakly in } B_{16} \times (-1,0),$$

where $\bar{\mathbf{A}}$ satisfies (3.4) over the indicated cylinder. The arguments of §3.1 are intended to lead to (3.8) so that j_* and hence ν could be chosen so as to satisfy the requirement (2.2) of Lemma 2.2, quantified by (2.1). Now the arguments of §3.1 can be repeated here with the proper modifications, where $\{Q_n\}$ now forms a nested collection of cubes in $B_8 \times (-1,0)$ with the same vertex at $(0,0)$, with k replaced by 1 and hence $k_j = 2^{-j}$. This leads to (3.8) with Q_{τ_o} replaced by $B_4 \times (-1,0)$. By Lemma 2.2 and (2.1), this would imply that $v \geq \frac{1}{2}2^{-j_*}$ in $B_2 \times (-\frac{1}{2},0)$ provided that

$$\left| [v < k_{j_*}] \cap B_4 \times (-1,0) \right| \leq \gamma_o \frac{2^{-j_*(2-p)}}{\left(1 + 2^{-j_*(2-p)}\right)^{\frac{N+p}{p}}}$$

for a constant $\gamma_o = \gamma_o(\text{data})$. From (3.8) with indicated modifications, this is possible if one can choose $j_* \in \mathbb{N}$ so that

$$\nu = \left(\frac{\gamma(\text{data},\alpha,\epsilon)}{j_*}\right)^{\frac{p-1}{p}} \leq \gamma_o \frac{2^{-j_*(2-p)}}{\left(1 + 2^{-j_*(2-p)}\right)^{\frac{N+p}{p}}}.$$

This is possible if p is sufficiently close to 2 by the following procedure. First choose $(2 - p_o) = 1/j_*$, and then j_* so large as to satisfy the indicated requirement for all $p_o \leq p \leq 2$.

5. Equations of porous-medium type

The techniques apply, by minor variants, to non-negative solutions of quasi-linear, singular, parabolic equations of the porous-medium type

$$(5.1) \qquad \begin{aligned} &u \in C_{\text{loc}}\big(0,T; L^{m+1}_{\text{loc}}(E)\big), \ |u|^m \in L^2_{\text{loc}}\big(0,T; W^{1,2}_{\text{loc}}(E)\big), \\ &u_t - \operatorname{div}\mathbf{A}(x,t,u,Du) = 0 \qquad \text{weakly in } E_T, \end{aligned} \qquad 0 < m < 1.$$

The functions $\mathbf{A} : E_T \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ are only assumed to be measurable and subject to the structure conditions

$$(5.2) \qquad \begin{cases} \mathbf{A}(x,t,u,Du) \cdot Du \geq C_o |u|^{1-m}\big|D|u|^m\big|^2 \\ |\mathbf{A}(x,t,u,Du)| \leq C_1\big|D|u|^m\big| \end{cases} \qquad \text{a.e. in } E_T,$$

where C_o and C_1 are given positive constants. The homogeneous prototype is

$$(5.1)_o \qquad u_t - \Delta|u|^{m-1}u = 0, \qquad 0 < m < 1, \qquad \text{weakly in } E_T.$$

Theorem 5.1. *Let u be a non-negative, local, weak solution to* (5.1)–(5.2) *satisfying*

$$(5.3) \qquad \left| [u(\cdot, t) > M] \cap B_\rho(y) \right| > \alpha |B_\rho|$$

for all times

$$(5.4) \qquad s - \epsilon M^{1-m} \rho^2 \leq t \leq s$$

for some $M > 0$ and some α and ϵ in $(0,1)$, and assume that

$$B_{16\rho} \times [s - \epsilon M^{1-m} \rho^2, s] \subset E_T.$$

There exist $\sigma \in (0,1)$ and $\varepsilon \in (0, \frac{1}{2}\epsilon]$, which can be determined a priori, quantitatively only in terms of the data, and the numbers α and ϵ, and independent of M, such that

$$(5.5) \qquad u(x,t) \geq \sigma M \quad \text{ for all } x \in B_{2\rho}(y)$$

for all times

$$(5.6) \qquad s - \varepsilon M^{1-m} \rho^2 < t \leq s.$$

Thus measure-theoretical information on the measure of the "positivity set" in $B_\rho(y)$ for all times in (5.4) implies that such a positivity set actually "expands" to $B_{2\rho}(y)$ for comparable times. Applications of such a "spread of positivity" to Harnack estimates for non-negative solutions of porous-medium equations of the type (5.1)–(5.2) are in [3, 4].

REFERENCES

1. Y.Z. Chen; E. DiBenedetto, Hölder estimates of solutions of singular parabolic equations with measurable coefficients. *Arch. Rational Mech. Anal.,* **118**(3) (1992), 257–271. MR1158938 (93a:35092)
2. E. DiBenedetto, *Degenerate Parabolic Equations.* Universitext, Springer–Verlag, New York, 1993. MR1230384 (94h:35130)
3. E. DiBenedetto; U. Gianazza; V. Vespri, Harnack Estimates for Quasi-Linear Degenerate Parabolic Differential Equations. *Acta Math.,* **200** (2008), 181–209. MR2413134 (2009g:35130)
4. E. DiBenedetto; U. Gianazza; V. Vespri, Forward, Backward and Elliptic Harnack Inequalities for Non-Negative Solutions to Certain Singular Parabolic Partial Differential Equations. *Ann. Scuola Norm. Super. Pisa Cl. Sci.* (5) (in press).
5. M. Miranda, Sul minimo dell'integrale del gradiente di una funzione. *Ann. Scuola Norm. Sup. Pisa,* **3**(19) (1965), 626–665. MR0188839 (32:6271)

DEPARTMENT OF MATHEMATICS, VANDERBILT UNIVERSITY, 1326 STEVENSON CENTER, NASHVILLE, TENNESSEE 37240
E-mail address: `em.diben@vanderbilt.edu`

DIPARTIMENTO DI MATEMATICA "F. CASORATI", UNIVERSITÀ DI PAVIA, VIA FERRATA 1, 27100 PAVIA, ITALY
E-mail address: `gianazza@imati.cnr.it`

DIPARTIMENTO DI MATEMATICA "U. DINI", UNIVERSITÀ DI FIRENZE, VIALE MORGAGNI 67/A, 50134 FIRENZE, ITALY
E-mail address: `vespri@math.unifi.it`