An introduction to the mathematical modelling of geothermal resources exploitation

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1. Introduction

This lecture aims at giving some ideas on a phenomenon that has a particular interest for its technological applications and for the implications in the econommic and ecological domains: the exploitation of geothermal resources for energy production.

As we will point out, a reliable mathematical model of the phenomenon could be an extremely useful tool to manage the industrial process and to ensure its sustainability. On the other hand, the process is so complex and the different mechanisms it involves are so many that it appears an almost desperate task to incorporate all of them in a tractable mathematical model. Indeed, heat conduction and convection, multiphase/multicomponent fluid motion in anisotropic porous media, vaporization, chemical reactions etc. are all entering the process ([3],[4]); thus the modelization consists primarily in identifying the phenomena that play the key role in the different practical situations that are encountered in different plants.

Let us start by describing the basic facts concerning geothermal energy ([1]).

It is well known that the earth mantle (i.e. the layer that is beneath the terrestrial crust) has temperatures of several hundreths of degrees Celsius. It is commonly believed that this temperature is maintained because of the nuclear reactions occuring in the mantle and in the earth core involving radioactive isotopes.

Figure 1:

Consequently, a thermal gradient always exists in the crust and its magnitude depends on the thickness of the crust itself which is not the same everywhere. Figure 1 shows the zones where the geothermal gradient is higher. These are the same zones where natural phenomena such as hot springs, geysers, etc. have been known (and sometimes used) for centuries. They often coincide with zones where volcanic activity is more intense. In the cases of hot springs and geysers, the common features are two: a source of thermal energy sufficiently close to the earth surface (stored in the rock at high temperature), and the presence of water as energy carrier.

The industrial use of geothermal energy is rather recent. The first (small) power plant exploiting geothermy was started one century ago in Larderello (Italy). Several "geothermal wells" were drilled afterwards in the same region so that they produce nowadays about $4 \cdot 10^9 \ Kwh$ of electric energy per year.

It is just in the late Fifties of past century that exploitation of geothermal energy starts outside Italy: in 1958 a plant was inaugurated in Wairakey (New Zealand). Today several plants are active throughout the world (see Figure 2) and electrical energy production by geothermal energy has increased by 300% over the last 20 years ([7]).

The corresponding reduction of CO_2 production (with respect to traditional plants) is of 100 million tons.

As we pointed out above in the case of natural phenomena, the exploitation of geothermal energy is only possible when a "carrier" is available to trans-
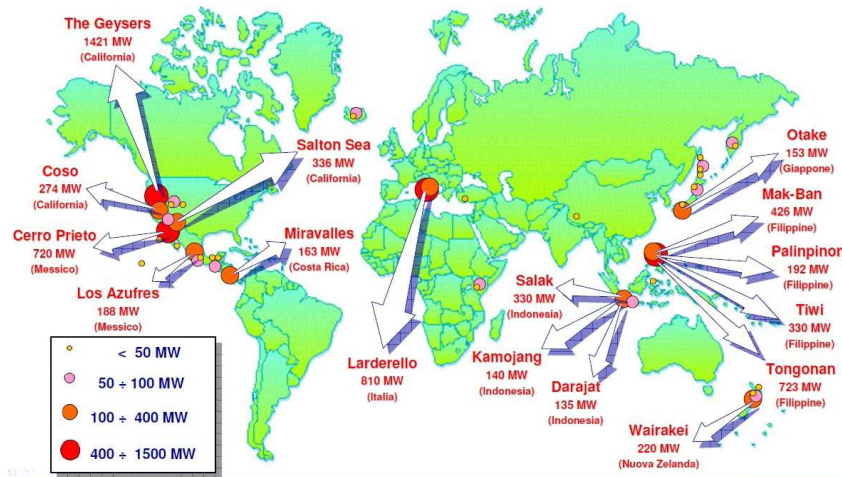
Figure 2:

port the energy to the earth surface. As a consequence the locations suitable for geothermal plants should be such that ([6]):

- the crust is sufficiently "thin" (so that the geothermal gradient is sufficiently "high" reaching values of several tenths of degrees per Km)

- there is a layer of porous rock (reservoir) where high pressure confined fluids circulate, and

- the reservoir is capped by an almost impermeable layer (clay) to prevent fluid and heat losses.

The depth at which exploitable reservoirs are found is between 2 and 4 Km. When a wall is drilled, i.e. when the impermeable layer is perforated and the reservoir is reached, hot vapour is extracted that operates turbines to produce electricity. This means that a geothermal plant can be defined as a renewable energy source only if the reservoir is suitably "recharged" by injecting water, since vapour extraction cannot be compensated just by atmospheric natural phenomena.

This is the reason why the regional government of Tuscany launched a research plan in order to get the scientific instruments enabling it to manage

and monitor the exploitation of the resource (that remain a public property) by the private societies that carry out the technology.

In this framework, our group was asked to:

1. set up a mathematical model to predict and to simulate the long-time behaviour of geothermal reservoirs in Tuscany under given industrial regimes of energy production;

2. to check the possible environmental impact of the extraction of deep geothermal fluids on the phreatic water;

3. to produce an user-friendly package to be used by governmental agencies in their control activity, such that it is suitably interfaced with the GIS database.

The members of the research group are: Fabio Rosso (also responsible for the coordination of the groups of experts in geology, ecology and information technology), Antonio Fasano, Angiolo Farina, Alessandro Speranza, Iacopo Borsi, Maurizio Ceseri, Luca Meacci, Matteo Cerminara.

2. Starting a simplified mathematical problem

The spirit of this meeting is aimed at presenting, mainly to young scientists and students, the variety of different applications of mathematical methods rather than very specialized results. Thus this presentation will isolate a single phenomenon from the complex interacting mechanisms entering the process and will display a model that is sufficiently simple so that formal complications do not hide some basic features, and sufficiently interesting from its mathematical aspects. Moreover, in this setting, some elements of the qualitative behaviour of the solution can be seen even without the help of numerical simulations.

With these goals in mind, we consider a one-dimensional geometry and assume that a homogeneous porous medium occupies the slab $0 < x < L$. Let ε be the porosity and denote by θ the liquid content and by p_s the saturation pressure. The latter is the pressure at which water and vapour coexist and depends on the temperature (Clapeiron's law). For temperature in the range $(500^o K, 700^o K)$, it can be found via the following approximating formula

$$p_s = 967.7 \exp\left[17.39\frac{T - 273}{T}\right] \tag{2.1}$$

(T in ^oK, p_s in Pa).

Let us consider an isothermal situation and express the water content (mass per unit volume of the porous medium) as a function of the pressure. It is

$$\theta_L = \varepsilon\phi(p - p_s)\rho^L \tag{2.2}$$

where ρ^L is the liquid density and ϕ is a function such that

$$\begin{cases} \phi(z) \in (0,1), & \phi'(z) > 0, \quad \text{for } z < 0 \\ \phi(x) = 1, & \text{for } z \geq 0 \end{cases} \tag{2.3}$$

From a practical point of view, as long as the width of the capillary fringe is negligible with respect to L, it is reasonable to approximate ϕ by a Heaviside jump function and thus assume that the porous medium is completely saturated by water for $p \geq p_s$ and occupied by vapour for $p < p_s$.

We will consider the following boundary value problem: for $t = 0$ the region $x \in (0, b)$ is occupied by liquid:

$$\theta_L(x, 0) = \varepsilon\rho^L(x, 0), \quad 0 < x < b, \tag{2.4}$$

while for $x \in (b, L)$ the vapour content is prescribed

$$\theta_V(x, 0) = \varepsilon\rho^V(x, 0) = \varepsilon\hat{\rho}^V(x), \quad b < x < L. \tag{2.5}$$

In (2.5) θ_V is the vapour content (mass per unit volume of the porous medium) and ρ_V is the vapour density.

On $x = 0$ water inflow (outflow) is prescribed for $t > 0$ as a linear function of the difference of the pressure at $x = 0$ and an external pressure p_0

$$p_x^L(0, t) = \gamma(p^L(0, t) - p_0), \quad t > 0, \tag{2.6}$$

where we implicitly assumed the validity of Darcy's law that expresses the volumetric flux as a linear function of the pressure gradient.

From now on, we neglect the influence of gravity although to take it into account just needs some formal complications. Of course this approximation can be misleading whenever the pressure gradient (in the liquid) is of the same order of magnitude of the gravimetric gradient; in such a case the (volumetric) flux is proportional to the gradient of the "hydraulic head"

$$h(x, t) = p(x, t) - \rho gx$$

if x is the vertical axis pointing upwards.

In any case, neglecting the influence of gravity on the motion of the fluid (besides of being an approximation that could be released with little additional work) is less restrictive that the assumption we did considering isothermal situation. This is why the problem we will study is just focussed on a single aspect of the problem we described in the introduction, whose modellization is much more complicated.

On $x = L$ we impose that the mass flow of the vapour is proportional to the difference between the vapour pressure p^V at $x = L$ and an external pressure p_1, and we write

$$\rho^V(L,t)p_x^V(L,t) = G(p_1 - p^V(L,t)), \quad t > 0. \tag{2.7}$$

We will look for a solution such that, for any t in a suitable interval $(0,T)$, there exists a function $s(t)$ with values in $(0,L)$ such that the surface $x = s(t)$ separates the regions occupied by liquid $(x < s(t))$ and by vapour $(x > s(t))$.

The pressure has to be continuous on $x = s(t)$:

$$p^V(s(t)^+,t) = p^L(s(t)^-,t) = p_S, \quad t \in (0,T). \tag{2.8}$$

Moreover the mass balance requires

$$\varepsilon[\rho^L(s(t)^-,t) - \rho^V(s(t)^+,t)]\dot{s}(t) = k[(\rho^V p_x^V)^+ - (\rho^L p_x^L)^-], t \in (0,T) \tag{2.9}$$

where k is the hydraulic conductivity, that will be assumed to be constant and we used the following notation:

$$\begin{aligned} (\rho^V p_x^V)^+ &= \rho^V(s(t)^+,t)p_x^V(s(t)^+,t); \\ (\rho^L p_x^L)^- &= \rho^L(s(t)^-,t)p_x^L(s(t)^-,t). \end{aligned} \tag{2.10}$$

In the liquid region

$$D_T^L = \{(x,t) : 0 < x < s(t),\ 0 < t < T\},$$

and in the vapour region

$$D_T^V = \{(x,t) : s(t) < x < L,\ 0 < t < T\},$$

we have to impose the respective mass balance equations.

If we impose that water is incompressible, we have that liquid pressure is divergence-free, i.e.

$$p_{xx}^L = 0, \quad \text{in } D_T^L. \tag{2.11}$$

On the other hand, mass conservation and Darcy's law in the vapour region are expressed by

$$\varepsilon \frac{\partial \rho^V}{\partial t} = k \frac{\partial}{\partial x}\left(\rho^V \frac{\partial p^V}{\partial x}\right), \quad \text{in } D_T^V. \qquad 2.12$$

Equation (2.12) has to be complemented with an equation of state for the vapour. If we take the perfect gas equation (which is a reasonable approximation) and we recall that the temperature is constant, we have

$$p^V(x,t) = rT_0 \rho^V(x,t), \text{in } D_T^V, \qquad (2.13)$$

where r is the universal gas constant divided by the molar mass of water. **3.**

Discussing time scales

First of all we have to give some necessary conditions on the data for the existence of physically meaningful solutions, i.e. of solutions like that

$$p^V(x,t) \le p_S, \text{ in } D_T^V, \qquad (3.1)$$

$$p^L(x,t) \ge p_S, \text{ in } D_T^L. \qquad (3.2)$$

To guarantee (3.2), we just need that p_x^L (that does not depend on x, because of (2.10)) is negative and thus:

$$p_0 \ge p_S \qquad (3.3)$$

(the case $p_0 = p_S$ corresponds to the particular case $p^L \equiv p_S$).

To guarantee (3.1) we need to impose conditions on p_1 and on the initial datum i.e. on $p^V(x,0)$.

We have

Proposition *Assume*

$$p_1 \le p^V(x,0) \le p_S, \quad s(0) < x < L. \qquad (3.4)$$

Then

$$p_1 \le p^V(x,t) \le p_S, \text{ in } D_T^V. \qquad (3.4)$$

P r o o f. If $p_1 = p_S$, (3.5) follows immediately from the (weak) maximum principle (see e.g. [5]).

For any $\delta \in (0, p_1)$ assume that is exists

$$\bar{t} = \sup\{t : p^V(x,t) > p_1 - \delta\}.$$

It would be, necessarily

$$p^V(L, \bar{t}) = p_1 - \delta,$$

but this would mean that p_v reaches in $(L, \bar{t})$ the minimum value in $D_{\bar{t}}^V$, where the equation (2.12) is uniformly parabolic. But, at the same point we would have, because of (2.7)

$$(p_1 - \delta)p_x^V(L, \bar{t}) = G\delta > 0,$$

a contradiction. Since δ is arbitrary, the first inequality in (3.5) is proved.

At this point the proof of the second inequality is straightforward, since if $p^V(L, t)$ would reach the value p_S, then p_x would be negative. ∎

Next, we find $p^L(x, t)$ in D_T^L from (2.11), (2.6), (2.8). It is

$$p^L(x, t) = p_S + \frac{\gamma(p_S - p_0)}{1 + \gamma s(t)}(x - s(t)), \quad 0 < x < T. \tag{3.6}$$

and we have to solve the following free boundary problem satisfied by the unknown density of vapour $\rho(x, t)$, where we dropped the suffix v to simplify notation:

Problem (P): To find

- $T > 0$;

- $s(t) \in C[0, T] \cup C^1(0, T)$, $0 < s(t) < L$, $t \in (0, T)$;

- $\rho \in C^{2,1}(D_T) \cup C(\bar{D}_T)$, $\rho_x \in C(\bar{D}_T - \{x = 0\})$

- the following equalities are satisfied:

$$\frac{\partial \rho}{\partial t} = rT_O \frac{k}{\varepsilon} \frac{\partial}{\partial x}\left(\rho \frac{\partial \rho}{\partial x}\right), \quad \text{in } D_T, \tag{3.7}$$

$$s(0) = b \in (0, L), \quad \rho(x, 0) = \hat{\rho}(x), \quad b < x < L, \tag{3.8}$$

$$\rho(L, t)\rho_x(L, t) = G\left(\frac{p_1}{rT_O} - \rho(L, t)\right), \quad 0 < t < T, \tag{3.9}$$

$$\rho(s(t), t) = \frac{p_S}{rT_O} \equiv \rho_O, \quad 0 < t < T \tag{3.10}$$

$$\dot{s}(t) = \frac{k}{\varepsilon(\rho_L - \rho_O)}\left(p_S\rho_x(s(t), t) + \rho_L\gamma\frac{p_O - p_S}{1 + \gamma s(t)}\right), \quad 0 < t < T. \tag{3.11}$$

We normalize x with resepct to L, ρ with respect to $\rho_o = p_S/(rT_O)$ and time with respect to a characteristic time $\hat{t}$ that will be chosen later. We call x^*, t^*, ρ^* the normalized quantities and we write:

$$\frac{\partial \rho^*}{\partial t^*} = \hat{t}\frac{k}{\varepsilon}\frac{p_S}{L^2}\frac{\partial}{\partial x^*}\left(\rho^*\frac{\partial \rho^*}{\partial x^*}\right) \tag{3.12}$$

in $D^* \equiv \{(x^*, t^*) : s(t^*) < x^* < 1, 0 < t^* < T^*\}$.

$$s^*(0) = b^* \equiv \frac{b}{L} \in (0,1), \rho^*(x,0) = \frac{\hat{\rho}(x^*)}{\rho_o} = \tilde{\rho}(x^*), b^* < x^* < 1, \tag{3.13}$$

$$2\rho^*(1,t^*)\frac{\partial \rho^*}{\partial x^*}(1,t^*) = H\left(\frac{p_1}{p_S} - \rho^*(1,t^*)\right), 0 < t^* < T^*, \tag{3.14}$$

$$\rho^*(s^*(t^*),t^*) = 1, \quad 0 < t^* < T^*, \tag{3.15}$$

$$\frac{ds^*}{dt^*} = \hat{t}\frac{k}{\varepsilon}\frac{p_S}{L^2}\frac{\rho_O}{\rho_L - \rho_O}\left(\frac{\partial \rho^*}{\partial x^*}(s^*(t^*),t^*) + \frac{\frac{\rho_L}{\rho_O}j\frac{p_O}{p_S} - 1}{1 + js^*(t^*)}\right), 0 < t^* < T^* \tag{3.16}$$

where we wrote

$$H = 2GL\rho_O, \tag{3.17}$$

$$j = \gamma L. \tag{3.18}$$

The problem has clearly two time scales, one for the diffusion of vapour, the other for the movement of the interphase. The ratio between them is $\rho_O/(\rho_L - \rho_O) \approx \rho_O/\rho_L$, i.e. about 10^{-3}. This makes it reasonable to take a quasi-steady approximation by choosing:

$$\frac{1}{\hat{t}} = \frac{k}{\varepsilon L^2}p_S\frac{\rho_O}{\rho_L - \rho_O}, \tag{3.19}$$

and writing

$$\frac{\partial}{\partial x^*}\left(\rho^*\frac{\partial \rho^*}{\partial x^*}\right) = 0, \text{ in } D^*, \tag{3.20}$$

From now on we drop the $*$ to simplify notation. We have the following problem:

$$\frac{\partial}{\partial x}\rho^2(x,t) = \phi(t), \text{ in } D^*, \tag{3.21}$$

$$2\rho(1,t)\rho_x(1,t) = H\left(\frac{p_1}{p_S} - \rho(1,t)\right), \quad t \in (0,T) \tag{3.22}$$

$$\rho(s(t),t) = 1, \quad t \in (0,T) \tag{3.23}$$

$$\dot{s}(t) = \rho_x(s(t), t) + \frac{A}{1 + js(t)}, \quad s(0) = b, \ t \in (0, T) \qquad (3.24)$$

where $\phi(t)$ is an unknown function of time only and

$$A = \frac{\rho_L}{\rho_O} \gamma L \frac{p_O - p_S}{p_S}. \qquad (3.25)$$

In the next section we will first analyze problem (3.21)-(3.24) in some special cases, and then we will discuss in general this quasi-steady approximation.

Recall that, according to assumptions (3.2), (3.3), the rescaled density satisfies the inequalities:

$$\frac{p_1}{p_S} \le \rho(x, t) \le 1. \qquad (3.26)$$

4. The quasi-steady case

A very simple situation corresponds to the case in which the second term in (3.24) is dominant. This means that the effect of the vapour pressure on the motion of the interface is negligible and this is in the same spirit of the Green-Ampt approximation.

In this case $s(t)$ is monotonically increasing and we have immediately

$$s(t) = \frac{-1 + \sqrt{(1 + jb)^2 + 2jAt}}{j}, \quad t \in (0, T). \qquad (4.1)$$

On the other hand, (3.21) and (3.22) give

$$\rho^2(x, t) = \phi(t)(x - s(t)) + 1, \qquad (4.2)$$

with

$$\phi(t) = H \left[\frac{p_1}{p_S} - \sqrt{\phi(t)(1 - s(t) + 1)} \right] \qquad (4.3).$$

Consequently, $\phi(t)$ is the negative solution of the equation

$$\phi^2 - \left[H^2(1 - s) + 2H \frac{p_1}{p_S} \right] \phi - H^2 \left(1 - \frac{p_1^2}{p_S^2} \right) = 0. \qquad (4.4)$$

It is worthwhile noting that the solution of (4.4) is such that

$$\phi(t)(1 - s(t)) > -1. \qquad (4.5)$$

Indeed, it is sufficient to evaluate the quadratic expression on the left hand side of (4.4) for $\phi = -1/(1 - s(t))$, obtaining $\left(\frac{1}{1-s(t)} + H\frac{p_1}{p_S}\right)^2$. Since this value is positive, we are on the left of the negative solution of the equation.

We can also note that the negative solution ϕ^- of (4.4) is on the right of the solution of the equation

$$x^2 - 2H\frac{p_1}{p_S}x - H^2\left(1 - \frac{p_1^2}{p_S^2}\right) = 0 \qquad (4.6),$$

i.e.

$$|\phi^-| < H\frac{p_S - p_1}{p_S}. \qquad (4.7)$$

Of course, the solution given by (4.1) and by (4.2) (where ϕ is the solution ϕ^- of (4.4)) exists in the time interval $(0, T)$ where

$$T = \sup(t : \ s(t) < 1). \qquad (4.8)$$

Moreover to be consistent with the assumption (Green-Ampt like) of negligible influence of vapour on the motion of the interface we also have to impose the additional requirement

$$|\rho_x(s(t), t)| << \frac{A}{1 + js(t)}, \quad 0 < t < T. \qquad (4.9)$$

Since $\rho(s(t), t) = 1$, this corresponds to

$$\phi^- << 2\frac{A}{1 + jb}. \qquad (4.10)$$

Just to play a little more with this solution we may note that for

$$t << \frac{(1 + js_0)^2}{2Aj} \qquad (4.11)$$

we have

$$s(t) \approx b + \frac{A}{1 + jb}t. \qquad (4.12)$$

Now we want to consider the case in which the first term in the r.h.s. of (3.24) is not negligible. Note that the order of magnitude of the term $\frac{A}{1+js(t)}$ depends on the product of $\frac{\rho_L}{\rho_O}$ and $\gamma L\frac{p_O - p_S}{p_S}$ and it can be $O(1)$ although $\rho_L/\rho_O = 10^3$. Thus this is not in contradiction with the assumption of quasi-steady process. A special interesting case is the one in which $\gamma = 0$ (i.e.

the wall $x = 0$ is impermeable) and the motion of the free boundary is governed by p_1.

Note that (4.2) and (4.3) are still valid and that

$$\rho_x(s(t), t) = \frac{\phi(t)}{2\rho(s(t), t)} = \frac{\phi(t)}{2}. \tag{4.13}$$

Thus, the motion of the free boundary is given by

$$\dot{s}(t) = \frac{\phi^-}{2} + \frac{A}{1 + js(t)}. \tag{4.14}$$

Since ϕ^- is given in terms of $s(t)$ (4.14) can be integrated to solve the problem.

It could be of some interest to note that if

$$H\frac{p_S - p_1}{p_S} << 1 \tag{4.15}$$

we can get an estimate of ϕ^- that is more accurate than (4.7). Indeed, (4.3) gives for $|\phi| << 1$

$$\phi^- \approx H\frac{p_1 - p_S}{p_S}\frac{2}{1 + 2H(1 - s(t))} \tag{4.16}$$

and thus the influence of vapour (governed by $p_S - p_1$) contrasts the influence of liquid (governed by $p_O - p_S$) and slows down the front, and possibly make $\dot{s}$ become negative.

Of course, the interval of existence $(0, T)$ is now defined as

$$T = \sup(t : \ s(t) \in (0, 1)).$$

5. Final remarks

If, instead of prescribing the condition (2.6) one prescribes the values of the pressure (or the flux) for $x = 0$, the solution is even simpler. Of course, in case of Neumann boundary conditions this corresponds to substitute the second term on the right hand side of (3.24) by a given constant (or a given function of time if the prescribed flux is variable).

In case of Dirichlet boundary conditions such term is replaced by a given function of time $B(t)$ divided by $s(t)$, so that in the Green-Ampt approximation we find, instead of (4.1)

$$s^2(t) = b^2 + 2 \int_0^t B(\tau)d\tau. \tag{5.1}$$

Also in the solution of the problem for the vapour pressure, prescribing Dirichlet or Neumann boundary conditions yield significant simplifications that are left to the reader.

We want to conclude the paper with some remarks on the complete problem (i.e. when the quasi-steady approximation is released) that has, in any case an interest from a purely mathematical point of view.

Just to show how this case can be dealt with, we confine our attention to the case in which the surface $x = 0$ is impermeable (i.e. $\gamma = 0$ or $p_O = p_S$), and Dirichlet datum is prescribed on $x = 1$.

Then we refer to the coordinate

$$y = 1 - x \tag{5.2}$$

and we write

$$\sigma(t) = 1 - s(t). \tag{5.3}$$

Thus the problem to be studied is (with obvious definitions of symbols) the following

$$\frac{\partial \rho}{\partial t} = a \frac{\partial}{\partial y} \left(\rho \frac{\partial \rho}{\partial y} \right) \quad 0 < y < \sigma(t),\ t > 0 \tag{5.4}$$

$$\sigma(0) = \sigma_0 > 0, \quad \rho(y, 0) = h(y), \quad 0 < y < \sigma(t) \tag{5.5}$$

$$\rho(0, t) = f(t), \quad 0 < \delta_0 \le f(t) \le 1,\ t > 0, \tag{5.6}$$

$$\rho(\sigma(t), t) = 1, \quad t > 0, \tag{5.7}$$

$$\dot{\sigma} = \rho_y(\sigma(t), t), \quad t > 0, \tag{5.8}$$

(we look for a classical solution in the usual sense).

Of course, the proposition we proved at the beginning of Sec. 3 is still valid, so that equation (5.4) is uniformly parabolic.

Thus (5.4)-(5.8) is a problem of Stefan type with monotonic free boundary, the only difference with respect to the classical Stefan problem being the nonlinearity of the equation.

Results of well-posedness can be obtained e.g. by ([2]). The cases with different boundary conditions can be dealt with as well.

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