CONVERGENCE IN SHAPE OF STEINER SYMMETRIZATIONS

GABRIELE BIANCHI, ALMUT BURCHARD, PAOLO GRONCHI, AND ALJOŠA VOLČIČ

Abstract. It is known that the iterated Steiner symmetrals of any given compact sets converge to a ball for most sequences of directions. However, examples show that Steiner symmetrization along a sequence of directions whose differences are square summable does not generally converge. Here we show that such sequences converge *in shape*. The limit need not be an ellipsoid or even a convex set.

We also consider uniformly distributed sequences of directions, and extend a recent result of Klain on Steiner symmetrization along sequences chosen from a finite set of directions.

1. Introduction

Steiner symmetrization is often used to identify the ball as the solution to geometric optimization problems. Starting from any given body, one can find sequences of iterated Steiner symmetrals that converge to the centered ball of the same volume as the initial body. If the objective functional improves along the sequence, the ball must be optimal.

Most constructions of convergent sequences of Steiner symmetrizations rely on auxiliary geometric functionals that decrease monotonically along the sequence. For example, the perimeter and the moment of inertia of a convex body decrease *strictly* under Steiner symmetrization unless the body is already reflection symmetric [15, 6], but for general compact sets, there are additional equality cases. The (essential) perimeter of a compact set decreases strictly under Steiner symmetrization in most, but not necessarily all directions $u \in S^{n-1}$, unless the set is a ball [7]. Steiner symmetrization in an arbitrary direction strictly decreases the moment of inertia, unless the set is already reflection symmetric up to a set of measure zero.

Recently, several authors have studied how a sequence of Steiner symmetrizations can fail to converge to the ball. This may happen, even if the sequence of directions is dense in S^{n-1}. Steiner symmetrizations of a convex body along any dense sequence of directions can be made to converge or diverge just by re-ordering [4], and any given sequence of Steiner symmetrizations (convergent or not) can be realized as a subsequence of a non-convergent sequence [5, Proposition 5.2].

In contrast, a sequence of Steiner symmetrizations that uses only finitely many distinct directions always converges [11]. The limit may be symmetric under all rotations or under a non-trivial subgroup, depending on the algebraic properties of those directions that appear infinitely often.

A number of authors have studied Steiner symmetrizations along random sequences of directions. If the directions are chosen independently, uniformly at random on the unit sphere, then the corresponding sequence of Steiner symmetrals converges almost surely to the ball simultaneously for all choices of the initial set [14, 16, 17]. Others have investigated the rate of convergence of random and non-random sequences [3, 5, 8, 12].

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We will address several questions that were raised in these recent papers. The examples of non-convergence presented there use sequences of Steiner symmetrizations along directions where the differences between successive angles are square summable. Our main result, Theorem 2.2, says that such sequences will converge if the Steiner symmetrizations are followed by suitable rotations. Convergence occurs both in Hausdorff distance and in symmetric difference. The limit is typically not an ellipsoid (or a convex set) unless the sequence starts from an ellipsoid (or a convex set, respectively). Some relevant examples and the statement of the theorem are contained in Section 2.

The proof of the theorem poses two technical challenges: to show convergence of a sequence of symmetrals to an unknown limit, rather than a ball; and to show convergence in Hausdorff distance for an arbitrary compact initial set. This is more delicate than convergence in symmetric difference, because Steiner symmetrization is not continuous and Lebesgue measure is only upper semicontinuous on compact sets (see for instance [9, p. 170]).

In Section 3, we collect the tools to address these challenges. Lemmas 3.1 and 3.2 relate convergence of a sequence of compact sets in Hausdorff distance to convergence of their parallel sets in symmetric difference. For sequences of Steiner symmetrals, convergence in Hausdorff distance implies convergence in symmetric difference; in particular, the limit has the same measure as the initial set. To address the geometric problem of identifying the limits of convergent subsequences, we use the functionals

$$\mathcal{I}_p(K) = \lambda_n(\{x : d(x, K) \leq \delta, |x - p| \geq r\}),$$

where K is a compact set, p a point in $\mathbb{R}^n$, and $r, \delta \geq 0$. We show that $\mathcal{I}_p(K)$ decreases under simultaneous Steiner symmetrization of K and p, with equality for all $\delta, r > 0$ only if K and p agree with their Steiner symmetrals up to a common translation, see (3.4) and Lemma 3.3. By allowing $p \neq o$, we obtain information about the intersections of the limiting shape with a family of non-centered balls and half-spaces. Lemma 3.4 implies that these intersections uniquely determine the shape.

In Section 4, we combine the lemmas to prove Theorem 2.2. It will be apparent from the proof that similar results should hold for other classical rearrangements. Lemmas 3.2 and 3.3 remain valid for every rearrangement that satisfies (2.1) and (2.2), including the entire family of cap symmetrizations studied by van Schaftingen [16], in particular polarization, spherical symmetrization, and the Schwarz rounding process.

Lemmas 3.2 and 3.3 are also useful for establishing convergence of Steiner symmetrals in Hausdorff distance in other situations, without the customary convexity assumption on the initial set. In the remaining two sections, we illustrate this with two more examples and pose some open questions.

In Section 5, we consider Steiner symmetrization in the plane along non-random sequences of directions that are uniformly distributed (in the sense of Weyl) on S^1, a property more restrictive than being dense. Theorem 5.1 shows that a sequence of Steiner symmetrals along a Kronecker sequence of direction always converges to a ball. In the opposite direction, we give examples where convergence to a ball fails for certain uniformly distributed sequences.

Finally, Section 6 is dedicated to a recent result of Klain [11] on Steiner symmetrization along sequences chosen from a finite set of directions. Klain proves that when K is a convex body the sequence of Steiner symmetrals always converges. We extend this result to compact sets.

2. Main Results

We start with some definitions. Let o denote the origin in $\mathbb{R}^n$. For $p \in \mathbb{R}^n$ and $r > 0$, let $B_{r,p}$ denote the closed ball of radius r centered at p. If $p = o$, we drop the second subscript and write simply B_r. We write λ_k for the Lebesgue measure on k-dimensional subspaces of $\mathbb{R}^n$. Directions in $\mathbb{R}^n$ are identified with unit vectors $u \in S^{n-1}$, and $u^\perp$ refers to the $(n-1)$-dimensional subspace orthogonal to u.

Let K and L be compact sets in $\mathbb{R}^n$. For $\delta > 0$, we denote by $K_\delta = K + B_\delta$ the *outer parallel set* of K. The distance between compact sets will be measured in the *Hausdorff metric*, defined by

$$d_H(K, L) = \inf\big\{\delta > 0 \mid K \subset L_\delta \text{ and } L \subset K_\delta\big\}\,.$$

Another measure of the distance between K and L is their *symmetric difference distance* defined as $\lambda_n(K \triangle L)$. Note that this distance function does not distinguish between sets whose symmetric difference has measure zero. We will say that a sequence of compact sets (K_m) *converges in symmetric difference* to a compact set L, if

$$\lim_{m\to\infty} \lambda_n(K_m \triangle L) = 0\,.$$

Given a direction $u \in S^{n-1}$, let $\mathcal{S}_u K$ denote the *Steiner symmetral* of K along u. The mapping $\mathcal{S}_u$ that sends each set to its symmetral is called *Steiner symmetrization.* We use here the variant that maps compact sets to compact sets, which is defined as follows. Denote by ℓ_y the line parallel to u through the point $y \in u^\perp$. If $\ell_y \cap K$ is non-empty, then $\ell_y \cap \mathcal{S}_u K$ is the closed line segment of the same one-dimensional measure centered at y; if the measure is zero, the line segment degenerates to a single point. Otherwise, ℓ_y intersects neither K nor $\mathcal{S}_u K$. Clearly, $\mathcal{S}_u K$ is symmetric under reflection at $u^\perp$.

By Cavalieri's principle, the symmetral $\mathcal{S}_u K$ has the same Lebesgue measure as the original set K. It is well known that Steiner symmetrization preserves convexity, compactness, and connectedness, and that it respects inclusions and reduces perimeter. For more information about Steiner symmetrization, we refer the reader to the book of Gruber [9, Chapter 9].

We have that $\mathcal{S}_u(K \cap L) \subset \mathcal{S}_u K \cap \mathcal{S}_u L$, since $\mathcal{S}_u$ respects inclusions. By writing $\lambda_n(\mathcal{S}_u K \setminus \mathcal{S}_u L)$ as $\lambda_n(\mathcal{S}_u K) - \lambda_n(\mathcal{S}_u K \cap \mathcal{S}_u L)$, this inclusion relation implies

$$\lambda_n(\mathcal{S}_u K \setminus \mathcal{S}_u L) \le \lambda_n(K \setminus L), \tag{2.1}$$

an inequality that we repeatedly use. It also implies that

$$\lambda_n(\mathcal{S}_u K \triangle \mathcal{S}_u L) \le \lambda_n(K \triangle L),$$

which means that Steiner symmetrization is continuous in the symmetric difference metric on the space of compact sets modulo sets of measure zero.

Moreover $\mathcal{S}_u K + \mathcal{S}_u L \subset \mathcal{S}_u\,(K + L)$, see [9, Proposition 9.1(iii)]. This in particular implies that for $\delta > 0$

$$\left(\mathcal{S}_u K\right)_\delta \subset \mathcal{S}_u K_\delta. \tag{2.2}$$

The following observation motivates our main result.

Example 2.1 (Non-convergence)**.** Let (α_m) be a sequence in $(0, \pi/2)$ with

$$\sum_{m=1}^{\infty} \alpha_m = \infty\,, \qquad \sum_{m=1}^{\infty} \alpha_m^2 < \infty\,, \tag{2.3}$$

and set $\gamma = \prod_{m=1}^{\infty} \cos\alpha_m$. For each positive integer m, let $\beta_m = \sum_{k=1}^{m} \alpha_k$ and $u_m = (\cos\beta_m, \sin\beta_m)$.

Note that $\gamma \in (0,1)$. Indeed (2.3) implies that $\alpha_m \in (0,1)$ for each m larger than a suitable ν. We have $\cos\alpha_m \geq 1-\alpha_m^2/2$, and we also have $\ln(1-x) \geq -(1+o(1))x$ when $x \in (0,1/2)$. Thus

$$\lim_{n\to\infty}\sum_{m=1}^{n}\ln\cos\alpha_m \geq \sum_{m=1}^{\nu}\ln\cos\alpha_m - (1+o(1))\lim_{n\to\infty}\sum_{m=\nu+1}^{n}\frac{\alpha_m^2}{2} > -\infty.$$

This inequality implies $\gamma > 0$ because $\gamma = \lim_{n\to\infty} e^{\sum_{m=1}^{n}\ln\cos\alpha_m}$.

Let K be a convex body that has area smaller than a disc of diameter γ and contains a vertical line segment ℓ of length 1. Apply the sequence of Steiner symmetrizations $\mathcal{S}_{u_m}$ to K and ℓ to obtain a sequence of convex bodies K_m and line segments ℓ_m. Each symmetrization $\mathcal{S}_{u_m}$ projects the previous line segment ℓ_{m-1} onto $u_m^\perp$, thereby multiplying its length by $\cos\alpha_m$. Since β_m diverges, the segments ℓ_m spin in circles forever while their length decreases monotonically to γ.

For each m, the diameter of K_m exceeds γ, because $K_m \supset \ell_m$. If the sequence converges, its limit must contain a disc of diameter γ. On the other hand its area equals that of K, a contradiction. □

It turns out that the sequences from Example 2.1 *converge in shape*, in the sense that there exist isometries $\mathcal{I}_m$ such that $(\mathcal{I}_m\mathcal{S}_{u_m}\dots\mathcal{S}_{u_1}K)$ converges for each compact set K. The sequence $(\mathcal{I}_m)$ depends only on (u_m).

Theorem 2.2. *Let (u_m) be a sequence in S^{n-1} with $u_{m-1}\cdot u_m = \cos\alpha_m$, where (α_m) is a sequence in $(0,\pi/2)$ that satisfies $\sum_{m=1}^\infty \alpha_m^2 < \infty$. There exists a sequence of rotations $(\mathcal{R}_m)$ such that for every non-empty compact set $K \subset \mathbb{R}^n$, the rotated symmetrals*

$$K_m = \mathcal{R}_m\mathcal{S}_{u_m}\dots\mathcal{S}_{u_1}K \tag{2.4}$$

converge in Hausdorff distance and in symmetric difference to a compact set L.

What can we say about the limit of the sequence (K_m) in (2.4)? Since Steiner symmetrization transforms ellipsoids into ellipsoids, one may wonder whether the limit is always an ellipsoid [4, 11]. The following examples show that this is not the case.

Example 2.3 (The limit need not be an ellipse)**.** Let the sequence (u_m) and $\gamma = \prod_{m=1}^\infty \cos\alpha_m$ be as in Example 2.1. Observe that dropping, if necessary, a few initial terms we can make $\sum_{m=1}^\infty \alpha_m^2$ arbitrarily small and hence γ arbitrarily close to 1. In particular we may suppose $\gamma > 2/\pi$. Let K be the convex envelope of the line segment ℓ from Example 2.1 and a centered ball B_r for some $r > 0$. If the sequence (2.4) converges, its limit contains both B_r and a line segment of length γ. Any ellipse that contains these sets has area at least $\pi\gamma r/2$. On the other hand its area agrees with the area of K, which is bounded from above by $r/\sqrt{1-4r^2}$, the area of the rhombus circumscribed to the circle centered at o whose longer diagonal is a segment of length 1. Since this is smaller than $\pi\gamma r/2$ if r is small enough, by our choice of γ, the limit cannot be an ellipse. □

Example 2.4 (The limit can be non-convex)**.** Take the sequence (u_m) as in Example 2.1, and let K be the union of a line segment ℓ and a ball B_r. The limit of the sequence (2.4) contains B_r and a line segment of length γ. Any convex set that contains these sets has area at least $\gamma r/2$. Since the area of K is πr^2, the limit cannot be a convex set if $\pi r < \gamma/2$. □

3. Some lemmas

The first lemma concerns the measure of parallel sets.

Lemma 3.1. *If K is a compact set, then $\lambda_n(K_\delta)$ is continuous in $\delta > 0$, and*

$$\lim_{\delta\to 0} \lambda_n(K_\delta) = \lambda_n(K)\,.$$

Proof. Since the parallel sets are nested compact sets, and

$$\bigcap_{\rho>\delta} K_\rho = K_\delta\,, \quad \bigcap_{\rho>0} K_\rho = K\,,$$

their measure is continuous from the right and converges to $\lambda_n(K)$ as δ tends to 0. Similarly, since

$$\bigcup_{\rho<\delta} K_\rho = \{x : d(x,K) < \delta\} =: \widetilde{K}_\delta\,,$$

we have $\lim_{\rho\to\delta^-} \lambda_n(K_\rho) = \lambda_n(\widetilde{K}_\delta)$. To conclude the proof it thus suffices to prove that $\widetilde{K}_\delta$ differs from K_δ by a set of measure zero.

Let $y \in K_\delta$. By definition of K_δ, there exists a point $p \in K$ with $y \in B_{\delta,p}$. Since $\widetilde{K}_\delta$ contains the open ball of radius δ about p, its (lower) Lebesgue density at y is at least $1/2$. Therefore, every point of K_δ is a point of positive density for $\widetilde{K}_\delta$. Since the Lebesgue differentiation theorem implies that almost every point in $\mathbb{R}^n \setminus \widetilde{K}_\delta$ is a point of density zero for $\widetilde{K}_\delta$, we have $\lambda_n(K_\delta \setminus \widetilde{K}_\delta) = 0$. □

The second lemma relates convergence in the Hausdorff metric to convergence in symmetric difference.

Lemma 3.2. *Let L and K_m, $m \geq 1$, be non-empty compact sets.*

(i) *The sequence (K_m) converges in Hausdorff distance to L if and only if*

$$\lim_{m\to\infty} \lambda_n((K_m)_\delta \triangle L_\delta) = 0$$

for each $\delta > 0$.

(ii) *If (K_m) converges in Hausdorff distance to L and each K_m is obtained from a compact set K via finitely many Steiner symmetrizations and Euclidean isometries, then*

$$\lim_{m\to\infty} \lambda_n(K_m \triangle L) = 0\,.$$

In particular, $\lambda_n(L) = \lambda_n(K)$.

Proof. For Claim (i), assume that K_m converges to L in Hausdorff distance. Fix $\delta > 0$, and let $\varepsilon > 0$ be given. Since L is compact, $\lambda_n(L_\rho)$ is continuous in $\rho > 0$. Choose $\rho \in (0,\delta)$ so small that $\lambda_n(L_{\delta+\rho}) - \lambda_n(L_\delta) < \varepsilon$ and $\lambda_n(L_\delta) - \lambda_n(L_{\delta-\rho}) < \varepsilon$, and let m be so large that $d_H(K_m, L) < \rho$. Then $d_H((K_m)_\delta, L_\delta) < \rho$ for each $\delta > 0$. It follows that

$$\lambda_n\left((K_m)_\delta \setminus L_\delta\right) \leq \lambda_n(L_{\delta+\rho} \setminus L_\delta) < \varepsilon$$

since $(K_m)_\delta \subset L_{\delta+\rho}$, and

$$\lambda_n\left(L_\delta \setminus (K_m)_\delta\right) \leq \lambda_n\left(L_\delta \setminus L_{\delta-\rho}\right) < \varepsilon$$

since $L_{\delta-\rho} \subset (K_m)_\delta$. Combining the two inequalities, we obtain that

$$\lambda_n((K_m)_\delta \triangle L_\delta)) < \varepsilon$$

for m sufficiently large. Since $\varepsilon > 0$ was arbitrary, convergence in symmetric difference follows.

To see the converse implication, assume that $d_H(K_m, L) \geq 2\rho > 0$. If $K_m \setminus L_{2\rho} \neq \emptyset$ then $(K_m)_\rho \setminus L_\rho$ contains $B_{\rho,p}$, where p is any point in $K_m \setminus L_{2\rho}$. Otherwise,

$L_\rho \setminus (K_m)_\rho$ contains $B_{\rho,p}$, where p is any point in $L \setminus (K_m)_{2\rho}$. In either case, $\lambda_n((K_m)_\rho \triangle L_\rho) \geq \lambda_n(B_\rho)$.

For Claim (ii), assume that (K_m) converges to L in Hausdorff distance. Given $\varepsilon > 0$, choose $\rho > 0$ so small that $\lambda_n(L_\rho) - \lambda_n(L) < \varepsilon$, and choose m so large that $d_H(K_m, L) < \rho$. Then $K_m \subset L_\rho$, and therefore

$$\lambda_n\left(K_m \setminus L\right) \leq \lambda_n(L_\rho \setminus L) < \varepsilon\,.$$

For the complementary inequality, construct $(K_\rho)_m$ by applying the same sequence of symmetrizations and isometries to the parallel set K_ρ that was used to produce K_m. Then, by (2.2) and since symmetrization does not change volume, we have

$$\begin{aligned}\lambda_n\left(L \setminus K_m\right) &\leq \lambda_n\left((K_m)_\rho \setminus K_m\right)\\ &\leq \lambda_n\left((K_\rho)_m\right) - \lambda_n\left(K_m\right)\\ &= \lambda_n\left(K_\rho\right) - \lambda_n\left(K\right)\\ &< \varepsilon\,.\end{aligned}$$

Combining the two preceding inequalities, we conclude as in the proof of the first claim that $\lambda_n(K_m \triangle L)$ converges to zero. □

The second part of Lemma 3.2 could have been proved by using (2.2) and a result of Beer [2, Theorem 1], who also, in [1, Lemma 4], proved the "only if" implication of the first part of the lemma.

The next lemma provides an equality statement for (2.1) in the case where one set runs through the family of parallel sets K_δ and the other set is a ball $B_{r,p}$.

Lemma 3.3. *Let $u \in S^{n-1}$, and let K be a non-empty compact set. If there exists a point $p \in u^\perp$ such that*

$$\lambda_n(\mathcal{S}_u K_\delta \setminus B_{r,p}) = \lambda_n(K_\delta \setminus B_{r,p}) \tag{3.1}$$

for all $\delta, r > 0$, then $\mathcal{S}_u K = K$.

Proof. Suppose $\mathcal{S}_u K \neq K$, and fix a point $q \in K \setminus \mathcal{S}_u K$. Let $y \in u^\perp$ be such that the line ℓ_y parallel to u and passing through y contains q. By definition, $\mathcal{S}_u K$ intersects ℓ_y in a centered line segment of the same one-dimensional measure as $K \cap \ell_y$. Since $p \in u^\perp$, we can choose $r < |p - q|$ such that $B_{r,p}$ contains $\mathcal{S}_u K \cap \ell_y$ in its interior.

We argue that the boundary of $B_{r,p}$ separates a neighborhood of q from $\mathcal{S}_u K \cap \ell_z$ for z close to y. For $\delta > 0$, consider the nested compact sets

$$A(\delta) = \mathcal{S}_u K_\delta \cap (\ell_y)_\delta.$$

By the compactness of K, and because the sets K_δ form a monotonically decreasing sequence with respect to inclusion, the one-dimensional measure of each cross section $K_\delta \cap \ell_z$ converges monotonically to the measure of $K \cap \ell_z$ as δ tends to zero, and hence $\bigcap_{\delta > 0} \mathcal{S}_u K_\delta = \mathcal{S}_u K$. It follows that

$$\bigcap_{\delta>0} A(\delta) = \left(\bigcap_{\delta>0} \mathcal{S}_u K_\delta\right) \cap \left(\bigcap_{\delta>0} (\ell_y)_\delta\right) = \mathcal{S}_u K \cap \ell_y\,.$$

Since $\mathcal{S}_u K \cap \ell_y$ does not meet $\{x : |x - p| \geq r\}$, by compactness there exists a set $A(\delta)$ that does not meet $\{x : |x - p| \geq r\}$ either. This means that the interior of $B_{r,p}$ contains $\mathcal{S}_u K_\delta \cap (\ell_y)_\delta$ for some $\delta > 0$. By choosing $\delta > 0$ small enough, we can further assume that $B_{r,p}$ does not intersect $B_{\delta,q}$.

By construction,

$$\lambda_n\big((\mathcal{S}_u K_\delta \cap (\ell_y)_\delta) \setminus \mathcal{S}_u B_{r,p}\big) < \lambda_n\big((K_\delta \cap (\ell_y)_\delta) \setminus B_{r,p}\big), \tag{3.2}$$

because the set on the left-hand side is empty, while the one on the right-hand side contains $B_{\delta,q}$. Let C be the closure of $\mathbb{R}^n \setminus (\ell_y)_\delta$. Since $\mathcal{S}_u K_\delta \cap C = \mathcal{S}_u(K_\delta \cap C)$, it follows from (2.1) that

$$\lambda_n((\mathcal{S}_u K_\delta \cap C) \setminus \mathcal{S}_u B_{r,p}) \le \lambda_n((K_\delta \cap C) \setminus B_{r,p})\,. \tag{3.3}$$

Adding (3.2) and (3.3) and using that $(\ell_y)_\delta$ and C form an almost disjoint partition of $\mathbb{R}^n$, we obtain that

$$\lambda_n(\mathcal{S}_u K_\delta \setminus \mathcal{S}_u B_{r,p}) < \lambda_n(K_\delta \setminus B_{r,p})\,,$$

negating (3.1). □

For the proof of the main result, we will combine (2.1) with (2.2) to obtain

$$\lambda_n((\mathcal{S}_u K)_\delta \setminus \mathcal{S}_u B_{r,p}) \le \lambda_n(K_\delta \setminus B_{r,p}) \tag{3.4}$$

for each $p \in \mathbb{R}^n$ and all $\delta, r > 0$. Lemma 3.3 implies that for every given $p \in u^\perp$, the inequality in (3.4) is strict for some $\delta, r > 0$ unless $\mathcal{S}_u K = K$.

The last lemma will be used to identify the limit of (2.4).

Lemma 3.4. *Let H_1, H_2 be compact sets in $\mathbb{R}^n$, and let $u \in S^{n-1}$. Assume that $\mathcal{S}_u H_j = H_j$ for $j = 1, 2$, and that*

$$\lambda_n(H_1 \cap \{x \cdot p > t\}) = \lambda_n(H_2 \cap \{x \cdot p > t\}) \tag{3.5}$$

for all non-zero $p \in u^\perp$ and all $t \in \mathbb{R}$. Then H_1 and H_2 agree up to a set of n-dimensional Lebesgue measure zero.

Proof. Denote by ℓ_y the line parallel to u through $y \in u^\perp$, and consider on $u^\perp$ the measurable functions $f_j(y) = \lambda_1(\ell_y \cap H_j)$ for $j = 1, 2$. By assumption, the difference $f_1 - f_2$ integrates to zero over every half-space $\{y \cdot p > t\} \subset u^\perp$. It follows from a standard argument that its integral over almost every $(n-2)$-dimensional subspace $\{y \cdot p = t\} \subset u^\perp$ vanishes as well [18]. In other words, the $(n-2)$-dimensional Radon transform of $f_1 - f_2$ is zero almost everywhere, and therefore $f_1 = f_2$ almost everywhere on $u^\perp$ [10, p.28]. Since $\mathcal{S}_u H_j = H_j$, the sets are uniquely determined by the functions f_j, and we conclude that H_1 and H_2 agree up to a set of measure zero. □

4. Proof of Theorem 2.2

We begin with some geometric considerations. Given $u \in S^{n-1}$, we want to compose a Steiner symmetrization $\mathcal{S}_u$ with a rotation $\mathcal{R}'$ so that the result is symmetric at the hyperplane $e_1^\perp$. Note that the commutation rule

$$\mathcal{R}\mathcal{S}_u = \mathcal{S}_{\mathcal{R}u}\mathcal{R} \tag{4.1}$$

holds for every rotation $\mathcal{R} \in O(n)$ and every $u \in S^{n-1}$.

Let $u \in S^{n-1}$, with $u \cdot e_1 = \cos\alpha$. Replacing u with $-u$, if necessary, we may take $\alpha \in (0, \pi/2)$. The Steiner symmetrization $\mathcal{S}_u$ projects subsets of $e_1^\perp$ linearly onto $u^\perp$. If $\mathcal{R}'$ is the rotation that maps u to e_1 and fixes $u^\perp \cap e_1^\perp$, then the composition $\mathcal{T} = \mathcal{R}'\mathcal{S}_u$ defines a linear transformation on $e_1^\perp$ that satisfies

$$|\mathcal{T}x - x| \le (1 - \cos\alpha)|x|\,, \quad |x| \ge |\mathcal{T}x| \ge |x|\cos\alpha \tag{4.2}$$

for all $x \in e_1^\perp$. More precisely, the restriction of $\mathcal{T}$ to $e_1^\perp$ is equivalent to a diagonal matrix with eigenvalues $\cos\alpha$ (simple) and 1 (of multiplicity $n-2$).

Given a non-empty compact set K, let $(\mathcal{S}_{u_m})$ be a sequence of Steiner symmetrizations as described in the statement of the theorem, and let (K_m) be the sequence of rotated symmetrals defined in the statement of the theorem. We construct the rotation $\mathcal{R}_m$ in (2.4) as a composition $\mathcal{R}_m = \mathcal{R}'_m \dots \mathcal{R}'_1$, where $\mathcal{R}'_m$ is recursively defined as the rotation that sends $\mathcal{R}_{m-1}u_m$ to e_1 and fixes the subspace orthogonal to these two vectors, and $\mathcal{R}'_0 = \mathcal{I}$. Note that $\mathcal{R}'_m$ is a rotation by α_m. By the commutation rule (4.1),

$$\mathcal{R}_{m+1}\mathcal{S}_{u_{m+1}} = \mathcal{R}'_{m+1}\mathcal{S}_{\mathcal{R}_m u_{m+1}}\mathcal{R}_m\,,$$

which gives for (K_m) the recursion relation

$$K_{m+1} = \mathcal{R}'_{m+1}\mathcal{S}_{\mathcal{R}_m u_{m+1}}K_m\,.$$

By Blaschke's selection principle, (K_m) has subsequences that converge in Hausdorff distance. Let L_1 and L_2 be limits of such subsequences. We want to prove that $L_1 = L_2$.

We will first show that

$$\lambda_n((L_1)_\delta \setminus B_{r,q}) = \lambda_n((L_2)_\delta \setminus B_{r,q}) \tag{4.3}$$

for all $q \in e_1^\perp$ and all $r, \delta \geq 0$. By the assumption that (α_m) is square summable, $\gamma = \prod_{m=1}^\infty \cos\alpha_m > 0$ (see Example 2.1 for a proof). Let $\mathcal{T}'_m$ be the linear transformation defined by $\mathcal{R}'_m\mathcal{S}_{\mathcal{R}_{m-1}u_m}$ on $e_1^\perp$ and consider the composition $\mathcal{T}_m = \mathcal{T}'_m \dots \mathcal{T}'_1$. By (4.2), the sequence $(\mathcal{T}_m)$ converges to a linear transformation $\mathcal{T}$ that satisfies $|\mathcal{T}x| \geq \gamma|x|$ for all $x \in e_1^\perp$. In particular, $\mathcal{T}$ is invertible on $e_1^\perp$. For each $m \geq 1$, let $p_m = \mathcal{T}_m\mathcal{T}^{-1}q$. Then $p_m = \mathcal{T}'_m p_{m-1}$, p_m converges to q and

$$\mathcal{R}'_m\mathcal{S}_{\mathcal{R}_{m-1}u_m}B_{r,p_{m-1}} = B_{r,p_m}\,.$$

Inequalities (2.1) and (2.2) imply that the sequence $\lambda_n\big((K_m)_\delta \setminus B_{r,p_m}\big)$ is monotonically decreasing,

$$\begin{aligned}
\lambda_n((K_m)_\delta \setminus B_{r,p_m}) &\geq \lambda_n(\mathcal{S}_{\mathcal{R}_m u_{m+1}}(K_m)_\delta \setminus \mathcal{S}_{\mathcal{R}_m u_{m+1}}B_{r,p_m})\\
&\geq \lambda_n((\mathcal{S}_{\mathcal{R}_m u_{m+1}}K_m)_\delta \setminus \mathcal{S}_{\mathcal{R}_m u_{m+1}}B_{r,p_m})\\
&= \lambda_n((K_{m+1})_\delta \setminus B_{r,p_{m+1}}),
\end{aligned}$$

hence convergent. In the last line, we have used the rotational invariance of Lebesgue measure and the recursion formula for K_m. Passing to the limit along the subsequences converging to L_1 and L_2 and using Lemma 3.2 yields (4.3).

Since half-spaces can be written as increasing unions of balls, we can take a monotone limit in (4.3) to obtain that (3.5) holds, with $H_i = (L_i)_\delta$, for all $q \in e_1^\perp$ and all $t \in \mathbb{R}$. Lemma 3.4 implies that the parallel sets $(L_1)_\delta$ and $(L_2)_\delta$ agree up to a set of measure zero. To complete the proof, suppose that $L_1 \neq L_2$. Then there exists a point x that lies in one of the two sets but not the other, say $x \in L_1 \setminus L_2$. If we choose $\delta = \frac{1}{2}\text{dist}(x, L_2)$, then $(L_1)_\delta \supset B_{\delta,x}$ while $(L_2)_\delta \cap B_{\delta,x} = \emptyset$. This is impossible since the parallel sets agree up to a set of measure zero, and we conclude that $L_1 = L_2$. □

5. Uniformly distributed directions in the plane

A sequence (u_m) in S^1 is called *uniformly distributed* in the sense of Weyl, if the fraction of terms in the initial segment $(u_m)_{m\leq N}$ that fall into any given arc I in S^1 converges to $\lambda_1(I)/(2\pi)$ as N tends to infinity, where $\lambda_1(I)$ is the length of I. A classical example is the Kronecker sequence $u_m = (\cos m\alpha, \sin m\alpha)$ for $m \geq 1$, which is uniformly distributed if α is not a rational multiple of π [13, Example 2.1].

Theorem 5.1 (The Kronecker sequence)**.** *Let $u_m = (\cos m\alpha, \sin m\alpha)$ for $m \geq 1$, and assume that α is not a rational multiple of π. Let $K \subset \mathbb{R}^2$ be a non-empty compact set. Then the symmetrals $\mathcal{S}_{u_m} \dots \mathcal{S}_{u_1}K$ converge in Hausdorff distance and in symmetric difference to the closed centered ball K^* equimeasurable with K.*

Proof. Let $\mathcal{R}$ be the rotation that sends $u = (\cos\alpha, \sin\alpha)$ to e_1, and let $\mathcal{S} = \mathcal{S}_{e_1}$ be the Steiner symmetrization in the direction of e_1. It suffices to show that

$$K_m = \mathcal{R}^m\mathcal{S}_{u_m} \dots \mathcal{S}_{u_1}K$$

converges to K^*.

By the commutation relation (4.1), $K_m = (\mathcal{SR})^m K$. Let (K_{j_m}) be a subsequence that converges in Hausdorff distance to a compact set L. By Lemma 3.2, the sequence $\big((K_{j_m})_\delta\big)$ converges in symmetric difference to L_δ for each $\delta > 0$. We estimate

$$
\begin{aligned}
\lambda_2(L_\delta \setminus B_r) &= \lambda_2(\mathcal{R}L_\delta \setminus B_r)\\
&\geq \lambda_2(\mathcal{SR}L_\delta \setminus \mathcal{S}B_r)\\
&= \lim_{m\to\infty} \lambda_2(\mathcal{SR}(K_{j_m})_\delta \setminus B_r)\\
&\geq \inf_m \lambda_2((K_{j_m+1})_\delta \setminus B_r)\,.
\end{aligned}
\tag{5.1}
$$

The inequality in the second line follows from (2.1). The third line uses the continuity of the Steiner symmetrization with respect to the symmetric difference metric and that $\mathcal{S}B_r = B_r$. The fourth line follows from (2.2) and the identity $\mathcal{SR}K_{j_m} = K_{j_m+1}$.

On the other hand, the sequence $\lambda_2((K_m)_\delta \setminus B_r)$ is monotone decreasing, for each $\delta, r > 0$, by (2.2) and (2.1), hence convergent along the entire sequence, and by Lemma 3.2, its limit is given by $\lambda_2\left(L_\delta \setminus B_r\right)$. This means that all inequalities in (5.1) hold with equality and, in particular,

$$\lambda_2(\mathcal{R}L_\delta \setminus B_r) = \lambda_2(\mathcal{SR}L_\delta \setminus \mathcal{S}B_r).$$

By construction, L is symmetric under reflection at $e_1^\perp$. By Lemma 3.3 and the fact that $\mathcal{R}L_\delta = (\mathcal{R}L)_\delta$, we have $\mathcal{SR}L = \mathcal{R}L$, i.e., L is also symmetric under reflection at $u^\perp$. Since α is incommensurable with π, these two reflections generate a dense subgroup of rotations, and we conclude that $L = K^*$. Since the subsequence was arbitrary, the entire sequence (K_m) converges to K^* in Hausdorff distance and in symmetric difference. □

One may wonder if every uniformly distributed sequence of directions gives rise to a convergent sequence of Steiner symmetrizations. Since a sequence of directions chosen independently and uniformly at random from S^1 is almost surely uniformly distributed [13, Theorem 3.2.2] and the corresponding Steiner symmetrizations almost surely converge to the ball [14, 16, 17], most uniformly distributed sequences of directions produce convergent sequence of Steiner symmetrizations.

Remarkably, there are exceptions. In the notation of Example 2.1, let (α_m) be a nonincreasing sequence of positive numbers and set $\beta_m = \sum_{k=1}^m \alpha_k$ for $m \geq 1$. If

$$\lim_{m\to\infty} \alpha_m = 0\,, \qquad \lim_{m\to\infty} m\alpha_m = \infty\,, \qquad \text{and} \quad \sum_{m=1}^{\infty} \alpha_m^2 < \infty\,,$$

then $(u_m) = (\cos\beta_m, \sin\beta_m)$ is uniformly distributed on S^1 (see [13, Theorem 2.5]). This includes in particular sequences of the form $\alpha_m = \vartheta m^{-\sigma}$ with $\sigma \in (1/2, 1)$ and $\vartheta > 0$. But the corresponding sequence of Steiner symmetrals of the compact set in Example 2.1 does not converge.

Uniformly distributed sequences play an important role in quasi-Monte Carlo methods, because they share many properties of random sequences. In some cases, they provide even better approximations to integrals than typical random sequences. The quality of the approximation defined by a sequence (u_m) in S^1 is determined by its *discrepancy*

$$D(N) = \sup_{I\subset S^1} \left| \frac{\#\{m \leq N : u_m \in I\}}{N} - \frac{\lambda_1(I)}{2\pi} \right|\,,$$

which describes how much the fraction of the initial segment $(u_m)_{m\leq N}$ that fall into any given arc I in S^1 differs from $\lambda_1(I)/(2\pi)$. The best approximations are provided by sequences of *minimal discrepancy*, that is by sequences for which $D(N)$ is proportional to $(\log N)/N$. Note that the discrepancy of a Kronecker sequence depends on the diophantine properties of α/π, and that the discrepancy of the sequence $(\cos\beta_n, \sin\beta_n)$ which corresponds to $\alpha_n = \vartheta n^{-\sigma}$, with $\sigma \in (1/2, 1)$ and $\vartheta > 0$, has asymptotic behavior $n^{-\sigma}$ [13].

Open Problem 5.2. *If (u_m) is uniformly distributed in S^1 and of minimal discrepancy, do the Steiner symmetrals $\mathcal{S}_{u_m} \dots \mathcal{S}_{u_1} K$ converge to K^* for each compact set K?*

6. Steiner symmetrization along a finite set of directions

Our final example concerns sequences of iterated Steiner symmetrization that use finitely many directions. Klain proved the elegant result that when the initial set is a convex body then such sequences always converge [11, Theorem 5.1]. The techniques developed in this paper allow us to extend his result to compact sets.

Theorem 6.1 (Klain's Theorem holds for compact sets)**.** *Let (u_m) be a sequence of vectors chosen from a finite set $F = \{v_1, \dots, v_k\} \subset S^{n-1}$. Then, for every compact set $K \subset \mathbb{R}^n$, the symmetrals*

$$K_m = \mathcal{S}_{u_m} \dots \mathcal{S}_{u_1} K$$

converge in Hausdorff distance and in symmetric difference to a compact set L. Furthermore, L is symmetric under reflection in each of the directions $v \in F$ that appear in the sequence infinitely often.

Proof. We follow Klain's argument. Dropping an initial segment $(K_m)_{m\le N}$ of the sequence and possibly substituting F with one of its subsets, we may assume, without loss of generality, that each direction in F appears infinitely often in the sequence (u_m). The main idea is to construct a subsequence along which the directions $v_i \in F$ appear in a particular order. With each index m, we associate a permutation π_m of the numbers $1, \dots, k$ that indicates the order in which the directions $v_1, \dots, v_k$ appear for the first time among the directions u_i with $i \ge m$. Since there are only finitely many permutations, we can pick a subsequence (u_{j_m}) such that the permutation π_{j_m} is the same for each m. By re-labeling the directions, we may assume that this permutation is the identity. Passing to a further subsequence, we may assume that every direction in F appears in each segment $u_{j_m}, u_{j_m+1}, \dots, u_{j_{m+1}}$.

By the Blaschke selection principle, there is a subsequence (again denoted by (K_{j_m})) that converges in Hausdorff distance to some compact set L. We note for later use that for each $\delta > 0$, the entire sequence $\big(\lambda_n((K_m)_\delta)\big)$ is decreasing by (2.2), hence convergent. By Lemma 3.2, the limit is given by

$$\inf_m \lambda_n((K_m)_\delta) = \lim_{m\to\infty} \lambda_n((K_{j_m})_\delta) = \lambda_n(L_\delta)\,. \tag{6.1}$$

We show by induction that $\mathcal{S}_{v_i} L = L$ for $i = 1, \dots, k$. For $i = 1$ observe that $u_{j_m} = v_1$. Therefore (K_{j_m}) is symmetric with respect to $v_1^\perp$ and the same is true for L. Suppose we already know that L is invariant under Steiner symmetrization in the directions $v_1, \dots v_{i-1}$. If j'_m is the index where v_i appears for the first time after j_m, then the inductive hypothesis implies that $\mathcal{S}_{u_{j'_m-1}} \dots \mathcal{S}_{u_{j_m+1}} L = L$. By (2.1) and (2.2) we have, for each $\delta > 0$,

$$\begin{aligned} \lambda_n((K_{j_m})_\delta \setminus L_\delta) &\ge \lambda_n(\mathcal{S}_{u_{j'_m-1}} \dots \mathcal{S}_{u_{j_m+1}} (K_{j_m})_\delta \setminus L_\delta) \\ &\ge \lambda_n((\mathcal{S}_{u_{j'_m-1}} \dots \mathcal{S}_{u_{j_m+1}} K_{j_m})_\delta \setminus L_\delta) \\ &= \lambda_n((K_{j'_m-1})_\delta \setminus L_\delta)\,. \end{aligned} \tag{6.2}$$

Since $((K_{j_m})_\delta)$ converges to L_δ in symmetric difference, the left hand side of the inequality converges to zero. Therefore, the right hand side converges to zero, and by (6.1), the sequence $(K_{j'_m-1})_\delta$ likewise converges to L_δ in symmetric difference. We continue the argument as in (5.1) and estimate

$$\begin{aligned} \lambda_n(L_\delta \setminus B_r) &\geq \lambda_n(\mathcal{S}_{v_i} L_\delta \setminus \mathcal{S}_{v_i} B_r) \\ &= \lim_{m\to\infty} \lambda_n(\mathcal{S}_{v_i}(K_{j'_m-1})_\delta \setminus B_r) \\ &\geq \inf_m \lambda_n((K_{j'_m})_\delta \setminus B_r)\,. \end{aligned} \tag{6.3}$$

The inequality in the first line follows from (2.1). In the second line we have used the convergence of $(K_{j'_m-1})_\delta$ in symmetric difference and the continuity of the Steiner symmetrization with respect to the symmetric difference distance. The inequality in the third line is a consequence of (2.2) and of the equality $\mathcal{S}_{v_i} K_{j'_m-1} = K_{j'_m}$.

Since $\big(\lambda_n((K_m)_\delta \setminus B_r)\big)$ is a decreasing sequence by (2.1), and since it contains the subsequence $\big(\lambda_n((K_{j_m})_\delta \setminus B_r)\big)$ which converges to $\lambda_n(L_\delta \setminus B_r)$, the first and last term in (6.3) are equal. In particular, the first line holds with equality for each $\delta > 0$. By Lemma 3.3 this implies $L = \mathcal{S}_{v_i} L$, which concludes the inductive step.

It remains to prove that the entire sequence converges. Since L is invariant under Steiner symmetrization in each of the directions $v_1, \ldots, v_k$ in F, we have, by the same reasoning as in (6.2) and in the lines following it, that

$$\lambda_n((K_{j_m})_\delta \setminus L_\delta) \geq \lambda_n((K_j)_\delta \setminus L_\delta)$$

for every $j \geq j_m$. We conclude that $(K_m)_\delta$ converges to L_δ in symmetric difference along the entire sequence, for each $\delta \geq 0$. By Lemma 3.2, (K_m) converges to L both in Hausdorff distance and in symmetric difference. □

Open Problem 6.2. *Do iterated Steiner symmetrals $\mathcal{S}_{u_m} \ldots \mathcal{S}_{u_1} K$ always converge in shape, without any assumptions on the sequence of directions?*

Open Problem 6.3. *Assume that a sequence of directions (u_m) is such that $(\mathcal{S}_{u_m} \ldots \mathcal{S}_{u_1} C)$ converges to C^* for each convex body C. Is it true that $(\mathcal{S}_{u_m} \ldots \mathcal{S}_{u_1} K)$ converges to K^* for each compact set K?*

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Dipartimento di Matematica e Informatica "U. Dini", Università di Firenze, Viale Morgagni 67/A, Firenze, Italy I-50134
E-mail address: gabriele.bianchi@unifi.it

Department of Mathematics, University of Toronto, 40 St. George Street, Toronto, Ontario, Canada M5S 2E4
E-mail address: almut@math.toronto.edu

Dipartimento di Matematica e Informatica "U. Dini", Università di Firenze, Viale Morgagni 67/A, Firenze, Italy I-50134
E-mail address: paolo.gronchi@unifi.it

Dipartimento di Matematica, Università della Calabria, Ponte Bucci, cubo 30B, Arcavacata di Rende, Cosenza, Italy I-87036
E-mail address: volcic@unical.it