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# Density lower bound estimates for local minimizers of the 2$d$ Mumford–Shah energy

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**Abstract.** We prove, using direct variational arguments, an explicit energy-treshold criterion for regular points of 2-dimensional Mumford-Shah energy minimizers. From this we infer an explicit constant for the density lower bound of De Giorgi, Carriero and Leaci.

## 1. Introduction

The Mumford–Shah model stands as a prototypical example of variational problem in image segmentation (see [14]). It consists in minimizing (adding either boundary or confinement conditions or fidelity terms) the energy

$$E(v, K) := \int_{\Omega \setminus K} |\nabla v|^2 \, dx + \mathcal{H}^1(K),$$

where $\Omega \subset \mathbb{R}^2$ is a fixed open set, $K$ is a rectifiable closed subset of $\Omega$, and $v \in C^1(\Omega \setminus K)$. This energy has been then borrowed and conveniently modified in Fracture Mechanics, mainly to model quasi-static irreversible crack-growth for brittle materials (see [2, Sect. 4.6.6]).

One of the first existence theories for minimizers of $E$ hinges upon a weak formulation in the space $SBV$ of Special functions of Bounded Variation, the subspace of $BV$ functions with singular part of the distributional derivative concentrated on a 1-rectifiable set. In this approach the set $K$ is substituted by the (Borel) set $S_v$ of approximate discontinuities of the function $v$ (throughout the paper we will use standard notations and results concerning $BV$ and $SBV$, following the book [2]). This is the reason for the terminology *free-discontinuity* problem introduced by De Giorgi. The Mumford–Shah energy of a function $v$ in $SBV(\Omega)$ on an open subset $A \subseteq \Omega$ then reads as

$$\mathrm{MS}(v, A) = \int_A |\nabla v|^2 dx + \mathcal{H}^1(S_v \cap A). \tag{1.1}$$

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In case $A = \Omega$ we drop the dependence on the set of integration. In what follows $u$ will always denote a *local minimizer*, that is any $u \in SBV(\Omega)$ with $\mathrm{MS}(u) < +\infty$ and such that

$$\mathrm{MS}(u) \leq \mathrm{MS}(w) \quad \text{whenever}\{w \neq u\} \subset\subset \Omega.$$

The class of all local minimizers shall be denoted by $\mathcal{M}(\Omega)$.

As established in [10] in all dimensions (and proved alternatively in [6] in dimension two), if $u \in SBV$ is a minimizer of the energy MS, then the pair $(u, \overline{S_u})$ is a minimizer of $E$. The main point is the identity $\mathcal{H}^1(\overline{S_u} \setminus S_u) = 0$, which holds for every $u \in \mathcal{M}(\Omega)$. The groundbreaking paper [10] proves this identity via the following density lower bound

$$\frac{\mathrm{MS}(u, B_r(z))}{2r} \geq \theta \quad \text{for all} \quad z \in \overline{S_u}, \quad \text{and all } r \in (0, \mathrm{dist}(z, \partial\Omega)) \tag{1.2}$$

with $\theta$ a dimensional constant independent of $u$. Building upon the same ideas, in [5] it is proved that for some dimensional constant $\theta_0$ independent of $u$ it holds

$$\frac{\mathcal{H}^1(S_u \cap B_r(z))}{2r} \geq \theta_0 \quad \text{for all} \quad z \in \overline{S_u}, \quad \text{and all } r \in (0, \mathrm{dist}(z, \partial\Omega)). \tag{1.3}$$

The argument for (1.2) used by De Giorgi et al. [10], and similarly in [5] for (1.3), is indirect: it relies on Ambrosio's $SBV$ compactness theorem, an $SBV$ Poincaré–Wirtinger type inequality and the asymptotic analysis of blow-ups of minimizers with vanishing Dirichlet energy. In this paper we give a simpler proof in 2 dimensions, which does not require any Poincaré–Wirtinger inequality, nor any compactness argument. Our argument differs from those used in [6] and [7] to derive (1.3) in the two dimensional case as well.

We first introduce some useful notation, which we borrow from [9]. Given $u \in \mathcal{M}(\Omega)$, $z \in \Omega$ and $r \in (0, \mathrm{dist}(z, \partial\Omega))$ let

$$e_z(r) := \int_{B_r(z)} |\nabla u|^2 dx, \quad \ell_z(r) := \mathcal{H}^1(S_u \cap B_r(z))$$

$$m_z(r) := \mathrm{MS}(u, B_r(z)), \quad \text{and} \quad h_z(r) := e_z(r) + \frac{1}{2}\ell_z(r).$$

Clearly $m_z(r) = e_z(r) + \ell_z(r) \leq 2h_z(r)$, with equality if and only if $e_z(r) = 0$.

**Theorem 1.1** *Let $u \in \mathcal{M}(\Omega)$. Then*

$$\frac{m_z(r)}{r} \geq 1 \quad \textit{for all} \quad z \in \overline{S_u} \quad \textit{and all } r \in (0, \mathrm{dist}(z, \partial\Omega)). \tag{1.4}$$

*More precisely, the set $\Omega_u := \{z \in \Omega : (1.4) \text{ fails}\}$ is open and $\Omega_u = \Omega \setminus \overline{J_u} = \Omega \setminus \overline{S_u}$.*

The quantity $m_z(\cdot)$ in Theorem 1.1 allows us to take advantage of a suitable monotonicity formula, discovered independently by David and Léger in [9] and Maddalena and Solimini in [13]. A simple iteration of Theorem 1.1 gives a density lower bound as in (1.3) with an explicit constant $\theta_0$.

**Corollary 1.2** *If $u \in \mathcal{M}(\Omega)$, then $\mathcal{H}^1(\overline{S_u} \setminus J_u) = 0$ and*

$$\frac{\ell_z(r)}{2r} \geq \frac{\pi}{2^{24}} \quad \textit{for all} \quad z \in \overline{S_u} \quad \textit{and all } r \in (0, \operatorname{dist}(z, \partial\Omega)). \tag{1.5}$$

A natural question is the sharpness of the estimates (1.4) and (1.5). The analysis performed by Bonnet [3] suggests that $\frac{\pi}{2^{24}}$ in (1.5) should be replaced by $\frac{1}{2}$ and 1 in (1.4) by 2. Note that the square root function $u(r, \theta) = \sqrt{\frac{2}{\pi} r} \cdot \sin(\theta/2)$ satisfies $\ell_0(r) = e_0(r) = r$ for all $r > 0$. Thus both the constants conjectured above would be sharp by [8, Sect. 62]. Unfortunately, we cannot prove any of them.

Instead, in Corollary 1.3 below we prove an infinitesimal version of (1.4) for quasi-minimizers of the Mumford–Shah energy, that is any function $v$ in $\mathrm{SBV}(\Omega)$ with $\mathrm{MS}(v) < +\infty$ and satisfying for some $\omega \geq 0$ and $\alpha > 0$ and for all balls $B_\rho(z) \subset \Omega$

$$\mathrm{MS}(v, B_\rho(z)) \leq \mathrm{MS}(w, B_\rho(z)) + \omega\, \rho^{1+\alpha} \quad \text{whenever } \{w \neq v\} \subset\subset B_\rho(z). \tag{1.6}$$

We denote the class of quasi-minimizers satisfying (1.6) by $\mathcal{M}_\omega(\Omega)$.

**Corollary 1.3** *Let $v \in \mathcal{M}_\omega(\Omega)$, then*

$$\overline{S_u} = \overline{J_u} = \left\{ z \in \Omega : \liminf_{r \downarrow 0^+} \frac{m_z(r)}{r} \geq \frac{2}{3} \right\}. \tag{1.7}$$

Let us finally mention that Bucur & Luckhaus, independently from us, have used a similar idea to the main one of Theorem 1.1 (see [4]). Moreover, in their paper they improve remarkably on this key idea obtaining some results in the spirit of Theorem 1.1 and Corollary 1.3 without our dimensional limitation.

**Plan of the paper.** In Sect. 2 we prove Theorem 1.1. The main ingredient, i.e. the David–Léger–Maddalena–Solimini monotonicity formula is proved in Appendix A. In section 3 we prove the Corollaries 1.2 and 1.3. The latter needs three additional tools: a Poincaré–Wirtinger type inequality, a technical lemma on sequences of MS minimizers and a decay lemma, proved in Appendices B, C and D, respectively. The technical lemma and the decay lemma are well-known facts. The Poincaré–Wirtinger inequality instead refines some results obtained in [12]: it is to our knowledge new and might be of independent interest.

## 2. Main result

As already mentioned, the main ingredient of Theorem 1.1 is the following monotonicity formula discovered independently in [9] and in [13] (cp. with [9, Proposition 3.5]).

**Lemma 2.1** *Let $u \in \mathcal{M}(\Omega)$, then for every $z \in \Omega$ and for $\mathcal{L}^1$ a.e. $r \in (0, \mathrm{dist}(z, \partial\Omega))$*

$$\int_{\partial B_r(z)} \left(\left(\frac{\partial u}{\partial \nu}\right)^2 - \left(\frac{\partial u}{\partial \tau}\right)^2\right) d\mathcal{H}^1 + \frac{\ell_z(r)}{r} = \frac{1}{r} \int_{J_u \cap \partial B_r(z)} |\langle \nu_u^\perp(x), x\rangle| d\mathcal{H}^0(x), \tag{2.1}$$

*$\frac{\partial u}{\partial \nu}$ and $\frac{\partial u}{\partial \tau}$ being the projections of $\nabla u$ in the normal and tangential directions to $\partial B_r(z)$, respectively.*

We will also need the following elementary well-known facts.

**Lemma 2.2** *Every $u \in \mathcal{M}(\Omega)$ is locally bounded and*

$$\mathrm{MS}(u, B_r(z)) \leq 2\pi r \quad \textit{for all } B_r(z) \subset \Omega. \tag{2.2}$$

We are now ready to prove the main result of the paper.
*Proof of Theorem 1.1* Introduce the set $J_u^\star$ of points $x \in J_u$ for which

$$\lim_{r \downarrow 0} \frac{\mathcal{H}^1(J_u \cap B_r(x))}{2r} = 1\,. \tag{2.3}$$

Since $J_u$ is rectifiable, $\mathcal{H}^1(J_u \setminus J_u^\star) = 0$. Next let $z \in \Omega$ be such that

$$m_z(R) < R \quad \text{for some} \quad R \in (0, \mathrm{dist}(z, \partial\Omega)) \tag{2.4}$$

We claim that $z \notin J_u^\star$. W.l.o.g. we take $z = 0$ and drop the subscript $z$ in $e, \ell, m$ and $h$.

In addition we can assume $e(R) > 0$. Otherwise, by the Co-Area formula and the trace theory of BV functions, we would find a radius $r < R$ such that $u|_{\partial B_r}$ is a constant. In turn, $u$ would necessarily be constant in $B_r$ because the energy decreases under truncations, thus implying $z \notin J_u^\star$. We can also assume $\ell(R) > 0$, since otherwise $u$ would be harmonic in $B_R$ and thus we would conclude $z \notin J_u^\star$.

We start next to compare the energy of $u$ with that of an harmonic competitor on a suitable disk. The inequality $\ell(R) \leq m(R) < R$ is crucial to select good radii.
*Step 1:* For any fixed $r \in (0, R - \ell(R))$, there exists a set $I_r$ of positive length in $(r, R)$ such that

$$\frac{h(\rho)}{\rho} \leq \frac{1}{2} \cdot \frac{e(R) - e(r)}{R - r - (\ell(R) - \ell(r))} \quad \text{for all } \rho \in I_r. \tag{2.5}$$

Define $J_r := \{t \in (r, R) : \mathcal{H}^0(S_u \cap \partial B_t) = 0\}$. We claim the existence of $J_r' \subseteq J_r$ with $\mathcal{L}^1(J_r') > 0$ and such that

$$\int_{\partial B_\rho} |\nabla u|^2 d\mathcal{H}^1 \leq \frac{e(R) - e(r)}{R - r - (\ell(R) - \ell(r))} \quad \text{for all } \rho \in J_r'. \tag{2.6}$$

Indeed, we use the Co-Area formula for rectifiable sets (see [2, Theorem 2.93]) to find

$$\mathcal{L}^1((r, R) \setminus J_r) \leq \int_{(r,R)\setminus J_r} \mathcal{H}^0(S_u \cap \partial B_t) dt = \int_{S_u \cap (B_R \setminus \overline{B_r})} \left| \left\langle \nu_u^\perp(x), \frac{x}{|x|} \right\rangle \right| d\mathcal{H}^1(x)$$
$$\leq \ell(R) - \ell(r).$$

In turn, this inequality implies $\mathcal{L}^1(J_r) \geq R - r - (\ell(R) - \ell(r)) > 0$, thanks to the choice of $r$. Then, define $J_r'$ to be the subset of radii $\rho \in J_r$ for which

$$\int_{\partial B_\rho} |\nabla u|^2 d\mathcal{H}^1 \leq \fint_{J_r} \left( \int_{\partial B_t} |\nabla u|^2 d\mathcal{H}^1 \right) dt.$$

Formula (2.6) follows by the Co-Area formula and the estimate $\mathcal{L}^1(J_r) \geq R - r - (\ell(R) - \ell(r))$.

We define $I_r$ as the subset of radii $\rho \in J_r'$ satisfying both (2.1) and (2.6). Therefore

$$\int_{\partial B_\rho} \left( \frac{\partial u}{\partial \tau} \right)^2 d\mathcal{H}^1 = \frac{1}{2} \int_{\partial B_\rho} |\nabla u|^2 d\mathcal{H}^1 + \frac{\ell(\rho)}{2\rho} \quad \forall \rho \in I_r. \tag{2.7}$$

Clearly, $I_r$ has full measure in $J_r'$, so that $\mathcal{L}^1(I_r) > 0$.

For any $\rho \in I_r$, we let $w$ be the harmonic function in $B_\rho$ with trace $u$ on $\partial B_\rho$. Then, as $\frac{\partial w}{\partial \tau} = \frac{\partial u}{\partial \tau}$ $\mathcal{H}^1$ a.e. on $\partial B_\rho$, the local minimality of $u$ entails

$$m(\rho) \leq \int_{B_\rho} |\nabla w|^2 dx \leq \rho \int_{\partial B_\rho} \left( \frac{\partial u}{\partial \tau} \right)^2 d\mathcal{H}^1 \overset{(2.7)}{=} \frac{\rho}{2} \int_{\partial B_\rho} |\nabla u|^2 d\mathcal{H}^1 + \frac{\ell(\rho)}{2}.$$

The inequality (2.5) follows from the latter inequality and from (2.6):

$$h(\rho) = e(\rho) + \frac{\ell(\rho)}{2} \leq \frac{\rho}{2} \int_{\partial B_\rho} |\nabla u|^2 d\mathcal{H}^1 \leq \frac{\rho}{2} \cdot \frac{e(R) - e(r)}{R - r - (\ell(R) - \ell(r))}.$$

*Step 2:* We now show that $0 \notin J_u^\star$.

Let $\varepsilon \in (0, 1)$ be fixed such that $m(R) \leq (1 - \varepsilon)R$, and fix any radius $r \in (0, R - \ell(R) - \frac{1}{1-\varepsilon} e(R))$. Step 1 and the choice of $r$ then imply

$$\frac{h(\rho)}{\rho} \leq \frac{1}{2} \frac{e(R) - e(r)}{R - r - (\ell(R) - \ell(r))} \leq \frac{e(R)}{2(R - \ell(R) - r)} < \frac{1 - \varepsilon}{2},$$

in turn giving $m(\rho) \leq 2h(\rho) < (1-\varepsilon)\rho$. Let $\rho_\infty := \inf\{t > 0 : m(t) \leq (1-\varepsilon)t\}$, then $\rho_\infty \in [0, \rho]$. Note that if $\rho_\infty$ were strictly positive then actually $\rho_\infty$ would be a minimum. In such a case, we could apply the argument above and find $\widetilde{\rho} \in (r_\infty, \rho_\infty)$, with $r_\infty \in (0, \rho_\infty - \ell(\rho_\infty) - \frac{1}{1-\varepsilon}e(\rho_\infty))$, such that $m(\widetilde{\rho}) < (1-\varepsilon)\widetilde{\rho}$ contradicting the minimality of $\rho_\infty$. Hence, there is a sequence $\rho_k \downarrow 0^+$ with $m(\rho_k) \leq (1-\varepsilon)\rho_k$. Then, clearly condition (2.3) is violated, so that $0 \notin J_u^\star$.

*Conclusion:* We first prove that $\Omega_u$ is open. Let $z \in \Omega_u$ and let $R > 0$ and $\varepsilon > 0$ be such that $m_z(R) \leq (1-\varepsilon)R$ and $B_{\varepsilon R}(z) \subset \Omega$. Let now $x \in B_{\varepsilon R}(z)$, then

$$m_x(R - |x-z|) \leq m_z(R) \leq (1-\varepsilon)R < R - |x-z|,$$

therefore $x \in \Omega_u$.

As $J_u^\star \cap \Omega_u = \emptyset$ by Step 2, we have $\mathcal{H}^1(J_u^\star \cap \Omega_u) = \mathcal{H}^1(J_u \cap \Omega_u) = \mathcal{H}^1(S_u \cap \Omega_u) = 0$. Hence, $u$ is in $W^{1,2}$ of the open set $\Omega_u$, and by minimality it is actually harmonic there. Thus, $S_u \cap \Omega_u = \emptyset$ and $\overline{S_u} \subseteq \Omega \setminus \Omega_u$. Moreover, let $z \notin \overline{J_u^\star}$ and $r > 0$ be such that $B_r(z) \subseteq \Omega \setminus \overline{J_u^\star}$. Since $\mathcal{H}^1(S_u \setminus \overline{J_u^\star}) = 0$, $u \in W^{1,2}(B_r(z))$ and thus $u$ is an harmonic function in $B_r(z)$ by minimality. Therefore $z \in \Omega_u$, and in conclusion $\Omega \setminus \Omega_u = \overline{J_u^\star} = \overline{J_u} = \overline{S_u}$. □

**Remark 2.3** *The same arguments of Theorem 1.1 complemented by Theorem 3.1 show that*

$$\Omega \setminus \overline{J_u} = \{z \in \Omega : m_z(R) \leq R \textit{ for some } R \in (0, d(z, \partial\Omega))\}. \tag{2.8}$$

*Indeed, assuming $z = 0$ and dropping the subscript $z$, if $e(R) = 0$ or $\ell(R) = 0$, then $0 \in \Omega \setminus \overline{J_u}$. In the former case, the assertion follows since $u$ is constant on $B_\rho$ for some $\rho \in (0, R)$ by Theorem 3.1; in the latter case, $u$ is harmonic on $B_R$ by minimality. Hence, both $e(R)$ and $\ell(R)$ are in $(0, R)$. By Step 1 in Theorem 1.1 we have $h(\rho) \leq \rho/2$ for some $\rho \in (0, R)$. If $e(\rho) = 0$ then $0 \in \Omega \setminus \overline{J_u}$, otherwise, $m(\rho) < 2h(\rho) \leq \rho$. In the last instance, we are back to Theorem 1.1, so that $0 \in \Omega \setminus \overline{J_u}$. In any case, the set on the rhs of (2.8) is contained in $\Omega \setminus \overline{J_u}$. The opposite inclusion is trivial.*

## 3. Proof of Corollaries 1.2 and 1.3

*Proof of Corollary* 1.2 Assume by contradiction that (1.5) fails for some $z \in \overline{S_u}$ and some $R_1 \in (0, \operatorname{dist}(z, \partial\Omega))$. W.l.o.g. we take $z = 0 \in S_u$ and drop the subscript $z$ in $e$, $\ell$, $m$ and $h$.

Note that $R_1/4 - \ell(R_1) > R_1/8$ since $\ell(R_1) < 2\pi R_1/2^{24} < R_1/8$. Then, choosing $r_1 \in (R_1/8, R_1/4 - \ell(R_1))$ we have $2(R_1 - \ell(R_1) - r_1) > 3R_1/2$, and by applying Step 1 in Theorem 1.1 we infer, by (2.2),

$$\frac{h(\rho_1)}{\rho_1} \leq \frac{1}{2(R_1 - \ell(R_1) - r_1)} e(R_1) < \frac{2}{3}\frac{e(R_1)}{R_1} \leq \frac{4}{3}\pi$$

for some $\rho_1 \in (r_1, R_1)$. Note that

$$\frac{\ell(\rho_1)}{2\rho_1} \leq \frac{R_1}{\rho_1}\frac{\ell(R_1)}{2R_1} < 8\frac{\ell(R_1)}{2R_1} < \frac{\pi}{2^{21}} < \frac{1}{16}.$$

Hence, we may use again Step 1 of Theorem 1.1 with the new radii $R_2 = \rho_1$, and $r_2$ satisfying $r_2 \in (R_2/8, R_2/4 - \ell(R_2))$ accordingly. Then, for some $\rho_2 \in (r_2, R_2)$ we get

$$\frac{h(\rho_2)}{\rho_2} \leq \frac{1}{2(R_2 - \ell(R_2) - r_2)} e(R_2) < \frac{2}{3}\frac{e(R_2)}{R_2} \Longrightarrow \frac{h(\rho_2)}{\rho_2} \leq \left(\frac{2}{3}\right)^2 2\pi.$$

In general, for $2 \leq k \leq 7$ given $R_{k-1}$, $r_{k-1}$ and $\rho_{k-1}$ set $R_k := \rho_{k-1}$, choose $r_k$ such that $r_k \in (R_k/8, R_k/4 - \ell(R_k))$, and use Step 1 of Theorem 1.1 to find $\rho_k \in (r_k, R_k)$ satisfying

$$\frac{h(\rho_k)}{\rho_k} \leq \left(\frac{2}{3}\right)^j 2\pi.$$

Note that for any $2 \leq k \leq 6$

$$\frac{\ell(\rho_k)}{2\rho_k} < 8\frac{\ell(\rho_{k-1})}{2\rho_{k-1}} < \frac{\pi}{2^{3(8-k)}} < \frac{1}{16},$$

and thus the construction is well defined. In addition,

$$\frac{h(\rho_7)}{\rho_7} \leq \left(\frac{2}{3}\right)^7 2\pi < \frac{1}{2} \Longrightarrow m(\rho_7) \leq 2h(\rho_7) < \rho_7.$$

From Theorem 1.1 we deduce that $0 \notin S_u$, which gives clearly a contradiction.

Eventually, standard density estimates imply $\mathcal{H}^1\left(\overline{S_u} \setminus S_u\right) = 0$ (cp. with [2, Theorem 2.56]), and being $\overline{S_u} = \overline{J_u}$ (see Theorem 1.1) we get $\mathcal{H}^1\left(\overline{S_u} \setminus J_u\right) = 0$.□

In the proof of Corollary 1.3 we will need a Poincaré–Wirtinger type inequality (see Appendix B), and a closure theorem for minimizers of the Mumford–Shah energy.

**Theorem 3.1** *Let $u \in \mathcal{M}(B_R)$ with $\mathcal{H}^1(S_u) < 2R$, and let $\lambda \in (0, 1)$. Then, $u \in L^\infty(B_\rho)$ for some $\rho \in (\lambda(R - \mathcal{H}^1(S_u)/2), R)$, and for any median* med$(u)$ *of $u$ on $B_R$ we have*

$$\|u - \mathrm{med}(u)\|_{L^\infty(B_\rho)} \leq \frac{2}{2R - \mathcal{H}^1(S_u)}\|\nabla u\|_{L^1(B_R,\mathbb{R}^2)}.$$

**Proposition 3.2** *Let $(u_k)_{k\in\mathbb{N}} \subset \mathcal{M}(\Omega)$ be a sequence converging to some $u \in SBV(\Omega)$ strongly in $L^2$. Then $u \in \mathcal{M}(\Omega)$ and for all open sets $A \subseteq \Omega$ we have*

$$\lim_k \int_A |\nabla u_k|^2 dx = \int_A |\nabla u|^2 dx, \quad \lim_k \mathcal{H}^1(J_{u_k} \cap A) = \mathcal{H}^1(J_u \cap A). \tag{3.1}$$

*Furthermore, $(\overline{J_{u_k}})_{k\in\mathbb{N}}$ converges locally in the Hausdorff distance to $\overline{J_u}$.*

We will also take advantage of the following decay lemma inspired by [10, Lemma 4.9] (cp. also with [2, Lemma 7.14, Theorem 7.21]) and proved in Appendix D.

**Lemma 3.3** *For all $\omega \geq 0$, $\beta \in (0,1]$ and $\tau \in (0,1)$ there exist $\varepsilon = \varepsilon(\beta,\tau) \in (0,1)$ and $R = R(\beta,\tau) > 0$ such that if $v \in \mathcal{M}_\omega(\Omega)$ satisfies*

$$\mathrm{MS}(v, B_\rho(z)) \leq \varepsilon\rho,$$

*for some $z \in \Omega$ and $\rho \in (0, (R/\omega^{1/\alpha}) \wedge \mathrm{dist}(z, \partial\Omega))$, then for all $k \geq 1$*

$$\mathrm{MS}(v, B_{\tau^k\rho}(z)) \leq \tau^{k+1-\beta}\,\varepsilon\rho.$$

*Proof of Corollary* 1.3 Denote by $\Omega_v$ the complement of the set on the rhs of (1.7). We first show that $\Omega_v = \Omega \setminus \overline{J_v^\star}$, where as usual $J_v^\star$ is the subset of points $z \in J_v$ for which

$$\lim_{r\downarrow 0} \frac{\mathcal{H}^1(J_u \cap B_r(z))}{2r} = 1.$$

Let $z \in \Omega \setminus \overline{J_v^\star}$, then $v \in W^{1,2}(B_R(z))$ for some $R > 0$. Observe $v|_{\partial B_\rho(z)} \in W^{1,2}(\partial B_R(z))$ for $\mathcal{L}^1$ a.e. $\rho \in (0,R)$. Testing the quasi-minimality condition (1.6) with the harmonic extension $\varphi$ of $v|_{\partial B_\rho(z)}$ to $B_\rho(z)$, Lemma 2.1 and the Co-Area formula yield

$$e_z(\rho) \leq \frac{\rho}{2}\, e_z'(\rho) + \omega\,\rho^{1+\alpha}.$$

Integrating this last inequality we get, for $\alpha \neq 1$,

$$e_z(\rho) \leq \left(\frac{\rho}{R}\right)^2 e_z(R) + \frac{2\omega}{\alpha-1}\,\rho^2\left(R^{\alpha-1} - \rho^{\alpha-1}\right), \tag{3.2}$$

from which we conclude $z \in \Omega_v$ since $m_z(\rho) = e_z(\rho) = o(\rho)$ as $\rho \downarrow 0^+$. Hence, $\Omega \setminus \overline{J_v^\star} \subseteq \Omega_v$. We can proceed analogously if $\alpha = 1$.

To prove the opposite inclusion, let $z \in \Omega_v$ and $r_k \downarrow 0^+$ be a sequence along which for some $\gamma \in (0, 2/3)$

$$\liminf_{r\downarrow 0^+} \frac{m_z(r)}{r} = \lim_{k\uparrow\infty} \frac{m_z(r_k)}{r_k} < \gamma. \tag{3.3}$$

Let $m_k$ be a median of $u$ on $B_{r_k}(z)$, and consider the functions $v_k : B_1 \to \mathbb{R}$ defined as $v_k(y) := r_k^{-1/2}(v(z + r_k y) - m_k)$. Note that $v_k \in \mathcal{M}_{\omega r_k^\alpha}(B_1)$. Let $\lambda \in (0,1)$ be a parameter whose choice will be specified later. Since $\mathcal{H}^1(J_{v_k}) < \gamma$ we apply Theorem B.6 to find functions $w_k : B_1 \to \mathbb{R}$ which are suitable truncations of $v_k$ and such that, for all $k$,

$$\|w_k\|_{L^\infty(B_{\lambda(1-\gamma/2)})} \leq 2\|\nabla v_k\|_{L^1(B_1,\mathbb{R}^2)} \leq 2\pi^{1/2}\|\nabla v_k\|_{L^2(B_1,\mathbb{R}^2)} \overset{(2.2)}{\leq} 4\pi.$$

In particular, up to a subsequence, $(w_k)_{k\in\mathbb{N}}$ converges in $L^2(B_{\lambda(1-\gamma/2)})$ to a function $w$ in $SBV(B_{\lambda(1-\gamma/2)})$ with $\mathrm{MS}(w, B_{\lambda(1-\gamma/2)}) < +\infty$ by Ambrosio's SBV compactness theorem (see [2, Theorems 4.7, 4.8]).

We claim that for all open subsets $A$ of $B_1$ it holds

$$0 \leq \mathrm{MS}(v_k, A) - \mathrm{MS}(w_k, A) \leq \omega\, r_k^\alpha. \tag{3.4}$$

Indeed, by the very definition of $w_k$ we have $\{w_k \neq v_k\} \subset\subset B_1$ (cp. with formula (B.3) in Theorem B.6). Then, as $v_k \in \mathcal{M}_{\omega r_k^\alpha}(B_1)$, we get

$$\mathrm{MS}(v_k, B_1) - \mathrm{MS}(w_k, B_1) \leq \omega r_k^\alpha.$$

We conclude (3.4) by the latter estimate and since $\mathrm{MS}(w_k, B) \leq \mathrm{MS}(v_k, B)$ for all Borel subsets $B$ of $B_1$ (recall that $w_k$ is obtained from $v_k$ by truncation).

Remark C.1 and (3.4) yield that $w \in \mathcal{M}(B_{\lambda(1-\gamma/2)})$, with

$$\mathrm{MS}(w, B_\rho) = \lim_{k\uparrow\infty} \mathrm{MS}(w_k, B_\rho) \quad \text{for all } \rho \in (0, \lambda(1-\gamma/2)]. \tag{3.5}$$

By collecting (3.3)–(3.5), we deduce for every $\rho \in (0, \lambda(1-\gamma/2)]$

$$\mathrm{MS}(w, B_\rho) = \lim_{k\uparrow\infty} \frac{m_z(\rho\, r_k)}{r_k} \leq \lim_{k\uparrow\infty} \frac{m_z(r_k)}{r_k} < \gamma \leq \lambda\left(1 - \frac{\gamma}{2}\right), \tag{3.6}$$

the last inequality holding true provided $\lambda \in (0, 1)$ is suitably chosen (recall that $\gamma \in (0, 2/3)$).

In particular, if $\rho = \lambda(1-\gamma/2)$ from (3.6) we infer that $0 \notin \overline{S_w}$ in view of Remark 2.3. Hence, being $w$ harmonic in $B_{\lambda(1-\gamma/2)}$ for every fixed $\rho \in (0, \lambda(1-\gamma/2)]$ we get

$$\frac{m_z(\rho\, r_k)}{\rho\, r_k} \leq 2\rho \quad \text{for all } k \geq k_\rho, \tag{3.7}$$

so that $z \in \Omega \setminus J_v^\star$. Moreover, if $\varrho > 0$ is such that $4\varrho \leq \varepsilon \wedge (\lambda(1-\gamma/2)) \wedge (2/3)$ then $B_{\varrho\, r_{k_\varrho}/2}(z) \subseteq \Omega_v$. For, if $x \in B_{\varrho\, r_{k_\varrho}/2}(z)$, by Lemma 3.3 applied with $\tau = 1/2$, any $\beta \in (0, 1)$ and $\rho = \varrho\, r_{k_\varrho}$, the choice of $\varrho$ yields that

$$\frac{m_x(\varrho\, r_{k_\varrho}/2)}{\varrho\, r_{k_\varrho}/2} \leq 2\frac{m_z(\varrho\, r_{k_\varrho})}{\varrho\, r_{k_\varrho}} \overset{(3.7)}{\leq} 4\varrho \leq \varepsilon,$$

and thus we deduce $x \in \Omega_v$ by iterating Lemma 3.3 along the sequence $(2^{-i}\varrho\, r_{k_\varrho})_{i\in\mathbb{N}}$. Hence, $\Omega_v$ is an open set and $\Omega_v \cap J_v^\star = \emptyset$, in turn this implies $\Omega \setminus \overline{J_v^\star} = \Omega_v$.

Finally, being $\Omega_v$ open and $v$ a quasi-minimizer of the Dirichlet energy on $\Omega_v$ then $v \in C^{1,1/2}(\Omega_v)$ by (3.2) and Campanato's estimates. In conclusion, $S_v \cap \Omega_v = \emptyset$, and then $\overline{S_v} = \overline{J_v} = \Omega \setminus \Omega_v$. □

## Appendix A: The David-Léger-Maddalena-Solimini monotonicity formula

*Proof of Lemma* 2.1 We start by recalling the first variation formula for local minimizers of the Mumford–Shah energy (see [2, Sect. 7.4]): for every vector field $\eta \in \mathrm{Lip} \cap C_c(\Omega, \mathbb{R}^2)$

$$\int_\Omega \Big(|\nabla u|^2 \mathrm{div}\eta - 2\langle \nabla u, \nabla u \nabla \eta\rangle\Big)\, dx + \int_{J_u} \mathrm{div}^{J_u} \eta \, d\mathcal{H}^1 = 0. \tag{A.1}$$

With fixed a point $z \in \Omega$, $r > 0$ with $B_r(z) \subseteq \Omega$, we consider special radial vector fields $\eta_{r,s} \in \mathrm{Lip} \cap C_c(B_r(z), \mathbb{R}^2)$, $s \in (0,r)$, in formula above. For the sake of simplicity we assume $z = 0$, and drop the subscript $z$ in what follows. Let

$$\eta_{r,s}(x) := x\, \chi_{[0,s]}(|x|) + \frac{|x|-r}{s-r}\, x\, \chi_{(s,r]}(|x|),$$

then routine calculations leads to

$$\nabla \eta_{r,s}(x) := \mathrm{Id}\, \chi_{[0,s]}(|x|) + \left( \frac{|x|-r}{s-r} \mathrm{Id} + \frac{1}{s-r} \frac{x}{|x|} \otimes x \right) \chi_{(s,r]}(|x|)$$

$\mathcal{L}^2$ a.e. in $\Omega$. In turn, from the latter formula we infer for $\mathcal{L}^2$ a.e. in $\Omega$

$$\mathrm{div}\eta_{r,s}(x) = 2\, \chi_{[0,s]}(|x|) + \left( 2\frac{|x|-r}{s-r} + \frac{|x|}{s-r} \right) \chi_{(s,r]}(|x|),$$

and, if $\nu_u(x)$ is a unit vector normal field in $x \in J_u$, for $\mathcal{H}^1$ a.e. $x \in J_u$

$$\mathrm{div}^{J_u} \eta_{r,s}(x) = \chi_{[0,s]}(|x|) + \left( \frac{|x|-r}{s-r} + \frac{1}{|x|(s-r)} |\langle x, \nu_u^\perp \rangle|^2 \right) \chi_{(s,r]}(|x|).$$

Consider the set $I := \{\rho \in (0, \mathrm{dist}(0, \partial\Omega)) : \mathcal{H}^1(J_u \cap \partial B_\rho) = 0\}$, then $(0, \mathrm{dist}\,(0, \partial\Omega)) \setminus I$ is at most countable being $\mathcal{H}^1(J_u) < +\infty$. If $\rho$ and $s \in I$, by inserting $\eta_{r,s}$ in (A.1) we find

$$\frac{1}{s-r} \int_{B_r \setminus B_s} |x| |\nabla u|^2 dx - \frac{2}{s-r} \int_{B_r \setminus B_s} |x| \langle \nabla u, \left( \mathrm{Id} - \frac{x}{|x|} \otimes \frac{x}{|x|} \right) \nabla u \rangle dx$$
$$= \ell(s) + \int_{J_u \cap (B_r \setminus B_s)} \frac{|x|-r}{s-r} d\mathcal{H}^1 + \frac{1}{s-r} \int_{J_u \cap (B_r \setminus B_s)} |x| |\langle \frac{x}{|x|}, \nu_u^\perp \rangle|^2 d\mathcal{H}^1.$$

Next we employ Co-Area formula and rewrite equality above as

$$\frac{1}{s-r} \int_s^r \rho\, d\rho \int_{\partial B_\rho} |\nabla u|^2 d\mathcal{H}^1 - \frac{2}{s-r} \int_s^r \rho\, d\rho \int_{\partial B_\rho} \left| \frac{\partial u}{\partial \tau} \right|^2 d\mathcal{H}^1$$
$$= \ell(s) + \int_{J_u \cap (B_r \setminus B_s)} \frac{|x|-r}{s-r} d\mathcal{H}^1 + \frac{1}{s-r} \int_s^r d\rho \int_{J_u \cap \partial B_\rho} |\langle x, \nu_u^\perp \rangle| d\mathcal{H}^0$$

where $\nu := x/|x|$ denotes the radial versor and $\tau := \nu^\perp$ the tangential one. Lebesgue differentiation theorem then provides a subset $I'$ of full measure in $I$ such that if $r \in I'$ and we let $s \uparrow r^-$ it follows

$$-r \int_{\partial B_r} |\nabla u|^2 d\mathcal{H}^1 + 2r \int_{\partial B_r} \left| \frac{\partial u}{\partial \tau} \right|^2 d\mathcal{H}^1 = \ell(r) - \int_{J_u \cap \partial B_r} |\langle x, \nu_u^\perp \rangle| d\mathcal{H}^0.$$

Formula (2.1) then follows straightforwardly. $\square$

## B. A Poincaré-Wirtinger type inequality

The arguments of this appendix refine a truncation procedure introduced by [12] (cp. with [12, Lemma 4.2, Theorem 4.1]). In what follows given any $\mathcal{L}^2$-measurable function $v : B_R \to \mathbb{R}$, for every $s \in \mathbb{R}$, we denote by $E_{v,s}$ the $s$ sub-level of $v$ in $B_R$, i.e.,

$$E_{v,s} := \{x \in B_R : v(x) \leq s\}, \tag{B.1}$$

and by $\mathrm{med}(v)$ a *median* of $v$ in $B_R$, for instance we can take

$$\mathrm{med}(v) := \sup\{s \in \mathbb{R} : \mathcal{L}^2(E_{v,s}) \leq \mathcal{L}^2(B_R)/2\}. \tag{B.2}$$

Let us begin with the truncation procedure for functions in $SBV$ with zero gradient.

**Lemma B.1** *For every $v \in SBV(B_R)$ with $\nabla v = 0$ $\mathcal{L}^2$ a.e. $B_R$ and $\mathcal{H}^1(S_v) < 2R$, the set $I = \{r \in (0, R) : \mathcal{H}^0(\partial B_t \cap S_v) = 0\}$ satisfies $\mathcal{L}^1(I) \geq R - \mathcal{H}^1(S_v)/2$.*

*In addition, for $\mathcal{L}^1$ a.e. $r \in I$ the trace of $v$ on $\partial B_R$ is constant.*

*Proof.* Set $J := \{r \in (0, R) : \mathcal{H}^0(\partial B_t \cap S_v) \geq 2\}$, and estimate $\mathcal{L}^1(J)$ by means of the Co-Area formula for rectifiable sets as follows

$$2\mathcal{L}^1(J) \leq \int_J \mathcal{H}^0(\partial B_t \cap S_v)\, dt \leq \mathcal{H}^1(S_v),$$

from which we infer $\mathcal{L}^1((0, R) \setminus J) \geq R - \mathcal{H}^1(S_v)/2$.

To conclude we prove the inequality $\mathcal{L}^1((0, R) \setminus J) \leq \mathcal{L}^1(I)$. To this aim note that for $\mathcal{L}^1$ a.e. $r \in (0, R) \setminus J$ the slice $v_r$ obtained by restricting $v$ to $\partial B_r$ belongs to $SBV(\partial B_r)$, it has zero approximate derivative and $\partial B_r \cap S_v = S_{v_r}$ (see [2, Section 3.11]). Finally, since $\#(\partial B_r \cap S_v) \leq 1$ as $r \in (0, R) \setminus J$, by taking into account that $v_r' = 0$ $\mathcal{H}^1$ a.e. on $\partial B_r$, we infer that actually $\partial B_r \cap S_v = \emptyset$. In conclusion, $\mathcal{L}^1((0, R) \setminus (I \cup J)) = 0$. □

**Remark B.2** *The estimate $\mathcal{L}^1(I) \geq R - \mathcal{H}^1(S_v)/2$ proved in Lemma B.1 above, clearly implies that $\mathcal{L}^1(I \cap (\lambda(R - \mathcal{H}^1(S_v)/2), R)) > 0$ for all $\lambda \in (0, 1)$.*

In what follows we identify any set of finite perimeter $E$ with its $\mathcal{L}^2$-measure theoretic interior defined by $E^{(1)} := \{x \in \mathbb{R}^2 : \lim_{t\to 0+}(\pi t^2)^{-1}\mathcal{L}^2(B_t(x) \cap E) = 1\}$. Recall that $\partial^* E$ denotes the *essential boundary* of $E$, satisfying $\mathrm{Per}(E) = \mathcal{H}^1(\partial^* E)$ (see [2, Definition 3.60, Theorem 3.61]).

In particular, from Lemma B.1 we immediately deduce the following corollary.

**Corollary B.3** *For every set of finite perimeter $E \subseteq B_R$ with $\mathrm{Per}(E) < 2R$ a set of positive $\mathcal{L}^1$ measure in $(0, R)$ exists such that either $\mathcal{H}^1(E \cap \partial B_t) = 0$ or $\mathcal{H}^1(E \cap \partial B_t) = \mathcal{H}^1(\partial B_t)$, for all $t$ in this set.*

Under an additional smallness condition on the $\mathcal{L}^2$ measure of $E$, the previous result can be further improved (cp. to [12, Lemma 4.2]). To this aim we recall that a set of finite perimeter $E \subset \mathbb{R}^2$ is said to be *decomposable* if there exists a partition of $E$ in two $\mathcal{L}^2$-measurable sets $A$, $B$ with strictly positive measure such that $\mathrm{Per}(E) = \mathrm{Per}(A) + \mathrm{Per}(B)$. Accordingly, a set of finite perimeter is *indecomposable* otherwise. Notice that the properties of being decomposable or indecomposable depend only on the $\mathcal{L}^2$-equivalence class of $E$.

**Lemma B.4** *If $E \subseteq B_R$ is such that $\mathcal{L}^2(E) \le \mathcal{L}^2(B_R)/2$ and* $\mathrm{Per}(E) < 2R$*, the set $\mathcal{I} := \{t \in (0, R) : \mathcal{H}^1(\partial B_t \cap E) = 0\}$ satisfies $\mathcal{L}^1(\mathcal{I}) \ge R - \mathrm{Per}(E)/2$.*

*Proof.* According to [1, Theorem 1] there exists a unique and at most countable family of pairwise disjoint (maximal) indecomposable sets $E_i$, $i \in I \subseteq \mathbb{N}$, with $\mathcal{L}^2(E_i) > 0$ such that

$$\mathcal{H}^1\left(E \setminus \bigcup_{i\in I} E_i\right) = 0 \quad \text{and} \quad \mathrm{Per}(E) = \sum_{i\in I} \mathrm{Per}(E_i).$$

An elementary projection argument shows that $2d_i := 2\mathrm{diam}(E_i) \le \mathrm{Per}(E_i)$, so that

$$2\sum_{i\in I} d_i \le \sum_{i\in I} \mathrm{Per}(E_i) = \mathrm{Per}(E) < 2R.$$

Let now $\mathcal{I}_i := \{t \in (0, R) : \mathcal{H}^1(\partial B_t \cap E_i) = 0\}$, and note that $\mathcal{I} = \cap_{i\in I}\mathcal{I}_i$. In addition, since for all $\varepsilon > 0$ the sets $E_i$ are contained in $B_{R_i+d_i+\varepsilon} \setminus \overline{B}_{R_i-\varepsilon}$ for some $R_i > 0$, we get that

$$\mathcal{L}^1((0, R) \setminus \mathcal{I}) \le \sum_{i\in I} \mathcal{L}^1((0, R) \setminus \mathcal{I}_i) \le \sum_{i\in I} d_i,$$

from which, finally, we infer

$$\mathcal{L}^1(\mathcal{I}) \ge R - \sum_{i\in I} d_i \ge R - \frac{\mathrm{Per}(E)}{2}.$$

□

**Remark B.5** *The estimate $\mathcal{L}^1(\mathcal{I}) \ge R - \mathrm{Per}(E)/2 > 0$ proved in Lemma B.4 above, clearly implies that $\mathcal{L}^1(\mathcal{I} \cap (\lambda(R - \mathrm{Per}(E)/2), R)) > 0$ for all $\lambda \in (0, 1)$.*

From Lemmata B.1 and B.4 we infer that $SBV$ functions with suitably quantified short jump set enjoy a Poincaré–Wirtinger type inequality.

**Theorem B.6** (A Poincaré–Wirtinger type inequality) *If $v \in SBV(B_R)$ with $\mathcal{H}^1(S_v) < 2R$, then there are truncation levels $s' \le s''$ and for all $\lambda \in (0, 1)$ radii $\rho' \le \rho''$ belonging to $(\lambda(R - \mathcal{H}^1(S_v)/2), R)$ in a way that the function*

$$w := \begin{cases} v \vee s' \wedge s'' & B_{\rho'} \\ v \wedge s'' & B_{\rho''} \setminus B_{\rho'} \\ v & B_R \setminus B_{\rho''}, \end{cases} \tag{B.3}$$

*satisfies* $\mathcal{H}^1(S_w \setminus S_v) = 0$ *and for any median* $\mathrm{med}(v)$ *of* $v$ *on* $B_R$

$$\|w - \mathrm{med}(v)\|_{L^\infty(B_{\rho'})} \leq \frac{2}{2R - \mathcal{H}^1(S_v)} \|\nabla v\|_{L^1(B_R,\mathbb{R}^2)}.$$

*Proof.* First note that if $\|\nabla v\|_{L^1(B_R,\mathbb{R}^2)} = 0$ we may apply Lemma B.1 and select $\rho \in (R/2 - \mathcal{H}^1(J_v)/4, R)$ (thanks to Remark B.2) such that the trace of $v$ on $\partial B_\rho$ is constant. In this case we take $s' = s''$ equal to such a value and $\rho = \rho' = \rho''$ to conclude.

Thus, we need to analyze only the case with $\|\nabla v\|_{L^1(B_R,\mathbb{R}^2)} > 0$. To this aim set $\alpha := 2R - \mathcal{H}^1(S_v) > 0$, then the $BV$ Co-Area Formula (see [2, Theorem 3.40]) implies

$$\int\limits_{\mathrm{med}(v)-2\|\nabla v\|_{L^1(B_R,\mathbb{R}^2)}/\alpha}^{\mathrm{med}(v)} \mathcal{H}^1(\partial^* E_s \setminus S_v)\, ds \leq \int\limits_{\mathbb{R}} \mathcal{H}^1(\partial^* E_s \setminus S_v)\, ds = \|\nabla v\|_{L^1(B_R,\mathbb{R}^2)},$$

where $E_s$ is the sub-level of $v$ in $B_R$ defined in (B.1) and $\mathrm{med}(v)$ is defined in (B.2). Hence, by the Mean Value Theorem there exists $s' \in (\mathrm{med}(v) - 2\|\nabla v\|_{L^1(B_R,\mathbb{R}^2)}/\alpha, \mathrm{med}(v))$ such that $\mathcal{H}^1(\partial^* E_{s'} \setminus S_v) \leq \alpha/2$, and so

$$\mathcal{H}^1(\partial^* E_{s'}) \leq \mathcal{H}^1(\partial^* E_{s'} \setminus S_v) + \mathcal{H}^1(S_v) < 2R. \tag{B.4}$$

Analogously, we can find $s'' \in (\mathrm{med}(v), \mathrm{med}(v) + 2\|\nabla v\|_{L^1(B_R,\mathbb{R}^2)}/\alpha)$ such that

$$\mathcal{H}^1(\partial^* E_{s''}) < 2R. \tag{B.5}$$

The definition of median (B.2) and the choice $s' < \mathrm{med}(v)$ yield $\mathcal{L}^2(E_{s'}) \leq \mathcal{L}^2(B_R)/2$, and by arguing similarly, the same inequality holds for the set $B_R \setminus E_{s''}$ as well. By taking into account inequalities (B.4), (B.5) we may apply Lemma B.4 separately to the two sets $E_{s'}$, $B_R \setminus E_{s''}$ and find radii $\lambda(R - \mathcal{H}^1(S_v)/2) < \rho' \leq \rho'' < R$ with $\mathcal{H}^1(E_{s'} \cap \partial B_{\rho'}) = 0$ and $\mathcal{H}^1((B_R \setminus E_{s''}) \cap \partial B_{\rho''}) = 0$ (thanks to Remark B.5).

The conclusion then follows at once by the very definition of $w$ in (B.3). □

In case $v$ is a local minimizer of the Mumford–Shah energy we deduce Theorem 3.1.

*Proof of Theorem* 3.1 By keeping the notation of Theorem B.6, the function $w$ defined in (B.3) turns out to be an admissible function to test the minimality of $u$ on $B_R$. By construction $\mathcal{H}^1(S_w \setminus S_u) = 0$ and $|\nabla w| \leq |\nabla u|$ $\mathcal{L}^2$ a.e. in $B_R$, from this we infer that $u = w$ $\mathcal{L}^2$ a.e. in $B_{\rho'}$ being the Mumford–Shah energy decreasing under truncation.

**Remark B.7** *If the length of the jump set exceeds* $2R$ *a similar Poincaré-Wirtinger type inequality does not hold. Take, for instance,* $v = 1$ *if* $y > 0$ *and* $-1$ *otherwise (see [2, Proposition 6.8] for a proof that such a function is in* $\mathcal{M}(B_R)$ *if* $R$ *is sufficiently small).*

## C. Limits of sequences of local minimizers

In this section we prove that limits of converging sequences of local minimizers are local minimizers as well (cp. with [2, Theorem 7.7] in case the measure of the jump sets is vanishing, and with [11, Proposition 5.1] if the Dirichlet energies are infinitesimal).

*Proof of Proposition* 3.2 Let $v$ be an admissible function to test the minimality of $u$, that is $v \in SBV(\Omega)$ and $\{v \neq u\} \subset\subset \Omega$. Moreover, let $\Omega'$ be an open set such that $\{v \neq u\} \subset\subset \Omega' \subset\subset \Omega$ and $\varphi \in C^1_c(\Omega)$ be such that $\varphi = 1$ on $\Omega'$ and $|\nabla\varphi| \leq 2/\mathrm{dist}(\Omega', \partial\Omega)$. Define $v_k := \varphi\, v + (1-\varphi)u_k$. Then $v_k \in SBV(\Omega)$ and it is an admissible test function for $u_k$. Thus, for some fixed constant $C > 0$, routine calculations lead to

$$\begin{aligned} \mathrm{MS}(u_k) &\leq \mathrm{MS}(v_k) \leq \mathrm{MS}(v) + C\,\mathrm{MS}(v, \Omega \setminus \overline{\Omega'}) + C\,\mathrm{MS}(u_k, \Omega \setminus \overline{\Omega'}) \\ &\quad + C \int_{\Omega\setminus\overline{\Omega'}} |u - u_k|^2 dx. \end{aligned} \tag{C.1}$$

To get the last term on the rhs above we have used the equality $v = u$ on $\Omega \setminus \overline{\Omega'}$.

Note that the sequence of Radon measures $(\mathrm{MS}(u_k, \cdot))_{k\in\mathbb{N}}$ is equi-bounded in mass in view of the energy upper bound (2.2). Hence, up to the extraction of a subsequence (not relabeled), $(\mathrm{MS}(u_k, \cdot))_{k\in\mathbb{N}}$ converges to some Radon measure $\mu$ on $\Omega$. Without loss of generality we may also assume that $\mu(\partial\Omega') = 0$. Furthermore, we recall that, by Ambrosio's lower semicontinuity theorem, we have, for every open set $A \subseteq \Omega$,

$$\liminf_k \int_A |\nabla u_k|^2 dx \geq \int_A |\nabla u|^2 dx, \quad \liminf_k \mathcal{H}^1(J_{u_k} \cap A) \geq \mathcal{H}^1(J_u \cap A), \tag{C.2}$$

(see [2, Theorems 4.7 and 4.8]). As $k \uparrow \infty$ in (C.1), thanks to condition $\mu(\partial\Omega') = 0$ and (C.2), we find

$$\begin{aligned} \mathrm{MS}(u) &\leq \liminf_k \mathrm{MS}(u_k) \leq \limsup_k \mathrm{MS}(u_k) \leq \mathrm{MS}(v) + C\,\mathrm{MS}(v, \Omega \setminus \overline{\Omega'}) \\ &\quad + C\,\mu(\Omega \setminus \overline{\Omega'}). \end{aligned}$$

Then, by letting $\Omega'$ increase to $\Omega$ (enforcing the condition $\mu(\partial\Omega') = 0$) we conclude

$$\mathrm{MS}(u) \leq \liminf_k \mathrm{MS}(u_k) \leq \limsup_k \mathrm{MS}(u_k) \leq \mathrm{MS}(v). \tag{C.3}$$

Hence, $u$ belongs to $\mathcal{M}(\Omega)$. In addition, by choosing $v$ equal to $u$ itself, we can perform the same construction above for every open set $A \subseteq \Omega$ (with $\Omega' \subset\subset A$) and infer (C.3) localized onto $A$, so that equalities in (3.1) follow at once.

Finally, the density lower bound in Corollary 1.2 and the equalities in (3.1) imply easily the claimed local Hausdorff convergence. □

**Remark C.1** *The same conclusion of Proposition 3.2 holds provided we are given a sequence* $(u_k)_{k\in\mathbb{N}}$ *converging in* $L^2(\Omega)$ *to* $u \in SBV(\Omega)$*, with* $u_k$ *satisfying, for some* $\vartheta_k \downarrow 0^+$,

$$\mathrm{MS}(u_k) \le \mathrm{MS}(w) + \vartheta_k \quad \textit{whenever } \{w \ne u_k\} \subset\subset \Omega.$$

## D. A decay Lemma

We start off by proving a preliminary decay property of the energy.

**Lemma D.1** *For all* $\beta \in (0, 2)$ *and* $\tau \in (0, 1)$ *there exist* $\varepsilon = \varepsilon(\beta, \tau)$ *and* $\vartheta = \vartheta(\beta, \tau)$ *in* $(0, 1)$ *such that if* $v \in SBV(\Omega)$ *satisfies, for some* $z \in \Omega$ *and* $\rho > 0$,

$$\mathrm{MS}(v, B_\rho(z)) \le \varepsilon\rho,$$

*and*

$$(1-\vartheta)\,\mathrm{MS}(v, B_\rho(z)) \le \mathrm{MS}(w, B_\rho(z)) \quad \textit{whenever } \{w \ne v\} \subset\subset B_\rho(z),$$

*then*

$$\mathrm{MS}(v, B_{\tau\rho}(z)) \le \tau^{2-\beta}\,\mathrm{MS}(v, B_\rho(z)).$$

*Proof.* We argue by contradiction and suppose that there are sequences $v_k \in SBV(\Omega)$, $\varepsilon_k \downarrow 0^+$, $\vartheta_k \downarrow 0^+$, $\rho_k \downarrow 0^+$ and $z_k \in \Omega$ with $B_{\rho_k}(z_k) \subset \Omega$ such that for some $\tau$ and $\beta \in (0, 2)$

$$\mathrm{MS}(v_k, B_{\rho_k}(z_k)) = \varepsilon_k\rho_k, \tag{D.1}$$

$$(1-\vartheta_k)\,\mathrm{MS}(v_k, B_{\rho_k}(z_k)) \le \mathrm{MS}(w, B_{\rho_k}(z_k)) \tag{D.2}$$

for all $w \in SBV(\Omega)$ with $\{w \ne v_k\} \subset\subset B_{\rho_k}(z_k)$, but

$$\mathrm{MS}(v_k, B_{\tau\rho_k}(z_k)) > \tau^{2-\beta}\,\mathrm{MS}(v_k, B_{\rho_k}(z_k)). \tag{D.3}$$

Denote by $w_k : B_1 \to \mathbb{R}$ the functions $w_k(y) = (\varepsilon_k\rho_k)^{-1/2}(v_k(z_k + \rho_k y) - m_k)$ and by $m_k$ a median of $v_k$ on $B_{\rho_k}(z_k)$, so that, if we set,

$$F_k(v, B_\rho) := \int_{B_\rho} |\nabla v|^2 dy + \frac{1}{\varepsilon_k}\mathcal{H}^1(S_v \cap B_\rho),$$

then (D.1)–(D.3) can be rewritten respectively as

$$F_k(w_k, B_1) = 1, \quad F_k(w, B_1) \ge 1 - \vartheta_k, \quad \text{and} \quad F_k(w_k, B_\tau) > \tau^{2-\beta}, \tag{D.4}$$

for all $w \in SBV(B_1)$ with $\{w \ne w_k\} \subset\subset B_1$.

In particular, from the first condition in (D.4) we infer that $\mathcal{H}^1(S_{w_k}) \le \varepsilon_k$. Thus, by applying Theorem B.6 to the $w_k$'s, we find functions $\widetilde{w}_k \in SBV(B_1)$ satisfying, for all $r \in (0, 1)$,

$$\{\widetilde{w}_k \ne w_k\} \subset\subset B_r, \quad \|\widetilde{w}_k\|_{L^\infty(B_r)} \le 2 \quad \text{for } k \ge k_r. \tag{D.5}$$

Then, Ambrosio's $SBV$ compactness theorem and a diagonal argument provide a subsequence (not relabeled) and a function $\widetilde{w} \in W^{1,2} \cap L^\infty(B_1)$ such that $(\widetilde{w}_k)_{k\in\mathbb{N}}$ converges to $\widetilde{w}$ in $L^2_{loc}(B_1)$. Note that by lower semicontinuity and (D.4), we have

$$\int_{B_1} |\nabla \widetilde{w}|^2 dx \le \liminf_k F_k(\widetilde{w}_k, B_1) \le 1. \tag{D.6}$$

Next, we claim that $\widetilde{w}$ is harmonic in $B_1$ and that for all $r \in (0,1)$

$$\lim_k F_k(w_k, B_r) = \int_{B_r} |\nabla \widetilde{w}|^2 dx. \tag{D.7}$$

Given this for granted, we get a contradiction, since from (D.4) and (D.7)

$$\tau^{2-\beta} \le \int_{B_\tau} |\nabla \widetilde{w}|^2 dx,$$

but on the other hand the harmonicity of $\widetilde{w}$ on $B_1$ and (D.6) yield that

$$\int_{B_\tau} |\nabla \widetilde{w}|^2 dx \le \tau^2.$$

To prove (D.7), let $r < s \in (0,1)$ and $\varphi \in C_c^\infty(B_s)$ be such that $\varphi = 1$ on $B_r$. Define $\zeta_k = \varphi\widetilde{w} + (1-\varphi)\widetilde{w}_k$, since $w_k = \widetilde{w}_k$ on $B_s$ for $k \ge k_s$ (see (D.5)), elementary computations, the first two conditions in (D.4), and the locality of the energy lead to

$$\begin{aligned} F_k(w_k, B_r) &= F_k(\widetilde{w}_k, B_r) \le F_k(\zeta_k, B_s) + \vartheta_k \le F_k(\widetilde{w}, B_r) + C\, F_k(\widetilde{w}_k, B_s \setminus \overline{B_r}) \\ &\quad + C\, F_k(\widetilde{w}, B_s \setminus \overline{B_r}) + C \int_{B_s\setminus\overline{B_r}} |\widetilde{w}_k - \widetilde{w}|^2 dx + \vartheta_k. \end{aligned}$$

The sequence of Radon measures $(F_k(\widetilde{w}_k, \cdot))_{k\in\mathbb{N}}$ is equi-bounded in mass in view of (D.4). Hence, up to a subsequence not relabeled for convenience, $(F_k(\widetilde{w}_k, \cdot))_{k\in\mathbb{N}}$ converges to some Radon measure $\mu$ on $B_1$. Assume that $\mu(\partial B_s) = 0$, by passing to the limit as $k \uparrow \infty$ and by Ambrosio's lower semicontinuity result we find

$$\begin{aligned} \int_{B_r} |\nabla \widetilde{w}|^2 dx &\le \liminf_k F_k(w_k, B_r) \le \limsup_k F_k(w_k, B_r) \\ &\le \int_{B_r} |\nabla \widetilde{w}|^2 dx + C\,\mu(B_s \setminus \overline{B_r}) + C \int_{B_s\setminus\overline{B_r}} |\nabla \widetilde{w}|^2 dx. \end{aligned}$$

Equality (D.7) then follows by letting $s \downarrow r^+$ along values satisfying $\mu(\partial B_s) = 0$.

Eventually, the harmonicity of $\widetilde{w}$ is easily deduced from its local minimality for the Dirichlet energy. This last property is obtained as above by modifying any test function $\zeta \in W^{1,2}(B_1)$ such that $\{\zeta \ne \widetilde{w}\} \subset\subset B_1$ into a test-function for $\widetilde{w}_k$ in order to exploit again the quasi-minimality condition satisfied by $w_k$ in (D.4).

□

We are now ready to prove Lemma 3.3.

*Proof of Lemma* 3.3 We argue as in [2, Theorem 7.21], and take $z = 0$ for the sake of simplicity. We claim that

$$\mathrm{MS}(v, B_{\tau\rho}) \leq \varepsilon\,\tau^{2-\beta}\rho \tag{D.8}$$

if we set $R := (\varepsilon\,\vartheta\,\tau^{2-\beta})^{1/\alpha}$, with $\varepsilon = \varepsilon(\beta, \tau)$ and $\vartheta = \vartheta(\beta, \tau)$ provided by Lemma D.1.

Indeed, either both the assumptions of Lemma D.1 are satisfied or not. In the former case the thesis of that lemma gives exactly inequality (D.8); otherwise for some $w \in SBV(\Omega)$ with $\{w \neq v\} \subset\subset B_\rho(z) \subset \Omega$ we have by the quasi-minimality of $v$

$$\mathrm{MS}(v, B_{\tau\rho}) \leq \mathrm{MS}(v, B_\rho) \leq \frac{1}{\vartheta}\big(\mathrm{MS}(v, B_\rho) - \mathrm{MS}(w, B_\rho)\big) \leq \frac{\omega}{\vartheta}\rho^{1+\alpha}.$$

Thus, (D.8) follows since $\rho \leq R/\omega^{1/\alpha}$.

Eventually, as $\tau \in (0, 1)$ we can repeat the previous argument, and conclude by induction. □

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