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An improvement of Massera's theorem for the existence and uniqueness of a periodic solution for the Liénard equation

Gabriele Villari

A Fabio Zanolin, "amico di una vita", per i suoi splendidi 60 anni.

Abstract. *In this paper we prove the existence and uniqueness of a periodic solution for the Liénard equation*

$$\ddot{x} + f(x)\,\dot{x} + x = 0.$$

The classical Massera's monotonicity assumptions, which are required in the whole line, are relaxed to the interval (α, δ), where α and δ can be easily determined. In the final part of the paper a simple perturbation criterion of uniqueness is presented.

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1. Preliminaries and well-known results

The problem of existence and uniqueness of a periodic solution for the Liénard equation,

$$\ddot{x} + f(x)\,\dot{x} + x = 0, \tag{1}$$

has been widely investigated in the literature. Among the uniqueness results, the most interesting and intriguing one is, without any doubt, the classical Massera's Theorem. This is due to the geometrical ideas and the fact that this result, despite several efforts, is in most cases no more valid for the generalized Liénard equation

$$\ddot{x} + f(x)\,\dot{x} + g(x) = 0. \tag{2}$$

For related results still valid for equation (2), we refer to [1], and to [3] for the equation

$$\ddot{x} + f(x, \dot{x})\dot{x} + x = 0.$$

Throughout this paper we assume that

(A) f is continuous and there exist $a < 0 < b$ such that $f(x)$ is negative for $a < x < b$, positive outside this interval. Moreover $xF(x) > 0$ for $|x|$ large.

It is well-known (see, for instance, [14, Theorem 1]), that such condition guarantees the existence of at least a stable limit cycle.

Equation (1) is equivalent to the phase-plane system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x)y - x. \end{cases} \tag{3}$$

We just notice that assumption **(A)** guarantees the property of uniqueness for the solutions to the Cauchy problem associated to system (3) and therefore the trajectories of such a system cannot intersect.

The phase-plane system is equivalent to the Liénard system

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -x \end{cases}, \qquad \text{where } F(x) = \int_0^x f(t)\, dt. \tag{4}$$

For equation (2) system (3) becomes

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x)y - g(x), \end{cases} \tag{5}$$

while system (4) becomes

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases}, \qquad \text{where } F(x) = \int_0^x f(t)\, dt. \tag{6}$$

It is well-known that the nonlinear transformation $(x, y + F(x))$ takes points of system (3) in points of system (4). Such a transformation preserves the x-coordinate and this will be crucial for the proof of the main result.

Now we define the property **(B)**

(B) $F(x)$ has three zeros at $\alpha < 0$, 0, $\beta > 0$. Moreover $xF(x)$ is negative for $\alpha < x < \beta$ and positive outside this interval, while F is monotone increasing for $x < \alpha$ and $x > \beta$ (see Figure 1).

We observe that property **(A)** implies property **(B)** and that property **(B)** can be assumed even if $f(x)$ changes sign several times in the interval (α, β),

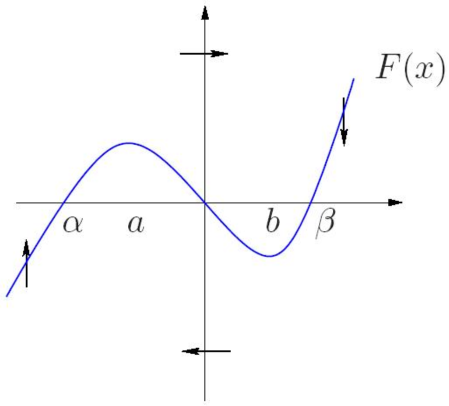

Figure 1:

which is not our case. Finally we notice that it is trivial to show that in system (4) and in system (6) the distance from the origin is increasing when $xF(x) < 0$, while is decreasing when $xF(x) > 0$.

We present the classical Massera Theorem which is a milestone among the results of limit cycles uniqueness for system (3).

THEOREM 1.1 (Massera [8]). *The system (3) has at most one limit cycle which is stable, and hence equation (1) has at most one non trivial periodic solution which is stable, provided that f is continuous and*

1. *$f(x)$ is monotone decreasing for $x < 0$,*
2. *$f(x)$ is monotone increasing for $x > 0$.*

The Theorem of Massera improved a previous result due to Sansone [12] in which there was the additional assumption $|f(x)| < 2$. This assumption comes from the fact that Sansone was using the polar coordinates. Such strong restriction on f is clearly not satisfied in the polynomial case and hence the Massera's result is much more powerful. We recall the recent paper [11] in which a discussion concerning these two results, as well as related results, may be found.

We must observe that in his paper, Massera was proving the uniqueness of limit cycles regardless the existence because only the monotonicity properties and the continuity were required. It is easy to prove that, in order to fulfill the necessary conditions for the existence of limit cycles, the only cases to be considered are

1. $f(x)$ has two zeros $a < 0 < b$. In this case property **(A)** is fulfilled and hence the existence of limit cycles is granted,

2. $f(x)$ remains negative for $x < 0$, (or for $x > 0$), while intersects the x axis once in $x > 0$ (or for $x < 0$).

In this case the existence of limit cycles is not granted. It is possible to produce examples in which, actually, there exists a unique limit cycle but, as far as we know, there is no existence result which can be applied in this situation. Moreover this case does not cover the crucial polynomial case, which is still the most important and it is related with the Lins-De Melo-Pugh conjecture [7], concerning the upper bound of limit cycles for equation (1) when $f(x)$ is a polynomial of degree n.

Now we recall another interesting result, which is due to Levinson-Smith for system (6) and to Sansone for system (4).

THEOREM 1.2 (Levinson-Smith [5] and Sansone [13], see also [15]). *If F has the property **(B)**, at most a limit cycle intersects both the lines $x = \alpha$ and $x = \beta$*

This is a very nice result, but it is abstract, because, in general, if there are no symmetry properties on f and g, such a situation is not easy to be verified. For system (6) there are sufficient conditions which guarantee that in the Liénard plane this situation actually occurs (see [2, 15] and, for more general cases, [1, 10, 17]). In the case of system (4) a sufficient condition is $|\alpha| = \beta$.

The aim of this paper is to relax the monotonicity assumptions, required by Massera, to a fixed interval given by the function f.

This will be achieved working both in the phase plane and in the Liénard plane and using property **(B)** and Theorem 1.2, together with Massera's Theorem.

Proofs are based on elementary phase plane analysis, but as far as we know, the result is original and this shows how still this classical problem deserves to be investigated.

In the final part of the paper, an existence and uniqueness result will be presented for the equation, depending on a parameter λ,

$$\ddot{x} + \lambda f(x)\dot{x} + x = 0.$$

2. The main result

We now present our result which improves the classical Massera Theorem when property **(A)** holds.

THEOREM 2.1 (Massera "revisited"). *Under the assumptions **(A)**, the Liénard system (4) has exactly one limit cycle, which is stable, provided that*

1. $|\alpha| > \beta$,

$f(x)$ is monotone decreasing for $\alpha < x < 0$,

$f(x)$ is monotone increasing for $0 < x < \delta$;

2. $|\alpha| < \beta$,

$f(x)$ is monotone decreasing for $\delta_1 < x < 0$,

$f(x)$ is monotone increasing for $0 < x < \beta$,

with

$$\delta = \sqrt{\left(1 + F(a) + \frac{\alpha^2}{2}\right)^2 + \beta^2}, \quad \delta_1 = -\sqrt{\left(-F(b) + 1 + \frac{\beta^2}{2}\right)^2 + \alpha^2},$$

where a and b are the zeros of $f(x)$ and α, β are the non trivial zeros of $F(x)$.

Proof. We preliminarly observe that, if $|\alpha| = \beta$, we can apply directly Theorem 1.2 and no monotonicity assumptions are required.

For sake of simplicity we are proving the theorem in several steps.

Step 1 We now consider the case $|\alpha| > \beta$.

*Under the assumption **(A)**, if $f(x)$ is monotone decreasing for $\alpha < x < 0$, and monotone increasing for $x > 0$, the Liénard system (4) has exactly a limit cycle, which is stable.*

In the Liénard plane any trajectory which intersects the line $x = \alpha$ in $y > 0$, also intersects the line $x = \beta$ because, as already mentioned, the distance from the origin is increasing in the strip $\alpha < 0 < \beta$.

If we keep the monotonicity properties of Massera's Theorem for $x > \alpha$, we know that, in the half plane $x > \alpha$, lies at most a stable limit cycle. This result is proved in the phase plane, but it also holds in the Liénard plane in virtue of the above mentioned property which preserves the x-coordinate, when one switches from one plane to the other. Hence in the Liénard plane there are only two possible configurations:

1. No limit cycle lies in the half plane $x > \alpha$. Hence all limit cycles must intersect both lines $x = \alpha$ and $x = \beta$ and, from Theorem 1.2, the limit cycle is unique.

2. We have a stable limit cycle in the half plane $x > \alpha$. Using again Theorem 1.2 we can have, at most, a second limit cycle intersecting both lines $x = \alpha$ and $x = \beta$. The sign conditions on f shows that such limit cycle must be semistable from his exterior. Using a perturbation argument, which may be found in [7] and [16], one can see that, with a suitable small perturbation of f near α and for $x < \alpha$, still keeping f positive and hence keeping the monotonicity properties of F required for property **(B)**, the semistable limit cycle bifurcates in two limit cycles, one

stable and one unstable, which is a contradiction because both limit cycles must intersect both the lines $x = \alpha$ and $x = \beta$. For the bifurcation from a semistable limit cycle in rotated vector fields, we refer also to the classical works of Duff [4] and Perko [9].

If $|\alpha| < \beta$ we easily get a dual result, namely:

Step 2 *Under the assumption* ***(A)****, if $f(x)$ is monotone decreasing for $x < 0$ and monotone increasing for $0 < x < \beta$, the Liénard system (4) has exactly a limit cycle, which is stable.*

In order to complete our proof, it is necessary to produce a fixed upper bound for the monotonicity assumptions for positive values of x.

Step 3 We consider, at first, the case $|\alpha| > \beta$.

Under assumption ***(A)****, a positive semitrajectory of the Liénard system (4), which starts at a point $P(\alpha, F(a) + 1)$, intersects the vertical isocline $y = F(x)$ in the half plane $x > 0$, at a point $S(x, F(x))$, with $x < \delta$, where $\delta = \sqrt{\left(1 + F(a) + \frac{\alpha^2}{2}\right)^2 + \beta^2}$.*

In the Liénard plane (4), the slôpe of a trajectory is given by

$$y'(x, y) = \frac{-x}{y - F(x)}.$$

At first, we observe that a positive semitrajectory, which starts at a point $P\,(\alpha, F(a) + 1)$, must intersect the y-axis at a point $Q\,(0, \bar{y})$, because the slôpe is positive, and the line $x = \beta$ at a point $R\,(\beta, \hat{y})$, due to the fact that, in the strip $\alpha < x < \beta$, the distance from the origin is increasing and $|\alpha| > \beta$ (see Figure 2).

$$y(Q) - y(P) = \int_\alpha^0 y'(x, y)\, dx = \int_\alpha^0 \frac{-x}{y - F(x)}\, dx.$$

In the strip $\alpha < x < 0$, $F(x) \le F(a)$, the slôpe is positive and, clearly, $y - F(x) \ge y - F(a) > 1$ and therefore

$$y(Q) - y(P) < \int_\alpha^0 -x\, dx = \frac{\alpha^2}{2},$$

that is

$$y(Q) = \bar{y} < 1 + F(a) + \frac{\alpha^2}{2}.$$

In the strip $0 < x < \beta$, the slôpe is negative; for this reason the positive semitrajectory intersects the β-line at a point $R\,(\beta, \hat{y})$, with $\hat{y} < \bar{y} < 1 + F(a) + \frac{\alpha^2}{2}$. For $x > \beta$, the distance from the origin is now decreasing. The positive semitrajectory intersects the vertical isocline $y = F(x)$ at a point $S\,(x, F(x))$, with

$$x < \sqrt{\left(1 + F(a) + \frac{\alpha^2}{2}\right)^2 + \beta^2} = \delta,$$

and this proves Step 3.

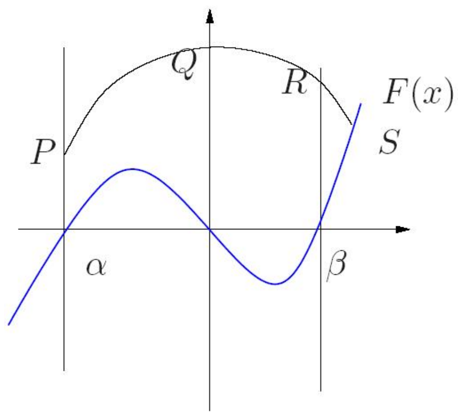

Figure 2:

From Step 3, we get that any negative semitrajectory intersecting the vertical isocline at $x > \delta$ intersects the line $x = \alpha$.

Now we require the monotonicity property of Massera Theorem just in the strip $\alpha < x < \delta$ and we can argue as in Step 1.

Again if $|\alpha| < \beta$, we can get the dual result:

Step 4 *Under assumption **(A)**, a positive semitrajectory of the Liénard system (4), which starts at a point $P(\beta, F(b)-1)$, intersects the vertical isocline $y = F(x)$ in the half plane $x < 0$, at a point $S(x, F(x))$, with $x > \delta_1$, where $\delta_1 = -\sqrt{\left(-F(b) + 1 + \frac{\beta^2}{2}\right)^2 + \alpha^2}$.*

This completes the proof of the Theorem. □

Remark 2.1. *Observe that it is easy to see that, actually, the value δ (δ_1) can be improved by $\hat{\delta} = F^{-1}\left(\sqrt{\delta^2 - x^2}\right)$ ($\hat{\delta_1} = F^{-1}\left(\sqrt{\delta_1^2 - x^2}\right)$). However, we prefer to keep the values δ and δ_1 becuse they explicitly contain the values a, b, α, β and this enlights the crucial role played by the zeros of f and F.*

Remark 2.2. *Notice that such result can also be viewed as a perturbation of the classical Massera Theorem, namely that we can perturb the function $f(x)$ outside the interval $[\alpha, \delta]$ ($[\delta_1, \beta]$), keeping only the sign conditions, and still having existence and uniqueness of a stable limit cycle.*

Remark 2.3. *Finally, as a side remark, we recall that outside the interval $[\alpha, \delta]$ the only restriction on $f(x)$ is the positivity. In the case of f tending, at 0^+, at infinity and F having a finite limit at infinity, still the above mentioned* *sufficient conditions for the existence of limit cycles are fulfilled [14] and the monotonicity assumptions on* $[\alpha, \delta]$ *give the uniqueness.*

We already noticed that the values $F(a)$, $F(b)$ play a crucial role in order to guarantee that the trajectories of system (3) intersect both lines $x = \alpha$ and $x = \beta$.

In the light of a result in [2], proved for equation (2), which now is more powerful due to the fact that $g(x) = x$, we prove the following simple perturbation result:

THEOREM 2.2. *Under the assumption* ***(A)*** *the equation*

$$\ddot{x} + \lambda f(x)\,\dot{x} + x = 0$$

has a unique non trivial periodic solution for every $\lambda \geq \hat{\lambda}$, *where* $\hat{\lambda} =$

$$\sqrt{\frac{\alpha^2 - \beta^2}{F^2(b)}}, \ \textit{if } |\alpha| > \beta,$$

$$\sqrt{\frac{\beta^2 - \alpha^2}{F^2(a)}}, \ \textit{if } |\alpha| < \beta,$$

any real number if $\alpha = \beta$.

Proof. We consider only the first case, being the second one treated in the same way and the result well-known if $|\alpha| = \beta$.

As usual we consider the Liénard system

$$\begin{cases} \dot{x} = y - \lambda F(x) \\ \dot{y} = -x. \end{cases}$$

We just notice that the parameter λ does not influence the values a, b, α, β. Assumption **(A)** gives the existence of at least a limit cycle. Any positive semitrajectory which intersects the line $x = \beta$ in $y < 0$, intersects the line $x = b$ at a point $P(b, y)$, with $y < \lambda F(b)$. Recalling again the fact that, in the strip $\alpha < x < \beta$, the distance from the origin is increasing, it is straightforward to observe that if

$$\sqrt{\lambda^2 F^2(b) + b^2} \geq |\alpha|,$$

such trajectory intersects the line $x = \alpha$. Hence all limit cycles must intersect both lines $x = \alpha$ and $x = \beta$ and we can use Theorem 1.2 again. $\square$

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Author's address:

Gabriele Villari
Dipartimento di Matematica "U. Dini"
Università di Firenze
viale Morgagni 67/a, 50134 Firenze, Italy
E-mail: villari@math.unifi.it

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