Chapter 2

SAR Image Statistics

From the physics of acquisition, the signal received by a single resolution cell can be expressed in its general complex form as the sum of all the scattering contributions inside the resolution cell:

$$Ae^{j\varphi} = \sum_{k=1}^{N} e^{j\varphi_k} \tag{2.1}$$

where A_k is the received attenuated amplitude and φ_k the received phase. Clearly, if the mean wavelength λ used in transmission is smaller than the resolution cell size, scatterers at different parts of the resolution cell will have very different phase values. Thus, the final sum can be considered in the complex plane as the sum of vectors having similar amplitude but very different direction. Considering the phase φ_k as uniformly distributed between $[-\pi, \pi]$, and presupposing a large number N of statistically identical scatterers, the previous behavior can be studied as a **random walk in the complex plane** [3]. In this case the in-phase ($Z_1 = A\cos\varphi$) and quadrature ($Z_2 = A\sin\varphi$) components are independent identically distributed (i.i.d.) Gaussian random variables (r.v.) with mean zero and variance $\sigma/2$ (i.e. $Z_1, Z_2 \sim \mathcal{N}(0, \sigma/2)$), where σ is the mean radar cross section (RCS) of the point targets (average backscattering coefficient), which is linked to the received amplitudes A_k (considered as a r.v.) with:

$$\mathrm{var}[Z_i] = \frac{N}{2}\mathrm{E}\left[A_k^2\right] = \frac{\sigma}{2} \tag{2.2}$$

The reason to pose the variance of Z_i equal to $\sigma/2$ is that (as we shall see in the following) in this way the mean value of the intensity (i.e. power) image becomes just equal to σ. It should be pointed out that in the rest of the document we will indicate the random variables with capitalized letters and their occurrence with lower-case ones.

2.1 Single-Look Data

2.1.1 Amplitude Data

Clearly, expressing the amplitude $A = \sqrt{Z_1^2 + Z_2^2}$ and considering Z_1, Z_2 Gaussian r.v. with mean zero and variance $\sigma/2$, it is straightforward to infer the Rayleigh distribution of A with parameter σ, indicated as $A \sim R(\sigma)$ (see Fig. 2.1):

$$f_A(a) = \frac{2a}{\sigma} e^{-\frac{a^2}{\sigma}}; \quad a \geq 0 \tag{2.3}$$

with mean, variance and squared coefficient of variation:

$$\mathrm{E}[A]=\frac{\sqrt{\pi\sigma}}{2};\ \ \mathrm{var}[A]=\left(1-\frac{\pi}{4}\right)\sigma;\ \ C_A^2=\frac{\mathrm{var}[A]}{\mathrm{E}[A]^2}=\left(\frac{4}{\pi}-1\right) \tag{2.4}$$

Now, exploiting the theorem for the pdf computation of a r.v. Y expressed as a monotonic and derivable transformation $Y=g(A)$ applied at A:

$$f_Y(y)=\left.\frac{f_A(a)}{g'(a)}\right|_{a=g^{-1}(y)} \tag{2.5}$$

many distributions (function of A) can be easily derived.

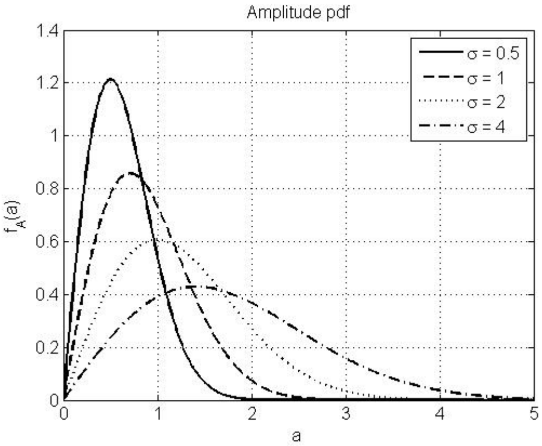

Fig. 2.1 - Single-Look amplitude pdf.

2.1.2 Intensity Data

Applying Eq. (2.5) the intensity $I=A^2$ is derived as an exponential distribution with parameter σ, i.e. $I\sim Exp(\sigma)$ (see Fig. 2.2):

$$f_I(t)=\frac{1}{\sigma}e^{-\frac{t}{\sigma}};\quad t\geq 0 \tag{2.6}$$

with mean, variance and squared coefficient of variation:

$$\mathrm{E}[I]=\sigma;\ \ \mathrm{var}[I]=\sigma^2;\ \ C_I^2=1 \tag{2.7}$$

Therefore, as said before, the mean received power (intensity) is σ indeed. It is worth noting that both amplitude and intensity data have a variance varying with the RCS σ.

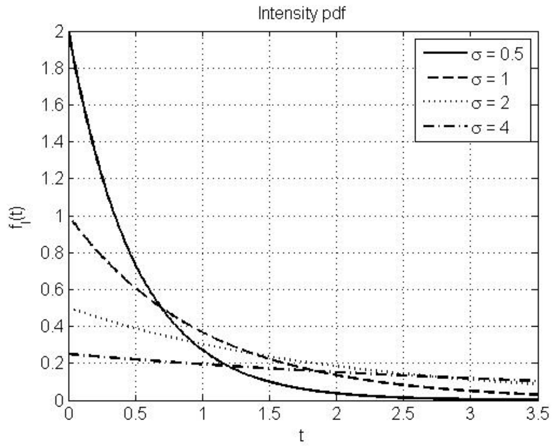

Fig. 2.2 - Single-Look intensity pdf.

2.1.3 Log-Intensity Data

For these reasons it is interesting to compute the pdf of the intensity data logarithm $I_{log} = \ln I$. In fact, applying Eq. (2.5) we have a Fisher-Tippet distribution:

$$f_{I_{log}}(t) = \frac{e^t}{\sigma} e^{-\frac{e^t}{\sigma}}; \quad t \geq 0 \tag{2.8}$$

with mean, variance and squared coefficient of variation [4]:

$$\mathrm{E}\left[I_{log}\right] = \ln(\sigma) - \gamma_E; \;\; \mathrm{var}\left[I_{log}\right] = \frac{\pi^2}{6}; \quad C^2_{I_{log}} = \frac{\pi^2/6}{(\ln(\sigma) - \gamma_E)^2} \tag{2.9}$$

where $\gamma_E \approx 0.57722$ is the Euler's constant. It should be pointed out that applying the logarithm to the intensity, the data variance becomes independent from the RCS σ. Moreover, we have to pay attention in estimating σ from the mean value of I_{log} since there exist a constant bias in $\mathrm{E}\left[I_{log}\right]$ equal to γ_E.

2.2 Multi-Look Data

As said in Section 1.3, SAR images could be processed to obtain a reduced data variance at the cost of a lower resolution than the nominal one. This **multilooking** process, which can be performed in the Doppler frequency domain, may be also obtained in spatial domain by simply averaging neighbor pixels (in intensity format). The reason to operate in such a way is that, presupposing L i.i.d. random variables I_i with variance $\mathrm{var}[I_i]$, the following multilooking operation:

$$I = \frac{1}{L}\sum_{i=1}^{L} I_i \tag{2.10}$$

reduces the variance of the resulting image as:

$$\mathrm{var}[I] = \frac{\mathrm{var}[I_i]}{L} \tag{2.11}$$

that is, it is reduced of a factor L. Clearly, if each pixel of the resulting image is computed by Eq. (2.10), the final data distribution is different from the ones descripted in Section 2.1.

2.2.1 Intensity Data

After applying Eq. (2.9) to data distributed as Eq. (2.6), the resulting observed intensity is Gamma distributed with scale parameter σ/L and shape parameter L, i.e. $I \sim \Gamma(L, \sigma/L)$ (see Fig. 2.3):

$$f_I(t) = \frac{t^{L-1}}{\Gamma(L)(\sigma/L)^L} e^{-\frac{t}{\sigma/L}}; \quad t \geq 0 \tag{2.12}$$

where Γ indicates the Gamma function. In this case the mean, variance and squared coefficient of variation are:

$$\mathrm{E}[I] = \sigma; \quad \mathrm{var}[I] = \frac{\sigma^2}{L}; \quad C_I^2 = \frac{1}{L} \tag{2.13}$$

Clearly, when $L = 1$, Eq. (2.12) becomes equal to Eq. (2.6), that is the Gamma distribution with $L = 1$ is equal to the exponential one.

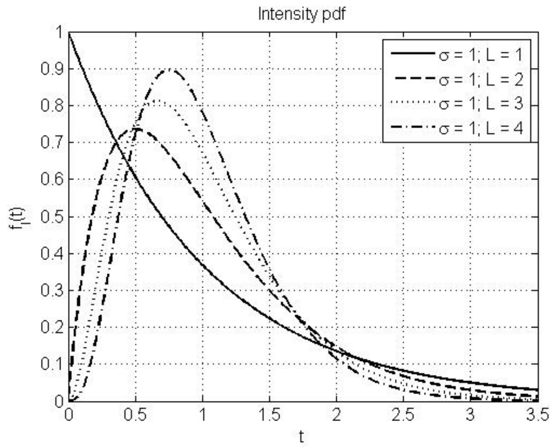

Fig. 2.3 - Multi-Look intensity pdf.

2.2.1.1 Multiplicative Speckle Model

From Eq. (2.12) we can see that, highlighting the dependence on the parameter σ, we can think at I as a r.v. conditioned to σ, with pdf $f_I(t|\sigma)$. The interesting property of the pdf in Eq. (2.12) is that it can be thought as composed by a noise r.v. n_I with pdf $f_{n_I}(t) = f_I(t|\sigma = 1)$ multiplied with a constant value σ, i.e.:

$$\mathrm{I} = \sigma n_I \tag{2.14}$$

with

$$f_{n_I}(t) = f_I(t|\sigma = 1) = \frac{t^{L-1}}{\Gamma(L)(1/L)^L} e^{-\frac{t}{1/L}}; \quad t \geq 0 \tag{2.15}$$

In fact, from Eq. (2.5), considering σ constant we have:

$$f_I(t) = \left.\frac{f_{n_I}(n)}{\sigma}\right|_{n=t/\sigma} = \left.\frac{f_I(n|\sigma = 1)}{\sigma}\right|_{n=t/\sigma} = \frac{t^{L-1}}{\Gamma(L)(\sigma/L)^L} e^{-\frac{t}{\sigma/L}}; \quad t \geq 0 \tag{2.16}$$

Given this property, the variable n_I is presupposed due to the speckle noise, which is therefore considered as a **multiplicative noise** type.

2.2.2 Amplitude Data (Square-Root Intensity)

Since usually the multi-look data are provided in amplitude format computed as the square-root of multi-look intensity data:

$$A = \sqrt{I} = \sqrt{\frac{1}{L}\sum_{i=1}^{L} I_i} \tag{2.17}$$

it could be useful to retrieve such distribution exploiting Eq. (2.5), which is a Nakagami pdf, indicated as $A \sim \text{Nakagami}(L, \sigma/L)$ (see Fig. 2.4):

$$f_A(a) = \frac{2a^{2L-1}}{\Gamma(L)(\sigma/L)^L} e^{-\frac{a^2}{\sigma/L}}; \quad a \geq 0 \tag{2.18}$$

with mean, variance and squared coefficient of variation:

$$\begin{aligned} \mathrm{E}[A] &= \frac{\Gamma(L+0.5)}{\Gamma(L)}\sqrt{\frac{\sigma}{L}}; \\ \mathrm{var}[A] &= \sigma - \mathrm{E}[A]^2 = \sigma\left(1 - \frac{\Gamma(L+0.5)^2}{\Gamma(L)^2 L}\right); \\ C_A^2 &= \frac{\mathrm{var}[A]}{\mathrm{E}[A]^2} = \frac{\Gamma(L)^2 L}{\Gamma(L+0.5)^2} - 1 \end{aligned} \tag{2.19}$$

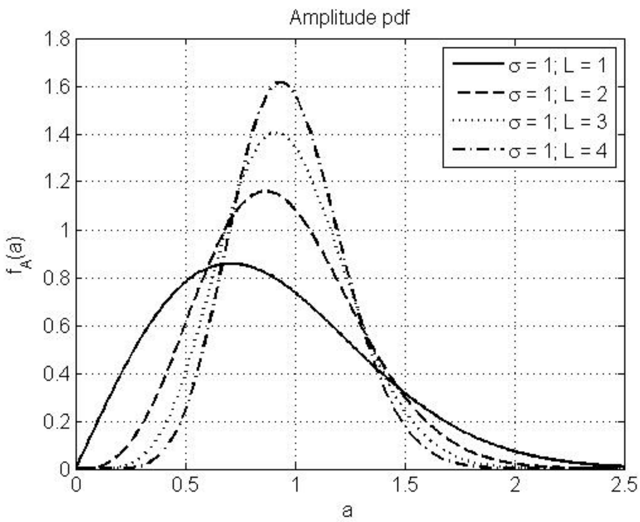

Fig. 2.4 - Multi-Look amplitude pdf.

2.2.2.1 Multiplicative Speckle Model

Clearly, even for this distribution the multiplicative speckle model hypothesis holds:

$$\mathrm{A} = \sqrt{\mathrm{I}} = \sqrt{\sigma n_I} = \sqrt{\sigma} n_A \tag{2.20}$$

with

$$f_{n_A}(a) = f_A(a|\sigma = 1) = \frac{2a^{2L-1}}{\Gamma(L)(1/L)^L} e^{-\frac{a^2}{1/L}}; \quad a \geq 0 \tag{2.21}$$

Consequently, $f_A(a)$ can be also computed by $f_{n_A}(a)$ by Eq. (2.5) as:

$$f_A(a) = \left.\frac{f_{n_A}(n)}{\sqrt{\sigma}}\right|_{n=a/\sqrt{\sigma}} = \left.\frac{f_A(n|\sigma=1)}{\sqrt{\sigma}}\right|_{n=a/\sqrt{\sigma}} = \frac{2a^{2L-1}}{\Gamma(L)(\sigma/L)^L} e^{-\frac{a^2}{\sigma/L}}; \quad a \geq 0 \tag{2.22}$$

2.2.3 Log-Intensity Data

From the multiplicative noise model can be immediately understood the advantage of using log-transformed data, i.e. to transform a multiplicative noise model to an additive one:

$$\mathrm{I}_{log} = \ln\mathrm{I} = \ln\sigma + \ln n_I = \ln\sigma + n_{I_{log}} \tag{2.23}$$

where, applying Eq. (2.5) to Eq. (2.12) we have:

$$f_{I_{log}}(t) = \frac{e^{t(L-1)}}{\Gamma(L)(\sigma/L)^L} e^{-\frac{e^t}{\sigma/L}}; \quad t \geq 0 \tag{2.24}$$

and

$$f_{n_{I_{log}}}(t) = f_{I_{log}}(\mathrm{t}|\sigma=1) = \frac{e^{t(L-1)}}{\Gamma(L)(1/L)^L} e^{-\frac{e^t}{1/L}}; \quad t \geq 0 \tag{2.25}$$

which has mean and variance [4]:

$$\mathrm{E}\left[n_{I_{log}}\right] = \psi(L) - \ln L; \;\; \mathrm{var}\left[n_{I_{log}}\right] = \psi(1, L) \tag{2.26}$$

where

$$\psi(x) = \frac{d\ln\Gamma(x)}{dx} \tag{2.27}$$

is the Diagamma function and $\psi(1, L)$ is known as the first-order Polygamma function of L. A general *n*th Polygamma function is defined as the *n*th derivative of the Diagamma function, i.e.:

$$\psi(n, x) = \frac{d^n}{dx^n}\psi(x) \tag{2.28}$$

Nevertheless, when L is integer we have:

$$\mathrm{E}\left[n_{I_{log}}\right] = \sum_{m=1}^{L-1}\frac{1}{m} + \psi(1) - \ln L; \;\; \mathrm{var}\left[n_{I_{log}}\right] = \psi(1,1) - \sum_{m=1}^{L-1}\frac{1}{m^2} \tag{2.29}$$

with $\psi(1) = \gamma_E \approx 0.577215$ the Euler's constant and $\psi(1,1) = \pi^2/6$. From these last equalities is clear that with $L = 1$ Eq. (2.26) becomes equal to Eq. (2.9).

2.3 Variable RCS Data

2.3.1 Intensity Data

Until now, we have considered σ as an intrinsic parameter of the data, which can be seen as a peculiar characteristic of data belonging at a same area. However, only very homogeneous areas follow the previous distributions so that a further complication has to be inserted in the model. One of the most useful distribution, which can be considered as a generalization of the previous Gamma Multi-Look distribution in intensity, is the K-distribution [3], [5]. In particular, this distribution comes out from several different hypotheses on data distribution. Physically, if the mean wavelength λ used in transmission is smaller than the resolution cell size and the number of scatterers does not tend to infinite (as in the Gamma case), but it is modeled as negative binomial r.v., the final distribution is a K-pdf indeed [5]. Moreover, K-distribution is often introduced modeling the natural RCS variation as a Gamma pdf, i.e. $\sigma \sim \Gamma(v, \langle\sigma\rangle/v)$:

$$f_\sigma(t) = \frac{t^{\nu-1}}{\Gamma(\nu)(\langle\sigma\rangle/\nu)^\nu} e^{-\frac{t}{\langle\sigma\rangle/\nu}}; \quad t \geq 0 \tag{2.30}$$

with mean value $\mathrm{E}[\sigma] = \langle\sigma\rangle$ and variance $\mathrm{var}[\sigma] = \langle\sigma\rangle^2/\nu$ controlled by the parameter ν. In this case, if $f_I(t|\sigma)$ is the one reported in Eq. (2.12):

$$f_I(t|\sigma) = \frac{t^{L-1}}{\Gamma(L)(\sigma/L)^L} e^{-\frac{t}{\sigma/L}}; \quad t \geq 0 \tag{2.31}$$

the final distribution of I becomes the following K-distribution:

$$f_I(t) = \int_0^\infty f_I(t|\sigma) f_\sigma(\sigma) d\sigma = \frac{2t^{(L+\nu-2)}}{\Gamma(L)\Gamma(\nu)(\langle\sigma\rangle/L\nu)^{\frac{L+\nu}{2}}} K_{\nu-L}\left(2\sqrt{\frac{L\nu t}{\langle\sigma\rangle}}\right); \quad t \geq 0 \tag{2.32}$$

where $K_{\nu-L}$ denotes the modified Bessel function of order $(\nu - L)$, whose moments are:

$$\mathrm{E}[I^m] = \frac{\langle\sigma\rangle^m \, \Gamma(L+m)\Gamma(\nu+m)}{L^m \nu^m \, \Gamma(L)\Gamma(\nu)} \tag{2.33}$$

from which mean, variance and squared coefficient of variation are:

$$\mathrm{E}[I] = \langle\sigma\rangle; \quad \mathrm{var}[I] = \langle\sigma\rangle^2 \left(\frac{1}{L} + \frac{1}{\nu} + \frac{1}{L\nu}\right); \; C_I^2 = \left(\frac{1}{L} + \frac{1}{\nu} + \frac{1}{L\nu}\right) \tag{2.34}$$

2.3.1.1 Multiplicative Speckle Model

Finally, K-distribution can be considered as coming out from the following product model:

$$\mathrm{I} = \langle\sigma\rangle \tau n_I \tag{2.35}$$

where $\langle\sigma\rangle$ is a constant peculiar of the acquired area, $n_I \sim \Gamma(L, 1/L)$ is the Gamma noise and $\tau \sim \Gamma(\nu, 1/\nu)$ is the statistical model of the present texture.

2.3.1.2 Texture Information

The RCS variation inside a region of the same type is called **texture** and it is an important feature often used in classification algorithm [6]. The model used in texture modeling is often the same as in Eq. (2.35), but no distribution of the speckle is presupposed and only the first two moments of the n_I are set. In particular, for pixels of a same area, the r.v. n_I and τ are considered independent and the following hypotheses on pdf moments are done:

$$\begin{aligned} \mathrm{I} &= \langle\sigma\rangle \tau n_I; \\ \langle\sigma\rangle &= \mathrm{const.} \\ \mathrm{E}[\tau] &= 1; \\ \mathrm{E}[n_I] &= 1; \mathrm{var}[n_I] = 1/L; \end{aligned} \tag{2.36}$$

Clearly, exploiting only pdf moments (first order statistics), the goal of the texture analysis is to compute the **variance of the texture** $\mathrm{var}[\tau]$. Now, from the following equality:

$$\mathrm{var[I]} = \mathrm{E}[I^2] - \mathrm{E}[I]^2 = \langle\sigma\rangle^2 (\mathrm{E}[\tau^2 n_I^2] - \mathrm{E}[\tau n_I]^2) \tag{2.37}$$

and exploiting all the previous hypotheses we have:

$$\mathrm{var}[\tau] = \frac{\left(\frac{\mathrm{var}[I]}{\mathrm{E}[I]^2}\right) - \mathrm{var}[n_I]}{\mathrm{var}[n_I] + 1} \tag{2.38}$$

so that estimating $\mathrm{var}[I]$ and $\mathrm{E}[I]$ from the data we can reach our scope. Naturally, since $\mathrm{E}[\tau] = 1$ and $\mathrm{E}[n_I] = 1$, Eq. (2.38) can be written in terms of coefficients of variations as:

$$\mathrm{C}_\tau^2 = \frac{\mathrm{C}_I^2 - \mathrm{C}_{n_I}^2}{\mathrm{C}_{n_I}^2 + 1} \tag{2.39}$$

conveniently expressed as:

$$\mathrm{C}_I^2 = \mathrm{C}_\tau^2 \mathrm{C}_{n_I}^2 + \mathrm{C}_\tau^2 + \mathrm{C}_{n_I}^2 \tag{2.40}$$

which makes clear that, in absence of texture ($\mathrm{C}_\tau^2 = 0$), we have $\mathrm{C}_I^2 = \mathrm{C}_{n_I}^2$. Therefore, we could think to estimate also $\mathrm{C}_{n_I}^2$ directly from the data in areas with no texture (e.g. water basins). Nevertheless, for classification aims, the first order statistics are usually not enough and relations between neighbor pixels have to be taken into account (second order statistics). One of the most used second order statistics is the autocovariance function that is often preferred to the autocorrelation for stochastic process with no zero mean. In particular, for an intensity image modeled as:

$$I(x,r) = \langle\sigma\rangle\, \tau(x,r)\, n_I(x,r) \tag{2.41}$$

and considering the hypothesis of wide sense stationary process (WSS), the autocovariance function becomes:

$$\begin{aligned} \mathrm{C}_I(x_1,r_1,x_2,r_2) &= \mathrm{E}\big\{ \{I(x_1,r_1) - \mathrm{E}[I(x_1,r_1)]\} \{I(x_2,r_2) - \mathrm{E}[I(x_2,r_2)]\}\big\} \\ &= \mathrm{E}[I(x,r)I(x+\Delta x, r+\Delta r)] - \mathrm{E}[I(x,r)]^2 \\ &= R_I(\Delta x,\Delta r) - \mu_I^2 \\ &= C_I(\Delta x,\Delta r) \end{aligned} \tag{2.42}$$

with μ_I the intensity mean and $R_I(\Delta x,\Delta r)$ the autocorrelation function. Following the model in Eq. (2.41), we have:

$$R_I(\Delta x,\Delta r) = \mu_I^2 R_\tau(\Delta x,\Delta r) R_{n_I}(\Delta x,\Delta r) \tag{2.43}$$

and considering the PSF shape of Eq. (1.53) we have:

$$C_{n_I}(\Delta x,\Delta r) = R_{n_I}(\Delta x,\Delta r) - \mu_{n_I}^2 = \mu_{n_I}^2 \mathrm{sinc}^2\left(\frac{\pi\Delta x}{r_x}\right) \mathrm{sinc}^2\left(\frac{\pi\Delta r}{r_r}\right) \tag{2.44}$$

with r_x and r_y respectively the azimuth and range resolution of the SAR system. Nevertheless, presupposing a homogeneous area with no texture ($R_\tau(\Delta x,\Delta r) = 1$), we have:

$$R_{n_I}(\Delta x,\Delta r) = \frac{R_I(\Delta x,\Delta r)}{\mu_I^2} \tag{2.45}$$

so that $R_{n_I}(\Delta x,\Delta r)$ can be estimated from the data in that homogeneous area. Now, the theoretical $R_{n_I}(\Delta x,\Delta r)$ in the case of **multi-look** data can be computed as:

$$R_{n_I}(\Delta x,\Delta r;L) = \frac{R_I(\Delta x,\Delta r;L)}{\mu_I^2} = \frac{1}{\mu_I^2}\mathrm{E}\left[\frac{1}{L}\sum_{i=1}^{L} I_i(x,r)\frac{1}{L}\sum_{i=1}^{L} I_i(x+\Delta x, r+\Delta r)\right] \tag{2.46}$$

and exploiting the independence hypotheses of the model in Eq. (2.41) we have:

$$C_{n_I}(\Delta x,\Delta r;L) = R_{n_I}(\Delta x,\Delta r;L) - \mu_{n_I}^2 = \frac{\mu_{n_I}^2}{L}\mathrm{sinc}^2\left(\frac{\pi\Delta x}{r_x}\right)\mathrm{sinc}^2\left(\frac{\pi\Delta r}{r_r}\right) = \frac{C_{n_I}(\Delta x,\Delta r;L=1)}{L} \tag{2.47}$$

where $C_{n_I}(\Delta x,\Delta r;L=1)$ is the one in Eq. (2.44). As can be seen from Eq. (2.47), multi-looking reduces the speckle correlation, i.e. the granular appearance of the SAR image. Now, the goal of this second order texture analysis is to compute the autocovariance function of texture. Nevertheless, the autocorrelation coefficient is sometime preferred due to its normalization property:

$$\rho_I(\Delta x,\Delta r) = \frac{C_I(\Delta x,\Delta r)}{\mathrm{var}[I]} \tag{2.48}$$

In particular, the autocorrelation coefficient in range direction in a homogeneous area with N_r pixels along the range and N_x pixels along the azimuth can be estimated by computing:

$$C_I(\Delta r, x) = \frac{1}{N_r - \Delta r} \sum_{r=1}^{N_r - \Delta r} [I(x,r) - \mu_I][I(x, r + \Delta r) - \mu_I]$$
$$C_I(\Delta r) = \frac{1}{N_x} \sum_{x=1}^{N_x} C_I(\Delta r, x) \tag{2.49}$$

with μ_I estimated from data and where the operations in Eq. (2.49) have to repeated for $d = 1, \cdots, d_M$ with $d_M \ll N_r$. Clearly applying the same reasoning in azimuth direction and estimating $\mathrm{var}[I]$ we can compute $\rho_I(\Delta x, \Delta r)$ exploiting Eq. (2.48). Now, using Eq. (2.43) that in the general case becomes:

$$R_I(\Delta x, \Delta r; L) = \mu_I^2 R_\tau(\Delta x, \Delta r) R_{n_I}(\Delta x, \Delta r; L) \tag{2.50}$$

we can compute the **autocorrelation coefficient of the texture** $\rho_\tau(\Delta x, \Delta r)$ as:

$$\rho_\tau(\Delta x, \Delta r) = \frac{1}{\mathrm{var}[\tau]} \left[\frac{\rho_I(\Delta x, \Delta r) \frac{\mathrm{var}[I]}{\mu_I^2} + 1}{R_{n_I}(\Delta x, \Delta r; L)} - 1 \right] \tag{2.51}$$

with $\mathrm{var}[\tau]$ estimated by Eq. (2.38) , $\mathrm{var}[I]$, μ_I estimated by data, and $R_{n_I}(\Delta x, \Delta r; L)$ theoretical (see (2.47)) or estimated.