UNIVERSITÀ DEGLI STUDI DI FIRENZE

Facoltà di Scienze Matematiche, Fisiche e Naturali

Dipartimento di Matematica "Ulisse Dini"

Dottorato di Ricerca in Matematica

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Unipotent Automorphisms of Soluble Groups

Candidato:
Marco Frati

Cotutore:
Prof. Orazio Puglisi

Tutore:
Prof. Carlo Casolo

Coordinatore del Dottorato:
Prof. Alberto Gandolfi

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Introduction

The notions of unipotent automorphism and stability group emerge from Linear Algebra in the following elementary manner.
Let V be a vector space over a field $\mathbb{F}$, and H a subgroup of the General Linear group $\mathrm{GL}(V, \mathbb{F})$. The automorphism $h \in H$ is called a *unipotent automorphism* if the endomorphism $h - 1$ is a nilpotent element of the ring $\mathrm{End}(V)$, that is there exists an integer $n = n(h)$ such that $(h-1)^n$ is the zero endomorphism. Furthermore, the whole group H is termed *n-unipotent* if it consists of unipotent automorphisms where the integer n can be chosen equal for all of them.
If the vector space V has finite dimension, an easy example of unipotent automorphism group is given by the *unitriangular group* $\mathrm{UT}(n, \mathbb{F})$, defined as the subgroup of $\mathrm{GL}(n, \mathbb{F})$ whose elements are upper triangular matrices with all the entries on the main diagonal equal to 1. This group is clearly n-unipotent, where n is the dimension of V over $\mathbb{F}$. Furthermore, it is a classical result that any other unipotent subgroup of $\mathrm{GL}(n, \mathbb{F})$ is conjugated to a subgroup of $\mathrm{UT}(n, \mathbb{F})$. From this embedding it follows easily that a n-unipotent automorphism group of a finite dimensional vector space V *stabilizes a finite series* in V, and in particular it is nilpotent of class at most $n - 1$. Here for series stabilized by an automorphism group H, we mean a series of subspaces of V such that H acts trivially on every factor.

Instead of taking a vector space, the concept of unipotent automorphism still makes sense for an arbitrary module M. Moreover, assuming some finiteness conditions on M, it is still possible to prove that a unipotent group of module automorphisms is a stability group, as shown by Theorem 13.6 of [14].

Theorem. *Let R be a commutative Noetherian ring and M a finitely generated R-module. If H is a group of R-automorphisms of M, then H is unipotent if and only if it stabilizes a finite series of submodules of M. In particular if H is unipotent then it is a nilpotent group.*

These ideas have been generalized also in Group Theory. Given a group G and a subgroup H of $\mathrm{Aut}(G)$, for every $g \in G$ and $h \in H$ we define their commutator as $[g, h] = g^{-1}g^h$, and we say that H is a n-unipotent automorphism group if $[g,_n h] = [[g,_{n-1} h], h]$ is the identity element of G, for every such pair of elements. Notice that this writing is nothing else than the usual group commutator in $G \rtimes H$.

The aim of this thesis is to investigate the action of unipotent automorphisms of groups. Precisely, we are interested in finding sufficient conditions on a group G which imply the nilpotence, or even the stability, of an unipotent subgroup of $\mathrm{Aut}(G)$. Since the whole theory of unipotent automorphisms has been proved to be very fruitful in the case of vector spaces and modules, and since as a consequence of the previous Theorem every unipotent automorphism group of a finitely generated abelian group is stable, it seems natural to restrict our attention to the class of soluble groups. In particular we will study the unipotent automorphism groups of two classes of soluble groups, namely the finitely generated abelian-by-polycyclic groups and the soluble groups with finite rank, proving that in both cases, a finitely generated group of unipotent automorphism is necessarily nilpotent.

For the reader's convenience, in the first chapter we shall give a brief account of the classes of soluble and nilpotent groups, which will be the object of our study, together with a description of their fundamental properties that will be used throughout the thesis. Most of the proofs are omitted, but they can be found in [9], [7] or [15]. Furthermore, we introduce the notion of Engel elements, n-Engel subgroups and residually hypercentral subgroups, and illustrate the structure Theorems, due to P.G. Crosby and G. Traustason, which inspired this work. Chapter 2 starts with the definition of stability group and the proof of its nilpotence when the series stabilized is finite, a result due to Kaluzhnin and P. Hall. After that, we give the definition of unipotent automorphism, one the most important for the aim of the thesis, and exhibit an example which shows that this property is weaker than the stability. Then we describe some sufficient conditions for a unipotent automorphism group acting on a soluble group G to stabilize a finite series in G, and finally we give a bound for the nilpotency class of such an automorphism group which is uniform, in the sense that it does not depend on the properties of G.
The first important result described in chapter 3 is the local nilpotence of an unipotent automorphism group of a finitely generated metabelian group. Then, using this fact as the first step of an inductive argument, we illustrate the main theorem of the thesis, namely the generalization of the previous result to a finitely generated unipotent automorphism group H acting on an arbitrary finitely generated abelian-by-polycyclic group. In this situation, thanks to the tools introduced in the previous chapter, we are also able to bound uniformly the nilpotency class

of such a group H. Finally, we end the chapter analyzing the situation even in the case of a finitely generated abelian-by-polycyclic-by-finite group, proving once again the local nilpotence of an its unipotent automorphism group.
Chapter 4 is dedicated to the study of soluble groups which satisfy some finite rank conditions. Here, the notions of total and reduced rank are firstly introduced for the class of abelian groups, and secondly they are generalized to an arbitrary soluble group. In the following section we illustrate the other main result of our work, that is the proof of the nilpotence of a unipotent automorphism group H acting on a soluble group G with finite Prüfer rank. If H is finitely generated, we settle once again an uniform bound for its nilpotency class. Finally, in the last section, the previous result is extended to the case where G is a residually soluble group of finite rank.

Chapter 1

Basic Results

The first chapter is dedicated to the introduction of some basic concepts of Group Theory and to fix the notation that will be used throughout the thesis. The only results which are not classical are those exposed in the last section.

1.1 Commutators

Let G be a group. We write

$$[x, y] = x^{-1}y^{-1}xy = x^{-1}x^y$$

for the *commutator* of the elements $x, y \in G$, and if $x_1, x_2, \ldots, x_n \in G$ we define recursively

$$[x_1, x_2, \ldots, x_n] = [[x_1, \ldots, x_{n-1}], x_n].$$

A very useful shorthand notation is

$$[x,_n y] = [x, \underbrace{y, \ldots, y}_{n \text{ times}}].$$

The basic properties of commutators are listed in the following Lemma, and they can be verified by direct calculations.

Lemma 1.1. *Let G be a group and $x, y, z \in G$. Then:*

1. $[x, y] = [y, x]^{-1}$;
2. $[xy, z] = [x, z]^y[y, z]$ *and* $[x, yz] = [x, z][x, y]^z$;
3. $[x, y^{-1}] = ([x, y]^{y^{-1}})^{-1}$ *and* $[x^{-1}, y] = ([x, y]^{x^{-1}})^{-1}$;
4. $[x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y]^x = 1$ *(the Hall-Witt identity).*

The *commutator subgroup* between the two subsets $X, Y \subseteq G$ is

$$[X, Y] = \langle [x, y] \mid x \in X,\ y \in Y \rangle.$$

Notice that clearly $[X, Y] = [Y, X]$ by the first assertion of Lemma 1.1.
More generally, if $X_1, X_2, \ldots, X_n \subseteq G$ we put

$$[X_1, \ldots, X_n] = [[X_1, \ldots, X_{n-1}], X_n].$$

Finally we define

$$[X,_n Y] = [X, \underbrace{Y, \ldots, Y}_{n \text{ times}}].$$

The following extremely useful fact is a corollary of the Hall-Witt identity.

Lemma 1.2 (The Three Subgroups Lemma)**.** *Let H, K, L be subgroups of a group G. If two of the commutator subgroups $[H, K, L], [K, L, H], [L, H, K]$ are contained in a normal subroup N of G, then so is the third.*

A very important case of the Three Subgroup Lemma is when $N = 1$, so that the identities $[H, K, L] = [K, L, H] = 1$ imply necessarily $[L, H, K] = 1$.
Next Lemma is another consequence of the elementary commutator properties. If X is a subset of a group G, writing X^G we mean the *normal closure* of X in G, that is the intersection of all the normal subgroups of G containing X.

Lemma 1.3. *Let H and K be subgroups of a group. If $H = \langle X \rangle$ and $K = \langle Y \rangle$, then $[H, K] = [X, Y]^{HK}$.*

The commutator subgroup $G' = [G, G]$ is called the *derived subgroup* of G. It is easy to prove that the derived subgroup is the smallest normal subgroup of G with abelian quotient, and it is a fully-invariant subgroup of G.

1.2 Soluble and Nilpotent Groups

Definition 1.4. *A group G is said to be* soluble *if there exists a finite series of subgroups $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$ such that for every $i = 0, \ldots, n-1$ the factor G_{i+1}/G_i is abelian. Such a series is called an* abelian series.
For a soluble group G the length of a shortest abelian series is called the derived length *of G.*

Obviously, the only group with derived length 0 is the trivial group, while a group is soluble with derived length 1 if and only if it is abelian. A soluble group with derived length at most 2 is also called *metabelian*.

Among all the abelian series of a soluble group, the shortest one is the *derived series*, which is defined recursively as

$$G' = [G, G] \quad \text{and} \quad G^{(i)} = [G^{(i-1)}, G^{(i-1)}] \quad \forall\, i \geq 2.$$

In literature, the second term of the derived series is usually indicated with G'' instead of $G^{(2)}$.
A group G is soluble if and only if its derived series reaches the identity subgroup in finitely many steps. If d is the smallest integer such that $G^{(d)} = 1$, then d is exactly the derived length of G.
The following proposition describes the closure properties of soluble groups and needs no proof.

Proposition 1.5. *Subgroups and images of a soluble group are soluble. Moreover the class of soluble groups is closed with respect to forming extensions. This means that if N is a normal subgroup of a group G such that N and G/N are soluble of derived length c and d respectively, then G itself is soluble with derived length at most $c + d$.*

Recall that a group is a *torsion group* if all its elements have finite order, while on the other hand is *torsion-free* if the only element of finite order is the identity. Furthermore, a group G has finite *exponent* n if n is the least positive integer such that $g^n = 1$ for all $g \in G$. We will need the following classical fact.

Proposition 1.6. *A finitely generated soluble torsion group is finite.*

Proof. Let G be a finitely generated soluble torsion group of derived length d.
If $d = 0$ then $G = 1$ and there is nothing to prove, so we can suppose $d \geq 1$ and put $A = G^{(d-1)}$. By induction on d, the finitely generated soluble torsion group G/A is finite. This implies that A is finitely generated since it has finite index in G. By the characterization of finitely generated abelian groups it follows that also A is finite, so that $|G| = [G : A]|A| < \infty$. □

Let's now introduce the definitions of central series and nilpotent group, which will be of fundamental relevance in our investigation.
Given an arbitrary subset S of a group G, we write

$$C_G(S) = \{g \in G \mid gs = sg \ \ \forall\, s \in S\}$$

to denote the *centralizer* of S in G, and

$$Z(G) = C_G(G) = \{g \in G \mid gx = xg \ \ \forall\, x \in G\}$$

for the *center* of G.

Definition 1.7. *A* finite central series *for a group G is a series of subgroups*

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_{n-1} \trianglelefteq G_n = G$$

such that for every $i = 0, \ldots, n-1$, we have the inclusion

$$\frac{G_{i+1}}{G_i} \leq Z\left(\frac{G}{G_i}\right).$$

It should be clear that the last condition is equivalent to $[G, G_{i+1}] \leq G_i$.

Definition 1.8. *A group G is called* nilpotent *if it has a finite central series. The length of a shortest central series for a nilpotent group G is the* nilpotency class *of G, and it will be denoted with $cl(G)$.*

A group is abelian if and only if its nilpotency class is less than or equal to 1. The trivial group has class 0. From the definition it follows immediately that nilpotent groups are soluble, while on the other hand the symmetric group on 3 objects, having trivial center, is the smallest example of a metabelian group which is not nilpotent.

Proposition 1.9. *Subgroups, images, and finite direct products of nilpotent groups are nilpotent.*

It is a worth mentioning fact that the class of nilpotent groups is not extension closed, and again the symmetric group on 3 objects provides an easy example.

Two important central series for a group G are the *upper central series* and the *lower central series*. They are built in the following way.
The upper central series is defined to be the series

$$1 = \zeta_0(G) \trianglelefteq \zeta_1(G) \trianglelefteq \zeta_2(G) \trianglelefteq \ldots$$

where $\zeta_1(G) = Z(G)$ and $\zeta_{i+1}(G)$ is the unique subgroup of G such that

$$\frac{\zeta_{i+1}(G)}{\zeta_i(G)} = Z\left(\frac{G}{\zeta_i(G)}\right), \quad \forall\ i \geq 1.$$

On the other hand, the lower central series is the series of subgroups

$$G = \gamma_1(G) \trianglerighteq \gamma_2(G) \trianglerighteq \gamma_3(G) \trianglerighteq \ldots$$

where

$$\gamma_2(G) = [G, G] = G^{'} \ \text{ and } \ \gamma_{i+1}(G) = [\gamma_i(G), G] = [G,_i G], \quad \forall\ i \geq 2.$$

Each term of the lower central series of a group G is a fully-invariant subgroup of G, while each term of the upper central series is a characteristic subgroup of G but not necessarily fully-invariant.

Proposition 1.10. *If G is a nilpotent group and $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$ is a finite central series for G then:*

1. *$G_i \leq \zeta_i(G) \ \ \forall\ i = 0, \dots, n,$ so that $\zeta_n(G) = G$;*
2. *$\gamma_i(G) \leq G_{n-i+1} \ \ \forall\ i = 1, \dots, n+1,$ so that $\gamma_{n+1}(G) = 1$;*
3. *the upper and lower central series of G always have the same length, which coincides with the nilpotency class of G.*

In the theory of nilpotent groups, a fundamental role is played by the following classical result.

Theorem 1.11 (Fitting's Theorem)**.** *Let M and N be normal nilpotent subgroups of a group G with nilpotency classes c and d, respectively. Then the subgroup MN is a normal nilpotent subgroup of G of class at most $c + d$.*

As a corollary of Fitting's Theorem, we have that in every group the product of a finite number of normal nilpotent subgroups is always normal and nilpotent. Unfortunately the same conclusion does not hold for the product of an infinite number of normal nilpotent subgroups.
Given a group G, the subgroup generated by all its normal nilpotent subgroups is called the *Fitting subgroup* of G and will be denoted with Fit$G$. Obviously this is always characteristic in G.
If $G \neq 1$ is a soluble group then Fit$G \neq 1$, because it contains the smallest non trivial term of the derived series of G.
Finally, G is said to be a *Fitting group* if $G = \mathrm{Fit}G$.

We end this section quoting an important fact concerning nilpotent groups.

Proposition 1.12. *If G is nilpotent and G/G' is finite, then G too is finite.*

1.3 Polycyclic Groups

Definition 1.13. *The group G satisfies the* maximal condition on subgroups *if every ascending chain of subgroups of G*

$$H_1 \leq H_2 \leq \cdots \leq H_i \leq \dots$$

is stationary, namely there exists an integer n such that $H_n = H_{n+1}$.
If this is the case, we will say that G satisfies Max*, for short.*

The property Max is inherited by subgroups, quotients and is closed with respect to the formation of group extensions.
It is an easy matter, but neverthless fundamental, to prove that G satisfies Max if and only if each of its subgroups is finitely generated. In particular, an abelian group satisisfies Max if and only if it is finitely generated.

Definition 1.14. *A group G is said to be* polycyclic *if it has a cyclic series, that is a finite series with cyclic factors.*

Describing the structure of polycyclic groups is an easy exercise.

Proposition 1.15. *A group is polycyclic if and only if it is soluble and satisfies the maximal condition on subgroups.*

Proof. Clearly every polycyclic group is soluble by definition. Since cyclic groups satisfy Max and the latter property is closed under forming extension, it follows that every polycyclic group has Max.
On the other hand, if G is a soluble group with Max, then the factors of its derived series are finitely generated abelian groups. It is then possible to refine this series to obtain a new one with cyclic factors. □

From this fact and the previous considerations on soluble groups, it follows immediately that even the class of polycyclic group is subgroup, quotient and even extension closed.

It is very important to notice that the number of infinite factors in a cyclic series of a polycyclic group G is independent of the series and hence is an invariant of G. In fact, if B is an infinite cyclic group and $A \leq B$, then either A is trivial and B/A is infinite cyclic, or A is infinite cyclic and B/A is finite. For this reason, any refinement of a cyclic series of a polycyclic group G has the same number of infinite cyclic factors than the original one. Since by the Schreier Refinement Theorem (see [9], Theorem 3.1.2) any two series of a group possess isomorphic refinements, if we choose two cyclic series for G, they must have the same number of infinite cyclic factors.
This numerical invariant is called the *Hirsch number* of G, and turns out to be a very useful tool for inductive proofs involving polycyclic groups. It is not difficult to prove that if G is polycyclic and $H \leq G$, then they have the same Hirsch number if and only if the index $[G : H]$ is finite.

Proposition 1.16. *Finitely generated nilpotent groups are polycyclic.*

Proof. Suppose $G = \langle g_1, g_2, \dots, g_t \rangle$ is nilpotent of class c, so that $\gamma_{c+1}(G) = 1$. By induction on c, we may assume that $G/\gamma_c(G)$ is polycyclic. Then its subgroup

$\gamma_{c-1}(G)/\gamma_c(G)$ is finitely generated, and we may suppose

$$\gamma_{c-1}(G) = \langle x_1, x_2, \ldots, x_m\rangle \gamma_c(G).$$

Remembering that $\gamma_c(G) \leq Z(G)$ and using Lemma 1.1 it is possible to write

$$\gamma_c(G) = [\langle g_1, \ldots, g_t\rangle, \langle x_1, \ldots, x_m\rangle] = \langle [g_i, x_j] \mid 1 \leq i \leq t,\ 1 \leq j \leq m\rangle,$$

thus $\gamma_c(G)$ is finitely generated. Finally, G is polycyclic by extension. □

An example of a polycyclic group which is not nilpotent is the so called *infinite dihedral* group

$$D_\infty = \langle a, x \mid x^2 = 1,\ xax = a^{-1}\rangle.$$

This group has Hirsch number 1 since it has the cyclic series $D_\infty \rhd \langle a\rangle \rhd 1$, but $\gamma_i(D_\infty) = \langle a^{2^{i-1}}\rangle \neq 1$ for each integer $i \geq 2$.

It is now necessary to get familiar with a useful notation which will be used through all the rest of this work.
A *group-theoretical property* $\mathcal{P}$ is a property pertaining to groups such that every group of order 1 has $\mathcal{P}$, and whenever a group has $\mathcal{P}$, the same is true for any other group isomorphic to it.
If $\mathcal{P}$ and $\mathcal{Q}$ are two group-theoretical properties, we will say that a group G is a *$\mathcal{P}$-by-$\mathcal{Q}$ group* if G has a normal subgroup N satisfying the property $\mathcal{P}$ such that G/N has the property $\mathcal{Q}$.
Furthermore, G is termed a *meta-$\mathcal{P}$ group* if it has a normal subgroup N such that both N and G/N have $\mathcal{P}$. For example, a group is *polycyclic-by-finite* if it contains a normal polycyclic subgroup of finite index.

For many purposes the class of polycyclic-by-finite groups turns out to be a more natural object to investigate than the class of polycyclic group itself. Here we just claim that subgroups and images of a polycyclic-by-finite group are polycyclic-by-finite, and the Hirsch number is still well defined for a group in this class.

The last result of this section is a fact concerning every finitely generated group.

Lemma 1.17. *Suppose G is a finitely generated group and n a positive integer. Then the number of normal subgroups of G with index n is finite.*

Proof. Let G be a finitely generated group and n a positive integer. A normal subgroup with index n in G is the kernel of a homomorphism from G to a finite group of order n. Since G is finitely generated and since a homomorphism is uniquely determined by the images of the generators of G, it follows that there are only finitely many such homomorphisms. □

1.4 Residually Finite Groups

In the study of infinite groups a very important role is played by the so-called *finiteness conditions*, that is group-theoretical properties which are possessed by all finite groups. For example, the Max condition is a finiteness condition. Here we introduce the notion of *residual finiteness*.

Definition 1.18. *A group G is called* residually finite *if for each $1 \neq g \in G$ there exists a normal subgroup of finite index N_g such that $g \notin N_g$.*

There are many conditions which are equivalent to the above definition.

Proposition 1.19. *For any group G the following conditions are equivalent:*

1. *G is residually finite;*
2. *For each $1 \neq g \in G$ there exist a finite group F_g and a homomorphism $\phi_g : G \to F_g$ such that $\phi_g(g) \neq 1$;*
3. *The intersection of all its subgroups of finite index is trivial;*
4. *The intersection of all its normal subgroup of finite index is trivial;*
5. *G embeds inside the cartesian product of a family of finite groups, and it surjects onto each factor.*

Every subgroup of a residually finite group is itself residually finite. Indeed, if H is a subgroup of a residually finite group G and $h \in H$ is a non trivial element, then by Propositon 1.19 there exists a finite group F_h and a homomorphism ϕ_h from G to F_h such that $\phi_h(h) \neq 1$. Then, the image of h under the composition of ϕ_h with the immersion of H in G is not trivial.
Residual finiteness is not inherited by quotients, in fact it is possible to prove that every free group is residually finite.

Proposition 1.20. *Polycyclic-by-finite groups are residually finite.*

A proof of Proposition 1.20, can be found in [15] or [9]. Since every finitely generated nilpotent group is polycyclic, next Corollary follows immediately.

Corollary 1.21. *Finitely generated nilpotent groups are residually finite.*

The notion of residual finiteness is a particular case of the following definiton, concerning any group-theoretical property $\mathcal{P}$. In fact, a group G can be termed a *residually-$\mathcal{P}$* group if for every non trivial element $g \in G$ there exists a normal subgroup N_g such that $g \notin N_g$ and G/N_g has $\mathcal{P}$. It is not difficult to prove that the class of residually-$\mathcal{P}$ groups has a characterization analogue to Proposition 1.19.

1.5 Locally Nilpotent Groups

The following definition represents the most natural generalization of the concept of nilpotence for infinite groups.

Definition 1.22. *A group G is* locally nilpotent *if every finitely generated subgroup of G is nilpotent.*

Once again, subgroups and images of locally nilpotent groups are locally nilpotent. One rather obvious way to construct a non-nilpotent locally nilpotent group is to take the direct product of an infinite family of nilpotent groups whose nilpotency classes are unbounded. Such a group is even a Fitting group, and in fact it is easy to prove that every Fitting group is locally nilpotent.
There exists an analogue of Fitting's Theorem which holds for the normal locally nilpotent subgroups of any group.

Theorem 1.23 (Hirsch-Plotkin Theorem)**.** *The product of a family of normal locally nilpotent subgroups of a group G is normal and locally nilpotent.*

From the Hirsch-Plotkin Theorem and Zorn's Lemma, it follows that in every group there is a unique maximal normal locally nilpotent subgroup, called the *Hirsch-Plotkin radical*, containing all normal locally nilpotent subgroups.
Notice that in any infinite group the Hirsch-Plotkin radical contains the Fitting subgroup, and is usually much larger. An important result concerning a particular class of groups for which these two subgroups coincide is the following Theorem of P. Hall, whose proof can be found in [9].

Theorem 1.24. *Let G be a finitely generated metanilpotent group. Then the Fitting subgroup of G is nilpotent and coincides with the Hirsch-Plotkin radical.*

We conclude this section warning the reader on the existence of many other subclasses of locally nilpotent groups which further generalize the notion of nilpotence. All of these types of groups have their own interest, but it is not possibile to illustrate them here. For a full description of the generalized nilpotent groups we refer the reader to Chapter 12 of [9].

1.6 Engel Elements

There is another interesting generalization of nilpotence which deserves attention.

Definition 1.25. *If G is a group, an element $g \in G$ is called a* right Engel element *if for each $x \in G$ there exists an integer $n = n(g, x)$ such that $[g,_n x] = 1$. In particular, if n can be chosen independently of x we will say that g is a* right

n-Engel element *or a* bounded right Engel element *of* G.
The set of right and bounded right Engel elements of G *are usually denoted with* $\mathrm{R}(G)$ *and* $\overline{\mathrm{R}}(G)$*, respectively.*

On the other hand, an element $g \in G$ *is called a* left Engel element *if for each* $x \in G$ *there exists an integer* $n = n(g,x)$ *such that* $[x,_n g] = 1$*. Exactly as in the previous case, if* n *can be chosen independently of* x*,* g *is called a* left n-Engel element *or a* bounded left Engel *element of* G.
We write $\mathrm{L}(G)$ *and* $\overline{\mathrm{L}}(G)$ *for the sets of left and bounded left Engel elements of* G.

For an arbitrary group G, the subsets $R(G)$, $\overline{R}(G)$, $L(G)$ and $\overline{L}(G)$ are always invariant under the action of automorphisms of G, but generally none of them is a subgroup. Indeed one of the major goal in the study of Engel elements is to find sufficient conditions on G in order that these four subsets are subgroups. For example, Gruenberg proved that for soluble groups, the sets of left and bounded left Engel elements have a very good structure. Here we only mention the part of Gruenberg's result which will be of interest for us, while we refer the reader to [9], Theorem 12.3.3., for a full account.

Theorem 1.26 (Gruenberg)**.** *In a soluble group* G*, the subsets* $\overline{L}(G)$ *and* $L(G)$ *are subgroups, and the second one coincides with the Hirsch-Plotkin radical of* G.

We will say that a group G is an *Engel group* if $G = L(G)$, a statement equivalent to $G = R(G)$. It is straightforward to check that every locally nilpotent group is an Engel group. Another interesting topic in Engel group theory, is to investigate under which hypothesis an Engel Group turns out to be nilpotent. For example the nilpotence holds for every finite Engel group (Zorn), or even for finitely generated soluble Engel groups (Gruenberg). One of the most important achievement in this direction is the following.

Theorem 1.27 (Baer)**.** *Let* G *be a group with* Max*. Then*

1. $L(G)$ *and* $\overline{L}(G)$ *coincide with the Hirsch-Plotkin radical, which is nilpotent.*
2. $R(G)$ *and* $\overline{R}(G)$ *coincide with a term of the upper central series of* G.

In particular, an Engel group with Max *is a finitely generated nilpotent group.*

1.7 Structure Theorems

In this section we illustrate three very recent Theorems from [3], which motivated this work. They will be important tools for the proofs of the main results of this thesis. First of all, following the same notation of that paper, we give a couple of definitions.

Definition 1.28. *A subgroup H of a group G is said to be a* right n-Engel subgroup *if all the elements of H are right n-Engel elements of G.*

Definition 1.29. *Let H be a subgroup of a group G. Then H is a* residually hypercentral *subgroup of G if*

$$\bigcap_{i=1}^{+\infty} [H,_i G] = 1.$$

Notice that subgroups of a nilpotent group of class c are always residually hypercentral and c-Engel.
For an arbitrary group G, if H is a residually hypercentral right n-Engel subgroup then every subgroup of H inherits these properties, and if N is a normal subgroup of G, then also HN/N is a residually hypercentral right n-Engel subgroup of G/N.
For the reader's convenience, we remark the following elementary fact which needs no proof.

Lemma 1.30. *If H is a subgroup of a group G and n is an integer, then*

$$H \leq \zeta_n(G) \quad \textit{if and only if} \quad [H,_n G] = 1.$$

From this it follows easily that in any group G, the normal subgroup $\zeta_n(G)$ consists of right n-Engel elements, while it is conversely not true that right n-Engel elements belong necessarily to a term of the upper central series of G. In fact, for every prime p it is possibile to construct a $(p+1)$-Engel group with trivial center.
The main Theorems of this section give conditions that force a right n-Engel subgroup to belong to some term of the upper central series. We are finally able to enunciate them.

Theorem 1.31. *Let G be a d-generated group and suppose H to be a normal right n-Engel subgroup of G that is residually hypercentral. Then there exists an integer $m = m(d, n)$, only depending on d and n, such that $H \leq \zeta_m(G)$.*

Theorem 1.32. *Let G be a group with a normal right n-Engel subgroup H that is torsion-free and belongs to some term of the upper central series. Then there exists an integer $m = m(n)$, only depending on n, such that $H \leq \zeta_m(G)$.*

Theorem 1.33. *Let G be a group with a normal right n-Engel and residually hypercentral subgroup H. Then there exist two integes $c = c(n)$ and $f = f(n)$, only depending on n, such that $H^f \leq \zeta_c(G)$.*

Writing H^f we mean the subgroup of G generated by all the elements of the form h^f, with $h \in H$.

Chapter 2

Stability Groups and Unipotent Automorphisms

There exists an important notion for an automorphism group of an arbitrary group G which is stronger than nilpotence, namely the property of *stabilizing a finite series* in G. Before embarking into the definition, we need to fix the notation.
We shall write $\mathrm{Aut}(G)$ for the full automorphism group of the group G.

Let G be a group and $h \in \mathrm{Aut}(G)$. For every $g \in G$ and $h \in H$ put

$$[g, h] = g^{-1}g^h \quad \text{and recursively} \quad [g,_i h] = [[g,_{i-1} h], h], \ \forall\ i \in \mathbb{N},$$

where g^h is the image of g under the automorphism h. Sometimes, in order to simplify the notation, it will be useful to write $h(g)$ instead of g^h.
Furthermore, if $K \leq G$ and $H \leq \mathrm{Aut}(G)$ we put

$$[K, H] = \langle [k, h] \mid k \in K, h \in H \rangle \leq G$$

Notice that $[g, h]$ is nothing else that the usual group commutator between the elements g and h seen as elements of the semidirect product $G \rtimes \mathrm{Aut}(G)$, thus the notation we have just established is consistent.

2.1 Stability Groups

Definition 2.1. *Let G be a group, $h \in \mathrm{Aut}(G)$ and*

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{i-1} \trianglelefteq G_i \trianglelefteq \dots$$

a series of subgroups of G. If

$$[g_i, h] = g_i^{-1}g_i^h \in G_{i-1} \quad \textit{for each } g_i \in G_i \textit{ and for each } i,$$

we will say that the automorphism h stabilizes *the given series.*
The set of all such automorphisms is a subgroup of $\mathrm{Aut}(G)$ *which is called* the stability group *of the series of the G_i.*

From the definition it follows at once that a group H of automorphisms of G stabilizes the series of the G_i if and only if $[G_i, H] \leq G_{i-1}$ for every i.
For sake of completeness, we prove shortly what has been claimed in the previous definition.

Lemma 2.2. *The stability group of a series of subgroup of G is always a subgroup of* $\mathrm{Aut}(G)$*.*

Proof. Let $\{G_i\}_{i\in I}$ be a series of subgroups of G and H its stability group.
Clearly the identity automorphism belongs to H.
If $g_i \in G_i$ and $h \in H$, using Lemma 1.1,

$$[g_i, h^{-1}]^{-1} = [g_i, h]^{h^{-1}} = [g_i, h][g_i, h]^{-1}[g_i, h]^{h^{-1}} = [g_i, h][g_i, h, h^{-1}]$$

and so $[g_i, h^{-1}] \in G_{i-1}$ since $[g_i, h] \in G_{i-1}$ and $[g_i, h, h^{-1}] \in G_{i-2}$. Thus $h^{-1} \in H$.
Finally, if $g_i \in G_i$ and $h, k \in H$, again by Lemma 1.1 we have

$$[g_i, hk] = [g_i, k][g_i, h]^k = [g_i, k][g_i, h][g_i, h]^{-1}[g_i, h]^k = [g_i, k][g_i, h][g_i, h, k],$$

and the last member belongs to the subgroup G_{i-1} since $[g_i, h], [g_i, k] \in G_{i-1}$ and $[g_i, h, k] \in G_{i-2}$. □

One of the reason why stability groups turns out to be interesting, is that they are always nilpotent if the series stabilized is normal and has finite length. Here for a *normal series* we mean a series in which every term is normal in the whole group. The first proof of this fact has been given by Kaluzhnin in 1950.

Theorem 2.3. *The stability group of a normal series of finite length r of a group G is nilpotent of class at most $r - 1$.*

Proof. Let H be the stability group of the series $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_r = G$, where every term is normal in G. To simplify the notation we also write γ_j instead of $\gamma_j(H)$ for every j.
Using induction on j we want to prove that

$$[G_{r-i}, \gamma_j] \leq G_{r-(i+j)},$$

for each couple of integers i, j such that $i + j \leq r$.
For $j = 1$ we obtain $[G_{r-i}, H] \leq G_{r-i-1} = G_{r-(i+1)}$, which is exactly the definition

of stability group.
By induction, for every i and some fixed $j > 1$ we have

$$[H, G_{r-i}, \gamma_j] \le [G_{r-(i+1)}, \gamma_j] \le G_{r-(i+j+1)} \quad \text{and}$$

$$[G_{r-i}, \gamma_j, H] \le [G_{r-(i+j)}, H] \le G_{r-(i+j+1)}.$$

Since every term of the series is normal in G, we can apply the Three Subgroup Lemma obtaining

$$[G_{r-i}, \gamma_{j+1}] = [\gamma_{j+1}, G_{r-i}] = [\gamma_j, H, G_{r-i}] \le G_{r-(i+j+1)},$$

as required.
Now, taking $i = 0$ and $j = r$ it follows that $[G, \gamma_r] = 1$, which implies $\gamma_r(H) = 1$, so that H is nilpotent of class at most $r - 1$. □

From the upper bound for the nilpotency class, it follows immediately that if $H \le \mathrm{Aut}(G)$ stabilizes a series of length 2 in G, then H is necessarily abelian.
If the terms of the series in the above Theorem are not required to be normal, it still turns out that the stability group of the series is nilpotent but we have a worse bound for the nilpotency class. These considerations are summed up in the following result due to Hall (1958).

Theorem 2.4. *The stability group of any series of length $r \ge 1$ is a nilpotent group of class at most $\binom{r}{2}$.*

Proof. Let H be the stability group of the series $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_r = G$.
We are going to use induction on r.
If $r = 1$ then clearly $H = 1$ and there is nothing to prove.
Consequentely, suppose $r \ge 2$ and define

$$K = C_H(G_{r-1}) = \{h \in H \mid g^h = g \quad \forall\, g \in G_{r-1}\}.$$

Since H stabilizes the series

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_{r-1},$$

by the inductive hypothesis we have that H/K is nilpotent of class at most $\binom{r-1}{2}$, which implies $\gamma_{1+\binom{r-1}{2}}(H) \le K$. Put $K_1 = K$ and $K_{i+1} = [K_i, H]$, $\forall\, i \ge 1$. Since

$$\left[G, \gamma_{1+\binom{r}{2}}(H)\right] = \left[G, \left[\gamma_{1+\binom{r-1}{2}}(H),_{r-1} H\right]\right] \le [G, [K,_{r-1} H]] = [G, K_r],$$

if we establish $[G, K_r] = 1$ then $\gamma_{1+\binom{r}{2}}(H) = 1$ and the thesis follows.
To achieve this we will prove that $[G, K_i] \le G_{r-i}$ by induction on i.

If $i = 1$ this is certainly true by the definition of stability group.
Suppose that $[G, K_i] \leq G_{r-i}$ for $i \geq 2$. In order to prove $[G, K_{i+1}] \leq G_{r-(i+1)}$ we choose an element $[g, k_{i+1}]$ where $g \in G$ and $k_{i+1} \in K_{i+1}$. By definition, k_{i+1} is a finite product of commutators of the form $[k_i, h^{-1}]$ where $k_i \in K_i$ and $h \in H$, and so k_{i+1}^g can be expressed as a product in elements

$$[k_i, h^{-1}]^g = [k_i, h^{-1}][k_i, h^{-1}, g].$$

Clearly $[k_i, h^{-1}, g]$ belongs to $[K_i, H, G]$.
We observe that

$$[h^{-1}, g, k_i] \in [H, G, K_i] \leq [G_{r-1}, K_i] \leq [G_{r-1}, K] = 1 \quad \text{and}$$

$$[g, k_i, h^{-1}] \in [G, K_i, H] \leq [G_{r-i}, H] \leq G_{r-i-1} \quad \text{by induction,}$$

thus, thanks to the Hall-Witt identity, we have $[k_i, h^{-1}, g] \in G_{r-i-1}$.
From this inclusion and the previous observations it follows that $[g, k_{i+1}]$ belongs to $[K_i, H, G] \leq G_{r-i-1}$, which implies $[G, K_{i+1}] \leq G_{r-(i+1)}$. □

Nilpotence is not the only property which can be stated for stability groups, since they are often influenced by the factors of the series stabilized. Before going through the details, it is necessary to expose a technical Lemma.

Lemma 2.5. *Let G be a group, N a normal subgroup and H the stability group of the series $1 \trianglelefteq N \trianglelefteq G$. If $Z = Z(N)$, then*

1. $[G, H] \leq Z$
2. *If $g \in G$, then $\theta_g : h \mapsto [g, h]$ is a homomorphism of H into Z.*
3. *If $G = \langle X \rangle N$, then $\theta = (\theta_x)_{x \in X} : h \mapsto ([x, h])_{x \in X}$ is an embedding of H into the cartesian product of $|X|$ copies of Z.*

Proof. 1. Clearly $[H, N, G] = [1, G] = 1$ and $[N, G, H] \leq [N, H] = 1$. Invoking Lemma 1.2, we have $[G, H, N] \leq 1$, that is $[G, H] \leq C_G(N)$. But $[G, H] \leq N$ since H stabilizes the series, and so $[G, H]$ is contained in $C_G(N) \cap N = Z(N)$.

2. If $g \in G$ and $h, k \in H$, by Lemma 1.1 and the previous assertion,

$$[g, hk] = [g, k][g, h]^k = [g, k][g, h][g, h, k] = [g, k][g, h] = [g, h][g, k],$$

that is $(hk)^{\theta_g} = h^{\theta_g} k^{\theta_g}$.

3. The function θ is certainly a homomorphim by *2*, so we only need to prove that it is injective.
If $h \in H$ is such that $h^\theta = 1$ then $[x, h] = 1$ for each $x \in X$. Since $[N, h] = 1$ and $\langle X, N \rangle = G$, it follows $[G, h] = 1$, hence $h = 1$. □

Proposition 2.6. *Let G be a group and suppose $H \leq \mathrm{Aut}(G)$ stabilizes a finite series of normal subgroups in G, say $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_r = G$. Then*

1. *If G_i/G_{i-1} is torsion-free for each $i < r$, then H itself is torsion-free.*
2. *If $G_i^q \leq G_{i-1}$ for each $i < r$ and some integer q, then H has exponent dividing q^{r-1}.*
3. *If G is a finite p-group for some prime p, then H is also a finite p-group.*

Proof. We prove *1* and *2* at the same time using induction on r.
If $r = 1$, then $H = 1$ and we are done in both cases.
Set $r > 1$, $C = C_H(G/G_1)$ and assume by induction that H/C is as required. By its very definition, C stabilizes the series $1 \trianglelefteq G_1 \trianglelefteq G$ and so by Lemma 2.5, C embeds into the cartesian product of $|G|$ copies of the center of G_1. Hence C is torsion-free in the first case, while has exponent q in the second one. Points *1* and *2* now follow by taking group extension.
For the last assertion, if G is finite then the same is certainly true for $\mathrm{Aut}(G)$. Furthermore, if $|G| = p^n$ where p is a prime and n a positive integer, then each element of H has order dividing $p^{n(r-1)}$ by point *2*, and so H is a finite p-group. □

Before ending this section we are going to give an example of a stability group.
For a fixed prime p, the *p-quasicyclic group* C_{p^∞} (also called *p^∞-group* or even *Prüfer p-group*) is the group, written in additive notation, generated by a countable set of generators $\{g_i\}_{i\in\mathbb{N}}$ with defining relations

$$pg_1 = 0, \quad pg_{i+1} = g_i \ \ \forall\, i \geq 1, \quad \text{and} \quad g_i + g_j = g_j + g_i \ \ \forall\, i, j \geq 1.$$

Now choose a positive integer n and write G for the direct sum of n copies of the p-quasicyclic group,

$$G = \bigoplus_{i=1}^{n} C_{p^\infty} \ .$$

It is a well known fact that the full automorphism group of G is the *general linear* group $\mathrm{GL}(n, \mathbb{Z}_p)$, that is the group of all the invertible matrices of order n over the ring of p-adic integers $\mathbb{Z}_p$.
Put $G_0 = 0$ and $G_i = \oplus_{k=1}^{i} C_{p^\infty}$ for each $i = 1, \ldots, n$. Then the stability group of the series $0 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$ is the *unitriangular* group $UT(n, \mathbb{Z}_p)$, defined as the group of the upper triangular matrices over $\mathbb{Z}_p$, whose entries in the main diagonal are all equal to 1. Notice that $UT(n, \mathbb{Z}_p)$ is nilpotent of class $n-1$. This example also shows that in part *3* of Proposition 2.6 the hypothesis of finiteness on G cannot be removed. In fact $G = \oplus_{k=1}^{n} C_{p^\infty}$ is an infinite p-group while $UT(n, \mathbb{Z}_p)$ is torsion-free.

2.2 Unipotent Automorphisms

Definition 2.7. *Let G be a group and $h \in \mathrm{Aut}(G)$. If there exists a positive integer n such that $[g,_n h] = 1$ for every $g \in G$, the automorphism h is called an* n-unipotent automorphism *of G.*
A subgroup $H \leq \mathrm{Aut}(G)$ is an n-unipotent automorphism group, *if its elements are n-unipotent automorphisms of G.*

Notice that a n-unipotent automorphism h is just a left n-Engel element in the semidirect product $G \rtimes \langle h \rangle$.
Furthermore if A is an abelian group, for every n-unipotent automorphism h of A, the endomorphism $h-1$ is a nilpotent element of the ring $\mathrm{End}(A)$. In fact, for a fixed $a \in A$, since $[a,h] = a^{-1}a^h = a^h a^{-1} = a^{h-1}$ we have

$$a^{(h-1)^n} = [a,_n h] = 1, \quad \text{hence} \quad (h-1)^n = 0 \text{ in End(A)}.$$

Let's see a very easy example of a unipotent automorphism group.
Consider $G = \langle g \rangle$ a cyclic group of order 2^n and $H = \mathrm{Aut}(G)$. Obviously, H is isomorphic to the group of units of the ring $\mathbb{Z}/2^n\mathbb{Z}$, and so its order is 2^{n-1}. Choose an element $h \in H$ and consider the commutator $[g,h]$. Since the image of g under h is a generator of G, there exists an integer k such that $g^h = g^{2k+1}$, and thus

$$[g,h] = g^{-1}g^{2k+1} = g^{2k}.$$

Hence, by induction on n, we have that

$$[g,_n h] = g^{2^n k^n} = 1^{k^n} = 1,$$

and so the whole $\mathrm{Aut}(G)$ is n-unipotent. Furthermore, $\mathrm{Aut}(G)$ stabilizes the series

$$1 = \langle g^{2^n} \rangle \lhd \langle g^{2^{n-1}} \rangle \lhd \cdots \lhd \langle g^2 \rangle \lhd \langle g \rangle = G.$$

The same conclusions still holds if $G = \langle g \rangle$ is cyclic of order p^n where p is an odd prime, with the only difference that the n-unipotent automorphisms have the form

$$h_k : g \mapsto g^k \quad \text{where} \quad k \equiv 1 \pmod{p}$$

and they constitute a proper subgroup of $\mathrm{Aut}(G)$.

It is clear that an automorphism group stabilizing a finite series of length n it is n-unipotent. However the converse is not generally true, so that for an automorphism group the property of being n-unipotent is weaker than that of stabilizing a finite series. We exhibit an example to support this assertion.

If p is a prime, a *Tarski p-group* G is a two generated infinite group such that every proper non-trivial subgroup of G has order p. The existence of such a group has been proved by A. Yu. Ol'shanskii for every prime $p > 10^{75}$, see [8].
Tarski groups have many interesting properties, for example they are *simple groups*, that is groups with no proper non trivial normal subgroups. In fact, if we suppose that a Tarski p-group G has a normal subgroup N with $1 \lneq N \lneq G$, for every element $g \in G \setminus N$ both N and $\langle g \rangle$ are finite of order p by the very definition of G, and since $N \cap \langle g \rangle = 1$, the subgroup $N\langle g \rangle$ would have p^2 elements, a contradiction.
Now choose a field $\mathbb{F}$ of characteristic p and set V for the group algebra $\mathbb{F}[G]$, where G is a Tarski p-group. There is a natural action of G upon V which derives from the group operation in G and, since the characteristic of $\mathrm{End}(V)$ is p, it follows that for every $g \in G$

$$(g-1)^p = g^p - 1^p = 1 - 1 = 0 \quad \text{in } \mathrm{End}(V),$$

thus G is a group of p-unipotent automorphisms of V.
However, Lemma 4 of [4] tells us that if G would stabilize a finite series inside the abelian group V then it would possess a series with abelian factors, which is impossible since G is simple.

At this point, the reader should be convinced that finding conditions on a group G which force the stability (or the nilpotency) of an its unipotent group of automorphisms turns out to be an interesting problem.
An important result in this direction is taken from [2].

Theorem 2.8. *If G satisfies* Max*, then every group of unipotent automorphisms of G stabilizes a finite series, and $[G, H]$ is contained in the Fitting subgroup of G. In particular H is nilpotent.*

From this it follows that for a group G with Max the presence of unipotent automorphisms influences the structure of G itself, which must have a large Fitting subgroup.
Since every finite group and every finitely generated abelian group satisfy Max, next two corollaries are straightforward consequences of Theorem 2.8.

Corollary 2.9. *Let G be a finite group and H a group of unipotent automorphisms of G. Then H stabilizes a finite series in G and $[G, H] \leq \mathrm{Fit}(G)$. In particular H is nilpotent.*

Corollary 2.10. *Every unipotent automorphism group of a finitely generated abelian group G stabilizes a finite series in G, and thus it is nilpotent.*

2.3 Nilpotent Unipotent Automorphism Groups

In the previuos section we have seen that, for an automorphism group, the notion of unipotence is generally much weaker than the one of stabilizing a finite series. However, we are able to illustrate some sufficient conditions for an automorphism group H acting on G which, together with the unipotence, guarantee the existence of a series stabilized by H. Our investigation will mainly concern the action of unipotent automorphisms upon soluble groups.

Lemma 2.11. *Let A be an abelian group and h an n-unipotent automorphism of A. Then the group $\langle h \rangle \leq \mathrm{Aut}(A)$ stabilizes a series of length at most n in A.*

Proof. Consider the subgroup of A given by $[A, h] = \langle [a, h] \mid a \in A \rangle$.
We claim that $[A, h] \lneq A$. Arguing by contradiction, if $[A, h] = A$ holds then for each $a \in A$ there exists an element $a_1 \in A$ such that $a = [a_1, h]$. Moreover, for the same reason there exists an $a_2 \in A$ with the property that $a_1 = [a_2, h]$, and iterating this process we would find an element $a_n \in A$ such that

$$a = [a_1, h] = [[a_2, h], h] = [a_{2,2}\, h] = \cdots = [a_{n,n}\, h] = 1$$

which is clearly a contradiction. Thus we have that the subgroups $[A_{,i}\, h]$ form a descending series in A which necessarily reaches 1 after n steps because, remembering that $h - 1$ is a nilpotent element in the ring $\mathrm{Aut}(A)$, we have

$$[A_{,n}\, h] = A^{(h-1)^n} = 1.$$

Obviously $\langle h \rangle$ stabilizes the series of the $[A_{,i}\, h]$ by construction. □

Needless to say that Lemma 2.11 is extremely restrictive, but neverthless it allows us to establish a more general result concerning group of automorphisms of abelian groups which are both nilpotent and unipotent.

Theorem 2.12. *Let A be an abelian group and H a finitely generated nilpotent n-unipotent subgroup of $\mathrm{Aut}(A)$. Then H stabilizes a finite series in A.*

Proof. Let's proceed by induction on the nilpotency class of H.

Suppose that $cl(H) = 1$, so that $H = \langle h_1, h_2, \ldots, h_t \rangle$ is a finitely generated abelian group of n-unipotent automorphisms of A. We will use induction on t, observing that the case $t = 1$ is exactly Lemma 2.11.
If $t \geq 2$, by Lemma 2.11 we know that the group $\langle h_1 \rangle$ stabilizes the series

$$A \trianglerighteq [A, h_1] \trianglerighteq [A_{,2}\, h_1] \trianglerighteq \cdots \trianglerighteq [A_{,n}\, h_1] = 1,$$

and since H is abelian, for every automorphism $h \in H$ we have

$$[A, h_1]^h = [A^h, h_1^h] = [A, h_1]$$

so that $[A, h_1]$ is an H-invariant subgroup of A. From this fact, it follows easily that each $[A,_i h_1]$ is also H-invariant for every integer i.
Thus if we put $C = C_H(A/[A, h_1])$, the group of n-unipotent automorphisms H/C acts faithfully on the quotient $A/[A, h_1]$ and has a number of generators strictly lower than t, so by the inductive hypothesis it stabilizes a finite series in $A/[A, h_1]$. This implies the existence of a series of subgroups running from A to $[A, h_1]$ which is stabilized by H.
Iterating this process for every section $[A,_i h_1]/[A,_{i+1} h_1]$, it follows that it is possible to refine the series of the $[A,_i h_1]$ with a new one which is stabilized by H.

Now set $c = cl(H) \geq 2$ and suppose the assertion true for all the finitely generated nilpotent n-unipotent automorphism groups of abelian groups whose nilpotency class is at most $c - 1$.
Finitely generated nilpotent groups satisfy Max, hence every subgroup of H is finitely generated. In particular $Z(H)$ is a non trivial finitely generated abelian group consisting of n-unipotent automorphisms of A and, by the previous case, we know that $Z(H)$ stabilizes the finite series of H-invariant subgroups

$$A = A_1 \trianglerighteq A_2 \trianglerighteq \cdots \trianglerighteq A_m = 1.$$

Consider the centralizer $C = C_H(A_1/A_2) \geq Z(H) \gneq 1$. Clearly we have the immersion

$$\frac{H}{C} \hookrightarrow \mathrm{Aut}\left(\frac{A}{A_2}\right)$$

where the faithful action of H/C on A/A_2 is n-unipotent. By the Isomorphism Theorem we have

$$\frac{H}{C} \cong \frac{\frac{H}{Z(H)}}{\frac{C}{Z(H)}}$$

and $H/Z(H)$ is nilpotent of class at most $c - 1$, so the same is true for H/C. Applying the inductive hypothesis to H/C, it follows that H stabilizes a finite series of subgroups running from $A = A_1$ to A_2.
Repeat this process for all the sections A_i/A_{i+1} to produce a refinement of the original series $\{A_i\}_{i=1,\ldots,m}$ which is stabilized by H. □

The following Corollary is a further generalization of Theorem 2.12.

Corollary 2.13. *Let G be a soluble group and H a finitely generated nilpotent n-unipotent subgroup of* $\mathrm{Aut}(G)$. *Then H stabilizes a finite series in G, obtained as a refinement of the derived series of G.*

Proof. We use induction on the derived length d of the soluble group G.
If $d = 1$ we are exactly in the case of Theorem 2.12 and so we are done.
Let $d > 1$ and put $A = G^{(d-1)}$, which is fully-invariant in G. By the inductive hypothesis, assume that the automorphism group $H/C_H(G/A)$ stabilizes a finite refinement of the derived series of G/A. Considering the action of $H/C_H(A)$ on A and using again Theorem 2.12, we conclude that H stabilizes a finite series inside A, and thus a finite refinement of the derived series of G. □

2.4 A Bound for the Nilpotency Class

Our next aim is to find an upper bound for the nilpotency class of a finitely generated nilpotent and n-unipotent automorphism group H acting on a soluble group G which is uniform, in the sense that it depends only on n and the number of generators of H but not on the properties of G.
We need the following two technical Lemmata. The first one is due to R. Baer.

Lemma 2.14. *If G is a soluble group, then G contains a characteristic subgroup U such that U is nilpotent of class at most 2 and $C_G(U) = Z(U)$.*

Proof. By Zorn's Lemma there exists a maximal abelian characteristic subgroup A of G. If $A = C_G(A)$ we are done setting $U = A$. Otherwise, once again Zorn's Lemma provides the existence of an abelian characteristic subgroup U/A of G/A which is maximal with respect to lying in $C_G(A)/A$. Since

$$\gamma_3(U) = [\gamma_2(U), U] \leq [A, U] \leq [A, C_G(A)] = 1,$$

the nilpotency class of U is at most 2.
Suppose by contradiction that $C_G(U)$ is not contained in U. Then $C_G(U)/A$ contains a non trivial abelian characteristic subgroup D/A of G/A. Thus UD/A is abelian, characteristic in G/A and contained in $C_G(A)/A$. By the maximality of U we have $D \leq U$, which implies that $D \leq U \cap C_G(U) = Z(U)$. Then $A \leq Z(U)$, which is characteristic in G, so by the choice of A it follows that $A = Z(U)$ and consequently $A = D$, a contradiction. □

The previous lemma still holds with weaker assumptions on G. It is enough for G to have an ascending series with characteristic terms and abelian factors, see [5].

The next result is due to C. Casolo and O. Puglisi, and appears in [2].

Lemma 2.15. *Let G be a group and H a group of n-unipotent automorphisms of G. Suppose that H has a normal abelian subgroup A such that we can write $H = A\langle h\rangle$ for an element $h \in H$, and assume also that $[G, A, A] = 1$. Then H is nilpotent of class at most $n^2 - 1$.*

Proof. Consider the normal subgroup $[G, A]$ of G. Since for every $g \in G$ and $a, b \in A$, we have $1 = [g, b, a] = [g, b]^{-1}[g, b]^a$ and $1 = [g, a, b] = [g, a]^{-1}[g, a]^b$, it follows that

$$[g, a][g, b] = [g, a][g, b]^a = [g, ba] = [g, ab] = [g, b][g, a]^b = [g, b][g, a],$$

and $[G, A]$ is abelian. Thus it can be regarded as a $\mathbb{Z}[h]$-module.
For each $k = 1, \dots n$, let $X_k = \{g \in G \mid [g,_k h] = 1\}$ and set $A_0 = A$, $A_k = C_A(X_k)$. Then $A_n = C_A(G) = 1$ and the subgroups A_k form a descending series in A. Choose an element $g \in X_1$ and consider the map

$$\begin{array}{rcl} \theta_g : A & \to & [G, A] \\ a & \mapsto & [g^{-1}, a] \ . \end{array}$$

For every $g \in X_1$, the map θ_g is a group homomorphism because A is abelian and $[G, A, A] = 1$. Furthermore it is also a $\mathbb{Z}[h]$-module homomorphism, since

$$[g^{-1}, a]^h = [(g^{-1})^h, a^h] = [[h, g]g^{-1}, a^h] = [[h, g], a^h]^{g^{-1}}[g^{-1}, a^h] = [g^{-1}, a^h].$$

Thus, $A/\mathrm{ker}\theta_g$ is $\mathbb{Z}[h]$-isomorphic to a submodule of $[G, A]$ and, in particular, h acts as an n-unipotent automorphism on $A/\mathrm{ker}\theta_g$. Since

$$A_1 = \bigcap_{g \in X_1} \mathrm{ker}\theta_g,$$

by the Isomorphism Theorem we have that A/A_1 embeds, as a $\mathbb{Z}[h]$-module, into a cartesian product of copies of $[G, A]$, and thus h acts as an n-unipotent automorphism also on A/A_1. Therefore the automorphism h stabilizes a series of length at most n in A/A_1.
Using induction, we may suppose that h stabilizes a series of length at most n in each factor A_i/A_{i+1} for each $i = 1, \dots, k < n$. Let S be the semigroup generated by $x = h^{-1}$, and $\mathbb{Z}[S]$ its semigroup algebra. Choose $g \in X_{k+1}$ and define the map

$$\begin{array}{rcl} \theta_g : A_k & \to & [G, A] \\ a & \mapsto & [g^{-1}, a] \ . \end{array}$$

Using the fact that $[g, h] \in X_k$, we see that $A_k/\mathrm{ker}\theta_g$ is $\mathbb{Z}[h]$-isomorphic to a submodule of $[G, A]$. In particular, being $(x - 1)^n \in \mathbb{Z}[S]$, for every $u \in A_k/\mathrm{ker}\theta_g$, we have that $[u,_n x] = 1$. It follows that the same identity still holds for the action on A_k/A_{k+1}, because this group is isomorphic to a $\mathbb{Z}[S]$-submodule of a cartesian product of copies of $[G, A]$. Therefore, h stabilizes a finite series of length at most n in A_k/A_{k+1}.
Thus we can find a series in A stabilized by $\langle h \rangle$ whose length is at most n^2, proving that $H = A\langle h \rangle$ is nilpotent of class at most $n^2 - 1$ by Theorem 2.4. □

Theorem 2.16. *If G is a soluble group and $H \leq \mathrm{Aut}(G)$ is t-generated, nilpotent and n-unipotent, then its nilpotency class is (n,t)-bounded.*

Proof. Let the groups G and H be as required in the hypothesis.
The soluble group G contains a nilpotent characteristic subgroup U, whose nilpotency class is at most 2 and such that $C_G(U) = Z(U)$, thanks to Lemma 2.14.

First of all we observe that $C = C_H(U) \trianglelefteq H$ stabilizes a series of length 2 in G. In fact

$$[C, U, G] = [1, G] = 1 \quad \text{and} \quad [U, G, C] \leq [U, C] = 1,$$

hence, by the Three Subgroup Lemma, it follows that $[G, C, U] = 1$. In other words, the subgroups $[G, C]$ and U commute. This implies

$$[G, C] \ \leq \ C_G(U) = Z(U) \ \leq \ U,$$

and consequentely $[G, C, C] = 1$. Moreover, by the upper bound for the nilpotency class of a stability group in Theorem 2.3, we have that C is abelian.

Firstly suppose that U is abelian and put $\overline{H} = H/C$.
Obviously, by the Isomorphism Theorem $\overline{H}$ embeds in $\mathrm{Aut}(U)$ as a finitely generated nilpotent group of n-unipotent automorphisms, hence Theorem 2.12 ensures us that $\overline{H}$ stabilizes a finite series in U. Thus we can assume the existence of an integer s such that $[U,_s \overline{H}] = 1$.
It should also be clear that the n-unipotent action of $\overline{H}$ upon U is equivalent for U to be a right n-Engel subgroup of $U \rtimes \overline{H}$.
Furthermore, since U is abelian it follows that

$$[U, \overline{H}] = [U, U \rtimes \overline{H}],$$

and consequentely $[U,_s U \rtimes \overline{H}] = [U,_s \overline{H}] = 1$. This implies at once

$$\bigcap_{i \in \mathbb{N}} [U,_i U \rtimes \overline{H}] = 1.$$

We can sum up these facts stating that U is a normal, right n-Engel and residually hypercentral subgroup of $U \rtimes \overline{H}$.
Now choose $u \in U$, and consider the finitely generated group $\langle u, \overline{H} \rangle \leq U \rtimes \overline{H}$.
It is easy to check that the normal closure $\langle u \rangle^{\langle u, \overline{H} \rangle}$ is a normal, right n-Engel and residually hypercentral subgroup of $\langle u, \overline{H} \rangle$, because it is contained in U. Since $\langle u, \overline{H} \rangle$ is finitely generated, it is possible to use Theorem 1.31 to find an integer $m = m(n,t)$, only depending on n and the number of generators of $\overline{H}$ (which is at most t), such that $[\langle u \rangle^{\langle u, \overline{H} \rangle},_m \langle u, \overline{H} \rangle] = 1$.

But then $[U,_m \overline{H}] = 1$ because it is generated by the subgroups $[\langle u \rangle^{\langle u, \overline{H} \rangle},_m \overline{H}]$ for every $u \in U$, in fact

$$[U,_m \overline{H}] \leq [U,_m \langle u, \overline{H} \rangle] = \prod_{u \in U} [\langle u \rangle^{\langle u, \overline{H} \rangle},_m \langle u, \overline{H} \rangle] = \prod_{u \in U} 1 = 1.$$

We have discovered that $\overline{H} = H/C$ stabilizes a series of length m in U and so it is nilpotent of class at most $m-1$. Hence the inclusion $\gamma_m(H) \leq C_H(U)$ holds.

If U is nilpotent of class 2 then clearly $U/Z(U)$ is abelian, and we can repeat the previous argument to find two integers $m_1 = m_1(n,t)$ and $m_2 = m_2(n,t)$ such that

$$\left[\frac{U}{Z(U)},_{m_1} \frac{H}{C_H\left(\frac{U}{Z(U)}\right)} \right] = 1, \quad \text{and} \quad \left[Z(U),_{m_2} \frac{H}{C_H(Z(U))} \right] = 1.$$

Thus, it follows that $[U,_{m_1} H] \leq Z(U)$ and therefore

$$\left[U,_{m_1+m_2} \frac{H}{C_H(U)} \right] = 1,$$

showing that $H/C_H(U)$ is nilpotent of class at most $m_1 + m_2 - 1$.
In conclusion, both in the case U is abelian or nilpotent of class 2, there exists a function $r = r(n,t)$, depending only on n and t, such that $\gamma_r(H) \leq C_H(U)$.

Now choose an element $h \in H$ and focus on the subgroup $C\langle h \rangle$ of H. This subgroup clearly satisfies the hypothesis of Lemma 2.15, and so $C\langle h \rangle$ is nilpotent of class at most $n^2 - 1$, that is $[C\langle h \rangle,_{n^2} C\langle h \rangle] = 1$. Obviously this implies

$$[C,_{n^2} h] = 1,$$

and so C is a normal, right n^2-Engel subgroup of H.
Since H is nilpotent, every of its subgroups is residually hypercentral and, in particular, this is true for C.
Invoking again Theorem 1.31, we have that there exists an integer $l = l(n,t)$ depending only on n and t such that

$$[C,_l H] = 1.$$

Thus we conclude with

$$\gamma_{r+l}(H) = [H,_{r+l} H] = [\gamma_r(H),_l H] \leq [C,_l H] = 1,$$

and H is nilpotent of class at most $r(n,t) + l(n,t) - 1$. □

Chapter 3

Finitely Generated Abelian-by-Polycyclic-by-Finite Groups

The aim of this chapter is to prove that a finitely generated group H of n-unipotent automorphisms of a finitely generated abelian-by-polycyclic-by-finite group G is nilpotent. Furthermore, if G is soluble the nilpotency class of H can be uniformly bounded; in other words, it depends only on n and on the minimal number of generators of H.

Our investigation on finitely generated abelian-by-polycyclic-by-finite groups relies on the following Theorem, due to Roseblade and Jategaonkar independently, which is a generalization of a result of Hall. The proof can be founded in [7].

Theorem 3.1. *Every finitely generated abelian-by-polycyclic-by-finite group is residually finite. In particular, finitely generated metabelian groups are residually finite.*

3.1 Right n-Engel and Residually Hypercentral Subgroups

The following Proposition gives sufficient conditions for the presence of a normal, right n-Engel and residually hypercentral subgroup inside the semidirect product $A \rtimes H$, where $H \leq \mathrm{Aut}(A)$ and A is abelian, and will be a very useful tool for the proofs of the main results of this chapter.

Proposition 3.2. *Let G be a finitely generated residually finite group, and H a group of n-unipotent automorphisms of G. If A is an abelian characteristic subgroup of G, then A is a normal, right n-Engel and residually hypercentral subgroup of $A \rtimes H$.*

Proof. Due to the fact that A is characteristic in G, it is possible to form the semidirect product $A \rtimes H$ which obviously contains A as a normal subgroup.

Since A is abelian, a direct calculation gives immediately $[A, A \rtimes H] = [A, H]$, and thus the n-unipotent condition on H is equivalent to say that A is a right n-Engel subgroup of $A \rtimes H$.

The only non-trivial part is to show that A is residually hypercentral in $A \rtimes H$. The group G is finitely generated, hence Lemma 1.17 ensures us that for every fixed $k \in \mathbb{N}$ there exists only a finite number of normal subgroups of G with index at most k. Thus if we put

$$L_k = \bigcap_{\substack{N \lhd G \\ [G:N] \leq k}} N ,$$

this is also a subgroup of G of finite index, for every $k \in \mathbb{N}$. From its definition it follows that each L_k is a characteristic subgroup of G, because every automorphism permutes subgroups which have the same index.
Furthermore G is residually finite, hence $\bigcap_{k \in \mathbb{N}} L_k = 1$.
For each integer k, define $A_k = A \cap L_k$. By the Isomorphism Theorem

$$\frac{A}{A_k} = \frac{A}{A \cap L_k} \cong \frac{AL_k}{L_k} \leq \frac{G}{L_k} ,$$

and so $[G : L_k] < +\infty$ implies immediately $[A : A_k] < +\infty$.
As $A_k \leq L_k$ for each k, it follows that

$$\bigcap_{k=1}^{+\infty} A_k \leq \bigcap_{k=1}^{+\infty} L_k = 1 ;$$

moreover, each A_k is H-invariant, since both A and L_k share this property.
For every $k \in \mathbb{N}$, A/A_k is a finite abelian group and if we put $C_k = C_H(A/A_k)$ then the group H/C_k acts faithfully on A/A_k as a group of n-unipotent automorphisms. Invoking Corollary 2.9, it is possible to find an integer m_k such that

$$1 = \left[\frac{A}{A_k} ,_{m_k} \frac{H}{C_k} \right] , \quad \text{which implies} \quad 1 = \frac{[A,_{m_k} H]A_k}{A_k}$$

and so $[A,_{m_k} H] \leq A_k$. But then

$$\bigcap_{i=1}^{+\infty} [A,_i H] \; \leq \; \bigcap_{i=1}^{+\infty} A_i = 1.$$

Finally, remembering that $[A, A \rtimes H] = [A, H]$, we have

$$\bigcap_{i=1}^{+\infty} [A,_i A \rtimes H] = \bigcap_{i=1}^{+\infty} [A,_i H] = 1,$$

and A is a residually hypercentral subgroup of $A \rtimes H$. □

3.2 Finitely Generated Metabelian Groups

In this section we will use Proposition 3.2 and Theorem 2.4 to prove one of the main achievements of this thesis, namely a result concerning the action of unipotent automorphisms on finitely generated metabelian groups. During the proof we will make use of the following Theorem due to Hall (1954), which gives information on the structure of group rings of polycyclic-by-finite groups over Noetherian rings. We quote it without proof, which can be found in [15], Theorem 3.7.

Theorem 3.3. *If J is a Noetherian ring and G is a polycyclic-by-finite group, then the group ring $J[G]$ is Noetherian.*

Now we are able to enunciate our result.

Theorem 3.4. *Let G be a finitely generated metabelian group and H a finitely generated n-unipotent automorphism group of G. Then H is a nilpotent group and its nilpotency class is bounded by a function depending only on n and the number of generators of H. Furthermore H stabilizes a finite series in G.*

Proof. First of all, notice that we only need to prove the nilpotency of H, since the upper bound for the nilpotency class will follow from Theorem 2.16, and the last assertion is a straightforward consequence of Corollary 2.13.

Let G be a finitely generated metabelian group, so that $G = \langle X \rangle$ where X is a finite subset of G. To simplify the notation we write $A = G^{'}$ for the abelian derived subgroup of G.
It is necessary to focus on the abelian quotient group $G_{ab} = G/A$.
There is a natural module action of the commutative group ring $\mathbb{Z}[G_{ab}]$ on the fully-invariant abelian subgroup A. In fact, we can define the action for the elements of G_{ab} as

$$a^{gA} = a^g \quad \forall \; a \in A \;\text{ and }\; gA \in G_{ab}$$

and extend it linearly to $\mathbb{Z}[G_{ab}]$.
We notice that A is not necessarily finitely generated as a subgroup of G, but surely it is finitely generated as $\mathbb{Z}[G_{ab}]$-module: in fact, by Lemma 1.3 we have

$$A = [G, G] = [\langle X\rangle, \langle X\rangle] = [X, X]^G,$$

and since X is finite then also $[X, X]$ is finite. Remarking that the module action is via coniugation, it is clear that A is finitely generated as a $\mathbb{Z}[G_{ab}]$-module, as claimed.

As A is a fully-invariant subgroup of G, for each automorphism $\phi \in \mathrm{Aut}(G)$ we can consider its restriction to A, that is $\widehat{\phi} = \phi_{|A} \in \mathrm{Aut}(A)$, and even the function $\overline{\phi} \in \mathrm{Aut}(G_{ab})$ which is defined as $\overline{\phi}(gA) = \phi(g)A,\ \forall\ gA \in G_{ab}$.
It is important to observe that

$$\overline{\phi} = id_{G_{ab}} \quad \Longrightarrow \quad \widehat{\phi} \in \mathrm{Aut}_{\mathbb{Z}[G_{ab}]}(A),$$

where $id_{G_{ab}}$ is the identity automorphism of G_{ab}.
In fact, if $\overline{\phi}$ is the identity automorphism for the quotient group G_{ab}, then $\forall\ a \in A$ and $\forall\ gA \in G_{ab}$ we have

$$\widehat{\phi}(a)^{gA} = \widehat{\phi}(a)^{\overline{\phi}(gA)} = \widehat{\phi}(a)^{\phi(g)A} = \phi(a)^{\phi(g)} = \phi(a^g) = \widehat{\phi}(a^{gA}),$$

and thus $\widehat{\phi}$ is a $\mathbb{Z}[G_{ab}]$-module automorphism for A.
Now remark that for an arbitrary ring R and any R-module M, the equality

$$\mathrm{End}_R(M) = \{\phi \in \mathrm{End}_{\mathbb{Z}}(M) \mid r\phi = \phi r,\ \forall\ r \in R\} = C_{\mathrm{End}_{\mathbb{Z}}(M)}(R)$$

holds by definition.
Considering only bijective homomorphisms, this implies $\mathrm{Aut}_R(M) = C_{\mathrm{Aut}_{\mathbb{Z}}(M)}(R)$.

Next step is to prove that H is a metanilpotent group, that is H has a normal subgroup N such that N and H/N are both nilpotent.
We have already checked the existence of the homomorphism

$$\begin{array}{rccc} \Phi: & H & \to & \mathrm{Aut}(G_{ab}) \\ & \phi & \mapsto & \overline{\phi} \end{array}$$

where $\overline{\phi}$ is defined as $\overline{\phi}(gA) = \phi(g)A$, for every $gA \in G_{ab}$.
Notice that from the previous assertion it follows

$$\ker\Phi = C_H(G_{ab}) = \{\phi \in H \mid \overline{\phi} = Id_{G_{ab}}\} \leq \{\phi \in H \mid \widehat{\phi} \in \mathrm{Aut}_{\mathbb{Z}[G_{ab}]}(A)\},$$

and furthermore we have the immersion

$$\frac{H}{C_H(G_{ab})} \hookrightarrow \mathrm{Aut}(G_{ab}),$$

which preserves the n-unipotent action on G_{ab}.
Since by Corollary 2.10 we know that a group of unipotent automorphisms of a finitely generated abelian group is nilpotent, then the image

$$\Phi(H) \cong \frac{H}{C_H(G_{ab})} \leq \mathrm{Aut}(G_{ab})$$

is nilpotent, say of class r, and consequently

$$\gamma_r\left(\frac{H}{C_H(G_{ab})}\right) = 1 \quad \Rightarrow \quad \gamma_r(H) \leq C_H(G_{ab}) \leq \mathrm{Aut}_{\mathbb{Z}[G_{ab}]}(A).$$

By Theorem 3.3 the ring $\mathbb{Z}[G_{ab}]$ is Noetherian, being the group ring of a finitely generated abelian group over the Noetherian ring $\mathbb{Z}$.
Since $\gamma_r(H) \leq \mathrm{Aut}_{\mathbb{Z}[G_{ab}]}(A)$, it follows that $\gamma_r(H)$ is a group of n-unipotent automorphisms of the $\mathbb{Z}[G_{ab}]-$module A, and so we can apply Theorem 13.6 of [14] which guarantees that $\gamma_r(H)$ stabilizes a finite series of submodules of A. Therefore $\gamma_r(H)$ is nilpotent, that is there exists an integer s such that $\gamma_s(\gamma_r(H)) = 1$.
We have found that H is metanilpotent, and in particular it is a soluble group.

Let's now proceed by contradiction, assuming H to be a counterexample of minimal derived length d. Thus every finitely generated soluble group of n-unipotent automorphisms with derived length strictly lower than d acting faithfully on a finitely generated metabelian group is nilpotent.

We focus on H', the derived subgroup of H, which is soluble of derived length $d-1$ and thus locally nilpotent, by the minimality of H. For this reason, H' is contained in the Hirsch-Plotkin radical of H which, by Theorem 1.24, coincides with Fit$H$ and is nilpotent. This implies that H' is a nilpotent group, and that the image $H/\mathrm{Fit}H$ is a finitely generated abelian group. Our minimal counterexample H is then necessarily nilpotent-by-abelian.

Among all the counterexamples of minimal derived length, we can choose H in such a way that it is minimal also for number of generators of $H/\mathrm{Fit}H$. We want to prove that with this choice $H/\mathrm{Fit}H$ is a cyclic group.
Arguing by contradiction, suppose that a minimal generating system for $H/\mathrm{Fit}H$ has t elements with $t \geq 2$, and put $\{x_i\mathrm{Fit}H\}$, where $i = 1, \ldots, t$, for such a

generating set. Consider the subgroups $H_i = \langle x_i,\ \mathrm{Fit}H\rangle \leq H$. These subgroups are clearly normal in H because they contain $H^{'}$, and we are going to prove that they are nilpotent. Fix an integer i and choose a finite subset $F \subseteq H_i$. Then,

$$\mathrm{Fit}\langle F\rangle \ \geq\ \langle F\rangle \cap \mathrm{Fit}H$$

which implies that

$$\frac{\langle F\rangle}{\langle \mathrm{Fit}F\rangle} \quad \text{is an image of} \quad \frac{\langle F\rangle}{\langle F\rangle \cap \mathrm{Fit}H}\ .$$

Invoking the Isomorphism Theorem,

$$\frac{\langle F\rangle}{\langle F\rangle \cap \mathrm{Fit}H} \cong \frac{\langle F\rangle\,\mathrm{Fit}H}{\mathrm{Fit}H} \leq \frac{H_i}{\mathrm{Fit}H}\ .$$

Since the abelian group $H_i/\mathrm{Fit}H$ has a number of generators strictly lower than $H/\mathrm{Fit}H$, the same holds for $\langle F\rangle/\mathrm{Fit}\langle F\rangle$, and by the minimality of H we have that $\langle F\rangle$ is nilpotent. Thus, for each $i = 1, \dots, t$ the subgroup H_i is locally nilpotent, and so by Theorem 1.23, the product $H_1 \dots H_t = H$ is a locally nilpotent group. In particular H is nilpotent because it is finitely generated. We have found a contradiction.
It is then possible to choose a counterexample of minimal derived length such that $H/\mathrm{Fit}H$ is cyclic. Thus H is nilpotent-by-cyclic, so there exists $h \in H$ with the property that

$$H = \langle h, \mathrm{Fit}H\rangle = \langle h\rangle \mathrm{Fit}H.$$

We focus on the index $[H : \mathrm{Fit}H]$ and we distinguish two cases.

Case 1 : $[H : \mathrm{Fit}H] < +\infty$
In this situation, being a subgroup of finite index of a finitely generated group, $\mathrm{Fit}H$ is a finitely generated nilpotent group. Furthermore, we notice that $\mathrm{Fit}H$ acts on G as a group of n-unipotent automorphisms, and so we can use Theorem 2.13 to find a finite refinement of the series $1 \lhd A \lhd G$ which is stabilized by $\mathrm{Fit}H$. Write

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \dots G_{j-1} \trianglelefteq A = G_j \trianglelefteq G_{j+1} \trianglelefteq \dots \trianglelefteq G_s = G$$

for such a refinement. Clearly each section G_{i+1}/G_i of this series is abelian, and we have the immersion

$$\frac{H}{C_H\left(\frac{G_{i+1}}{G_i}\right)} \ \hookrightarrow\ \mathrm{Aut}\left(\frac{G_{i+1}}{G_i}\right)$$

which preserves the n-unipotent action of H. Since $\mathrm{Fit}H \leq C_H(G_{i+1}/G_i)$ and $H/\mathrm{Fit}H$ is cyclic, it follows that for every i the quotient $H/C_H(G_{i+1}/G_i)$ is a

cyclic group, and invoking Lemma 2.11 we have that it stabilizes a finite series which starts at G_{i+1} and ends with G_i.
Thus, repeating this argument for every i, it turns out that $H = \langle h\rangle \mathrm{Fit}H$ stabilizes a finite series in G, and so it is nilpotent by Theorem 2.4. This is a contradiction.

Case 2 : $[H : \mathrm{Fit}H] = +\infty$
Unlike the previous case, this situation will require the use of Proposition 3.2.
First of all, remember that G_{ab} is a finitely generated abelian group, hence by Corollary 2.10 its unipotent automorphism group $H/C_H(G_{ab})$ stabilizes a finite series inside it. This implies the existence of a positive integer s such that

$$\left[\frac{G}{A},_s \frac{H}{C_H(G_{ab})}\right] = 1 \quad \text{and so} \quad [G,_s H] \leq A.$$

Now, notice that the finitely generated metabelian group G is residually finite by Theorem 3.1; thus G and its fully-invariant abelian subgroup A satisfy the conditions of Proposition 3.2, which ensure us that A is a normal, right n-Engel and residually hypercentral subgroup of $A \rtimes H$.

Choose an arbitrary element $a \in A$ and consider the finitely generated group $\langle a, H\rangle$ seen as a subgroup of the semidirect product $A \rtimes H$.
Clearly, being contained in A, the normal closure $\langle a\rangle^{\langle a,H\rangle}$ consists of right n-Engel elements for $A \rtimes H$, and so it is a right n-Engel subgroup for $\langle a, H\rangle$.
Furthermore, $\langle a\rangle^{\langle a,H\rangle}$ is also residually hypercentral in $\langle a, H\rangle$, since

$$\bigcap_{i=1}^{+\infty} \left[\langle a\rangle^{\langle a,H\rangle},_i \langle a, H\rangle\right] \leq \bigcap_{i=1}^{+\infty} [A,_i A \rtimes H] = 1.$$

From Theorem 1.31 applied to $\langle a\rangle^{\langle a,H\rangle} \leq \langle a, H\rangle$ there exists $m \in \mathbb{N}$, only depending on n and the number of generators of H, such that

$$\langle a\rangle^{\langle a,H\rangle} \leq Z_m(\langle a, H\rangle),$$

which is equivalent to

$$[\langle a\rangle^{\langle a,H\rangle},_m \langle a, H\rangle] = 1.$$

Thanks to the fact that A abelian, the subgroup $[A,_m \langle a, H\rangle]$ is generated by all the subgroups of the form $[\langle a\rangle^{\langle a,H\rangle},_m \langle a, H\rangle]$ with $a \in A$ and, since m does not depend on the chosen element a, this leads to

$$[A,_m H] \leq [A,_m \langle a, H\rangle] = \prod_{a\in A}[\langle a\rangle^{\langle a,H\rangle},_m \langle a, H\rangle] = \prod_{a\in A} 1 = 1,$$

hence H stabilizes a finite series in A.
In conclusion, for the chosen non-nilpotent counterexample H we have the equality $[G,_{s+m} H] = 1$, so that the unipotent automorphism group H is the stability group of a finite series inside the whole group G. Invoking once again Theorem 2.4 it follows the nilpotency of H, the final contradiction. □

3.3 Finitely Generated Abelian-by-Polycyclic Groups

This section contains one of the main result of the thesis, and is dedicated to the investigation of the action of unipotent automorphisms upon finitely generated abelian-by-polycyclic groups. First of all, we prove the elementary closure properties of this class of groups which will be used in the following.

Lemma 3.5. *Subgroups and images of an abelian-by-polycyclic group G are still abelian-by-polycyclic.*

Proof. Let G be polycyclic-by-finite and A a normal abelian subgroup of G such that G/A is polycyclic.
If $H \leq G$ then $A \cap H$ is abelian and normal in H thus by the Isomorphism Theorem

$$\frac{H}{A \cap H} \cong \frac{AH}{A} \leq \frac{G}{A},$$

so that $H/A \cap H$ is polycyclic. Thus H is abelian-by-polycyclic.
Now if N is normal in G, we consider the subgroup $AN/N \trianglelefteq G/N$ which is abelian since it is isomorphic to an image of A, and by the Isomorphism Theorem

$$\frac{\frac{G}{N}}{\frac{AN}{N}} \cong \frac{G}{AN},$$

where the right side member is polycyclic since it is an image of the polycyclic group G/A. This implies that G/N is abelian-by-polycyclic. □

We now quote without proof a very deep result, which is an analogue of Theorem 1.24. For a proof we refer the reader to Chapter 7 of [7].

Theorem 3.6. *Let G be a finitely generated abelian-by-polycyclic-by-finite group. Then the Fitting subgroup of G coincides with the Hirsch-Plotkin radical and it is nilpotent.*

The main result of this chapter is a natural generalization of Theorem 3.4.

Theorem 3.7. *Let G be a finitely generated abelian-by-polycyclic group and H a finitely generated n-unipotent automorphism group of G. Then H is a nilpotent group and its nilpotency class is (n,t)-bounded, where t is the minimal number of generators for H. Furthermore H stabilizes a finite series in G, and $[G,H]$ is contained in the Fitting subgroup of G.*

Proof. Let G be a finitely generated abelian-by-polycyclic group of derived length d, and H a t-generated and n-unipotent subgroup of $\mathrm{Aut}(G)$.
First of all we deal with the nilpotence of H and, arguing by contradiction, assume the theorem false choosing a counterexample H which is not nilpotent. We shall use induction on d to find a contradiction.
Write the derived series of G

$$1 = G^{(d)} \lhd G^{(d-1)} \lhd \cdots \lhd G'' \lhd G' \lhd G$$

and, for shortness, put $A = G^{(d-1)}$ which is a fully-invariant (and in particular H-invariant) abelian subgroup of G.
The image G/A is a finitely generated abelian-by-polycyclic group of derived length $d-1$. Consider the natural homomorphism

$$\begin{array}{rccc} \Phi: & H & \to & \mathrm{Aut}(G/A) \\ & \phi & \mapsto & \overline{\phi} \end{array}$$

where the automorphism $\overline{\phi}$ is defined as $\overline{\phi}(gA) = \phi(g)A,\ \forall\, gA \in G/A$.
Let K be the kernel of Φ, which can be explicitely written as

$$K = \{\phi \in H \mid \phi(g)A = gA\} = \{\phi \in H \mid [g,\phi] \in A,\ \forall\, g \in G\};$$

the inductive hypothesis ensures us that H/K is a finitely generated nilpotent group.

By Theorem 3.1 the finitely generated abelian-by-polycyclic group G is residually finite, hence we can apply Proposition 3.2 to its characteristic subgroup A, which turns out to be a normal, residually hypercentral, n-Engel subgroup of the semidirect product $A \rtimes H$.

We need to study two different cases.

Case 1 : The group H/K is finite.
In this situation, among all the counterexamples we can choose H in such a way that $|H/K|$ is as small as possible.
Firstly notice that the quotient H/K cannot be the identity group.

Indeed, from the equality $H = K$ it would follows that $[G, H] \leq A$. Now, if we choose an element $a \in A$, since A is an abelian normal residually hypercentral and n-Engel subgroup of $A \rtimes H$, the same is true for $\langle a \rangle^{\langle a,H \rangle}$ seen as a subgroup of $\langle a, H \rangle$. Invoking Theorem 1.31 it follows that there exists an integer m, which does not depend on the choice of a, such that

$$[\langle a \rangle^{\langle a,H \rangle},_m H] \leq [\langle a \rangle^{\langle a,H \rangle},_m \langle a, H \rangle] = 1,$$

and consequentely

$$[A,_m H] = \prod_{a \in A} [\langle a \rangle^{\langle a,H \rangle},_m \langle a, H \rangle] = \prod_{a \in A} 1 = 1.$$

The hypothesis $H = K$ thus leads to $[G,_{m+1} H] = 1$. Thanks to Theorem 2.4, this proves that H is nilpotent, against our assumption.

For this reason K is a proper subgroup of H and, being nilpotent, the automorphism group H/K strictly contains its derived subgroup $H^{'}K/K$, hence

$$\left| \frac{H^{'}K}{K} \right| \ < \ \left| \frac{H}{K} \right| .$$

Since $H^{'}$ is a group of n-unipotent automorphisms of G, thanks to the minimality of $|H/K|$ we can assume that $H^{'}$ is locally nilpotent.

Case 2 : The group H/K is infinite.
In this situation we choose the counterexample H such that H/K has minimal Hirsch number. Then the Hirsch number of its derived subgroup $H^{'}K/K$ is strictly lower than that of H/K. In fact if they had the same Hirsch number, a condition equivalent to $[H/K : H^{'}K/K] < +\infty$, it would follow by Proposition 1.12 that the nilpotent group H/K is finite, a fact which is in contrast with our assumption.
Thus, thanks to the minimality of H, it is once again possible to suppose that $H^{'}$ is locally nilpotent.

We have just proved that without loss of generality we can always choose a counterexample H which possesses a locally nilpotent derived subgroup $H^{'}$.
Denoting with R the Hirsch-Plotkin Radical of H, it follows easily that $H^{'}$ is contained in R and consequently H/R is a finitely generated abelian group.
Repeating what we have done in Theorem 3.4, we can also choose H in such a way that the number of generators of H/R is minimal.
We claim that with this choice H/R is a cyclic group. In fact, if we suppose by contradiction that $\{x_i R\}$ is a generating system for H/R of minimal cardinality

consisting of $t \geq 2$ elements, then for every $i = 1 \dots t$, the subgroup $H_i = \langle x_i, R \rangle$ would be normal in H and locally nilpotent. Then, being the product of the subgroups H_i and due to Theorem 1.23, H would be necessarly locally nilpotent, which is a contradiction thanks to the fact that H is also finitely generated.
Consequently we can write $H = R\langle h \rangle$ for some $h \in H$.

If we suppose $[H : R] < +\infty$ then the Hirsch-Plotkin radical R of H is finitely generated and nilpotent, and then it coincides with the Fitting subgroup of H. Following the proof of Theorem 3.4, we discover that R stabilizes a finite series $\mathbf{S}$ in G, which is obtained refining the derived series of G. Thanks to Lemma 2.11 and to the fact that every section of $\mathbf{S}$ is abelian, we can further refine $\mathbf{S}$ to find a finite series which is stabilized by $H = R\langle h \rangle$. This proves that H is a nilpotent group by Theorem 2.4, which is a contradiction.

The above argument necessarily implies $[H : R] = +\infty$.
Once again, we remark that A is an abelian, normal and residually hypercentral n-Engel subgroup of $A \rtimes H$ thanks to Proposition 3.2, thus the same properties still hold for A seen as a subgroup of $A \rtimes R$. Thus by Theorem 1.33, there exist two integers $f = f(n)$ and $c = c(n)$, which both depend only on n, such that

$$[A^f,_c R] = 1.$$

Hence the Hirsch-Plotkin Radical R of H stabilizes a series of length c in A^f, the subgroup of A generated by the f-th powers of its elements.
Now, we check the action of R on the quotient group $\overline{A} = A/A^f$, and our aim is to prove that R stabilizes a finite series also in $\overline{A}$. We will use the additive notation for $\overline{A}$ and our analysis will be divided in two cases.

Case 1 : The exponent f is a power of a prime number, that is $f = p^m$ with p a prime and m a positive integer.
We want to use induction on m. If $m = 1$ then $\overline{A}$ is an abelian group of exponent p, that is a direct product of cyclic groups of order p and also a vector space over the field $\mathbb{Z}/p\mathbb{Z}$. The ring $\mathrm{End}(\overline{A})$ has characteristic p, because if $v \in \overline{A}$ and ϕ is an endomorphism of $\overline{A}$, then

$$p\phi(v) = \phi(pv) = \phi(0) = 0 \;\Rightarrow\; p\phi = 0.$$

Using this fact and setting $C = C_H(\overline{A})$, we prove easily that the group of unipotent automorphisms H/C of $\overline{A}$ has finite exponent. In fact, if we define k to be the least positive integer such that $p^k \geq n$, then for every $\phi \in H/C$ and for every $v \in \overline{A}$ we have

$$0 = [v,_n \phi] = (\phi - 1)^n(v) = (\phi - 1)^{p^k}(v) \quad \Rightarrow \quad (\phi - 1)^{p^k} = 0,$$

that is $(\phi - 1)^{p^k}$ is the zero endomorphism; thus using the Frobenius endomorphism we have

$$0 = (\phi - 1)^{p^k} = \phi^{p^k} - 1^{p^k} = \phi^{p^k} - 1 \quad \Rightarrow \quad \phi^{p^k} = 1.$$

The last equality proves that H/C has finite exponent bounded by p^k. Hence H/C is a finitely generated group of finite exponent.

Focus on the subgroup RC/C of H/C. Since $R \trianglelefteq RC \trianglelefteq H$ and H/R is an infinite cyclic group, we have that also H/RC is cyclic. Furthermore the image

$$\frac{H}{RC} \cong \frac{\frac{H}{C}}{\frac{RC}{C}}$$

has finite exponent because is a quotient of H/C.
In conclusion, H/RC is a cyclic group of finite exponent, that is a finite cyclic group. It is then clear that RC/C is finitely generated, since it is a subgroup of finite index of the finitely generated group H/C. Finally, the local nilpotence of R implies at once the nilpotence of its finitely generated quotient

$$\frac{RC}{C} \cong \frac{R}{R \cap C} .$$

We have just discovered that H/C is nilpotent-by-cyclic, and in particular it is soluble. Applying Proposition 1.6 to the finitely generated soluble group of finite exponent H/C, it follows that it is a finite automorphism group, and a fortiori the same is true for its subgroup RC/C.
Invoking Theorem 2.12, the unipotent action of the finite nilpotent group RC/C upon the abelian group $\overline{A}$ guarantees the existence of a finite series stabilized in $\overline{A}$ by RC/C.

Suppose $m > 1$, and consequently $f = p^m > p$.
Consider the fully-invariant subgroup $A^{p^{m-1}}$ of A. By the inductive hypothesis, being $A/A^{p^{m-1}}$ an abelian group of exponent p^{m-1}, we can suppose that there exists a finite series stabilized by R running from A to $A^{p^{m-1}}$. It remains only to check what happens on the section $A^{p^{m-1}}/A^{p^m}$ due to the unipotent action of R. But this section has exponent p by definition, and so by repeating the same techniques seen above, it follows easily the existence of a finite series stabilized by R running from $A^{p^{m-1}}$ to A^{p^m}.

Case 2 : The exponent f is not a power of a prime.
Write $f = p_1^{\alpha_1} \dots p_t^{\alpha_t}$, where the p_is are distinct prime numbers and α_is are positive integers. The abelian torsion group $\overline{A}$ can be decomposed into the direct sum of its p_i-components, which are characteristic in $\overline{A}$, and in particular R-invariant. By the

previous case we can state that R stabilizes a finite series in every p_i-component. Since these components are in a finite number, if we take the direct product of the terms of the t series stabilized, we obtain a finite series in $\overline{A}$ which is stabilized by R.

From the above analysis it follows that the Hirsch-Plotkin radical R stabilizes a finite series running from A to A^f.
We can repeat these arguments for every section of the derived series of the soluble group G, and thus it is possible to find a finite series with abelian factors inside G which is stabilized by R. With Theorem 2.4, this proves that the Hirsch-Plotkin radical of H is nilpotent and consequently it coincides with the Fitting subgroup.
Write $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_s = G$ for the series stabilized by R and remember that $H = R\langle h \rangle$.
For every $i = 0, \ldots, s-1$, we have the inclusion $R \leq C_i = C_H(G_{i+1}/G_i)$ and the immersion preserving the unipotent action

$$\frac{H}{C_i} = \frac{H}{C_H\left(\frac{G_{i+1}}{G_i}\right)} \hookrightarrow \operatorname{Aut}\left(\frac{G_{i+1}}{G_i}\right).$$

It is then clear that, for every integer i, the cyclic group H/C_i acts faithfully on the abelian section G_{i+1}/G_i as a n-unipotent group of automorphisms and so by Lemma 2.11, it stabilizes a finite series inside G_{i+1}/G_i. Thus it follows that we can further refine the series $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_s = G$ to find another one which is stabilized by H. Once again, Theorem 2.4 guarantees the nilpotence of H. This is the final contradiction.

Now that we have proved that the t-generated and n-unipotent automorphism group H is necessarily nilpotent, using Theorem 2.16 it follows immediately that its nilpotency class is bounded by a number depending only on n and t.
Furthermore, by Corollary 2.13 the finite series stabilized by H is a refinement of the derived series of G.

It only remains to prove the inclusion $[G, H] \leq \operatorname{Fit}(G)$.
Choose an element $h \in H$ and consider the semidirect product $G \rtimes \langle h \rangle$, which is soluble by extension. Clearly h is a left bounded Engel element of $G \rtimes \langle h \rangle$, hence by Theorem 1.26, it is contained in its Hirsch-Plotkin radical. It follows that $[G, h]$ lies inside the Hirsch-Plotkin radical of G which, by Theorem 3.6, is nilpotent and coincides with the Fitting subgroup of G. Therefore $[G, H] = \langle [G, h] \mid h \in H \rangle$ is contained in $\operatorname{Fit}(G)$. □

3.4 Finitely generated abelian-by-polycyclic-by-finite groups

Theorem 3.7 can be further generalized even for finitely generated abelian-by-polycyclic-by-finite groups.

Theorem 3.8. *Let G be a finitely generated abelian-by-polycyclic-by-finite group and H a finitely generated, n-unipotent group of automorphisms of G. Then H stabilizes a finite series in G and is therefore nilpotent.*

Proof. Let the group G be as requested and $Q \trianglelefteq G$ an abelian-by-polycyclic subgroup such that G/Q is finite. Since the index of Q in G is finite, Q is clearly finitely generated. Suppose also that $H = \langle h_1, h_2, \ldots, h_t \rangle$. Set

$$P = \bigcap_{i=1}^{t} Q^{h_i} \quad \text{and} \quad F = G/P,$$

so that P is still of finite index in G, being the intersection of a finite number of subgroups and, unlike Q, it is H-invariant.
The automorphism group $\overline{H} = H/C_H(F)$ acts faithfully and n-unipotently on F, thus by Corollary 2.9 there exists an integer m such that $[F,_m \overline{H}] = 1$ and consequentely $[G,_m H] \leq P$. Applying Theorem 3.7 to the finitely generated abelian-by-polyciclic group P, the finitely generated n-unipotent automorphism group $H/C_H(P)$ turns out to be nilpotent. Hence, by Corollary 2.13 we know that $H/C_H(P)$ stabilizes a finite series in P, say of length s. From this, the equality $[G,_{m+s} H] = 1$ follows easily, and the nilpotence of H is once again a consequence of Theorem 2.4. □

Unfortunately, in the situation of the previous Theorem, it is an open question whether it is possible to set a uniform bound for the nilpotency class of the unipotent group H. In fact, following the notation used in the proof, the finitely generated abelian-by-polycyclic-by-finite group G is not soluble and thus we are not able to apply Theorem 2.16; furthermore the finite image F of G could be arbitrarily large.

Chapter 4

Soluble Groups of Finite Rank

This chapter is dedicated to the study of unipotent automorphisms acting on soluble groups of finite rank. Since in Group Theory there exist many different definitions of rank, first of all we shall give a short account of such finiteness conditions for abelian groups, and then we will describe the main classes of soluble groups of finite rank.
Throughout the chapter, every abelian group will be written additively.

4.1 Finite Rank Conditions

Let A be an abelian group and $a_1, \ldots, a_k \in A$ distinct non zero elements. The a_is are said to be *linearly independent* if, whenever the relation

$$m_1 a_1 + m_2 a_2 + \cdots + m_k a_k = 0$$

holds with $m_i \in \mathbb{Z}$, it necessarily follows that $m_i a_i = 0$ for each $i = 1, \ldots, k$.
Given a subset S of A consisting only of linearly independent elements, thanks to Zorn's Lemma it is possibile to prove that S is contained in a linearly independent subset of maximal cardinality. Furthermore, instead of speaking of arbitrary linearly independent elements it is more useful to restrict our attention to elements of infinite order or elements of p-power order, for every prime number p.

Definition 4.1. *Let A be an abelian group.*
The 0-rank *or* torsion-free rank *of A is the cardinality of a maximal independent subset of elements of infinite order.*
If p is a prime, the p-rank *of A is the cardinality of a maximal independent subset of elements whose order is a power of p.*

We will write $r_0(A)$ and $r_p(A)$ to denote the torsion-free rank and the p-rank of the abelian group A, respectively.

As one may expects, these definitions do not depend on the choice of the maximal independent subset, as shown by the following Lemma.

Lemma 4.2. *If A is an abelian group, p is a prime and $\mathbb{F}_p$ is the field with p elements, then*

$$\begin{aligned} r_0(A) &= \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}} \mathbb{Q}), \\ r_p(A) &= \dim_{\mathbb{F}_p}(\mathrm{Hom}(\mathbb{F}_p, A)). \end{aligned}$$

In particular, any two maximal independent subsets of elements of infinite order or p-power order have the same cardinality.

Proof. Let S be a maximal independent subset of elements of infinite order. If T is the torsion subgroup of A, that is the set of all the elements of finite order, define $\overline{S} = \{s + T \mid s \in S\}$. This is an independent subset of A/T: in fact, if $\sum_i m_i(s_i + T) = 0_{\frac{A}{T}} = T$ with $s_i \in S$, then there exists a positive integer n such that

$$0 = n\left(\sum_i m_i s_i\right) = \sum_i m_i n s_i,$$

hence it follows that $m_i n s_i = 0$ and $m_i = 0$ for every i.
Suppose that $\overline{U}$ is an independent subset of A/T strictly containing $\overline{S}$. For every $u+T \in \overline{U}\setminus\overline{S}$, the subset $S \cup \{u\}$ is independent in A, contradicting the maximality of S. Thus $\overline{S}$ is a maximal independent subset of A/T, and $|S| = |\overline{S}|$. Hence, without loss of generality, to prove the first assertion we may assume $T = 0$.
Since A is torsion-free, the map

$$\begin{aligned} \theta : A &\rightarrow A \otimes_{\mathbb{Z}} \mathbb{Q} \\ a &\mapsto a \otimes 1 \end{aligned}$$

is a monomorphism. If $a \in A$, then $S \cup \{a\}$ is dependent and there exists a positive integer m such that $ma \in \langle S \rangle$. Thus, setting $\theta(S)$ for the image of S under θ, we have that $m(a \otimes 1) = ma \otimes 1 \in \langle s \otimes 1 \mid s \in S \rangle = \langle \theta(S) \rangle$, and the independent subset $\theta(S)$ generates $A \otimes_{\mathbb{Z}} \mathbb{Q}$ as a rational vector space. Thus

$$r_0(A) = |S| = |\theta(S)| = \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}} \mathbb{Q}).$$

Now, let $R \subseteq A$ be a maximal independent subset of elements of p-power order. By replacing each element of R with order larger than p with a suitable multiple, we obtain an independent subset R^* consisting of elements of order p, such that $|R^*| = |R|$. Consider the subgroup

$$A[p] = \{a \in A \mid pa = 0\} \supseteq R^*.$$

If $a \in A[p]$, then $R \cup \{a\}$ is not an independent subset and thus we can write $ma + \sum_i m_i r_i = 0$, where $r_i \in R$, $m_i \in \mathbb{Z}$ for every i, and m is a suitable integer such that $ma \neq 0$. Since $pa = 0$, it follows that

$$0 = p\left(ma + \sum_i m_i r_i\right) = pma + \sum_i pm_i r_i = \sum_i pm_i r_i,$$

and consequently $pm_i r_i = 0$ for every i, since R is independent. Hence $m_i r_i \in \langle R^* \rangle$ and $a \in \langle R^* \rangle$. This implies that $A[p] = \langle R^* \rangle$, proving that R^* is a basis for the $\mathbb{F}_p$-vector space $A[p]$. Finally,

$$r_p(A) = |R| = |R^*| = \dim_{\mathbb{F}_p} A[p] = \dim_{\mathbb{F}_p}(\mathrm{Hom}(\mathbb{F}_p, A[p])) = \dim_{\mathbb{F}_p}(\mathrm{Hom}(\mathbb{F}_p, A))$$

and the proof is complete. □

Definition 4.3. *The* total rank *of an abelian group A is defined as*

$$\overline{r}(A) = r_0(A) + \sum_{p \text{ prime}} r_p(A),$$

and the reduced rank *of A is*

$$\hat{r}(A) = r_0(A) + \max\{r_p(A) \mid p \textit{ is a prime}\}.$$

It is possible to fully describe the classes of abelian groups for which one of the numerical invariants $r_0(A), r_p(A), \overline{r}(A)$ or $\hat{r}(A)$ is finite. Remember that the torsion subgroup of an abelian group can be decomposed as the direct sum of its p-components, where p are prime numbers.

Theorem 4.4. *Let A be an abelian group, T its torsion subgroup and p a prime.*

1. *$r_0(A)$ is finite if and only if A/T is isomorphic with a subgroup of a finite dimensional vector space over the field $\mathbb{Q}$.*

2. *$r_p(A)$ is finite if and only if the p-component of A is the direct sum of a finite number of cyclic or quasicyclic groups.*

3. *$\overline{r}(A)$ is finite if and only if A is the direct sum of finitely many cyclic and quasicyclic groups and a torsion-free group of finite rank.*

4. *$\hat{r}(A)$ is finite if and only if the torsion subgroup T is the direct sum of a boundedly finite number of cyclic p-groups or p^∞-groups for each prime p, and A/T has finite torsion-free rank.*

For a proof of Theorem 4.4 we refer the reader to [7].

Definition 4.5. *A group has finite* Prüfer rank r *if* r *is the least positive integer such that every finitely generated subgroup can be generated by* r *elements.*

In an arbitrary group such an integer r may not exist, and in this case the Prüfer rank of the group is infinite.
However if G has finite Prüfer rank r, then every subgroup and quotient of G has Prüfer rank at most r. Morever, if G is a group with a normal subgroup N such that N and G/N have finite Prüfer ranks s and t respectively, then G has Prüfer rank at most $s+t$.

Proposition 4.6. *An abelian group has finite Prüfer rank if and only if it has finite reduced rank, and in this situation they coincide.*

Proof. Let A be an abelian group with Prüfer rank $r(A)$ and reduced rank $\hat{r}(A)$, respectively. Throughout the proof, the minimal number of generators for a group X will be denoted with $t(X)$, so that

$$r(A) = \max\{t(H) \mid H \leq A,\ H \text{ is finitely generated}\}.$$

Choose a finitely generated subgroup B of A. It is an easy matter to verify that $r(B) = \hat{r}(B)$, and thus we have

$$t(B) = r(B) = \hat{r}(B) \leq \hat{r}(A).$$

By the arbitrariety of B, it follows that $r(A) \leq \hat{r}(A)$.

On the other hand, assume by contradiction that $r(A) < \hat{r}(A)$.
Firstly suppose that A is torsion-free, so that $\hat{r}(A) = r_0(A)$. Choose a finitely generated subgroup B of A such that $t(B) = r(A)$. A minimal set of generators for B is linearly independent and, by our assumption, there exists at least an element $a \in A$ such that $B \cup \{a\}$ is still independent. Hence the subgroup $\langle B, a\rangle$ has a minimal number of generators strictly greater than B, against the maximality of B. Thus if A is torsion-free, we have that $r_0(A) \leq r(A)$.

Now, suppose that T is the torsion subgroup of A. If $\mathbb{P}$ is the set of prime numbers, we can write $T = \oplus_{p\in\mathbb{P}} T_p$, where T_p is the p-component of T. For each $p \in \mathbb{P}$, our assumption simply means that $r(T_p) < r_p(T_p)$, and arguing exactly as in the torsion-free case, once again we find a contradiction.
Thus

$$r_p(T_p) \leq r(T_p) \leq r(T) \quad \text{for every } p \in \mathbb{P},$$

and this implies that

$$\max\{r_p(T) \mid p \in \mathbb{P}\} = \max\{r_p(T_p) \mid p \in \mathbb{P}\} \leq r(T).$$

Hence, we have that

$$\hat{r}(A) = r_0(A/T) + \max\{r_p(T) \mid p \in \mathbb{P}\} \leq r(A/T) + r(T) = r(A),$$

and we are done. □

Finally, we are able to generalize these notions of rank also for soluble groups. The following definitions are taken from Chapter 5 of [7].

Definition 4.7. *A soluble group has* finite abelian total (reduced) rank *if it has an abelian series where each factor has finite total (reduced) rank .*

It is worth mentioning that the class of soluble groups with finite abelian total rank is subgroup and extension-closed, but unlike the one of soluble groups with finite abelian reduced rank, it is not closed by taking quotient. For example the abelian group $\mathbb{Q}$ has total rank 1, since it is torsion-free and $r_0(\mathbb{Q}) = 1$, but its torsion quotient $\mathbb{Q}/\mathbb{Z}$ is isomorphic to the direct sum of the p^∞-groups for every prime p, and hence has p-rank 1 for every p.
Moreover, it is not difficult to prove the following Proposition, which is the analogue of the abelian case, concerning the structure of soluble groups with finite abelian reduced rank.

Proposition 4.8. *A soluble group has finite abelian reduced rank if and only if it has finite Prüfer rank.*

This follows from the closure properties of these two classes of groups, invoking Proposition 4.6 as the first step of an inductive argument on the derived length.

Definition 4.9. *A group is called* minimax *if it has a series of finite length for which each factor satisfies the Max or Min condition.*

Obviously the Min condition is defined as the analogue of Max for descending chains of subgroups. The easiest example of a minimax group which does not satisfy itself Max or Min is given by

$$\mathbb{Q}_\pi = \left\{ \frac{a}{b} \in \mathbb{Q} \;\middle|\; \text{the prime divisors of } b \text{ are in } \pi \right\}$$

where π is a finite set of prime numbers.
This group does not satisfy the Max condition because it is not finitely generated, and neither has Min since it is torsion-free. Neverthless, $\mathbb{Q}_\pi$ is minimax since $\mathbb{Z} \lhd \mathbb{Q}_\pi$ obviously has Max while $\mathbb{Q}_\pi/\mathbb{Z}$ has Min. This fact can be easily generalized.

Proposition 4.10. *An abelian group A is a minimax group if and only if it is Max-by-Min.*

Proof. Clearly a Max-by-Min group is minimax by definition.
On the other hand, given a finite series in A whose factors satisfy either Max or Min, we can choose a finite set of generators from each factor with Max, and the preimages in A of all these elements generate a finitely generated abelian subgroup X of A. Then A/X is necessarily a Min group. □

Because of the classification of the abelian groups with Max and those ones with Min, it is possibile to choose the subgroup X in the previous proof in such a way that X is free abelian of finite rank, or eventually A/X is divisible with Min, that is a finite direct product of quasicyclic groups.

Clearly, every soluble minimax group is a soluble group of finite abelian total rank, but the converse is not true, being $\mathbb{Q}$ an easy counterexample.
In Group Theory, the notion of minimax group turns out to be important especially in the soluble case. The first thing to notice is that the class of soluble minimax groups is subgroup, quotient and extension-closed. In particular, this implies that every term of the derived series of a soluble minimax group is still a soluble minimax group, and every factor is an abelian Max-by-Min group.

4.2 Soluble Groups of Finite Prüfer Rank

In this section we will prove that a unipotent group of automorphisms of a soluble group with finite Prüfer rank is nilpotent. The difficulties to reach this achievement mainly rely on the abelian case, which requires a technical Lemma. Its proof can be found in [11] Lemma 7.44.

Lemma 4.11. *Let A be an abelian p-group with finite Prüfer rank r. Then the Sylow p-subgroups of* $\mathrm{Aut}(A)$ *are finite and have Prüfer rank at most $\frac{1}{2}r(5r-1)$.*

For the reader's convenience we shall include some relevant definitions.

Definition 4.12. *An abelian group A is called* divisible *if for each element a and for each positive integer m there exists $b \in A$ such that $a = mb$.
On the other hand, an abelian group is said to be* reduced *if it has no nontrivial divisible subgroups.*

The only thing we need for the main theorem of this section are the following proposition and the characterization of abelian divisible groups. They are quoted without proof (see [9]).

Proposition 4.13. *If A is an abelian group, there exists a unique largest divisible subgroup D of A. Moreover, $G = D \oplus E$ where E is a reduced group.*

Theorem 4.14. *An abelian group A is divisible if and only if it is a direct sum of isomorphic copies of $\mathbb{Q}$ and of quasicyclic groups.*

We are ready to prove the nilpotence of unipotent automorphism groups acting on abelian groups of finite Prüfer rank.

Theorem 4.15. *Let A be an abelian group with finite reduced rank $r = r(A)$ and H a n-unipotent group of automorphisms of A. Then H stabilizes a series in A whose length is bounded by a function $f(n,r)$, depending only on n and r. Consequentely H is nilpotent of class at most $f(n,r)-1$.*

Proof. Let A be an abelian group of finite reduced rank $r = r(A)$ and put

$$M = \max\{r_p(A) \mid p \in \mathbb{P}\} \leq r.$$

From a straightforward application of Theorem 4.4, it follows that the group A and its torsion subgroup T have a well known structure.
First of all, since $r_0(A)$ is finite, notice that the torsion-free quotient A/T embeds as a subgroup of a finite dimensional rational vector space, namely $V = A/T \otimes_{\mathbb{Z}} \mathbb{Q}$, and it is clear that $\dim_{\mathbb{Q}}(V) = r_0(A/T) = r_0(A)$.
It is possible to extend the action of $\overline{H} = H/C_H(A/T)$ to the whole vector space V, obtaining a subgroup $\widehat{H}$ of $\mathrm{Aut}(V)$ by setting

$$\hat{h}(aT \otimes q) = \overline{h}(aT) \otimes q, \quad \forall\, a \in A,\ \forall\, q \in \mathbb{Q} \text{ and } \forall\, \overline{h} \in \overline{H}.$$

The action of H on A is n-unipotent, and this condition is inherited by $\overline{H}$. Consequentely the same is still true for $\widehat{H}$, since

$$[aT \otimes q,_n \hat{h}] = [aT,_n \overline{h}] \otimes q = 0 \otimes q = 0, \quad \forall\, a \in A,\ \forall\, q \in \mathbb{Q} \text{ and } \forall\, \hat{h} \in \widehat{H}.$$

It follows that $\widehat{H}$ is isomorphic to a subgroup of $\mathrm{UT}(r_0(A), \mathbb{Q})$, the group of upper triangular matrices with rational entries and all the elements on the main diagonal equal to 1, and so it stabilizes in V a finite series of subspaces whose length is at most the dimension of V, that is $r_0(A)$.

Now we have to deal with the torsion subgroup T. Denoting with $\mathbb{P}$ the set of all the prime numbers, by Theorem 4.4, for every $p \in \mathbb{P}$ there exists a positive integer n_p such that $n_p \leq M$ and

$$T = \bigoplus_{p \in \mathbb{P}} \left[\left(\bigoplus_{i=1}^{n_p} C_{p^\infty} \right) \oplus F_p \right] \ ,$$

since every p-component of T is the direct sum of a bounded number of quasicyclic p-groups and a finite abelian p-group F_p.
For our purposes it is more convenient to write the previuos decomposition as

$$T = \left[\bigoplus_{p \in \mathbb{P}} \left(\overset{n_p}{\underset{i=1}{\oplus}} C_{p^\infty} \right) \right] \oplus R$$

where $R = \oplus_{p \in \mathbb{P}} F_p$, since this way it is clear that the first member of the sum is the largest divisible subgroup of T, in the sense of Proposition 4.13, while R is reduced. If we define

$$D = \bigoplus_{p \in \mathbb{P}} \left(\overset{n_p}{\underset{i=1}{\oplus}} C_{p^\infty} \right),$$

this is a characteristic subgroup of T, hence it is also characteristic in A.

Consider the image $T/D \cong R$. The abelian reduced group R is the direct sum of its p-components, and each of these is a finite abelian p-group of rank at most M. Put $\widetilde{H} = H/C_H(T/D)$ and $\widetilde{H}_{|p}$ for the restriction of $\widetilde{H}$ to the p-component R_p of R. As a consequence of Lemma 4.11 and Proposition 2.6, we have that for each prime p the restriction $\widehat{H}_{|p}$ is a finite p-group of Prüfer rank at most $\frac{1}{2}M(5M-1)$, and so the semidirect product $R_p \rtimes \widehat{H}_{|p}$ is a finite p-group of finite rank at most $M + \frac{1}{2}M(5M-1) = \frac{1}{2}M(5M+1)$ by extension. In particular, it is a finite nilpotent group.
Now, it is an easy matter to check that R_p is an abelian normal right n-Engel and residually hypercentral subgroup of the finitely generated group $R_p \rtimes \widehat{H}_{|p}$, thanks to Proposition 3.2. Thus by Theorem 1.31, there exists an integer $m = m(n, M)$, depending only on M and n, such that $[R_{p,m}\, R_p \rtimes \widehat{H}_{|p}] = 1$.
Since m does not depend on the prime p, we conclude that $[R,_m \widetilde{H}] = 1$, which implies that $[T,_m H] \leq D$.

It only remains to study the action of H on D. For each prime p, write D_p for the p-component of D, namely

$$D_p = \overset{n_p}{\underset{i=1}{\oplus}} C_{p^\infty}.$$

It is a well known fact that

$$\mathrm{Aut}(D_p) \cong \mathrm{GL}(n_p, \mathbb{Z}_p),$$

where the last one is the General Linear group of degree n_p over the ring of p-adic integers $\mathbb{Z}_p$.
The action of $H/C_H(D_p)$ on D_p is faithful and unipotent, so it is clear that

$H/C_H(D_p)$ embeds as a subgroup of $UT(n_p, \mathbb{Z}_p)$; hence it follows that $H/C_H(D_p)$ stabilizes a finite series running from D_p to 1 of length at most n_p.
Since this is true for every p-component, and since $n_p \leq M$ for every prime p, we have that $H/C_H(D)$ is the stability group of a finite series of subgroups of D whose length is at most M.

By what has been just proved above, setting

$$f = f(n, r) = r_0(A) + m(n, M) + M = m(n, M) + r$$

it follows that

$$[A,_f H] = 1$$

and so H stabilizes a finite series in A whose length depends only on n and the Prüfer rank of A.
Applying once again Theorem 2.4 it follows that H is nilpotent of class at most $f(n, r(A)) - 1$. □

At this point, the main Theorem of this section derives easily from the previous result and the fact that the class of soluble group with finite reduced rank is subgroup, quotient and extension closed.

Theorem 4.16. *If H is a n-unipotent automorphism group of a soluble group G of derived length d and finite Prüfer rank $r = r(G)$, then there exists a function $f = f(n, r, d)$ such that H stabilizes a finite series in G of length f and is therefore nilpotent of class at most $f - 1$.*
Furthermore, if H is also finitely generated and a minimal subset of generators has cardinality t, then its nilpotency class can be bounded by a number depending only on n and t.

Proof. Let G be a soluble group as required, and consider its derived series

$$1 = G^{(d)} \trianglelefteq G^{(d-1)} \trianglelefteq \cdots \trianglelefteq G^{'} \trianglelefteq G^{(0)} = G.$$

Once again, notice that every term of this series is fully-invariant and, in particular, characteristic in G. For this reason it is possible to consider its abelian factors and the action of H on each of them. For every integer $i = 1, \ldots, d$, the quotient

$$\overline{H}_i = \frac{H}{C_H\left(\frac{G^{(i-1)}}{G^{(i)}}\right)}$$

act as an n-unipotent group of automorphisms of G^{i-1}/G^i, which is abelian of finite reduced rank at most $r(G)$. Thus by Theorem 4.15 there exists an integer

$f_i = f_i(n, r(G))$ such that $\overline{H}_i$ stabilizes a finite series in $G^{(i-1)}/G^{(i)}$ of length at most f_i, that is

$$\left[\frac{G^{(i-1)}}{G^{(i)}},_{f_i} \overline{H}_i\right] = 1.$$

The last equality clearly implies the inclusion $[G^{(i-1)},_{f_i} H] \leq G^{(i)}$, thus if we repeat this argument for every integer $i = 1, \dots, d$ and if we set

$$f = f(n, r(G), d) = \sum_{i=1}^{d} f_i,$$

it follows at once that $[G,_f H] = 1$ and H is nilpotent of class at most $f - 1$ by Theorem 2.4.

Now suppose that H is generated by a finite subset consisting of t elements. Then H is a finitely generated, nilpotent, and n-unipotent group of automorphisms of the soluble group G, hence Theorem 2.16 can be invoked to find a bound for its nilpotency class depending only on n and t. □

Due to the description of the groups of finite rank, as a corollary we obtain

Corollary 4.17. *Let G be a soluble group with finite abelian total rank and suppose that H is a t-generated and n-unipotent automorphism group of G. Then H stabilizes a finite series in G and is nilpotent of class (n, t)-bounded.*
In particular this is true if G is a soluble minimax group.

4.3 Residually Soluble of Finite Rank Groups

Following the notation adopted in the first chapter, we will say that a group G is *residually soluble of finite Prüfer rank* if for every non trivial element $g \in G$ there exists a normal subgroup N_g such that $g \notin N_g$ and G/N_g is a soluble group of finite Prüfer rank.

Theorem 4.18. *Let G be a residually soluble of finite Prüfer rank group.*
If $H \leq Aut(G)$ is n-unipotent and t-generated automorphism group of G, then it is nilpotent of class (n, t)-bounded.

Proof. Let $\{N_i\}_{i \in I}$ be a family of normal subgroups of G such that $\bigcap_{i \in I} N_i = 1$ and each image G/N_i is a soluble group with finite Prüfer rank.
Notice that since $\bigcap_{i \in I} N_i = 1$, H embeds as a subgroups of $\mathrm{Car}_{i \in I}(G/N_i)$.

Thanks to Theorem 4.16, we know that there exists an integer $f = f(n,t)$, only depending on n and t, such that for each $i \in I$ the finitely generated n-unipotent group of automorphisms $H/C_H(G/N_i)$ is nilpotent of class at most f, that is

$$\gamma_f(H) \leq C_H\left(\frac{G}{N_i}\right) \qquad \text{for each positive integer } i.$$

However it is easy to see that

$$\bigcap_{i \in \mathbb{N}} C_H\left(\frac{G}{N_i}\right) = 1 \ .$$

In fact, by definition

$$h \in \bigcap_{i \in \mathbb{N}} C_H\left(\frac{G}{N_i}\right) \iff \forall\, g \in G \ \text{ we have } \ g^{-1}g^h \in N_i,\ \forall\, i \in \mathbb{N}$$

and the last condition is equivalent to $[G,h] \in \bigcap_{i \in \mathbb{N}} N_i = 1$, which implies $h = 1$. Thus $\gamma_f(H) = 1$ and H is nilpotent of class at most $f - 1 = f(n,t) - 1$. □

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