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Analysis and Control of the Hopf bifurcation

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Ai miei gentori e ai miei nonni
per tutto il sostegno.

Contents

Introduction

Studying the real world systems, it is a pretty common experience to observe situations where a steady constant regime turns into a periodic behaviour as a consequence of the modification of a system parameter. This phenomenon behaves as a continuous process, since at the beginning the oscillation has a small amplitude, while it grows wider with a further variation of the parameter.
The interest in this kind of dynamics is due to its generality, highlighted by the fact that it is common to a large variety of systems, such as civil engineering structures, ecologic communities and economic situations. This has produced a rich literature, starting from H. Poincaré, which was the first to study the onset of a stable periodic motion from a constant stable regime.
It is interesting to observe that the same mechanism that produces such a dynamics can be responsible of a dual situation, where an unstable periodic motion arises from an unstable steady regime. Although the latter behaviour can not be highlighted by the real life experience, because of its instability property, it is present as well.

The first rigorous proof of the general phenomenon is due to Andronov and Leontovich [1937], which studied the two dimensional problem. The extension to the case of dimension n, instead, was realized by Hopf [1942] and after these early publications a wide literature has been produced, for example by Marsden and McCracken [1976], Arnold [1983] and Guckenheimer and Holmes [1983], just to recall some famous authors. The phenomenon is named as the *Andronov-Leontovich-Hopf bifurcation*, even though it is commonly referred to as the Hopf bifurcation only.

One of the most interesting features of this phenomenon is that the related limit cycle can be completely characterized analytically. Unfortunately, the application of these results turns out to be complicated by a large amount of difficult computations. Even for low dimensional systems the analytical approach usually results unfeasible. However, when this situation does not happen, the exact mathematical results can be exploited in several interesting topics such as, in particular, the bifurcation control problem. To this regard, see the papers by Fu and Abed [1993] for the classical state space approach and by Di Marco et al. [2002] for the frequency one.

In this work, we will develop mathematically rigorous tools for the study of the Hopf bifurcation. We will be concerned with systems represented in the differential equation form. Then, exploiting the features of this class, we will design a computationally efficient method, which turns out to be more suitable for the analytical approach than the classic techniques. Finally, we will introduce some extensions to the theory to enlarge the class of the systems, which can be studied with our tools.

In particular, Chapter 1 will be devoted to the introduction of the main mathematical tools. Some sufficient conditions to transform different models into the differential equation form will be presented. Moreover, the Harmonic Balance technique for the study of the periodic solutions will be introduced as well. In Chapter 2 the classical state space approach to the Hopf bifurcation will be recalled and some specific tools for its analysis in the differential equation case will be derived. In Chapter 3 the frequency approach will be presented with emphasis on the Harmonic Balance technique. Starting from these results, we will develop some analytical tools, which turn out to be effective in the study of the Hopf bifurcation in differential equation systems. A criterion to state the nature of the bifurcation will be presented along with a procedure to define an approximation of the real limit cycle. In Chapter 4 these mathematical tools will be employed to approach the Hopf bifurcation control problem. Here, the main idea is suggested by the observation that such a phenomenon is local and that it can be completely disclosed just studying a proper truncation of the power development of the system. Exploiting this result, we will extend our technique to every system, that can be locally represented into the differential equation form. To provide a general framework for this approach, we will exploit the *controller normal form theory* developed by Kang and Krener [1992]. According to the authors, every state space system can be locally described by its normal form. Hence, we will check the conditions for the local transformation into the differential equation form just looking at the related normal form. In particular, we will provide some classes of normal forms, which can be studied employing the analytical tools, developed for the differential equation model. Finally, we will extend our theory to a larger class of systems exploiting a proper state feedback control law to improve the degrees of freedom of the local transformation into the differential equation form.

Notation

$\mathbb{R}$: real space;
$\mathbb{C}$: complex space;
$\mathbb{N}$: natural numbers set;
$\mathbb{Z}$: integer numbers set;
$\jmath$: imaginary unit;
$\Re[x]$: real part of $x \in \mathbb{C}$;
$\Im[x]$: imaginary part of $x \in \mathbb{C}$;
$\mathcal{D}$: derivative operator;
^{T}: transpose operator;
adj: adjoint operator;
vec: vectorization operator;
$f^{[k]}(x)$: homogeneous function of order k in x.

Chapter 1

The models

The development of mathematical models is a crucial step in the behaviour analysis of real-world systems. Indeed, also the tools that we may employ for the system analysis strictly depend on the adopted model. In the following we will introduce some different mathematical representations stressing the relations among them and studying the conditions, which make possible the transformation from a model into an other.

1.1 The State Equation Model

A large variety of continuous-time autonomous systems admits the following representation:

$$\dot{x} = F(x) \ , \tag{1.1}$$

where $x \in \mathbb{R}^n$ is the *state* of the system and $F : \mathbb{R}^n \to \mathbb{R}^n$ a possibly nonlinear function. The form (1.1) is known as the *state space equation model* for a n-th dimensional system. It is a pretty general representation and many real and artificial processes may be described according to this model. Nonetheless, the analysis of these systems is deeply affected by the nature of the law F and by its complexity. In particular, for our purposes it is worth to introduce the equivalent form

$$\dot{x} = Ax + f(x) \ , \tag{1.2}$$

where $A \in \mathbb{R}^{n \times n}$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ are respectively the linear and the pure nonlinear component of the function F, so to highlight the different degree of complexity of the single parts of the map F. It is worth to remark, that on one hand the state space model is able to describe a wide range of possibly nonlinear systems, but on the other only a small set of mathematical tools are available for the analysis of (1.1).

1.2 The Differential Equation Model

Consider the equation:

$$y^{(n)} + G\left(y^{(n-1)}, \ldots, \dot{y}, y\right) = 0 \; , \tag{1.3}$$

where $y \in \mathbb{R}$ is a scalar signal and $G : \mathbb{R}^n \to \mathbb{R}$ is a possibly nonlinear function. In literature this is referred as the *ordinary differential equation form* for a n-th dimensional system. Since in the following we will devote a particular attention to this model, let us introduce some results which will be extensively used in its analysis.
Let us first derive an equivalent form of (1.3) by separating the linear component of the function G from the nonlinear one. Then, we derive the expression:

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1}\dot{y} + a_n y + g\left(y^{(n-1)}, \ldots, \dot{y}, y\right) = 0 \; , \tag{1.4}$$

where $a_k \in \mathbb{R}$, $k = 1, \ldots, n$, and $g : \mathbb{R}^n \to \mathbb{R}$ is a pure nonlinear map. It is straightforward to check that the introduction of the derivative operator $\mathcal{D}$ in this equation leads to

$$\left(\mathcal{D}^n + a_1\mathcal{D}^{n-1} + \ldots + a_{n-1}\mathcal{D} + a_n\right) y(t) + g\left(\mathcal{D}^{n-1}y, \ldots, \mathcal{D}y, y\right) = 0 \; . \tag{1.5}$$

Moreover, the definition of the operators

$$\begin{aligned} L(\mathcal{D}) &\doteq \frac{1}{\mathcal{D}^n + a_1\mathcal{D}^{n-1} + \ldots + a_{n-1}\mathcal{D} + a_n} \; , \\ \mathcal{N} \circ y &\doteq -g\left(\mathcal{D}^{n-1}y, \ldots, \mathcal{D}y, y\right) \end{aligned} \tag{1.6}$$

transforms the equation (1.5) into the compact and evocative form

$$y(t) = L(\mathcal{D})\left(\mathcal{N} \circ y\right)(t) \; , \tag{1.7}$$

which is an alternate description of the original model (1.3). In particular, observe that the nonlinearity can be rewritten as:

$$(\mathcal{N} \circ y)(t) = \mathcal{M}(\mathcal{D}) \left(\tilde{\mathcal{N}} \circ y\right)(t) \ ,$$

where

$$\mathcal{M}(\mathcal{D}) \doteq b_1 \mathcal{D}^{n-1} + \ldots + b_{n-1}\mathcal{D} + b_n$$

is a polynomial in $\mathcal{D}$, with $b_i \in \mathbb{R}$ $i = 1, \ldots, n$, and $\tilde{\mathcal{N}}$ a proper nonlinear operator. Thus, the representation (1.7) can be manipulated so to assume the form:

$$y(t) = \tilde{L}(\mathcal{D}) \left(\tilde{\mathcal{N}} \circ y\right)(t) \ ,$$

where $\tilde{L}(\mathcal{D})$ is the rational function:

$$\tilde{L}(\mathcal{D}) = \frac{b_1 \mathcal{D}^{n-1} + \ldots + b_{n-1}\mathcal{D} + b_n}{\mathcal{D}^n + a_1 \mathcal{D}^{n-1} + \ldots + a_{n-1}\mathcal{D} + a_n} \ .$$

Therefore, this result shows that in general the model (1.7) admits linear operators $L(\mathcal{D})$ described by rational functions with numerators of the proper order in $\mathcal{D}$.

1.3 The Block Diagram Representation

In the previous models the description focused on the mathematical properties of the system. On the contrary, when the modeling process can take advantage of the knowledge of the general structure of the system, the *block diagram representation* turns out to be particularly effective. This kind of description is especially used in the engineering field and it is based on the decomposition of the system in several interconnected operators, or “blocks”, each of them representing an input-output process.

For our purposes, let us introduce the *feedback interconnection model.* The basic structure of this representation is composed of two operators, each modeling a different process, connected in a “feedback loop”, so that the input of the one is the the output of the other. To depict the corresponding model,

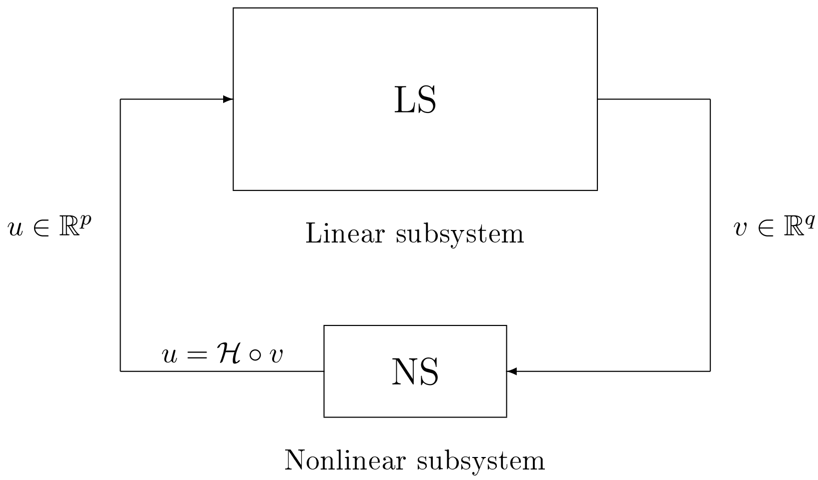

Figure 1.1: the general feedback scheme.

let us suppose that the first block is a linear model (LS) and that the second is a nonlinear subsystem (NS), according to the following mathematical descriptions:

$$\begin{aligned} \text{LS}\ &: \quad \begin{cases} \dot{x} = Ax + Bu \\ v = Cx \end{cases} \quad , \\ \text{NS}\ &: \quad u = \mathcal{H} \circ v \ , \end{aligned} \tag{1.8}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$ and $C \in \mathbb{R}^{q \times n}$ are constant matrices, while $\mathcal{H} : \mathbb{R}^q \to \mathbb{R}^p$ is a general nonlinear functional operator. Observe, that the model (1.8) can be interpreted as the feedback interconnection of Fig. 1.1 just exploiting the input-output description of LS and NS. It is worth to observe that both the state space model and the differential equation form can be suitably described as feedback systems. Indeed, consider the equation (1.2) and define:

$$\begin{aligned} v &\doteq x \ , \\ \mathcal{H} \circ v &\doteq f(v) \ , \\ B &\doteq I \ . \end{aligned} \tag{1.9}$$

Then, it is straightforward to check that (1.2) boils down to (1.8). Let us stress that the dimensions p and q depend on the nature of the function f, but in the general case the feedback interconnection scheme of the system

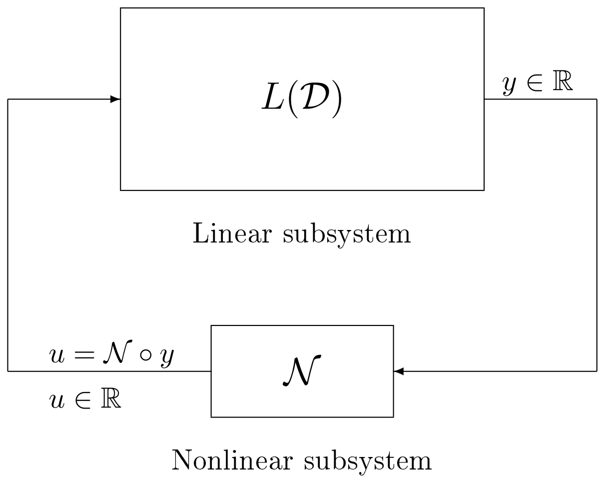

Figure 1.2: the feedback representation for a differential equation system.

(1.2) requires $p = q = n$, as it follows from the equations (1.9).
Then, consider the differential equation system (1.4). Making use of the alternate form (1.7), we can derive a feedback block diagram interpretation just choosing:

$$v \doteq y \ ,$$
$$\mathcal{H} \doteq \mathcal{N} \ ,$$

and using the operator $L(\mathcal{D})$ defined in (1.6) to model the linear subsystem LS. Consequently, in this case $p = q = 1$, as it is stressed in the corresponding Fig. 1.2. Therefore, while a general state space representation leads to a block diagram with a loop made of multidimensional connections, the differential equation form gives rise to a feedback scheme with only scalar signals. This feature is particularly interesting and it can be exploited to develop simplified mathematical tools specific for the differential equation form. For instance, if a certain procedure can be manipulated so to be applied only to the quantity v or u, such a technique will gain an advantage when such a signal is scalar.

It is worth to observe that the block diagram representation can describe dynamical systems, which do not admit the state space model. Indeed, it is known that the time-delayed models and the distributed parameter systems need an infinite-dimensional state. However, it turns out that they can be modeled in the feedback scheme just employing a description similar to (1.7), where $L(\mathcal{D})$ is substituted by a proper transcendental operator.

1.4 Model transformations

The classes represented by the two models (1.1) and (1.3) are closely related and it is known that the state space form is more general than the differential equation one [Isidori, 1995; Khalil, 2002; Vidyasagar, 1993]. In particular, it is straightforward to observe that a system described as (1.3) can be always put in the form (1.1). To this aim, consider the equation (1.4) and chose the so-called "phase coordinates" as the new set of variables, that is define $x_1 \doteq y$, $x_2 \doteq \dot{y}$, ... , $x_n \doteq y^{(n-1)}$. Thus, the system equation boils down to:

$$\begin{cases} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_n x_1 - \ldots - a_1 x_n - g(x_1, \ldots, x_n) \ , \end{cases} \tag{1.10}$$

which is indeed a state space model. This example makes immediately clear that the equation (1.1) defines a wider class of systems and illustrates a general transformation to change the differential equation form into the state space representation.

The inverse transformation from (1.1) to (1.3), instead, can not be always performed and in general it is a formidable problem even to check if such a transformation exists. However, making use of the differential geometry, a set of necessary and sufficient conditions can be formulated [Isidori, 1995; Nijmeijer and Mareels, 1997]. Unfortunately, their complexity is such that these relations can be checked only numerically.

In the following, on the contrary, we will introduce some simplified results about the transformation from the state space model to the differential equation form. In particular, since we are interested in finding analytical results, we will develop some sufficient conditions, which are easy to check. Moreover, they will turn out to be defined directly by the structure of the nonlinear part of the function F in (1.1), that is by the function f of the equation (1.2).

For a general differential equation system, the following statement holds.

Proposition 1. *The system* (1.2) *can be transformed into the differential equation form* (1.3) *if at least one of the following conditions is satisfied:*

1)

$$f(x) = \Phi(y) \ , \quad y = C^T x \tag{1.11}$$
$$C \in \mathbb{R}^n \ , \quad \Phi : \mathbb{R} \to \mathbb{R}^n \ ;$$

2)

$$f(x) = H\phi(x) \tag{1.12}$$
$$H \in \mathbb{R}^n \ , \quad \phi : \mathbb{R}^n \to \mathbb{R} \ .$$

Proof. Let us briefly report an outline of the proof. Similarly to the procedure followed to derive the (1.7) from the (1.4), the derivative operator $\mathcal{D}$ will be used to obtain an alternate formulation of the original state space model. Then, this representation will be manipulated till it assumes a form equivalent to (1.5), which can be interpreted as a differential equation (1.4).

Consider the first condition (1.11) and define

$$y \doteq C^T x \ .$$

Making use of the derivative operator, the system model (1.2) can be expressed as:

$$\mathcal{D}x(t) = Ax(t) + \Phi(y(t)) \ .$$

In turn, this equation can be rewritten as:

$$(\mathcal{D}I - A)\, x(t) = \Phi(y(t))$$

and it is straightforward to find that:

$$y(t) = C^T \left(\mathcal{D}I - A\right)^{-1} \Phi(y(t)) = \frac{1}{\det\left(\mathcal{D}I - A\right)} C^T \mathrm{adj}\ \left(\mathcal{D}I - A\right) \Phi(y(t)) \ .$$

Thus, if we consider

$$\det\left(\mathcal{D}I - A\right) y(t) = C^T \mathrm{adj}\ \left(\mathcal{D}I - A\right) \Phi(y(t)) \ ,$$

we obtain exactly an expression equivalent to (1.4), where the terms a_i are the coefficients of the characteristic polynomial of A and g is a polynomial combination of the time derivatives of Φ up to the order $(n-1)$.

Consider now the second condition (1.12). Proceeding as before, the original system equation can be rewritten by means of the derivative operator in the following form:

$$x(t) = (\mathcal{D}I - A)^{-1} H\phi(x(t)) = \frac{1}{\det(\mathcal{D}I - A)}\text{adj }(\mathcal{D}I - A)\, H\phi(x(t)) \ . \tag{1.13}$$

Let us define the quantity $V \in \mathbb{R}^n$ such that

$$V \doteq \frac{1}{\|\text{adj }(\mathcal{D}I - A)\, H\|}\text{adj }(\mathcal{D}I - A)\, H \ .$$

Therefore, it follows that:

$$y(t) \doteq V^T x(t) = \frac{1}{\det(\mathcal{D}I - A)}\phi(x(t))$$

and so we obtain that:

$$\phi(x(t)) = \det(\mathcal{D}I - A)\, y(t) \ . \tag{1.14}$$

Hence, from (1.13) we can rewrite the relation between x and y as:

$$x(t) = (\mathcal{D}I - A)^{-1} H\phi(x(t)) = \text{adj }(\mathcal{D}I - A)\, Hy(t) \ .$$

Resolving the derivative operator in the latter, we obtain the equivalent expression:

$$x(t) = \text{adj }(\mathcal{D}I - A)\, Hy(t) \doteq \tilde{G}\left(y^{(n-1)}, \ldots, \dot{y}, y\right) \ , \tag{1.15}$$

being $\tilde{G}$ a polynomial combination of y and its derivatives up to the $(n-1)$ order. Then, from (1.14) we derive the representation of the original system in the unique variable y:

$$\det(\mathcal{D}I - A)\, y(t) = \phi\left(\tilde{G}(t)\right) \ , \tag{1.16}$$

which leads to the differential equation form:

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1}\dot{y} + a_n y - \phi\left(\tilde{G}\left(y^{(n-1)}, \ldots, \dot{y}, y\right)\right) = 0 \ .$$

□

Example 1. *Consider the Hindmarsh-Rose neuron model [Hindmarsh and Rose, 1984]:*

$$\begin{cases} \dot{x}_1 = -a x_1^3 + b x_1^2 + x_2 - x_3 + I \\ \dot{x}_2 = c - d x_1^2 - x_2 \\ \dot{x}_3 = r\left(k\left(x_1 - x_0\right) - x_3\right) \end{cases}, \tag{1.17}$$

where the parameters a, b, c, d and k are fixed according to biological considerations, r and x_0 depend on the fast and slow subsystems and I represents the external current, which is supposed to be a constant input signal. It is straightforward to note that (1.17) *satisfies the condition* (1.11) *with:*

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 0 \\ rk & 0 & -r \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \Phi(y) = \begin{bmatrix} -a y^3 + b y^2 + I \\ c - d y^2 \\ -r k x_0 \end{bmatrix},$$

where the nonlinear function f include also the two constant terms c and I. Then, according to the mainline of the proof of the Prop. 1, the system can be represented as:

$$\begin{aligned} &\left(\mathcal{D}^3 + (1+r)\mathcal{D}^2 + r(1+k)\mathcal{D} + rk\right) y(t) = \\ &\quad = \begin{bmatrix} \mathcal{D}^2 + (1+r)\mathcal{D} + r \\ \mathcal{D} + r \\ -(\mathcal{D}+1) \end{bmatrix}^T \Phi(y(t)) \ . \end{aligned}$$

Hence, the equivalent differential equation system is:

$$\begin{aligned} &\dddot{y} + (1+r)\ddot{y} + r(1+k)\dot{y} + rky = \\ &\quad = \left[\mathcal{D}^2 + (1+r)\mathcal{D} + r\right]\left(-a y^3 + b y^2\right) + (\mathcal{D}+r)\left(-d y^2\right) + rI + rc + p x_0 \ . \end{aligned}$$

Example 2. *Consider the famous Rossler system*

$$\begin{cases} \dot{x}_1 = -x_2 - x_3 \\ \dot{x}_2 = x_1 + a x_2 \\ \dot{x}_3 = b + x_3(x_1 - c) \end{cases},$$

where a, b and c are positive parameters [Rössler, 1976]. One can check that the system satisfies the second sufficient condition (1.12) *with:*

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ 0 & 0 & -c \end{bmatrix}, \quad H = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \phi(x) = b + x_1 x_3 \ ,$$

where ϕ contains also a constant term. According to (1.15) *let us compute the quantity:*

$$\text{adj}\,(\mathcal{D}I - A)\,Hy(t) = \begin{bmatrix} a - \mathcal{D} \\ -1 \\ \mathcal{D}^2 - a\mathcal{D} + 1 \end{bmatrix} y(t) = \tilde{G}\left(y^{(n)}, \ldots, \dot{y}, y\right) \ .$$

Therefore, from the relation (1.16) *we derive the equivalent system in the differential equation form:*

$$\dddot{y} + (c - a)\ddot{y} + (1 - ac)\dot{y} + cy - b - (ay - \dot{y})\,(\ddot{y} - a\dot{y} + y) = 0 \ .$$

Remark 2. *When both condition* (1.11) *and* (1.12) *are satisfied, the system assumes the form*

$$\dot{x} = Ax + D\phi\left(C^T x\right) \ , \tag{1.18}$$

which is known in the engineering field with the name of Lur'e system.

Proposition 3. *Every system in the n-th order ordinary differential equation form admits the Lur'e representation and vice versa.*

Proof. The proof follows directly from the Prop. 1 and from the observation, that the state space representation with the phase coordinates (1.10) satisfies (1.18). □

In the following we will focus our attention on the differential equation system class. As observed before, this kind of description is more restrictive that the state space representation. However, the interest for this form is justified by the existence of a large variety of processes, which admit this model, as the Lur'e systems. Moreover, the analysis of the differential equation form can be performed with more effective mathematical tools, just exploiting the simplified internal connections which have been remarked in Fig. 1.2 with respect to the general scheme of Fig. 1.1. In the following we will introduce an important technique for the study of the limit cycles, which takes advantage from the above reasoning.

1.5 The Harmonic Balance technique

The Harmonic Balance (HB) technique has been developed to study the periodic solutions of a system. It is a quite general method and it can be applied to a large variety of models. However, in the following we will introduce that theory for system represented in the differential equation form (1.4), because in that case the HB technique results particularly effective. The starting point of this method is the Fourier series. Hence, let us briefly recall the main points of this theory.

Let us consider a complex valued periodic function $z : \mathbb{R} \to \mathbb{C}$ and let be T its period. Moreover, suppose that the following regularity properties, called "Dirichlet conditions", hold:

- z has a finite number of discontinuities in a single period;
- z has a finite number of maxima and minima inside a single period;
- $z \in L^1(0,T)$, that is

$$\int_0^T |z(t)|\, dt < \infty \ .$$

Then, the periodic function z can be developed in the Fourier series as follows:

$$z(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega t) + b_n \sin(n\omega t)\right) = \sum_{n=-\infty}^{+\infty} c_n e^{\jmath n\omega t} \ , \tag{1.19}$$

being $\omega \doteq 2\pi/T$ the *fundamental harmonic* and $(n\omega)$ the higher harmonics. The coefficients in (1.19) are defined as:

$$\begin{aligned} a_n &= \frac{1}{T}\int_0^{2T} z(\tau)\cos(n\omega\tau)\, d\tau \qquad &, b_n &= \frac{1}{T}\int_0^{2T} z(\tau)\sin(n\omega\tau)\, d\tau \ , \\ a_0 &= \frac{1}{2T}\int_0^{2T} z(\tau) d\tau &, c_n &= \frac{1}{T}\int_0^{2T} z(\tau) e^{-\jmath n\omega\tau} d\tau \ . \end{aligned}$$

In particular, if z is a real valued function, that is $z(t) \in \mathbb{R}\ \forall t \in \mathbb{R}$, the coefficients a_0, a_n and b_n are real and the following relations hold:

$$\begin{cases} a_0 = c_0 & n = 0 \\ \frac{1}{2}\left(a_n + \jmath b_n\right) = c_n & n \in \mathbb{N} \\ \frac{1}{2}\left(a_n - \jmath b_n\right) = c_{-n} & n \in \mathbb{N} \ , \end{cases}$$

In such a case c_{-n} is the complex conjugate of c_n and from (1.19) we can write:

$$z(t) = \sum_{n=0}^{+\infty} \left(c_n e^{\jmath n \omega t} + c_{-n} e^{-\jmath n \omega t} \right) = \sum_{n=0}^{+\infty} 2\Re \left[c_n e^{\jmath n \omega t} \right] \ . \tag{1.20}$$

For our purposes, it is worth to recall some important features of the Fourier series. In particular, the regularity of z affects the convergence properties of the series and thus, if z is sufficiently smooth, the derivative of z can be developed in a Fourier series as well. Moreover, let us recall that the set $\{e^{\jmath n \omega t}\}_{n\in\mathbb{Z}}$ can be interpreted as an orthonormal basis for the space $L^2(0,T)$ of all the square-integrable periodic functions. Therefore, if z is sufficiently smooth, it is univocally determined by the coefficients of its Fourier series.

Now we can briefly introduce the main outline of the HB technique.
Let us consider the differential equation system (1.4) and its alternative formulation (1.5), based on the derivative operator $\mathcal{D}$. Then, suppose that it admits a sufficiently smooth periodic solution y_p of period T, whose Fourier series is

$$y_p(t) = \sum_{n=-\infty}^{+\infty} \alpha_n e^{\jmath \omega n t} \ . \tag{1.21}$$

The series (1.21) is still a solution of the system equation and so, it can be substituted in (1.5). Then, consider the nonlinear part of the problem. Because all the derivatives of y_p are periodic functions, if g is a sufficiently smooth nonlinearity, its output is periodic and in turn can be developed in a Fourier series:

$$g\left(\mathcal{D}^{n-1} y_p, \ldots, \mathcal{D} y_p, y_p\right) = \sum_{n=-\infty}^{+\infty} \beta_n e^{\jmath \omega n t} \ ,$$

where the coefficients β_n depend on y_p, that is:

$$\beta_n \left(\omega, \alpha_0, \alpha_{\pm 1}, \alpha_{\pm 2}, \ldots\right) \ .$$

Hence, (1.5) becomes:

$$\left(\mathcal{D}^n + a_1 \mathcal{D}^{n-1} + \ldots + a_{n-1}\mathcal{D} + a_n\right) \sum_{n=-\infty}^{+\infty} \alpha_n e^{\jmath \omega n t} + \sum_{n=-\infty}^{+\infty} \beta_n e^{\jmath \omega n t} = 0 \ .$$

Under the appropriate regularity conditions, we can apply the derivative operators and so, balancing the coefficients of each harmonic, the original system boils down to the equivalent representation:

$$\begin{cases} ((\jmath n\omega)^n + a_1(\jmath n\omega)^{n-1} + \ldots + a_{n-1}(\jmath n\omega) + a_n)\,\alpha_n = \\ \qquad = -\beta_n\,(\omega, \alpha_0, \alpha_{\pm 1}, \alpha_{\pm 2}, \ldots) \\ n \in \mathbb{Z}\ . \end{cases} \tag{1.22}$$

It is important to stress that in the real valued case, the equation related to $n = -k$ is the complex conjugate of the k-th one and thus the system reduces only to the equations defined for $n \in \mathbb{N}$. Hence, finding the periodic solution y_p is equivalent to solve the infinite set of algebraic equations (1.22).

The HB approach relies on the intuitive idea that a periodic solution can be suitably approximated by a finite number of harmonics and that the system (1.4) acts as a "low-pass filter", as usually happens when the model describes a physical process.

For the sake of simplicity, let us now consider the real valued problem. The HB technique provides for $y_p \in \mathbb{R}$ an approximation of the form:

$$y_p(t) \approx \tilde{y}_p(t) \doteq \sum_{n=-k}^{k} \tilde{\alpha}_n e^{\jmath\tilde{\omega}nt} = \sum_{n=0}^{k} \Re\left[\tilde{\alpha}_n e^{\jmath\tilde{\omega}nt}\right]\ , \tag{1.23}$$

where $\tilde{\omega}$ and the coefficients $\tilde{\alpha}_n$ are the solution of the finite-dimensional approximation of the system (1.22). This is obtained just supposing null all the harmonics higher than the k-th, that is:

$$\begin{cases} ((\jmath n\tilde{\omega})^n + a_1(\jmath n\tilde{\omega})^{n-1} + \ldots + a_{n-1}(\jmath n\tilde{\omega}) + a_n)\,\tilde{\alpha}_n = \\ \qquad = -\beta_n\,(\tilde{\omega}, \tilde{\alpha}_0, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k, 0, \ldots) \\ n = 0, 1, \ldots, k\ , \end{cases} \tag{1.24}$$

where we have exploited that $\tilde{\alpha}_n$ and $\tilde{\alpha}_{-n}$ are complex conjugate as well as β_n and β_{-n}. Since the time origin is arbitrary, we can choose it so to set $\Im[\tilde{\alpha}_1] = 0$. Consequently, it is straightforward to observe that (1.24) is a system of $2k+1$ algebraic equations in the $2k+1$ unknowns $\tilde{\omega}$, $\tilde{\alpha}_0$, $\Re[\tilde{\alpha}_1]$, $\Re[\tilde{\alpha}_n]$, $\Im[\tilde{\alpha}_n]$, for $n = 2, \ldots, k$. Such a problem is known as the *k-th order Harmonic Balance*.

Of course, (1.23) is expected to be a better approximation of y_p, when higher is the number k of the considered harmonics. In particular, rigorous arguments can be exploited to state the reliability of this result and to quantify the approximation error. However, in general (1.24) can be solved only numerically, with the remarkable exception of the first and second order HB approach, where often that solution can be analytically computed. In particular, the first order HB problem is known in the engineering field as the *Describing Function Method* and its empirical evidence along the years confirms the significant power of this approach. Moreover, it is worth to observe that the HB tool has been widely and effectively exploited in several situations and we want to highlight its employment in the control problems [Tesi et al., 1996].

As a final remark, we want to stress that the HB approach can be extended also to system of the form (1.2), which allows for the graphical description of Fig. 1.1. In such a case, however, one must consider multiple Fourier series, nominally one for each component of x. Hence, it is clear that, balancing the harmonics, one obtains a number of algebraic equations, which grows too rapidly with the system dimension to be analytically handled.

1.6 The second order HB problem

In this section, we want to derive a different form of the second order HB problem, taking advantage from the feedback block diagram interpretation of Fig. 1.1
Consider the system (1.4) and its feedback formulation (1.7), depicted in Fig. 1.2. According to the above general theory and to the equation (1.23), consider the following second order HB approximation of period $T = 2\pi/\omega$:

$$\begin{aligned}\tilde{y}_p(t) &= A + B\cos(\omega t) + P\cos(2\omega t) + Q\sin(2\omega t) = \\ &= \Re\left[A + Be^{\jmath\omega t} + (P - \jmath Q)\, e^{\jmath 2\omega t}\right] = \\ &= \Re[\hat{y}_p(t)]\ ,\end{aligned} \tag{1.25}$$

where A, B, P and Q are real values and the time origin has been chosen, without loss of generality, so to have $B > 0$ and no $\sin(\omega t)$ component. Therefore, consider the output of the feedback nonlinearity when the operator $\mathcal{N}$ is driven by $\tilde{y}_p$. Since that output is still a periodic function, if the original function g is sufficiently smooth, we can assume that it can be developed in a Fourier series as well:

$$\left(\mathcal{N} \circ \tilde{y}_p\right)(t) = \sum_{n=0}^{+\infty} 2\Re\left[\tilde{\beta}_n e^{\jmath n\omega t}\right] . \tag{1.26}$$

Then, let us introduce the following quantities:

$$\hat{z}_p(t) = \sum_{n=0}^{2} 2\tilde{\beta}_n e^{\jmath n\omega t} , \tag{1.27}$$

$$\Delta z(t) = \sum_{n=3}^{+\infty} 2\Re\left[\tilde{\beta}_n e^{\jmath n\omega t}\right] ,$$

so that (1.26) can be represented equivalently as:

$$\left(\mathcal{N} \circ \tilde{y}_p\right)(t) = \Re\left[\hat{z}_p(t)\right] + \Delta z(t) .$$

Then, we define the following functions depending on the parameters of $\tilde{y}_p$:

$$N_0 = N_0(A, B, P, Q, \omega) \doteq \frac{\tilde{\beta}_0}{A} \in \mathbb{R} ,$$

$$N_1 = N_1(A, B, P, Q, \omega) \doteq \frac{\tilde{\beta}_1}{B} \in \mathbb{C} ,$$

$$N_2 = N_2(A, B, P, Q, \omega) \doteq \frac{1}{P^2 + Q^2}\left(P + \jmath Q\right)\tilde{\beta}_2 \in \mathbb{C}$$

and so the periodic response of the feedback nonlinearity up to the second harmonic assumes the form:

$$\hat{z}_p(t) = \tilde{\beta}_0 + \tilde{\beta}_1 e^{\jmath\omega t} + \tilde{\beta}_2 e^{\jmath 2\omega t} = N_0 A + N_1 B e^{\jmath\omega t} + N_2\left(P - \jmath Q\right) e^{\jmath 2\omega t} .$$

Observe that in the Describing Function Method only N_0 and N_1 are employed. Therefore, balancing the harmonics up to the second and neglecting Δz, is equivalent to set up the following equivalence:

$$\hat{y}_p(t) = L(\mathcal{D})\hat{z}_p(t) , \tag{1.28}$$

since the operator L in (1.7) is linear and since the real part extractor $\Re$ can be interpreted as in (1.20). Then, balancing the single harmonics, we finally derive the expression of the second order HB problem:

$$\begin{cases} A = L(0)N_0(A, B, P, Q, \omega)A \\ B = L(\jmath\omega)N_1(A, B, P, Q, \omega)B \\ (P - \jmath Q) = L(\jmath 2\omega)N_2(A, B, P, Q, \omega)(P - \jmath Q) \end{cases} , \tag{1.29}$$

which results in five algebraic equations, that is one real and two complex, in the five unknowns A, B, P, Q and ω.

Chapter 2

The Hopf bifurcation: the state space approach

2.1 The Hopf Theorem

In the previous chapter we have introduced some different mathematical models to describe a dynamical system. In general, we may suppose that the adopted laws depend on a certain set of parameters, let say $p \in \mathbb{R}^m$. We want to stress that a large amount of problems in several scientific fields deals with the changes of the system dynamics as p varies. This kind of study is referred as the *bifurcation analysis*. In general, such problems concern situations where the process undergoes a deep and sudden modification of its behaviour, usually due to the appearance or the disappearance of a stable solution [Kuznetsov, 1998]. The values of p at which this happens are referred to as *bifurcation points*, because they divide the parametric space in two or more regions related to different system behaviours. Thus, since it will be useful for the following developments, let us stress the presence of such parameters by the introduction of a proper notation. Then, recalling (1.1) and (1.2), let

$$\dot{x} = F(x; p) = A(p)x + f(x; p) \tag{2.1}$$

be the state space model for a parametric system, where we have also divided the linear part of the system from the pure nonlinear one. In particular, we are interested in the study of the system (2.1), when the parameter vector p

varies along a one-dimensional manifold. This hypothesis boils down to:

$$p = p(\mu) \ ,$$

where $\mu \in \mathbb{R}$ is the *bifurcation parameter*, which determines the system behaviour. Thus, the equation (2.1) reduces to the form:

$$\dot{x} = F(x; \mu) = A(\mu)x + f(x; \mu) \ . \tag{2.2}$$

For our purposes, let us suppose that the system has an equilibrium point, which locally is not affected by the μ-parametrization of p. Without loss of generality, we can assume the fixed point to be at the origin. Then, the above reasoning can be stated as:

$$f(0_n, \mu) = 0_n \quad \forall \mu \in (-\hat{\mu}, \hat{\mu}) \ , \tag{2.3}$$

for some given $\hat{\mu} \in \mathbb{R} : \ \hat{\mu} > 0$. Let $\lambda_k(\mu) \in \mathbb{R}$, $k = 1, \ldots, n$, denote the eigenvalues of $A(\mu)$ and suppose that the following conditions hold:

i) two complex conjugate eigenvalues, nominally $\lambda_{1,2}(\mu) \doteq h(\mu) \pm \jmath\omega(\mu)$, with $h,\ \omega \in \mathbb{R}$, are purely imaginary at $\mu = 0$, that is:

$$\lambda_{1,2}(0) = \pm \jmath\omega_0 \ ,$$

being $\omega(0) = \omega_0 > 0$;

ii) the couple $\lambda_{1,2}(\mu)$ transversely crosses the imaginary axis at $\mu = 0$, that is:

$$h'(0) \doteq \left. \frac{d}{d\mu} h(\mu) \right|_{\mu=0} \neq 0 \ ;$$

iii) all the other $(n-2)$ eigenvalues $\lambda_k(\mu)$, $k = 3, \ldots, n$, have negative real part in $(-\hat{\mu}, \hat{\mu})$, that is:

$$\Re\left[\lambda_k(\mu)\right] < 0 \quad \forall \mu \in (-\hat{\mu}, \hat{\mu}) \ .$$

Then, we can introduce a rigorous formulation of the *Hopf bifurcation theorem*, which states the birth of a limit cycle from an equilibrium point, as the latter changes its stability property [Farkas, 1994; Marsden and McCracken, 1976; Hassard et al., 1981].

Theorem 4. *Consider the parametric system* (2.2)*. Suppose that* $f \in C^{k+1}$ *jointly in* x *and* μ*, with* $k \geq 4$*, and that it has a fixed point in the origin, according to the condition* (2.3)*. Moreover, suppose that the eigenvalues of the equilibrium in the origin satisfy the conditions i), ii) and iii). Then, there exists a* $\hat{\varepsilon} \in \mathbb{R} : \hat{\varepsilon} > 0$ *and a function* $\mu(\varepsilon) \in C^{k-2}(-\hat{\varepsilon}, \hat{\varepsilon})$ *with* $\mu(0) = 0$*, such that the system* (2.2) *has a periodic solution* $\theta(t, \varepsilon) \in \mathbb{R}^n$ *for* $\varepsilon \in (-\hat{\varepsilon}, \hat{\varepsilon})$ *whose period is* $T(\varepsilon) > 0$*. In addition,* $T(\varepsilon) \in C^{k-2}(-\hat{\varepsilon}, \hat{\varepsilon})$*,* $T(0) = 2\pi/\omega(0) = 2\pi/\omega_0$ *and it results that the amplitude of* $\theta(t, \varepsilon)$ *varies proportionally to* $\sqrt{|\varepsilon|}$*, with* $\theta(t, 0) \equiv 0_n$*. Moreover, there is a neighborhood of* $(x, \mu) = (0_n, 0)$ *that does not contain any periodic solution of* (2.2) *but those of the family* $\theta(t, \varepsilon)$*,* $\varepsilon \in (-\hat{\varepsilon}, \hat{\varepsilon})$*. Finally, if the equilibrium in the origin is asymptotically stable (respectively unstable) for* $\varepsilon \in (-\hat{\varepsilon}, 0) \cup (0, \hat{\varepsilon})$*, then* $h'(0)\mu(\varepsilon) < 0$ *(respectively* $h'(0)\mu(\varepsilon) > 0$*) and the periodic solution* $\theta(t, \varepsilon)$ *is unstable (respectively stable).*

Proof. For a detailed proof of the theorem, see [Farkas, 1994]. □

Theorem 4 gives a complete characterization of the local dynamics as the bifurcation parameter crosses zero. According to the hypothesis, there exists a branch of values of μ such that the equilibrium is asymptotically stable and another where it is unstable, due to a couple of complex conjugate eigenvalues. If the nonlinearity f is sufficiently smooth, as the fixed point changes its stability property in the transition from a branch to the other, a locally unique periodic solution arises at the origin and it grows depending on the variations of μ. This limit cycle exists only for one of the branches of the bifurcation parameter. It turns out to be stable, if it is present when the fixed point is unstable. That situation is known in literature as the *supercritical Hopf bifurcation.* Conversely, the periodic solution results unstable, if it corresponds to the asymptotically stable branch of the equilibrium. This is the *subcritical Hopf bifurcation.*

2.2 The coefficient of curvature

It is worth to observe that, according to Theorem 4, the presence of a Hopf bifurcation can be checked by the only analysis of the linearized problem, provided that f is sufficiently smooth. This result turns out to be particularly useful in the study of the bifurcation. Nonetheless, the theorem does not provide any direct technique to state the super or subcritical nature of the bifurcation. Therefore, let us introduce an effective method, originally developed by Howard [1979]. To this aim, we refer to a slightly different formulation of the problem. Consider the system (2.2) and suppose that it depends on the bifurcation parameter just for the linear part, that is:

$$\dot{x} = F(x;\mu) = A(\mu)x + f(x) \ . \tag{2.4}$$

Moreover, let us suppose that the matrix A depends on μ according to the law:

$$A(\mu) = A_0 + \mu A_1 \ . \tag{2.5}$$

Recalling the conditions i), ii) and iii) on A, let $r(\mu) \in \mathbb{R}^{n\times 1}$ and $l(\mu) \in \mathbb{R}^{1\times n}$ denote a right and left eigenvector of $A(\mu)$ associated to the eigenvalue $\lambda_1(\mu)$, respectively, that is:

$$\begin{cases} A(\mu)r(\mu) &= \lambda_1(\mu)r(\mu) \\ l(\mu)A(\mu) &= \lambda_1(\mu)l(\mu) \end{cases} \quad , \tag{2.6}$$

and define

$$l_0 \doteq l(0) \ ,$$
$$r_0 \doteq r(0) \ .$$

Moreover, suppose that at the bifurcation point $\mu = 0$ the normalization condition

$$l_0 r_0 = 1 \tag{2.7}$$

be satisfied. Finally, let us introduce the notation

$$r'(\mu) \doteq \frac{d}{d\mu}r(\mu) \ , \quad l'(\mu) \doteq \frac{d}{d\mu}l(\mu) \ , \quad \lambda_1'(\mu) \doteq \frac{d}{d\mu}\lambda_1(\mu)$$

for the derivative with respect to the bifurcation parameter. Then, the following statement holds.

Proposition 5. *At the Hopf bifurcation point, the variation rate with respect to the parameter μ of the real part of the eigenvalues $\lambda_{1,2}$ assumes the value:*

$$h'(0) = \Re\left[l_0 A_1 r_0\right] \ . \tag{2.8}$$

Proof. Derive the first of the equations (2.6) with respect to μ and then left-multiply the equation for $l(\mu)$. Hence, by (2.5) it follows that:

$$l(\mu)A_1r(\mu) + l(\mu)A(\mu)r'(\mu) = \lambda_1'(\mu)l(\mu)r(\mu) + \lambda_1(\mu)l(\mu)r'(\mu) \ .$$

Then, by using the second equation of the (2.6) we obtain:

$$l(\mu)A_1r(\mu) = (h'(\mu) + \jmath\omega'(\mu))\, l(\mu)r(\mu) \ . \tag{2.9}$$

Therefore, evaluating (2.9) at the bifurcation point $\mu = 0$ and considering the normalization hypothesis (2.7) we have the statement. □

Consider the problem formulation (2.4) along with the hypothesis (2.5) and perform a local power development of f in a neighborhood of the origin:

$$\dot{x} = (A_0 + \mu A_1)x + f^{[2]}(x) + f^{[3]}(x) + O\left(x\right)^4 \ .$$

Since $f^{[2]}$ and $f^{[3]}$ are respectively a quadratic and a cubic function, there exist $\hat{f}^{[2]} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\hat{f}^{[3]} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that:

$$\begin{aligned} \hat{f}^{[2]}(x,x) &\equiv f^{[2]}(x) \ , \\ \hat{f}^{[3]}(x,x,x) &\equiv f^{[3]}(x) \ . \end{aligned}$$

Hence, let us refer directly to the system

$$\dot{x} = (A_0 + \mu A_1)x + \hat{f}^{[2]}(x,x) + \hat{f}^{[3]}(x,x,x) + O\left(x\right)^4 \ . \tag{2.10}$$

According to results of Theorem 4, if we consider the relation between the existence of the limit cycle and the bifurcation parameter, we have that the periodic solution locally exists only for the positive or only for the negative branch of μ. Therefore, if we recall the description of the bifurcation parameter as a function of ε, it follows that:

$$\mu(\varepsilon) = \mu_2\varepsilon^2 + O\left(\varepsilon^3\right) \ .$$

Moreover, let us consider the power development of the period T with respect to ε:

$$T(\varepsilon) = \frac{2\pi}{\omega_0}\left(1 + T_1\varepsilon + T_2\varepsilon^2\right) + O\left(\varepsilon^3\right)$$

Then, we can formulate the following result.

Proposition 6. *The* ε*-developments of* μ *and* T *satisfy the relation:*

$$\mu_2 l_0 A_1 r_0 + \jmath\omega_0 T_2 = -2 l_0 \hat{f}^{[2]}(r_0, \xi) - l_0 \hat{f}^{[2]}(\overline{r}_0, \eta) - \frac{3}{4} l_0 \hat{f}^{[3]}(r_0, r_0, \overline{r}_0) \ , \tag{2.11}$$

where:

$$\xi = -\frac{1}{2} A_0^{-1} \hat{f}^{[2]}(r_0, \overline{r}_0) \ , \tag{2.12}$$

$$\eta = \frac{1}{2}(\jmath 2\omega_0 I - A_0)^{-1} \hat{f}^{[2]}(r_0, r_0) \ . \tag{2.13}$$

Proof. For a detailed proof, see [Howard, 1979]. □

Exploiting Propositions 5 and 6, the following result can be stated.

Proposition 7. *Define the* coefficient of curvature *of the Hopf bifurcation as the quantity*

$$\beta_2 \doteq -2\mu_2 \Re\left[\lambda'_{1,2}(0)\right] = -2\mu_2 h'(0) \ . \tag{2.14}$$

Then, if $\beta_2 < 0$ *the limit cycle is stable and consequently the Hopf bifurcation is supercritical. On the contrary, if* $\beta_2 > 0$ *the limit cycle is unstable and the bifurcation subcritical.*

Proof. According to the Hopf theory, the bifurcation is supercritical (subcritical) if and only if a stable (unstable) limit cycle arises when the equilibrium point becomes unstable (stable). Moreover, observe that the fixed point stability depends on $\lambda_{1,2}$ and its relation with ε depends on the sign of $h'(0)$. In particular, the limit cycle exists for the positive branch of the bifurcation parameter if $\mu_2 > 0$ and for the negative otherwise. Moreover, the equilibrium becomes unstable by increasing μ if $h'_{1,2}(0) > 0$ and stable if $h'(0) < 0$. Therefore, the conditions to have the bifurcation to be supercritical (subcritical) boil down to $\mu_2 h'(0) > 0$ (respectively $\mu_2 h'(0) < 0$). □

Remark 8. *According to the Howard's results [Howard, 1979], the quantity* (2.14) *turns out to be the primal coefficient of the* ε*-development of the maximal Floquet exponent* ν *associated to the limit cycle, that is:*

$$\nu(\varepsilon) = \beta_2 \varepsilon^2 + O\left(\varepsilon^3\right) \ .$$

Corollary 9. *The coefficient of curvature satisfies the relation:*

$$\beta_2 = 2\Re\left[2l_0\hat{f}^{[2]}(r_0,\xi) + l_0\hat{f}^{[2]}(\overline{r}_0,\eta) + \frac{3}{4}l_0\hat{f}^{[3]}(r_0,r_0,\overline{r}_0)\right] , \tag{2.15}$$

where ξ *and* η *are defined according to* (2.12) *and* (2.13) *respectively.*

Proof. The proof follows directly from the definition (2.14), exploiting the (2.8) and (2.11):

$$\begin{aligned}\beta_2 &= -2\mu_2 h'(0) = \\ &= -2\mu_2\Re\left[l_0 A_1 r_0\right] = -2\Re\left[\mu_2 l_0 A_1 r_0 + \jmath\omega_0 T_2\right] \\ &= 2\Re\left[2l_0\hat{f}^{[2]}(r_0,\xi) + l_0\hat{f}^{[2]}(\overline{r}_0,\eta) + \frac{3}{4}l\hat{f}_0^{[3]}(r_0,r_0,\overline{r}_0)\right] .\end{aligned}$$

□

2.3 The coefficient of curvature for the differential equation systems class

Let us apply the state space approach based on the coefficient of curvature to the parametric differential equation system

$$y^{(n)} + a_1(\mu)y^{(n-1)} + \ldots + a_{n-1}(\mu)\dot{y} + a_n(\mu)y + g\left(y^{(n-1)},\ldots,\dot{y},y\right) = 0 , \tag{2.16}$$

where only the linear part of the problem depends on the bifurcation parameter μ. Adopting the phase variables $x_1 = y,\ x_2 = \dot{y},\ldots,\ x_n = y^{(n-1)}$ defined in Chapter 1 (see section 1.4, page 6), the system (2.16) assumes the form (2.4), with:

$$A(\mu) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 & 0 \\ 0 & \ldots & 0 & 0 & 1 \\ -a_n(\mu) & \ldots & -a_3(\mu) & -a_2(\mu) & -a_1(\mu) \end{bmatrix}$$

and:

$$f(x) \doteq -e_n g(x) \ . \tag{2.17}$$

Moreover, supposing that

$$a_k(\mu) = \alpha_k + \tilde{\alpha}_k \mu \ , \tag{2.18}$$

the condition (2.5) turns out to be satisfied for:

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \\ -\alpha_n & \dots & -\alpha_3 & -\alpha_2 & -\alpha_1 \end{bmatrix} , \tag{2.19}$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 \\ -\tilde{\alpha}_n & \dots & -\tilde{\alpha}_3 & -\tilde{\alpha}_2 & -\tilde{\alpha}_1 \end{bmatrix} . \tag{2.20}$$

Then, consider the power development of g

$$g(x) = g^{[2]}(x) + g^{[3]}(x) + O\left(\|x\|^4\right)$$

and let be $\hat{g}^{[2]} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $\hat{g}^{[3]} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ the quadratic and cubic extensions of $g^{[2]}$ and $g^{[3]}$, so that:

$$\hat{g}^{[2]}(x,x) \equiv g^{[2]}(x) \ ,$$
$$\hat{g}^{[3]}(x,x,x) \equiv \hat{g}^{[3]}(x) \ .$$

Then, the system (2.16) assumes the form (2.10) for:

$$\hat{f}^{[2]}(x,x) \doteq -e_n \hat{g}^{[2]}(x,x) \ , \tag{2.21}$$
$$\hat{f}^{[3]}(x,x,x) \doteq -e_n \hat{g}^{[3]}(x,x,x) \ . \tag{2.22}$$

For our purposes, we find useful the introduction of the matrix

$$B \doteq \begin{bmatrix} b_1 & 0 & 0 & \dots & 0 \\ b_2 & b_{n+1} & 0 & \dots & 0 \\ b_3 & b_{n+2} & b_{2n} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & b_{2n-1} & b_{3n-3} & \dots & b_{n(n+1)/2} \end{bmatrix} ,$$

such that:

$$\hat{g}^{[2]}(x, y) = x^T B y = \left(y^T \otimes x^T\right) \operatorname{vec} B \ .$$

Then, if we define the quantity:

$$b \doteq \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n(n+1)/2} \end{bmatrix} \in \mathbb{R}^{n(n+1)/2} \ ,$$

it is straightforward to observe that it exists a matrix $V \in \mathbb{R}^{n \times n(n+1)/2}$ such that:

$$\operatorname{vec} B = V b \ .$$

Thus, the vector b completely identifies the quadratic part of the nonlinearity g:

$$\hat{g}^{[2]}(x, y) = \left(y^T \otimes x^T\right) V b \ . \tag{2.23}$$

Similarly, we introduce the following matrices:

$$C_k = \begin{bmatrix} c_{n(k-1)(n-1)/2+1} & 0 & \cdots & 0 \\ c_{n(k-1)(n-1)/2+2} & c_{n(k-1)(n-1)/2+n+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_{n(k-1)(n-1)/2+n} & c_{n(k-1)(n-1)/2+2n-1} & \cdots & c_{nk(n+1)/2} \end{bmatrix} ,$$

so that:

$$\begin{aligned} \hat{g}^{[3]}(x, y, z) &= \left[x^T C_1 y, \ldots, x^T C_n y\right] z = \\ &= \left(y^T \otimes x^T\right) \left[\operatorname{vec} C_1, \ldots, \operatorname{vec} C_n\right] z = \\ &= \left(y^T \otimes x^T\right) H z = \\ &= \left(z^T \otimes y^T \otimes x^T\right) \operatorname{vec} H \ , \end{aligned}$$

being:

$$H = \left[\operatorname{vec} C_1, \ldots, \operatorname{vec} C_n\right] \in \mathbb{R}^{n(n+1)/2 \times n}$$

and:

$$\operatorname{vec} C_k = \begin{bmatrix} c_{n(k-1)(n-1)/2+1} \\ c_{n(k-1)(n-1)/2+2} \\ \vdots \\ c_{kn(n-1)/2} \end{bmatrix} \in \mathbb{R}^{n(n+1)/2} .$$

As before, the definition of the vector

$$c \doteq \begin{bmatrix} c_1 \\ \vdots \\ c_{n(n+1)/2} \\ \vdots \\ c_{n^2(n+1)/2} \end{bmatrix} \in \mathbb{R}^{n^2(n+1)/2}$$

allows us to state the existence of a suitable matrix $W \in \mathbb{R}^{n(n+1)/2 \times n^2(n+1)/2}$ such that:

$$\operatorname{vec} H = Wc .$$

Therefore, the cubic component of the nonlinearity is completely defined by c:

$$\hat{g}^{[3]}(x, y, z) = \left(z^T \otimes y^T \otimes x^T\right) Wc . \tag{2.24}$$

With the introduction of the above notation, for the differential equation systems the coefficient β_2 of Corollary 9 turns out to depend quadratically on b and linearly on c, according to the result reported in the following statement.

Theorem 10. *Consider the differential equation system* (2.16) *and its state space form* (2.4) *obtained by using the phase coordinates, along with the hypothesis* (2.18)*. Hence, the linear part is defined by* (2.19)*, while the nonlinearity satisfies* (2.17) *and its power development* (2.21) *and* (2.22)*. Moreover, let be b and c the parameters, which define the quadratic and cubic part of g according to the equation* (2.23) *and* (2.24)*. Finally, suppose that the system satisfies the Hopf bifurcation conditions of the Theorem 4. Then, there exist $M_2 \in \mathbb{R}^{n(n+1)/2 \times n(n+1)/2}$ and $M_3 \in \mathbb{R}^{1 \times n^2(n+1)/2}$ such that the coefficient of curvature β_2 assumes the form:*

$$\beta_2 = b^T M_2 b - M_3 c . \tag{2.25}$$

Proof. To prove the statement we first need to introduce some preliminary result.

Lemma 11. *Consider the matrix A_0 as defined in the equation* (2.19) *and suppose that it has a couple of pure imaginary eigenvalues $\pm \jmath\omega_0$, $\omega_0 > 0$. Then, the vectors*

$$r_0 = \begin{bmatrix} 1 \\ \jmath\omega_0 \\ \vdots \\ (\jmath\omega_0)^{n-1} \end{bmatrix} ,$$

$$l_0 = \gamma_0 \begin{bmatrix} \alpha_n \\ \frac{1}{\jmath\omega_0}(\alpha_n + \alpha_{n-1}(\jmath\omega_0)) \\ \vdots \\ \frac{1}{(\jmath\omega_0)^{n-2}} \left(\alpha_n + \alpha_{n-1}(\jmath\omega_0) + \ldots + \alpha_2(\jmath\omega_0)^{n-2}\right) \\ -\jmath\omega_0 \end{bmatrix}^T$$

are respectively a right and a left eigenvector of $\jmath\omega_0$, for every $\gamma_0 \in \mathbb{C}$. Moreover, if

$$\gamma_0 = \frac{1}{n\alpha_n + \alpha_{n-1}(n-1)(\jmath\omega_0) + \ldots + \alpha_1(\jmath\omega_0)^{n-1}} ,$$

r_0 and l_0 satisfy the normalization condition (2.7).

Proof. Consider the equation:

$$A_0 x = \jmath\omega_0 x \ . \tag{2.26}$$

Due to the companion form of the matrix A_0, (2.26) is equivalent to the n-th dimensional algebraic system:

$$\begin{cases} x_2 &= \jmath\omega_0 x_1 \\ x_3 &= \jmath\omega_0 x_2 = (\jmath\omega_0)^2 x_1 \\ &\ldots \\ x_n &= \jmath\omega_0 x_{n-1} = (\jmath\omega_0)^{n-1} x_1 \\ \jmath\omega_0 x_n &= -\alpha_n x_1 - \alpha_{n-1} x_2 - \ldots - \alpha_1 x_n = \\ &= -\alpha_n x_1 - \alpha_{n-1}(\jmath\omega_0) x_1 - \ldots - \alpha_1(\jmath\omega_0)^{n-1} x_1 = \\ &= -\left(\alpha_1(\jmath\omega_0)^{n-1} + \ldots + \alpha_{n-1}(\jmath\omega_0) + \alpha_n\right) x_1 \ . \end{cases} \tag{2.27}$$

Exploiting the previous equations, the latter assumes the form:

$$(\jmath\omega_0)^n x_1 = -\left(\alpha_1(\jmath\omega_0)^{n-1} + \ldots + \alpha_{n-1}(\jmath\omega_0) + \alpha_n\right) x_1 \ ,$$

which turns out to be an identity, since $\jmath\omega_0$ is an eigenvalue of A_0, whose characteristic polynomial indeed is:

$$\lambda^n + \alpha_1\lambda^{n-1} + \ldots + \alpha_{n-1}\lambda + \alpha_n = 0 \ .$$

Hence, by direct substitution it is straightforward to check that r_0 satisfies the algebraic equations system (2.27). Then, consider the equation

$$x^T A_0 = \jmath\omega_0 x^T \ , \tag{2.28}$$

that is equivalent to

$$\begin{cases} -\alpha_n x_n &= \jmath\omega_0 x_1 \\ x_1 - \alpha_{n-1} x_n &= \jmath\omega_0 x_2 \\ & \ldots \\ x_{n-1} - \alpha_1 x_n &= \jmath\omega_0 x_n \ . \end{cases}$$

Exploiting the hypothesis $\omega_0 > 0$, it is clear that this system can be formulated as:

$$\begin{cases} x_1 &= -\frac{\alpha_n}{\jmath\omega_0} x_n \\ x_2 &= -\frac{1}{(\jmath\omega_0)^2}(\alpha_n + \jmath\omega_0\alpha_{n-1}) x_n \\ \ldots \\ x_n &= -\frac{1}{(\jmath\omega_0)^n}(\alpha_n + \jmath\omega_0\alpha_{n-1} + \ldots + (\jmath\omega_0)^{n-1}\alpha_1) x_n \ , \end{cases} \tag{2.29}$$

where the latter equation is an identity, since it boils down to:

$$(\jmath\omega_0)^n x_n = -(\alpha_n + \jmath\omega_0\alpha_{n-1} + \ldots + (\jmath\omega_0)^{n-1}\alpha_1) x_n \ .$$

Therefore, the direct substitution of l_0 in the (2.29) leads immediately to the statement. Finally, let us consider the normalization condition (2.7):

$$\begin{aligned} l_0 r_0 &= \gamma_0 \left(\alpha_n + (\alpha_n + \alpha_{n-1}(\jmath\omega_0)) \ldots + \left(\alpha_n + \alpha_{n-1}(\jmath\omega_0) + \ldots + \alpha_1(\jmath\omega_0)^{n-1}\right)\right) = \\ &= \gamma_0 \left(n\alpha_n + \alpha_{n-1}(n-1)(\jmath\omega_0) + \ldots + \alpha_1(\jmath\omega_0)^{n-1}\right) = 1 \ . \end{aligned}$$

Then, it is straightforward to check that it is satisfied if the hypothesis on γ_0 holds. ■

Lemma 12. *Consider the matrix A_0 as defined in the equation (2.19) and suppose that it has a couple of pure imaginary eigenvalues $\pm \jmath\omega_0$, $\omega_0 > 0$, while all the others have strict negative real part. Then, the following equations hold:*

$$A_0^{-1} e_n = \frac{1}{\alpha_n} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \frac{1}{\alpha_n} e_1 \tag{2.30}$$

and:

$$(\jmath 2\omega_0 I - A_0)^{-1} e_n = \gamma_1 z_1 \ , \tag{2.31}$$

being:

$$\gamma_1 = \frac{1}{(2\jmath\omega_0)^n + \alpha_1 (2\jmath\omega_0)^{n-1} + \ldots + \alpha_{n-1}(\jmath 2\omega_0) + \alpha_n} \ ,$$

$$z_1 = \begin{bmatrix} 1 \\ \jmath 2\omega_0 \\ (\jmath 2\omega_0)^2 \\ \vdots \\ (\jmath 2\omega_0)^{n-1} \end{bmatrix} \ .$$

Proof. Consider the following identity:

$$(\lambda I - A_0)(\lambda I - A_0)^{-1} e_n = e_n \tag{2.32}$$

and define:

$$\tilde{x} \doteq (\lambda I - A_0)^{-1} e_n \ ,$$

so that (2.32) becomes:

$$(\lambda I - A_0)\tilde{x} = e_n \ .$$

Then, the explicit form of the latter equation turns out to be:

$$\begin{cases} \lambda \tilde{x}_1 - \tilde{x}_2 = 0 \\ \lambda \tilde{x}_2 - \tilde{x}_3 = 0 \\ \cdots \\ \alpha_n \tilde{x}_1 + \alpha_{n-1}\tilde{x}_2 + \ldots + \alpha_1 \tilde{x}_n + \lambda \tilde{x}_n = 1 \ . \end{cases}$$

It is straightforward to observe that the k-th algebraic equation of this system can be developed by the recursive application of the previous $(k-1)$ ones, so to obtain:

$$\begin{cases} \tilde{x}_2 = \lambda \tilde{x}_1 \\ \tilde{x}_3 = \lambda^2 \tilde{x}_1 \\ \dots \\ \alpha_n \tilde{x}_1 + \alpha_{n-1} \lambda \tilde{x}_1 + \ldots + \alpha_1 \lambda^{n-1} \tilde{x}_1 + \lambda^n \tilde{x}_1 = 1 \ , \end{cases}$$

which can be formulated as:

$$\begin{cases} \tilde{x}_2 = \lambda \tilde{x}_1 \\ \tilde{x}_3 = \lambda^2 \tilde{x}_1 \\ \dots \\ \tilde{x}_1 = \frac{1}{\lambda^n + \alpha_1 \lambda^{n-1} + \ldots + \alpha_{n-1} \lambda + \alpha_n} \ , \end{cases}$$

if λ is not an eigenvalue of A_0. Therefore, the following equation:

$$(\lambda I - A_0)^{-1} e_n = \frac{1}{\lambda^n + \alpha_1 \lambda^{n-1} + \ldots + \alpha_{n-1} \lambda + \alpha_n} \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \end{bmatrix} \tag{2.33}$$

holds for each λ that does not belong to the spectrum of A_0. Then, the statement follows directly from (2.33) along with the choices $\lambda = 0$ and $\lambda = \jmath 2\omega_0$, which according to the hypothesis are not eigenvalues of A_0. ■

Then, according to Proposition 6 let us compute the vectors ξ and η exploiting the equations (2.23), (2.24), (2.30), (2.31) and the the previous Lemmas:

$$\begin{aligned} \xi &= -\frac{1}{2} A_0^{-1} e_n \hat{g}^{[2]}(r_0, \overline{r}_0) \\ &= -\frac{1}{2\alpha_n} \left(r_0^T \otimes \overline{r}_0^T \right) V b e_1 \ , \\ \eta &= \frac{1}{2} (2\jmath\omega_0 I - A_0)^{-1} e_n \hat{g}^{[2]}(r_0, r_0) = \\ &= \frac{1}{2} \gamma_1 \left(r_0^T \otimes r_0^T \right) V b z_1 \ . \end{aligned}$$

Moreover, consider the quantities:

$$\begin{aligned}
l_0\hat{f}^{[2]}(r_0,\xi) &= l_0 e_n \hat{g}^{[2]}(r_0,\xi) = \\
&= -\jmath\omega_0\gamma_0\left(r_0^T\otimes\xi^T\right)Vb = \\
&= \frac{1}{2a_n}\jmath\omega_0\gamma_0\left(r_0^T\otimes\overline{r}_0^T\right)Vb\left(r_0^T\otimes e_1^T\right)Vb = \\
&= \frac{1}{2a_n}\jmath\omega_0\gamma_0 b^T V^T\left(r_0\otimes\overline{r}_0\right)\left(r_0^T\otimes e_1^T\right)Vb\ , \\
l_0\hat{f}^{[2]}(\overline{r}_0,\eta) &= l_0 e_n \hat{g}^{[2]}(\overline{r}_0,\eta) = \\
&= -\jmath\omega_0\gamma_0\left(\overline{r}_0^T\otimes\eta^T\right)Vb = \\
&= -\frac{1}{2}\jmath\omega_0\gamma_0\gamma_1\left(r_0^T\otimes r_0^T\right)Vb\left(\overline{r}_0^T\otimes z_1^T\right)Vb = \\
&= -\frac{1}{2}\jmath\omega_0\gamma_0\gamma_1 b^T V^T\left(r_0\otimes r_0\right)\left(\overline{r}_0^T\otimes z_1^T\right)Vb\ , \\
l_0\hat{f}^{[3]}(r_0,r_0,\overline{r}_0) &= l_0 e_n \hat{g}^{[3]}(r_0,r_0,\overline{r}_0) = \\
&= -\jmath\omega_0\gamma_0\left(r_0^T\otimes r_0^T\otimes\overline{r}_0^T\right)Wc\ .
\end{aligned}$$

Therefore, from the Corollary 9 we obtain that:

$$\begin{aligned}
\beta_2 &= 2\Re\left[2l_0\hat{f}^{[2]}(r_0,\xi)+l_0\hat{f}^{[2]}(\overline{r}_0,\eta)+\frac{3}{4}l_0\hat{f}^{[3]}(r_0,r_0,\overline{r}_0)\right] = \\
&= 2\Re\Bigg[2\frac{1}{2a_n}\jmath\omega_0\gamma_0 b^T V^T\left(r_0\otimes\overline{r}_0\right)\left(r_0^T\otimes e_1^T\right)Vb+ \\
&\quad -\frac{1}{2}\jmath\omega_0\gamma_0\gamma_1 b^T V^T\left(r_0\otimes r_0\right)\left(\overline{r}_0^T\otimes z_1^T\right)Vb-\jmath\omega_0\gamma_0\left(r_0^T\otimes r_0^T\otimes\overline{r}_0^T\right)Wc\Bigg] = \\
&= b^T\Re\left[\jmath\omega_0\gamma_0 V^T\left(\frac{2}{a_n}\left(r_0\otimes\overline{r}_0\right)\left(r_0^T\otimes e_1^T\right)-\gamma_1\left(r_0\otimes r_0\right)\left(\overline{r}_0^T\otimes z_1^T\right)\right)V\right]b+ \\
&\quad -\Re\left[2\jmath\omega_0\gamma_0\left(r_0^T\otimes r_0^T\otimes\overline{r}_0^T\right)W\right]c \\
&= b^T M_2 b - M_3 c\ .
\end{aligned}$$

□

Corollary 13. *The matrix M_2 and the vector M_3 in the expression* (2.25) *of the coefficient of curvature satisfy:*

$$\begin{aligned}
M_2 &= \Re\left[\jmath\omega_0\gamma_0 V^T\left(\frac{2}{a_n}\left(r_0\otimes\overline{r}_0\right)\left(r_0^T\otimes e_1^T\right)-\gamma_1\left(r_0\otimes r_0\right)\left(\overline{r}_0^T\otimes z_1^T\right)\right)V\right]\ , \\
M_3 &= \Re\left[2\jmath\omega_0\gamma_0\left(r_0^T\otimes r_0^T\otimes\overline{r}_0^T\right)W\right]\ ,
\end{aligned}$$

where V is as in (2.23)*,* W *as in* (2.24)*,* r_0*,* l_0 *and* γ_0 *as in Lemma 11 and finally* z_1 *and* γ_1 *as in Lemma 12.*

Proof. See the proof of Theorem 10. □

Chapter 3

The Hopf bifurcation: the frequency approach

The state space approach to the Hopf bifurcation turns out to be a powerful theoretical method to state its existence, but it does not provide easy to use analytical tools either to compute the limit cycle or to check its nature. Also, the restriction to the differential equation systems, based on the Howard's procedure, leads to results which are quite hard to handle, because of the many Kronecker products.

In the following, exploiting the Harmonic Balance method, we will develop an approach that provides an effective analytical tool to study Hopf bifurcations of systems in the differential equation form. The theoretical background of our results is the frequency approach originally introduced by Allwrigth and Mees [Allwright, 1977; Mees, 1981; Moiola and Chen, 1996].

3.1 The existence of the second order HB solution

Consider the parametric differential equation system:

$$y^{(n)} + G\left(y^{(n-1)}, \ldots, \dot{y}, y; \mu\right) = 0$$

and let us divide the linear from the pure nonlinear part as follows:

$$\begin{aligned} & y^{(n)} + a_1(\mu)y^{(n-1)} + \ldots + a_{n-1}(\mu)\dot{y} + a_n(\mu)y + \\ & \quad + g\left(y^{(n-1)}, \ldots, \dot{y}, y; \mu\right) = 0 \ . \end{aligned} \tag{3.1}$$

Let be

$$y^{(n-1)} = 0 \ , \ldots , \quad \dot{y} = 0 \ , \quad y = y_e(\mu)$$

an equilibrium point of (3.1) and, without loss of generality, assume that it satisfies:

$$y_e(0) = 0 \ . \tag{3.2}$$

We are interested in studying the Hopf bifurcation of (3.1) related to the fixed point (3.2) when the bifurcation parameter crosses zero. Then, for the sake of simplicity, let us say that (3.1) undergoes a Hopf bifurcation at the origin when $\mu = 0$, if the state space representation obtained through the phase coordinates satisfies the Hopf bifurcation conditions of Theorem 4.

Therefore, assume that (3.1) as a Hopf bifurcation in the origin at $\mu = 0$. According to the previous theory, a limit cycle of (3.1) locally exists only for one branch of the values of the bifurcation parameter, that is just for $\mu > 0$ or vice versa. In particular, let us denote this periodic solution through its Fourier series:

$$\begin{aligned} y_p(t;\mu) &= \sum_{k=1}^{\infty} \Re\left[\alpha_k(\mu)e^{\jmath k\omega(\mu)t}\right] = \\ &= \Re\left[\alpha_0(\mu) + \alpha_1(\mu)e^{\jmath\omega(\mu)t} + \alpha_2(\mu)e^{\jmath 2\omega(\mu)t}\right] + \\ &+ \sum_{k=3}^{+\infty} \Re\left[\alpha_k(\mu)e^{\jmath k\omega(\mu)t}\right] = \\ &= \Re\left[\sigma_2(t;\mu)\right] + \Delta y_p(t;\mu) \ , \end{aligned} \tag{3.3}$$

being

$$\sigma_2(t;\mu) \doteq \alpha_0(\mu) + \alpha_1(\mu)e^{\jmath\omega(\mu)t} + \alpha_2(\mu)e^{\jmath 2\omega(\mu)t} \tag{3.4}$$

the complex second harmonic truncation of $y_p(t;\mu)$. Observe that, according to the Theorem 4 and since the origin of the time scale is arbitrary, without loss of generality we can always assume that:

$$\begin{cases} \alpha_1 \in \mathbb{R} : \ \alpha_1 > 0 \\ \omega > 0 \ . \end{cases} \tag{3.5}$$

The following statement describes the relevance of the lower harmonics of the real limit cycle in the study of the Hopf bifurcation.

Proposition 14. *For a sufficiently small $|\mu|$, the real limit cycle $y_p(t;\mu)$ defined as in* (3.3)-(3.5) *can be approximated only by its second order component* $\Re\left[\sigma_2(t;\mu)\right]$ *with a negligible error due to $\Delta y_p(t;\mu)$.*

Outline of the proof. This result can be found as a part of the Mees theory [Mees, 1981] and it is based on the contraction mapping method originally developed by Allwrigth [Allwright, 1977]. He proves that $\Delta y_p(t;\mu)$ can be described as a unique function of $\sigma_2(t;\mu)$, when the latter is sufficiently small. Observe that according to the Hopf bifurcation theorem, this situation is met when the system is approaching the bifurcation, that is if $|\mu|$ is sufficiently small. In particular, it turns out that $\Delta y_p(t;\mu)$ is $O\left(|\sigma_2(t;\mu)|^3\right)$. Therefore, while $|\mu| \to 0$ the contribution of the harmonics higher than the second become a negligible error and, in turn, the real limit cycle $y_p(t;\mu)$ is essentially identified only by its second order harmonic truncation. □

For the following developments, it is important to compare (3.3), and in particular $\sigma_2(t;\mu)$, to the solution of the second order HB problem. Then, let us recall the results introduced in Paragraph 1.6, emphasizing the dependence from μ. Defining the parametric operators

$$\begin{aligned} L(\mathcal{D};\mu) &\doteq \frac{1}{\mathcal{D}^n + a_1(\mu)\mathcal{D}^{n-1} + \ldots + a_{n-1}(\mu)\mathcal{D} + a_n(\mu)} \ , \\ \mathcal{N}_\mu \circ y &\doteq -g\left(y^{(n-1)}, \ldots, \dot{y}, y; \mu\right) \end{aligned}$$

and following the same procedure developed in Paragraph 1.6, we achieve the parametric feedback form of the system (3.1):

$$y(t;\mu) = L(\mathcal{D};\mu)\left(\mathcal{N}_\mu \circ y\right)(t;\mu) \ . \tag{3.6}$$

Then, according to the HB problem formulation, let us denote

$$\tilde{y}_p(t;\mu) = A + B\cos(\omega t) + P\cos(2\omega t) + Q\sin(2\omega t)\ , \tag{3.7}$$
$$\hat{y}_p(t;\mu) = A + Be^{\jmath\omega t} + (P - \jmath Q)e^{\jmath 2\omega t}\ , \tag{3.8}$$
$$\hat{z}_p(t;\mu) = \left(\mathcal{N}_\mu \circ \hat{y}_p\right)(t;\mu) = N_0 A + N_1 B e^{\jmath\omega t} + N_2(P - \jmath Q)e^{\jmath 2\omega t} \tag{3.9}$$

respectively the prototype second order periodic solution, its complex representation and the complex second order harmonic development of the related nonlinearity response, as defined in the equations (1.25) and (1.27). Observe that, in accordance with the conditions (3.5), we can look for $B \in \mathbb{R} :\ B > 0$ and $\omega > 0$, without any loss of generality. We will always assume these conditions, but for the sake of simplicity, in the following we will explicitly report them only when they turn out to be necessary for the comprehension of the result. Due to the presence of the bifurcation parameter, the quantities which define the amplitudes of the harmonics in the latter equations result functions of μ, that is:

$$\begin{aligned}
&A = A(\mu) \in \mathbb{R}\ , \quad B = B(\mu) \in \mathbb{R}\ ,\\
&P = P(\mu) \in \mathbb{R}\ , \quad Q = Q(\mu) \in \mathbb{R}\ ,\\
&\omega = \omega(\mu) \in \mathbb{R}\ ,
\end{aligned}$$

$$\begin{aligned}
N_0 &= N_0\left(A, B, P, Q, \omega; \mu\right) \in \mathbb{R}\ ,\\
N_1 &= N_1\left(A, B, P, Q, \omega; \mu\right) \in \mathbb{C}\ ,\\
N_2 &= N_2\left(A, B, P, Q, \omega; \mu\right) \in \mathbb{C}\ ,
\end{aligned}$$

Then, substituting the prototypes (3.8) and (3.9) in the equation (3.6) and balancing the harmonics, the second order HB problem assumes the form:

$$\begin{cases}
A(\mu) = L(0;\mu)N_0\left(A(\mu), B(\mu), P(\mu), Q(\mu), \omega(\mu); \mu\right) A(\mu)\\
B(\mu) = L(\jmath\omega;\mu)N_1\left(A(\mu), B(\mu), P(\mu), Q(\mu), \omega(\mu); \mu\right) B(\mu)\\
\left(P(\mu) - \jmath Q(\mu)\right) =\\
\quad = L(\jmath 2\omega;\mu)N_2\left(A(\mu), B(\mu), P(\mu), Q(\mu), \omega(\mu); \mu\right)\left(P(\mu) - \jmath Q(\mu)\right)\ .
\end{cases} \tag{3.10}$$

For a fixed value of μ, the solution of (3.10) with respect to A, B, P, Q and ω leads to $\hat{y}_p(t;\mu)$, that is to $\tilde{y}_p(t;\mu)$. The following result states the relation between the second order HB solution (3.8) and the second harmonic truncation (3.4) of the power development (3.3) of the real limit cycle arising from the Hopf bifurcation.

Proposition 15. *For a sufficiently small* $|\mu|$, *the second harmonic truncation* $\sigma_2(t;\mu)$ *of the real limit cycle* $y_p(t;\mu)$, *defined as in* (3.3)-(3.5), *is equivalent to a second order HB non constant solution* $\hat{y}_p(t;\mu)$, *defined as in* (3.8).

Outline of the proof. The proof is based on the results reported by Mees in [Mees, 1981]. Consider the real limit cycle $y_p(t;\mu)$ defined in (3.3), which can be seen as the solution of the infinite dimensional HB problem. From the proof of the Proposition 14, it turns out that $\Delta y_p(t;\mu)$ is univocally defined as a function $O\left(|\sigma_2(t;\mu)|^3\right)$, provided that $|\mu|$ is sufficiently small. Then, the substitution of this development of $y_p(t;\mu)$ in the general HB problem leads it to assume the form of a second order HB problem to be solved in $\sigma_2(t;\mu)$, being the neglecting error $O\left(|\sigma_2(t;\mu)|^4\right)$. Mees shows that this problem is the same as the second HB problem associated to the system. Moreover, he proves that $\sigma_2(t;\mu)$ satisfies such a problem with the same neglecting error of a second order HB solution $\hat{y}_p(t;\mu)$ collapsing to zero as $|\mu| \to 0$. Thus, for a sufficiently small $|\mu|$, $\sigma_2(t;\mu)$ is essentially close to a second order HB solution $\hat{y}_p(t;\mu)$, defined as in (3.8), which in turn results non constant. □

The following result deals with the solvability of the second order HB problem.

Proposition 16. *For sufficiently small values of* $|\mu|$, *the second order HB problem* (3.10) *has a locally unique non constant solution* $\hat{y}_p(t;\mu)$, *defined as in* (3.8), *which exists for just one of the branches of the values of* μ *and collapses to zero as* $|\mu| \to 0$.

Outline of the proof. This important statement can be derived as a partial result from the approach to the Hopf bifurcation developed by Allwrigth and Mees. Consider the real limit cycle $y_p(t;\mu)$ defined in (3.3). According to the proof of the Proposition 15, if $|\mu|$ is sufficiently small, the infinite dimensional HB problem related to $y_p(t;\mu)$ can be developed in a reduced second order HB problem to be solved in $\sigma_2(t;\mu)$ only. This problem turns out to be identical to the second order HB problem of the system and $\sigma_2(t;\mu)$

is equivalent to a solution $\hat{y}_p(t;\mu)$, defined as in (3.8). On the other hand, consider a second order HB solution $\hat{y}_p(t;\mu)$ and suppose that it collapses to the equilibrium in the origin when $|\mu| \to 0$. Mees shows that it solves also the reduced second order HB problem related to $y_p(t;\mu)$, that is it defines a suitable $\sigma_2(t;\mu)$, which in turn identifies a real limit cycle. Thus, for sufficiently small values of $|\mu|$, the second order HB problem is equivalent to the infinite dimensional one and the related $\hat{y}_p(t;\mu)$ corresponds to the real periodic solution $y_p(t;\mu)$. Then, according to the Hopf bifurcation theorem, it follows that the second order HB problem has a locally unique non constant solution defined for just one branch of the values of μ. □

The previous results can be collected in the following theorem.

Theorem 17. *For a sufficiently small* $|\mu|$*, the second order HB problem* (3.10) *has a locally unique non constant solution* $\tilde{y}_p(t;\mu)$ *as in* (3.7)*, which is defined for only one branch of the values of the bifurcation parameter and which is essentially close to the real limit cycle arising from the Hopf bifurcation.*

Proof. This result directly follows from the Propositions 14-16 and their proofs. □

3.2 The relation between the second order HB solution and the Hopf bifurcation nature

According to the Theorem 17, the limit cycle can be located just solving the second order HB problem. Thus, checking the values of μ, which make (3.10) solvable, allows us to state the super or subcritical nature of the bifurcation. Indeed, for the Hopf theory, it is sufficient to compare the range of existence of $\tilde{y}_p(t;\mu)$ and the stability property of the equilibrium point, which is known from the computation of its eigenvalues. Moreover, the non constant solution of (3.10) gives us a local approximation of the real limit cycle. Hence, in the following we will develop a general analytical procedure to check the solvability of the second order HB problem and to compute an approximation of its solution.

Consider the five unknowns of the algebraic problem (3.10). Since locally $B(\mu) > 0$ due to Proposition 16, without loss of generality, we find useful to consider $B^2(\mu)$ rather than the first harmonic amplitude. Then, let us introduce

$$S(\mu) \doteq \begin{bmatrix} A(\mu) \\ B^2(\mu) \\ C(\mu) \\ \omega(\mu) \end{bmatrix} , \tag{3.11}$$

where we have defined

$$C(\mu) \doteq P(\mu) + \jmath Q(\mu) .$$

Moreover, let us introduce the polynomials:

$$\Gamma(\jmath k\omega; \mu) \doteq L^{-1}(\jmath k\omega; \mu) . \tag{3.12}$$

Then, it is straightforward to observe that the computation of the second order HB solution $\tilde{y}_p(t; \mu)$ is equivalent to find $S(\mu)$ such that:

i) it is a locally unique solution of the algebraic system

$$\begin{cases} \left(N_0\left(S(\mu); \mu\right) - \Gamma(0; \mu)\right) A(\mu) = 0 \\ \left(N_1\left(S(\mu); \mu\right) - \Gamma(\jmath\omega; \mu)\right) B^2(\mu) = 0 \\ \left(N_2\left(S(\mu); \mu\right) - \Gamma(\jmath 2\omega; \mu)\right) C(\mu) = 0 , \end{cases} \tag{3.13}$$

ii) at the Hopf bifurcation point it satisfies

$$S(0) \doteq S_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \omega_0 \end{bmatrix} , \tag{3.14}$$

where $\lambda_{1,2}(0) \doteq \pm\jmath\omega_0$, with $\omega_0 > 0$, are the eigenvalues of the equilibrium at the origin crossing the imaginary axis at $\mu = 0$, according to Theorem 4;

iii) it locally satisfies the constraints

$$\begin{cases} A(\mu) \in \mathbb{R} \\ B^2(\mu) \in \mathbb{R} : \; B^2(\mu) > 0 \\ C(\mu) \in \mathbb{C} \\ \omega(\mu) \in \mathbb{R} : \; \omega(\mu) > 0 \end{cases} \tag{3.15}$$

on just one side of the Hopf bifurcation, i.e. for a single branch of the values of μ.

The computation of such a $S(\mu)$ allows one to state the super or subcritical nature of the Hopf bifurcation, as explained in the following theorem.

Theorem 18. *Consider the parametric system* (3.6) *and suppose that it has a Hopf bifurcation in the origin when* $\mu = 0$. *Moreover, define* $\chi \in \{-1, +1\}$ *so that* $\chi\mu > 0$ *if the equilibrium is stable and* $\chi\mu < 0$ *otherwise. Let be* $S(\mu)$ *such that it satisfies the above conditions i), ii) and iii) and consider its* μ*-development:*

$$S(\mu) = S_0 + S_1\mu + O\left(\mu^2\right) \ , \tag{3.16}$$

where:

$$S_1 \doteq \left[\begin{array}{cccc} \xi_1 & \xi_2 & \xi_3 & \xi_4 \end{array} \right]^T \ , \tag{3.17}$$

Then, $\xi_2 \in \mathbb{R}$ *and the Hopf bifurcation is supercritical if* $\chi\xi_2 < 0$ *and subcritical if* $\chi\xi_2 > 0$.

Proof. Consider the truncated power development of $S(\mu)$:

$$\hat{S}(\mu) \doteq S_0 + S_1\mu \ . \tag{3.18}$$

Since $S(\mu)$ satisfies iii), for a sufficiently small $|\mu|$, $\hat{S}(\mu)$ must satisfy the same condition. It is straightforward to observe that this is possible only if:

$$\begin{cases} \xi_1 \in \mathbb{R} \\ \xi_2 \in \mathbb{R} \\ \xi_3 \in \mathbb{C} \\ \xi_4 \in \mathbb{R} : \ \xi_4 > 0 \ . \end{cases}$$

Moreover, the constraints (3.15) are fulfilled on just one side of the bifurcation according to the sign of ξ_2. Indeed, the condition iii) is satisfied for the positive branch of μ if $\xi_2 > 0$, while for the negative one if $\xi_2 < 0$. Therefore, the second order HB solution and so the real limit cycle (locally) exist either for $\mu > 0$, if $\xi_2 > 0$, or for $\mu < 0$, if $\xi_2 < 0$. Then, consider the coefficient χ. According to the definition, its sign states the stability property of the equilibrium in the origin. Since the nature of the Hopf bifurcation depends on the relation between the existence of the limit cycle and the stability of the fixed point, it is straightforward to check that the supercritical case corresponds to $\chi\xi_2 < 0$ and the subcritical to $\chi\xi_2 > 0$. □

3.3 The local approximation of the second order HB solution

According to Theorem 18, the behaviour of the system at the Hopf bifurcation is completely disclosed by the knowledge of just an approximation of $S(\mu)$. Therefore, in the following we will provide a method to efficiently compute (3.18).
For the sake of simplicity we will only study the case of asymmetric oscillations around the origin, that is equivalent to assume $A(\mu) \not\equiv 0$ in (3.13). However, we want to stress that analogous results can be developed for the symmetric limit cycles, as well. Indeed, the second order HB problem related to this kind of solutions corresponds to the four equations subsystem obtained from (3.10) by placing $A(\mu) \equiv 0$. Therefore, to illustrate the following method we find useful to consider only the asymmetric oscillations, which require the solution of the complete equations system.
Moreover, it is worth to recall that a system of the form (3.1) may exhibit symmetric oscillations only if its nonlinearity is odd. Hence, in all the other cases we can assume $A(\mu) \not\equiv 0$ without loss of generality.
Then, let us define the quantities:

$$M(S_0) \doteq \begin{bmatrix} \Delta N_0^T - \Delta\Gamma_0^T \\ \Delta N_1^T - \Delta\Gamma_1^T \\ \Delta N_2^T - \Delta\Gamma_2^T \end{bmatrix} , \qquad W(S_0) \doteq \begin{bmatrix} \tilde{\Gamma}_0 - \tilde{N}_0 \\ \tilde{\Gamma}_0 - \tilde{N}_0 \\ \tilde{\Gamma}_0 - \tilde{N}_0 \end{bmatrix} ,$$

where:

$$\Delta\Gamma_k^T \doteq \begin{bmatrix} 0 & 0 & 0 & \frac{\partial\Gamma}{\partial\omega}(\jmath k\omega_0;0) \end{bmatrix} , \quad \tilde{\Gamma}_k \doteq \frac{\partial\Gamma}{\partial\mu}(\jmath k\omega_0;0) ,$$

$$\Delta N_k^T \doteq \begin{bmatrix} \frac{\partial N_k}{\partial A}(S_0;0) & \frac{\partial N_k}{\partial B^2}(S_0;0) & \frac{\partial N_k}{\partial C}(S_0;0) & \frac{\partial N_k}{\partial\omega}(S_0;0) \end{bmatrix} ,$$

$$\tilde{N}_k \doteq \frac{\partial N_k}{\partial\mu}(S_0;0) .$$

Theorem 19. *Consider the parametric system* (3.6) *and suppose that it undergoes a Hopf bifurcation in the origin at* $\mu = 0$*. Assume that* $S(\mu)$ *is an asymmetric oscillation, which satisfies the conditions i), ii) and iii), and let be* (3.18) *its truncated power development. Then, if* $\det M(S_0) \neq 0$*, it turns out that:*

$$S_1 = M^{-1}(S_0)W(S_0) \ . \tag{3.19}$$

Proof. Consider the power development of $S(\mu)$ along with the definition (3.17) and let us define its second order term as:

$$S_2 \doteq \begin{bmatrix} \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 \end{bmatrix}^T \ .$$

Moreover, let

$$\begin{aligned}\Gamma(\jmath k\omega;\mu) &= \Gamma(\jmath k\omega_0;0) + \frac{\partial\Gamma}{\partial\omega}(\jmath k\omega_0;0)(\omega-\omega_0)+\\ &+\frac{\partial\Gamma}{\partial\mu}(\jmath k\omega_0;0)\mu + O^{[2]}(\omega,\mu) =\\ &= \Gamma(\jmath k\omega_0;0) + \frac{\partial\Gamma}{\partial\omega}(\jmath k\omega_0;0)\xi_4\mu + \frac{\partial\Gamma}{\partial\mu}(\jmath k\omega_0;0)\mu + O(\mu^2) =\\ &= \Gamma(\jmath k\omega_0;0) + \left(\Delta\Gamma_k^T S_1 + \tilde{\Gamma}_k\right)\mu + O(\mu^2)\end{aligned}$$

and

$$\begin{aligned}N_k(S(\mu);\mu) &= N_k(S_0;0) + \frac{\partial N_k}{\partial A}(S_0;0)A(\mu) + \frac{\partial N_k}{\partial B^2}(S_0;0)B^2(\mu)+\\ &+\frac{\partial N_k}{\partial C}(S_0;0)C(\mu) + \frac{\partial N_k}{\partial\omega}(S_0;0)(\omega(\mu)-\omega_0)+\\ &+\frac{\partial N_k}{\partial\mu}(S_0;0)\mu + O^{[2]}\left(A,B^2,C,\omega,\mu\right) =\\ &= N_k(S_0;0) + \frac{\partial N_k}{\partial A}\xi_1\mu + \frac{\partial N_k}{\partial B^2}\xi_2\mu + \frac{\partial N_k}{\partial C}\xi_3\mu+\\ &+\frac{\partial N_k}{\partial\omega}\xi_4\mu + \frac{\partial N_k}{\partial\mu}\mu + O\left(\mu^2\right) =\\ &= N_k(S_0;0) + \left(\Delta N_k^T S_1 + \tilde{N}_k\right)\mu + O\left(\mu^2\right)\end{aligned}$$

be the μ-developments of $\Gamma(\jmath k\omega;\mu)$ and $N_k(S(\mu);\mu)$. Thus, the algebraic equations system (3.13) assumes the form:

$$\left\{\begin{array}{l}(N_0(S(\mu);\mu)-\Gamma(0;\mu))\,A(\mu)=(N_0(S_0;0)-\Gamma(0;0))\,\xi_1\mu+\\ \quad+\left[\left(\tilde{N}_0-\tilde{\Gamma}_0\right)+\left(\Delta N_0^T-\Delta\Gamma_0^T\right)S_1\right]\xi_1\mu^2+\\ \quad+(N_0(S_0;0)-\Gamma(0;0))\,\zeta_1\mu^2+O\left(\mu^3\right)=0\\ (N_1(S(\mu);\mu)-\Gamma(\jmath\omega;\mu))\,B^2(\mu)=(N_1(S_0;0)-\Gamma(\jmath\omega_0;0))\,\xi_2\mu+\\ \quad+\left[\left(\Delta N_1^T-\Delta\Gamma_1^T\right)S_1+\left(\tilde{N}_1-\tilde{\Gamma}_1\right)\right]\xi_2\mu^2+\\ \quad+(N_1(S_0;0)-\Gamma(\jmath\omega_0;0))\,\zeta_2\mu^2+O\left(\mu^3\right)=0\\ (N_2(S(\mu);\mu)-\Gamma(\jmath2\omega;\mu))\,C(\mu)=(N_2(S_0;0)-\Gamma(\jmath2\omega_0;0))\,\xi_3\mu+\\ \quad+\left[\left(\Delta N_2^T-\Delta\Gamma_2^T\right)S_1+\left(\tilde{N}_2-\tilde{\Gamma}_2\right)\right]\xi_3\mu^2+\\ \quad+(N_2(S_0;0)-\Gamma(\jmath2\omega_0;0))\,\zeta_3\mu^2+O\left(\mu^3\right)=0\ .\end{array}\right.$$

Then, balancing the first and the second power of μ and considering only the asymmetric oscillations ($A(\mu)\not\equiv 0$) along with the constraints of condition iii), we find the two following equation systems:

$$\left\{\begin{array}{l}(N_0(S_0;0)-\Gamma(0;0))=0\\ (N_1(S_0;0)-\Gamma(\jmath\omega_0;0))=0\\ (N_2(S_0;0)-\Gamma(\jmath2\omega_0;0))=0\ ,\end{array}\right. \tag{3.20}$$

$$M(S_0)S_1=W(S_0)\ , \tag{3.21}$$

in the only unknown S_1, since S_0 is already defined by condition ii). Observe that it exists at least one couple (S_0,S_1), that solves both (3.20) and (3.21), because the Proposition 16 states the existence of the second order HB solution $\tilde{y}_p(t;\mu)$ and then of a related $S(\mu)$ satisfying i), ii) and iii). Therefore, if $\det M(S_0)\neq 0$, the system (3.21) admits only one solution S_1, which necessarily turns out to be the first order component of the desired $S(\mu)$. Such a solution can be directly computed as in (3.19). □

It is worth to observe that $A(\mu)\not\equiv 0$ does not necessary mean $\xi_1\neq 0$. Indeed, let us consider the proof of Theorem 19 and let be $\tau>1$ the order of the first term of the power development of $A(\mu)$, which is not null. Then, it is straightforward to check that the first equations of system (3.20) and (3.21) are obtained by balancing the power terms of order τ and $(\tau+1)$ in the first equation of the HB problem.

Corollary 20. *Under the hypothesis of the Theorem 19, the* $\hat{S}(\mu)$ *defined in* (3.18) *along with the condition* (3.19) *provides the following local approximation of the real periodic solution* $y_p(t;\mu)$*:*

$$y_{app}(t;\mu) = \xi_1\mu + \sqrt{\xi_2\mu}\cos(\omega_0 + \xi_4\mu)t + \\ + \Re[\xi_3]\,\mu\cos 2(\omega_0 + \xi_4\mu)t + \Re[\xi_3]\,\mu\sin 2(\omega_0 + \xi_4\mu)t \ .$$

Proof. It sufficient to observe that $\hat{S}(\mu)$ is a local approximation of $S(\mu)$, which identifies $\tilde{y}_p(t;\mu)$, being the latter essentially close to $y_p(t;\mu)$ as stated in the Theorem 17. □

Remark 21. *Since* $S(\mu)$ *is a locally unique solution for a sufficiently small* $|\mu|$*, it is straightforward to note that the coefficients of its power development* (3.16) *have to be univocally defined as well. This observation means that, fixed the proper* S_0*, there is just one* S_1 *solving the system* (3.21)*. Therefore, the condition* $\det M(S_0) \neq 0$ *of Theorem 19 is always satisfied in the case of asymmetric oscillations. Then, in particular, such a result turns out to be true for every system whose nonlinearity is not odd.*

The above Theorem 19 provides an effective tool for the study of the nature of the Hopf bifurcation. Its main idea is that one can check the solvability of the second order HB problem through a local approximation. Hence, with an analogous approach, one can think to locally approximate the whole system (3.6). Then, according to this reasoning, it turns out to be sufficient to consider a proper truncation of the power development of the nonlinearity g:

$$g\left(y^{(n-1)},\dots,\dot{y},y\right) = \sum_{i=2}^{\infty} g^{[i]}\left(y^{(n-1)},\dots,\dot{y},y\right) \ . \tag{3.22}$$

The crucial point of this approach is the number of terms that one has to pick up to properly approximate the system behaviour at the bifurcation. This problem is studied in the following result.

Proposition 22. *If the nonlinearity power development* (3.22) *is such that* $g^{[2]}$ *is different from the null function, then the bifurcation can be studied just employing the following nonlinear truncation:*

$$\hat{g}\left(y^{(n-1)},\dots,\dot{y},y\right) = g^{[2]}\left(y^{(n-1)},\dots,\dot{y},y\right) + g^{[3]}\left(y^{(n-1)},\dots,\dot{y},y\right) \ .$$

Proof. This results can be derived from the theory of Allwrigth and Mees [Allwright, 1977; Mees, 1981]. Indeed, their procedure, which puts the second order HB problem into correspondence with the general one, outlines the degree of the negligible error and it is set up for exactly the above case. □

Chapter 4

The Hopf bifurcation in control systems

In the previous chapters we have developed some analytical tools to effectively study the nature of the Hopf bifurcation at the origin in autonomous differential equation systems of the form (2.16) and (3.1). Starting from the Howard's state space approach, we have extended that result to the differential equation class, obtaining a direct correspondence between the coefficient of curvature and the parameters which (locally) describe the system. It is worth to underline that the complexity of this method quickly grows up depending on the system dimension n, because of the many Kronecker products needed to compute β_2.
Besides this result, we have also developed an original and effective method based on the second order HB problem. Observe that this procedure turns out to be more suitable for the analitycal approach, since it has a minor computational demand. Indeed, the inverse matrix $M^{-1}(S_0)$ in (3.19) is not affected by the dimension n. However, the higher the system dimension is, the more complex the N_i, $i = 0, 1, 2$, are. Nonetheless, if we consider the local development introduced in Proposition 22, the computation of such functions can be systematically tackled, since they depends on quadratic and cubic powers of the periodic function and its derivatives. Moreover, it is worth to recall that the HB approach not only gives information about the bifurcation's nature, but also it provides a local approximation of the real limit cycle, thus resulting more appropriate when the features of the arising

periodic solution turn out to be important.
In this chapter we are interested in the application of the above results to the control systems, so to provide effective tools to induce and control the Hopf bifurcation.

4.1 The problem set up

The methods presented in the previous chapters refer to autonomous models, which do not have any kind of exogenous input. It is known that a control system can be transformed into such a form just choosing as the input a proper feedback, that in general can be a generic function of the internal signals of the system.
Moreover, the techniques developed in Chapters 2 and 3 can be applied only to systems in the differential equation form, and then in particular to the state space models, which satisfy the sufficient conditions presented in Proposition 1. Hence, since the feedback can play an important role in satisfying such constraints, in the following we will focus our attention into the state space systems, which are feedback connected with a possibly nonlinear operator of their state and which admit the differential equation form.
A further enlargement of the systems class, that can be studied with our techniques, can be achieved by observing that the Hopf bifurcation is a local phenomenon and that its features can be completely disclosed via a local analysis. To this regard, refer to the problem formulation (2.10) and to Proposition 22. In other words, our approaches can be employed not only to the differential equation systems, but also to all the state space models which can be locally transformed into the differential equation form, being the latter operation influenced by the nature of the feedback input. We want to stress this point, since the state space systems class does not have any tools, which turn out to be so effective as the method developed in the previous chapters.
In the following, to provide a comprehensive description of this approach without bothering the reader with too many computations, we will only refer

to the third order state space systems with scalar input:

$$\begin{cases} \dot{x} = F(x,u) = Ax + Bu + \hat{f}(x,u) \\ u = H(x) = Cx + h(x) \end{cases} , \tag{4.1}$$

where: $x \in \mathbb{R}^n$ is the state of the system, $u \in \mathbb{R}$ is the feedback input, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$, $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ the state function, $H : \mathbb{R}^n \to \mathbb{R}$ the feedback operator and $\hat{f} : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ are pure nonlinear functions. According to the above reasoning, the functions $\hat{f}$ and h will be approximated by their local power developments, so to have only quadratic and cubic nonlinear terms in (4.1).
Since here the interest is in the development of the main line of the approach, we will consider only *linearly controllable* systems. For this class the eigenvalues of the matrix $(A + BC)$ of the linearization at the origin can be freely set, focusing the attention on the nature of the Hopf bifurcation rather than on its birth.

4.2 The control normal forms

The first normal forms theory was due to the Poincaré studies on the equivalent representations of a system, obtained by the application of homogeneous transformations. The Poincaré's results have been employed successfully in the nonlinear vector fields area and many different normal forms have been developed in several frameworks [Arnold, 1983; Baider, 1989; Kuznetsov, 1998; Wiggins, 2003]. In the control system field the *Brunovsky form* turns out to be particularly useful in the realization of nonlinear control actions [Brunovsky, 1970; Kailath, 1980]:

$$\dot{x} = Ax + Bu , \tag{4.2}$$

being:

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix} , \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} . \tag{4.3}$$

If a nonlinear control system can be put into the form (4.2)-(4.3) through a change of coordinates and input, the design of the controller becomes particularly easy.
Unfortunately, only few systems admit this normal form. However, the linear part of every linearly controllable system (4.1) can always assume the form (4.3) by mean of a linear transformation. Therefore, in the following we will suppose that the linearized system is already in the Brunovsky form.
Since the form (4.2)-(4.3) turns out to be too restrictive, in [Kang and Krener, 1992] the authors developed a quadratic normal form that results an "extension" of the Brunovsky form and that can be locally assumed by every linearly controllable system (4.1) affine in the control, i.e. such that:

$$\hat{f}(x,u) = f(x) + g(x)u \; . \tag{4.4}$$

In [Kang, 1994] Kang improved that result, introducing *extended controller normal forms* of arbitrary degree.

Proposition 23. *Consider a linearly controllable system* (4.1) *satisfying* (4.3) *and* (4.4). *Moreover, let be* $f \in C^{d+1}$ *and* $g \in C^d$. *Then, the system can be (locally) transformed into the extended controller normal form of order* d*:*

$$\dot{z} = Az + Bv + \sum_{k=2}^{d} \tilde{f}^{[k]}(z) + O\left(z,v\right)^{d+1} \quad , \tag{4.5}$$

where

$$\tilde{f}_i^{[k]}(z) \doteq \begin{cases} \sum_{j=i+2}^{n} p_{ij}^{[k-2]}(z_1,\ldots,z_j)z_j^2 & ,1 \le i \le n-2 \\ 0 & ,i = n-1, n \; , \end{cases} \tag{4.6}$$

being $p_{ij}^{[k-2]}(z_1,\ldots,z_j) : \mathbb{R}^j \to \mathbb{R}$ *an homogeneous polynomial of degree* $(k-2)$.

Proof. For a detailed proof see Kang [1994]. □

Remark 24. *The normal form* (4.5) *derives from the initial control system through the application of proper homogeneous transformations of order* $k \geq 2$*:*

$$\begin{cases} x = z + \xi^{[k]}(z) \\ u = v + \mu^{[k]}(z,v) \; . \end{cases} \tag{4.7}$$

Each transformation (4.7) *can change only the terms strictly greater than* $k-1$*, leaving the others unchanged.*

Observe that the extended controller normal form theory perfectly fits our approach. Indeed, it performs a local analysis of the linearly controllable systems and provides a suitable power development which can be always satisfied. Therefore, the normal forms turns out to be a powerful tool that we can employ to settle our approach in a general and comprehensive framework. In such a way, we can apply all the mathematics developed in the previous chapters to specific system equations without any loss of generality, since every system can be locally represented as a normal form.

Before proceeding, it is worth to introduce a further result in the theory of the extended controller normal forms, which allows one to consider a wider class of dynamical systems, just relaxing condition (4.4) on the scalar input.

Proposition 25. *Consider a linearly controllable system* (4.1) *satisfying* (4.3). *Moreover, suppose that* $\hat{f} \in C^{d+1}$ *jointly on its arguments. Then, by suitable changes of coordinates and input of the type* (4.7)*, the system can be locally transformed into the following normal form of degree* d*:*

$$\dot{z} = Az + Bv + \sum_{k=2}^{d} \tilde{f}^{[k]}(z) + O\left(z,v\right)^{d+1} \ , \tag{4.8}$$

where we have defined

$$z_{n+1} \doteq v \tag{4.9}$$

and

$$\tilde{f}_i^{[k]}(z) \doteq \begin{cases} \sum_{j=i+2}^{n+1} p_{ij}^{[k-2]}(z_1,\ldots,z_j) z_j^2 & ,1 \le i \le n-1 \\ 0 & ,i = n \ , \end{cases} \tag{4.10}$$

being $p_{ij}^{[k-2]}(z_1,\ldots,z_j) : \mathbb{R}^j \to \mathbb{R}$ *an homogeneous polynomial of degree* $(k-2)$.

Proof. For a detailed proof see Kang and Krener [2005]. ☐

4.3 The Hopf bifurcation in the normal form systems

In this section we want to introduce the study of the Hopf bifurcation in controller normal form systems by the application of the mathematical tools previously developed. As already anticipated, we will describe the overall procedure for the third order linearly controllable systems class, being this choice suitable to illustrate the method without having to handle singular cases and plenty of computations.
Then, consider the result of Proposition 22. Although here introduced for systems in the differential equation form, the original theory by Allwrigth and Mees [Allwright, 1977; Mees, 1981] is formulated in the more general state space case. Hence, if we assume the non restrictive hypothesis that in the system local power development at least one between the quadratic and the cubic part is present, we can study the Hopf bifurcation just through the analysis of the terms up to the cubic one.

Then, consider the third order cubic normal form affine in the control

$$\begin{cases} \dot{z}_1 = z_2 + b_1 z_3^2 + (c_1 z_1 + c_2 z_2 + c_3 z_3) z_3^2 \\ \dot{z}_2 = z_3 \\ \dot{z}_3 = v \ . \end{cases} \tag{4.11}$$

Our aim is to build up a state feedback control input such that the system can be transformed into the differential equation form. In particular, we are interested in finding a direct correspondence between the parameters of the model (4.11) and the second order approximation of its limit cycle arising at the Hopf bifurcation. Moreover, we want to study the degrees of freedom of such a control input. Observe that in our framework the normal forms are used as local representations, thus, since they are reached by means of the homogeneous transformation (4.7), in general we can assume that:

$$v(z) = v^{[1]}(z) + v^{[2]}(z) + v^{[3]}(z) \ . \tag{4.12}$$

Then, the following result holds.

Proposition 26. *Consider the normal form* (4.11) *and choose the state feedback control input*

$$v(z) = -a_3 z_1 - a_2 z_2 - a_1 z_3 + \gamma b_1 z_3^2 + \gamma (c_1 z_1 + c_2 z_2 + c_3 z_3) z_3^2 \ , \tag{4.13}$$

being $\gamma \in \mathbb{R}$ and $a_i \in \mathbb{R}$, $i = 1, 2, 3$. Then, the feedback system (4.11) *and* (4.13) *admits the the differential equation form*

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = \\ = \left(\gamma\ddot{y} - a_3\dot{y}\right)^2 \big(b_1 + (\gamma c_1 + a_2 c_1 - a_3 c_2)y + \\ + (\gamma c_2 + a_1 c_1 - a_3 c_3)\dot{y} + (\gamma c_3 + c_1)\ddot{y}\big) \ . \tag{4.14}$$

Proof. It is straightforward to check that the system satisfies the second sufficient transformation condition (1.12) of Paragraph 1.4, where:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} , \quad H = \begin{bmatrix} 1 \\ 0 \\ \gamma \end{bmatrix} ,$$

$$\phi(z) = b_1 z_3^2 + (c_1 z_1 + c_2 z_2 + c_3 z_3) z_3^2 \ .$$

Thus, according to (1.15):

$$\left(\tilde{G} \circ y\right)(t) = \operatorname{adj}\left(\mathcal{D}I - A\right)Hy(t) = \begin{bmatrix} \mathcal{D}^2 + a_1\mathcal{D} + \gamma + a_2 \\ \gamma\mathcal{D} - a_3 \\ \gamma\mathcal{D}^2 - a_3\mathcal{D} \end{bmatrix} y(t) \ .$$

Finally, from the (1.16) (1.16) it follows

$$\det(\mathcal{D}I - A)y(t) = \dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = \phi\left((\tilde{G} \circ y)(t)\right) \ ,$$

that is equivalent to (4.14). □

Remark 27. *According to the Proposition 26, the cubic normal form* (4.11) *of a third order system affine in the control can be always transformed into the differential equation form just exploiting a control input with one degree of freedom. Observe that some specific normal forms of this class may be transformed by means of inputs with more degrees of freedom, but in general it always exists a non restrictive control (i.e. with at least one degree of freedom), which realizes the transformation.*

It is worth to recall that the parameters of the normal form are the *invariants* of the system under the homogeneous transformations (4.7) (see Kang [1994]). Therefore, the computation of the limit cycle features with respect to these coefficients gives a direct connection between the periodic solution and the original system.
In the following, we will provide an example to illustrate how the invariants can be related to the limit cycle just exploiting the mathematical tools developed in the previous chapters.

Example 3. *Consider the normal form* (4.11) *and the control input* (4.13). *For the sake of simplicity, let us assume that* $\gamma = 0$. *Thus, the equivalent differential equation form* (4.14) *becomes:*

$$\begin{aligned} &\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = \\ &= (-a_3\dot{y})^2 \left(b_1 + (a_2c_1 - a_3c_2)y + (a_1c_1 - a_3c_3)\dot{y} + c_1\ddot{y}\right) = \\ &= b_1a_3^2\dot{y}^2 + a_3^2(a_2c_1 - a_3c_2)\dot{y}^2 y + a_3^2(a_1c_1 - a_3c_3)\dot{y}^3 + c_1a_3^2\ddot{y}\dot{y}^2 \ . \end{aligned}$$

Observe that in such a situation the control input can only "activate" the Hopf bifurcation by setting the coefficients a_i, $i = 1, 2, 3$, *of the linear part. This is equivalent to determine the eigenvalues of the equilibrium at the origin. Then, let us suppose the following dependences on the bifurcation parameter* μ*:*

$$\begin{cases} b_1 = b_{10} \\ c_i = c_{i0}\,, \quad i = 1, 2, 3 \\ a_3 = a_{20}a_{10} \\ a_2 = a_{20} - \mu\,, \quad a_{20} > 0 \\ a_1 = a_{10} > 0 \ . \end{cases} \tag{4.15}$$

We want to highlight that the (4.15) *describe the transversal passage of a complex pair of conjugate eigenvalues of the origin through the imaginary axis at* $\mu = 0$. *We neglect to report this computation, since it is not our primary aim. Moreover, it is straightforward to check that the equilibrium in the origin is stable for* $\mu < 0$ *and unstable for* $\mu > 0$. *Hence, it follows from Theorem 18 that* $\chi = -1$.
Then, let us compute the second order HB problem (3.10). *The first equation*

is pure real and it assumes the form:

$$
\begin{aligned}
a_3 A = {} & a_3 b_1 a_3 \left(\frac{1}{2} B^2 + 2P^2 + 2Q^2 \right) \omega^2 + \\
& - \left(a_3^2 c_2 a_3 - a_3^2 c_1 a_2 \right) \left(\frac{1}{2} AB^2 + \frac{3}{4} B^2 P + 2AP^2 + 2AQ^2 \right) \omega^2 + \\
& + \frac{3}{2} \left(a_3^2 c_3 a_3 - a_3^2 c_1 a_1 \right) B^2 Q \omega^3 \ .
\end{aligned} \tag{4.16}
$$

The second one, instead, is complex and it can be separated into two scalar equations:

$$
\begin{cases}
-a_1 \omega^2 + a_3 = 2 a_3 b_1 a_3 P \omega^2 - (a_3^2 c_2 a_3 - a_3^2 c_1 a_2) \cdot \\
\qquad \cdot \left(\frac{1}{4} B^2 + 2AP + 2P^2 + 2Q^2 \right) \omega^2 + \\
\qquad - a_3^2 c_1 \left(\frac{1}{4} B^2 + 2P^2 + 2Q^2 \right) \omega^4 \\
\omega^3 - a_2 \omega = 2 a_3 b_1 a_3 Q \omega^2 - 2 \left(a_3^2 c_2 a_3 - a_3^2 c_1 a_2 \right) AQ \omega^2 + \\
\qquad + \left(a_3^2 c_3 a_3 - a_3^2 c_1 a_1 \right) \left(\frac{3}{4} B^2 + 6P^2 + 6Q^2 \right) \omega^3 \ .
\end{cases} \tag{4.17}
$$

Finally, the third one is complex too and it can be divided into the two scalar equations:

$$
\begin{cases}
-8 \omega^3 Q - 4 a_1 \omega^2 P + 2 a_2 \omega Q + a_3 P = \\
\qquad = -\frac{1}{2} a_3 b_1 a_3 B^2 \omega^2 - (a_3^2 c_2 a_3 - a_3^2 c_1 a_2) \cdot \\
\qquad \cdot \left(-\frac{1}{2} AB^2 + \frac{1}{2} B^2 P + P^3 + PQ^2 \right) \omega^2 + \\
\qquad - \left(a_3^2 c_3 a_3 - a_3^2 c_1 a_1 \right) \left(3B^2 Q + 6P^2 Q + 6Q^3 \right) \omega^3 + \\
\qquad + a_3^2 c_1 \left(2B^2 P - 4P^3 - 4PQ^2 \right) \omega^4 \\
8 \omega^3 P - 4 a_1 \omega^2 Q - 2 a_2 \omega P + a_3 Q = \\
\qquad = - \left(a_3^2 c_2 a_3 - a_3^2 c_1 a_2 \right) \left(\frac{1}{2} B^2 Q + P^2 Q + Q^3 \right) \omega^2 + \\
\qquad + \left(a_3^2 c_3 a_3 - a_3^2 c_1 a_1 \right) \left(3B^2 P + 6P^3 + 6PQ^2 \right) \omega^3 + \\
\qquad + a_3^2 c_1 \left(2B^2 Q - 4P^2 Q - 4Q^3 \right) \omega^4 \ .
\end{cases} \tag{4.18}
$$

Exploiting the relations (4.15) *between the parametric set and* μ*, we can compute the matrix* $M(S_0)$ *and the vector* $W(S_0)$ *of Theorem 19 of Paragraph 3.3:*

$$M(S_0)=\begin{bmatrix} a_{20}a_{10} & -\frac{1}{2}b_{01}a_{20}^3a_{10}^2 & 0 & 0 & 0 \\ 0 & \frac{1}{4}c_{20}a_{20}^3a_{10}^2 & -2b_{01}a_{20}^3a_{10}^2 & 0 & -2a_{10}\sqrt{a_{20}} \\ 0 & -\frac{3}{4}a_{20}^3a_{10}^2(c_{30}a_{20}a_{10}-c_{10}a_{10})\sqrt{a_{20}} & 0 & -2b_{01}a_{20}^3a_{10}^2 & 2a_{20} \\ 0 & \frac{1}{2}b_{01}a_{20}^3a_{10}^2 & -3a_{20}a_{10} & -6a_{20}\sqrt{a_{20}} & 0 \\ 0 & 0 & 6a_{20}\sqrt{a_{20}} & -3a_{20}a_{10} & 0 \end{bmatrix},$$

$$W(S_0)=\begin{bmatrix} 0 \\ 0 \\ -\sqrt{a_{20}} \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, when

$$\begin{aligned} &\det M(S_0)= \\ &= 9a_{20}^5a_{10}^2\Big[\left(2a_{20}+\frac{1}{2}a_{10}^2\right)\left(a_{20}^3a_{10}^2c_{20}+3a_{20}^2a_{10}^3c_{10}-3a_{20}^3a_{10}^3c_{30}\right)+ \\ &-2b_{01}^2a_{20}^4a_{10}^4\Big]\neq 0 \end{aligned}$$

the condition of Theorem 19 is satisfied and we can compute the vector S_1 *defined in Theorem 18, which provides the information for a local description of the limit cycle:*

$$\begin{aligned} S_1 &= M^{-1}(S_0)W(S_0)=\begin{bmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 \end{bmatrix}^T = \\ &= \frac{3a_{20}^4a_{10}^2}{\det M(S_0)}\begin{bmatrix} -3a_{20}^2a_{10}b_{10}\left(4a_{20}+a_{10}^2\right) \\ -6\left(4a_{20}+a_{10}^2\right) \\ -a_{20}^2a_{10}^3b_{10} \\ -2a_{20}^2a_{10}^2b_{10}\sqrt{a_{20}} \\ a_{20}\sqrt{a_{20}}\left(3a_{20}^3a_{10}^2c_{20}-\frac{3}{4}a_{10}^2+a_{20}^3a_{10}^4b_{10}^2\right) \end{bmatrix}. \end{aligned}$$

According to Theorem 18, we can state the nature of the Hopf bifurcation by studying the sign of $\chi\xi_2$*:*

$$\chi\xi_2=\frac{1}{\det M(S_0)}18a_{20}^4a_{10}^2\left(4a_{20}+a_{10}^2\right).$$

Then, the bifurcation turns out to be supercritical if $\chi\xi_2 < 0$, *i.e. when*

$$a_{20}c_{20} + 3a_{10}c_{10} - 3a_{20}a_{10}c_{30} < \frac{2b_{01}^2 a_{20}^2 a_{10}^2}{2a_{20} + \frac{1}{2}a_{10}^2} \ ,$$

and subcritical if $\chi\xi_2 > 0$, *i.e. when*

$$a_{20}c_{20} + 3a_{10}c_{10} - 3a_{20}a_{10}c_{30} > \frac{2b_{01}^2 a_{20}^2 a_{10}^2}{2a_{20} + \frac{1}{2}a_{10}^2} \ .$$

Proposition 26 refers to the normal forms affine in the control, that is satisfying the condition (4.4). In the following we want to extend this approach to the general case. Therefore, let us consider the generic third order cubic normal form:

$$\begin{cases} \dot{z}_1 = z_2 + p_{13}(z) z_3^2 + p_{14}(z, v) v^2 \\ \dot{z}_2 = z_3 + p_{24}(z, v) v^2 \\ \dot{z}_3 = v \end{cases} , \tag{4.19}$$

where:

$$\begin{aligned} p_{13}(z) &\doteq b_1 + c_1 z_1 + c_2 z_2 + c_3 z_3 \ , \\ p_{14}(z, v) &\doteq b_2 + c_4 z_1 + c_5 z_2 + c_6 z_3 + c_7 v \ , \\ p_{24}(z, v) &\doteq b_3 + c_8 z_1 + c_9 z_2 + c_{10} z_3 + c_{11} v \ , \end{aligned} \tag{4.20}$$

Then, the following statement holds.

Proposition 28. *Consider the controller normal form* (4.19)-(4.20) *and the homogeneous state feedback control input* (4.12). *Then, the system locally satisfies the transformation condition* (1.11) *or* (1.12), *during any possible bifurcation process of the equilibrium in the origin, only if it belongs to at least one of the following classes:*

i)

$$\begin{cases} \dot{z}_1 = z_2 + h_1 \varphi(z, v) v^2 \\ \dot{z}_2 = z_3 + h_2 \varphi(z, v) v^2 \\ \dot{z}_3 = v \ , \end{cases}$$

$$\begin{aligned} v(z) &= v^{[1]}(z) + h_3 \varphi\left(z, v^{[1]}(z)\right) \left(v^{[1]}(z)\right)^2 + \\ &\quad + 2h_3^2 \left(\varphi^{[0]}(z, v(z))\right)^2 \left(v^{[1]}(z)\right)^3 \ , \\ \varphi(z, v) &= \varphi^{[0]}(z, v) + \varphi^{[1]}(z, v) \ ; \end{aligned}$$

ii)

$$\begin{cases} \dot{z}_1 = z_2 + h_1\psi(z)z_3^2 + h_1\varphi(z,v)v^2 \\ \dot{z}_2 = z_3 \\ \dot{z}_3 = v \ , \end{cases}$$

$$\begin{aligned} v(z) &= v^{[1]}(z) + h_3\psi(z)z_3^2 + h_3\varphi\left(z, v^{[1]}(z)\right)\left(v^{[1]}(z)\right)^2 + \\ &\quad + 2h_3^2\varphi^{[0]}\left(z, v^{[1]}(z)\right)v^{[1]}(z) \ , \left(\psi^{[0]}(z)z_3^2 + \varphi^{[0]}\left(z, v^{[1]}(z)\right)\left(v^{[1]}(z)\right)^2\right) \\ \psi(z) &= \psi^{[0]}(z) + \psi^{[1]}(z) \ , \\ \varphi(z) &= \varphi^{[0]}(z,v) + \varphi^{[1]}(z,v) \ ; \end{aligned}$$

iii)

$$\begin{cases} \dot{z}_1 = z_2 + \varphi(v)v^2 \\ \dot{z}_2 = z_3 + \psi(v)v^2 \\ \dot{z}_3 = v \ , \end{cases}$$

$$\begin{aligned} v(z) &= v^{[1]}(z) + \gamma_2\left(v^{[1]}(z)\right)^2 + \gamma_3\left(v^{[1]}(z)\right)^3 \ , \\ \varphi(v) &= \varphi^{[0]}(v) + \varphi^{[1]}(v) \ , \\ \psi_1(v) &= \psi^{[0]}(v) + \psi^{[1]}(v) \ . \end{aligned}$$

Proof. First, observe that the transformation conditions must be satisfied for any possible choice of $v^{[1]}(z)$. Indeed, the eigenvalues of the equilibrium depend directly on $v^{[1]}(z)$. Thus, the system must be transformable for any possible value of the linear component of the feedback control.

Consider the first sufficient condition (1.11).
Then, the quadratic part of the system (4.19) driven by the feedback control input (4.12) must satisfy

$$\begin{cases} h_1\phi^{[2]}(z) = p_{13}^{[0]}(z)z_3^2 \quad + p_{14}^{[0]}(z, v(z))\left(v^{[1]}(z)\right)^2 \\ h_2\phi^{[2]}(z) = \qquad\qquad\quad p_{24}^{[0]}(z, v(z))\left(v^{[1]}(z)\right)^2 \\ h_3\phi^{[2]}(z) = v^{[2]}(z) \end{cases}$$

for any possible $v^{[1]}(z)$. The following scenarios may happen.

1) Let $p_{13}^{[0]}(z) = 0$. In such a case from the first two equations it follows that:

$$\begin{aligned} p_{14}^{[0]}(z,v) &= h_1\varphi^{[0]}(z,v) \ , \\ p_{24}^{[0]}(z,v) &= h_2\varphi^{[0]}(z,v) \ , \end{aligned}$$

being:

$$\phi^{[2]}(z) = \varphi^{[0]}(z, v(z)) \left(v^{[1]}(z)\right)^2 ,$$

If $h_1 = h_2 = 0$, then the third equation does not provide any constraint on the choice of $v^{[2]}(z)$, otherwise:

$$v^{[2]}(z) = h_3 \varphi^{[0]}(z, v(z)) \left(v^{[1]}(z)\right)^2 .$$

2) Let $p_{13}^{[0]}(z) \neq 0$, $p_{14}^{[0]}(z, v) \neq 0$ and $p_{24}^{[0]}(z, v) = 0$ and define:

$$\begin{aligned} p_{13}^{[0]}(z) &= h_1 \psi^{[0]}(z) , \\ p_{14}^{[0]}(z, v) &= h_1 \varphi^{[0]}(z, v) . \end{aligned}$$

From the second equation, neglecting the case $\phi(z) \equiv 0$, that corresponds to the trivial case where the system is linear, it follows $h_2 = 0$. Thus, we have

$$\phi^{[2]}(z) = \psi^{[0]}(z) z_3^2 + \varphi^{[0]}(z, v(z)) \left(v^{[1]}(z)\right)^2$$

and the input turns out to be constrained to the form:

$$v^{[2]}(z) = h_3 \psi^{[0]}(z) z_3^2 + h_3 \varphi^{[0]}(z, v(z)) \left(v^{[1]}(z)\right)^2$$

3) Let $p_{13}^{[0]}(z) \neq 0$, $p_{14}^{[0]}(z, v(z)) \neq 0$ and $p_{24}^{[0]}(z, v(z)) \neq 0$. Then, the conditions can be satisfied only for some specific choice of $v^{[1]}(z)$. Therefore, such a case can not be considered valid.

Then, consider the cubic part of the system. It must satisfy the conditions:

$$\begin{cases} h_1 \phi^{[3]}(z) = p_{13}^{[1]}(z) z_3^2 + 2 p_{14}^{[0]}(z, v(z)) v^{[1]}(z) v^{[2]}(z) + p_{14}^{[1]}\left(z, v^{[1]}(z)\right) \left(v^{[1]}(z)\right)^2 \\ h_2 \phi^{[3]}(z) = 2 p_{24}^{[0]}(z, v(z)) v^{[1]}(z) v^{[2]}(z) + p_{24}^{[1]}\left(z, v^{[1]}(z)\right) \left(v^{[1]}(z)\right)^2 \\ h_3 \phi^{[3]}(z) = v^{[3]}(z) . \end{cases}$$

Let us check any possible situation deriving from the cases highlighted before.

1) There are some subcases.

1.a) Consider the special case of $h_1 = h_2 = 0$. Then, the equations assume the form:

$$\begin{cases} 0 = p_{13}^{[1]}(z)z_3^2 + p_{14}^{[1]}\left(z, v^{[1]}(z)\right)\left(v^{[1]}(z)\right)^2 \\ 0 = p_{24}^{[1]}\left(z, v^{[1]}(z)\right)\left(v^{[1]}(z)\right)^2 \\ h_3\phi^{[3]}(z) = v^{[3]}(z) \ . \end{cases}$$

Since these conditions must be satisfied for any $v^{[1]}(z)$, it follows that the only solution is

$$p_{13}^{[1]}(z) \equiv p_{14}^{[1]}(z, v) \equiv p_{24}^{[1]}(z, v) \equiv 0 \ .$$

Summing up all the constraints, it is straightforward to check that the starting system turns out to be linear.

1.b) If at least one between h_1 and h_2 is not null, then the constraints assume the form:

$$\begin{cases} h_1\phi^{[3]}(z) & = p_{13}^{[1]}(z)z_3^2 + \Big(2h_1h_3\left(\varphi^{[0]}(z, v(z))\right)^2 v^{[1]}(z) + \\ & \quad +p_{14}^{[1]}\left(z, v^{[1]}(z)\right)\Big)\left(v^{[1]}(z)\right)^2 \\ h_2\phi^{[3]}(z) & = \Big(2h_2h_3\left(\varphi^{[0]}(z, v(z))\right)^2 v^{[1]}(z) + \\ & \quad +p_{24}^{[1]}\left(z, v^{[1]}(z)\right)\Big)\left(v^{[1]}(z)\right)^2 \\ h_3\phi^{[3]}(z) & = v^{[3]}(z) \ . \end{cases}$$

There are some different subcases to study.

1.b.1) Let $p_{13}^{[1]}(z)$ be the constantly null polynomial. Then, the conditions are satisfied for each $v^{[1]}(z)$ only if

$$\frac{p_{14}^{[1]}(z, v)}{p_{24}^{[1]}(z, v)} = \frac{h_1}{h_2} \ ,$$

that is:

$$p_{14}^{[1]}(z, v) = h_1\varphi^{[1]}(z, v) \ ,$$
$$p_{24}^{[1]}(z, v) = h_2\varphi^{[1]}(z, v) \ ,$$

for some polynomial $\varphi^{[1]}(z, v)$. In such a case

$$\phi^{[3]}(z) = \Big(2h_3\left(\varphi^{[0]}(z, v(z))\right)^2 v^{[1]}(z) + \\ + \varphi^{[1]}\left(z, v^{[1]}(z)\right)\Big)\left(v^{[1]}(z)\right)^2$$

and the related input turns out to be:

$$v^{[3]}(z) = h_3\Big(2h_3\left(\varphi^{[0]}(z,v(z))\right)^2 v^{[1]}(z) + \\ + \varphi^{[1]}\left(z, v^{[1]}(z)\right)\Big)\left(v^{[1]}(z)\right)^2 \ .$$

The solution corresponds to the class i).

1.b.2) Let $p_{13}^{[1]}(z)$ be different from the constantly null polynomial. Then, the second equation must be null, because the transformation has to be independent from the choice of $v^{[1]}(z)$. Since we neglect the trivial case $\phi^{[3]}(z) \equiv 0$, this situation can happen only if $h_2 = 0$. Then it follows that:

$$p_{24}^{[1]}(z,v) \equiv 0 \ .$$

Hence, defining

$$p_{13}^{[1]}(z) = h_1\psi^{[1]}(z) \ , \\ p_{14}^{[1]}(z,v) = h_1\varphi^{[1]}(z,v) \ ,$$

we have

$$\phi^{[3]}(z) = \psi^{[1]}(z)z_3^2 + \Big(2h_3\left(\varphi^{[0]}(z,v(z))\right)^2 v^{[1]}(z) + \\ + \varphi^{[1]}\left(z, v^{[1]}(z)\right)\Big)\left(v^{[1]}(z)\right)^2$$

and the related input assumes the form:

$$v^{[3]}(z) = h_3\psi^{[1]}(z)z_3^2 + h_3\Big(2h_3\left(\varphi^{[0]}(z,v(z))\right)^2 v^{[1]}(z) + \\ + \varphi^{[1]}\left(z, v^{[1]}(z)\right)\Big)\left(v^{[1]}(z)\right)^2 \ .$$

Thus, the solution belongs to the class ii), along with the condition $\psi^{[0]}(z) = 0$.

2) From the condition on the quadratic part of the problem, it follows that:

$$\begin{cases} h_1\phi^{[3]}(z) &= \Big(p_{13}^{[1]}(z) + 2h_1h_3\psi^{[0]}(z)\varphi^{[0]}(z,v(z))v^{[1]}(z)\Big) z_3^2 + \\ & + \Big(2h_1h_3\left(\varphi^{[0]}(z,v(z))\right)^2 v^{[1]}(z) + p_{14}^{[1]}\left(z,v^{[1]}(z)\right)\Big)\left(v^{[1]}(z)\right)^2 \\ 0 &= p_{24}^{[1]}\left(z,v^{[1]}(z)\right)\left(v^{[1]}(z)\right)^2 \\ h_3\phi^{[3]}(z) &= v^{[3]}(z) \ . \end{cases}$$

Therefore, the system can be solved for each choice of $v^{[1]}(z)$ only if

$$p_{24}^{[1]}(z,v) \equiv 0 \ .$$

Then, the solution assumes the form:

$$\begin{aligned} p_{13}^{[1]}(z) &= h_1\psi^{[1]}(z) \ , \\ p_{14}^{[1]}(z,v) &= h_1\varphi^{[1]}(z,v) \ , \end{aligned}$$

for two generic polynomial $\psi^{[1]}(z)$ and $\varphi^{[1]}(z,v)$. Consequently,

$$\begin{aligned} \phi^{[3]}(z) = &\left(\psi^{[1]}(z) + 2h_3\psi^{[0]}(z)\varphi^{[0]}(z,v(z))v^{[1]}(z)\right) z_3^2 + \\ &+ \left(2h_3\left(\varphi^{[0]}(z,v(z))\right)^2 v^{[1]}(z) + \varphi^{[1]}(z,v)\right)\left(v^{[1]}(z)\right)^2 \end{aligned}$$

and the related input turns out to be:

$$\begin{aligned} v^{[3]}(z) = &\, h_3\left(\psi^{[1]}(z) + 2h_3\psi^{[0]}(z)\varphi^{[0]}(z,v(z))v^{[1]}(z)\right) z_3^2 + \\ &+ h_3\left(2h_3\left(\varphi^{[0]}(z,v(z))\right)^2 v^{[1]}(z) + \varphi^{[1]}(z,v)\right)\left(v^{[1]}(z)\right)^2 \ , \end{aligned}$$

The solution identifies the class ii).

Consider now the second sufficient condition.
Then, let us first highlight that we are looking for a pure nonlinearity of the form:

$$\begin{aligned} \Phi\left(C^T z\right) &= \Phi^{[2]}\left(C^T z\right) + \Phi^{[3]}\left(C^T z\right) + O(z)^4 = \\ &= K_2 \cdot \left(\varphi^{[1]}(z)\right)^2 + K_3 \cdot \left(\varphi^{[1]}(z)\right)^3 + O(z)^4 \ , \end{aligned}$$

being $K_{2,3} \in \mathbb{R}^{3\times 1}$ and $\varphi^{[1]}(z) \doteq C^T z$. Then, the quadratic part of the problem assumes the form:

$$\begin{cases} k_{21}\left(\varphi^{[1]}(z)\right)^2 = p_{13}^{[0]}(z)z_3^2 \quad + p_{14}^{[0]}(z,v(z))\left(v^{[1]}(z)\right)^2 \\ k_{22}\left(\varphi^{[1]}(z)\right)^2 = \qquad\qquad\quad p_{24}^{[0]}(z,v(z))\left(v^{[1]}(z)\right)^2 \\ k_{23}\left(\varphi^{[1]}(z)\right)^2 = v^{[2]}(z) \ . \end{cases}$$

Therefore, since the latter condition must be satisfied for each choice of $v^{[1]}(z)$, we have the following necessary condition:

$$p_{13}^{[0]}(z) = 0 \ .$$

Thus, the solution assumes the form:

$$p_{14}^{[0]}(z,v) = k_{21} \ ,$$
$$p_{24}^{[0]}(z,v) = k_{22} \ ,$$

being:

$$\varphi^{[2]}(z) = \left(v^{[1]}(z)\right)^2 \ .$$

The related input is

$$v^{[2]}(z) = k_{23}\left(v^{[1]}(z)\right)^2 \ .$$

Let us consider the cubic part along with the constraints defined above:

$$\begin{cases} k_{31}\left(\varphi^{[1]}(z)\right)^3 = p_{13}^{[1]}(z)z_3^2 \quad +2k_{21}k_{23}\left(v^{[1]}(z)\right)^3 + p_{14}^{[1]}\left(z, v^{[1]}(z)\right)\left(v^{[1]}(z)\right)^2 \\ k_{32}\left(\varphi^{[1]}(z)\right)^3 = \qquad\qquad 2k_{22}k_{23}\left(v^{[1]}(z)\right)^3 + p_{24}^{[1]}\left(z, v^{[1]}(z)\right)\left(v^{[1]}(z)\right)^2 \\ k_{33}\left(\varphi^{[1]}(z)\right)^3 = v^{[3]}(z) \ . \end{cases}$$

Because of the independence from the choice of $v^{[1]}(z)$, it turns out the following necessary condition:

$$p_{13}^{[1]}(z) \equiv 0 \ .$$

Then, the solution assumes the form:

$$p_{14}^{[1]}(z,v) = k_{14}v \ ,$$
$$p_{24}^{[1]}(z,v) = k_{24}v \ .$$

Consequently,

$$\left(\varphi^{[1]}(z)\right)^3 = \frac{2k_{22}k_{23} + k_{24}}{k_{32}}\left(v^{[1]}(z)\right)^3 = \frac{2k_{21}k_{23} + k_{14}}{k_{31}}\left(v^{[1]}(z)\right)^3 \ .$$

The related input turns out to be:

$$v^{[3]}(z) = k_{33}\frac{2k_{22}k_{23} + k_{24}}{k_{32}}\left(v^{[1]}(z)\right)^3 = k_{33}\frac{2k_{21}k_{23} + k_{14}}{k_{31}}\left(v^{[1]}(z)\right)^3 \ .$$

This structure corresponds to the class iii). □

4.4 Nonlinear homogeneous transformations

In the previous sections we have highlighted that the Hopf bifurcation is a local phenomenon and that it can be analytically investigated just employing a truncation of the power development of the system. In particular, under the conditions of the Proposition 22, the quadratic and cubic parts of the problem turn out to be sufficient to detect the nature of the limit cycle. Moreover, we have noticed that the mathematical tools for the study of the differential equation systems are more effective than the methods related to the state space models. Therefore, instead of looking for a global transformation into the differential equation form, we can limit our study to the systems which only locally admit this representation. In such a framework, the controller normal form theory turns out to be an effective approach, once we have supposed that they are driven by a state feedback control input. Indeed, the controller normal forms allow us to treat the systems in a uniform and general way, since every state space equation locally admits this model.

On this basis, we have identified a set of normal form systems, which can be studied employing the differential equation form tools, developed in the previous chapters. Moreover, we have highlighted the nature of the state feedback input, that makes possible the transformation, and its degrees of freedom.

It is worth to observe that the conditions (1.11) and (1.12) are only sufficient to grant the transformation into the differential equation form. This implies that, given two equivalent systems with different state space models, one could satisfy the transformation conditions while the other could do not. Consequently, it could happen that a controller normal form, that does not satisfy (1.11) and (1.12), could be transformed into an equivalent local representation, which in turn can be put in the differential equation form with our procedure. Therefore, we can extend our approach just focusing on the transformation into equivalent forms.

The main idea is to use the homogeneous transformations (4.7) of the state and the input. Indeed, we are looking for a local power development of the system and it is known that a transformation (4.7) of order k can change only the k-th term and the higher ones. Then, consider the system (4.1) and suppose that it is linearly controllable and that A and B are in the Brunovsky form. Moreover, let the local development of the system satisfy a sufficient transformation condition up to the order $(k-1)$, being $\hat{u}(x)$ the pure nonlinear part of the state feedback control input, that makes possible the transformation:

$$\begin{cases} \dot{x} = \tilde{A}x + f(x) + O(x)^k \\ u(x) = \sum_{i=1}^{k-1} u^{[i]}(z) = u^{[1]}(z) + \hat{u}(z) \ , \end{cases}$$

where

$$f(x) \doteq B\hat{u}(x) + \hat{f}^{[2]}(x, \hat{u}(x)) + \ldots + \hat{f}^{[k-1]}(x, \hat{u}(x)) = \begin{cases} \sum_{i=2}^{k-1} \Phi^{[i]} \left(C^T x\right) \\ \\ H \sum_{i=2}^{k-1} \phi^{[i]}(x) \end{cases} ,$$

$$\tilde{A}z = Az + Bv^{[1]}(z) \ .$$

Then, apply the homogeneous transformation of order k:

$$\dot{x} = \dot{z} + \frac{\partial \xi^{[k]}}{\partial z}\dot{z} = \left(I + \frac{\partial \xi^{[k]}}{\partial z}\right)\dot{z} \ .$$

It is straightforward to derive that the local representation of the transformed system satisfies:

$$\begin{aligned} \dot{z} &= \left(I + \frac{\partial \xi^{[k]}}{\partial z}\right)^{-1} \dot{x} = \left(I - \frac{\partial \xi^{[k]}}{\partial z} + \ldots\right)\left(Ax + Bu + \hat{f}(x,u)\right) = \\ &= \left(I - \frac{\partial \xi^{[k]}}{\partial z} + \ldots\right)\left(Az + Bv + A\xi^{[k]}(z) + B\mu^{[k]}(z,v) + \tilde{f}(z,v)\right) = \\ &= \left(Az + Bv + \hat{f}^{[2]}(z,v) + \ldots + \hat{f}^{[k-1]}(z,v)\right) + \qquad (4.21) \\ &+ \left(A\xi^{[k]}(z) + B\mu^{[k]}(z,v) - \frac{\partial \xi^{[k]}}{\partial z}Az - \frac{\partial \xi^{[k]}}{\partial z}Bv + \hat{f}^{[k]}(z,v)\right) + O(z,v)^{k+1} \ . \end{aligned}$$

Indeed, the first k terms of the function

$$\tilde{f}(z,v) \doteq \hat{f}\left(z + \xi^{[k]}(z), v + \mu^{[k]}(z,v)\right)$$

are equal to those of $\hat{f}(x,u)$. Thus, the terms up to $(k-1)$ are left unchanged.

In the normal form theory, no other elements play a role in the transformation. On the contrary, since we want to deal with autonomous systems, we can spend the hypothesis on the state feedback control input to improve the situation. Therefore, since we want to preserve the transformation condition for the lower terms, we must choose an input of the form:

$$v(z) \doteq u^{[1]}(z) + \hat{u}(z) + v^{[k]}(z) \ ,$$

that performs a change only in the k-th term and on the higher ones. Hence, the equation (4.21) becomes:

$$\begin{aligned}\dot{z} = \left(\tilde{A}z + f(z)\right) + &+\Big(A\xi^{[k]}(z) + Bv^{[k]}(z) + B\mu^{[k]}\left(z, u^{[1]}(z)\right) + \\ &- \frac{\partial \xi^{[k]}}{\partial z}Az - \frac{\partial \xi^{[k]}}{\partial z}Bu^{[1]}(z) + g^{[k]}(z)\Big) + O(z)^{k+1} \ ,\end{aligned}$$

where $g^{[k]}(z)$ is defined according to the development:

$$\sum_{i=1}^{k} \hat{f}^{[i]}\left(z, v(z)\right) = \sum_{i=1}^{k-1} g^{[i]}(z) + g^{[k]}(z) + O(z)^{k+1} \ .$$

Therefore, to extend up the order k the local transformation into the differential equation form, the following condition must be satisfied:

$$\begin{aligned}&A\xi^{[k]}(z) + B\mu^{[k]}\left(z, u^{[1]}(z)\right) + \\ &\quad - \frac{\partial \xi^{[k]}}{\partial z}Az - \frac{\partial \xi^{[k]}}{\partial z}Bu^{[1]}(z) + g^{[k]}(z) = \begin{cases} \Phi^{[k]}\left(C^T z\right) \\ \\ H\phi^{[k]}(z) \end{cases} .\end{aligned} \tag{4.22}$$

For example, if we consider the second sufficient condition, the problem (4.22) assumes the form:

$$\begin{cases} h_1\phi^{[k]}(z) & = \xi_2^{[k]}(z) - \frac{\partial \xi_1^{[k]}}{\partial z}z_2 + g_1^{[k]}(z) \\ \quad \cdots & \\ h_{n-1}\phi^{[k]}(z) & = \xi_n^{[k]}(z) - \frac{\partial \xi_{n-1}^{[k]}}{\partial z}z_n + g_{n-1}^{[k]}(z) \\ h_n\phi^{[k]}(z) & = v^{[k]}(z) + \mu^{[k]}\left(z, u^{[1]}(z)\right) - \frac{\partial \xi_n^{[k]}}{\partial z}u^{[1]}(z) + g_n^{[k]}(z) \ . \end{cases}$$

When we are looking for the systems which admit the differential equation form, we have to study the constraint on $g^{[k]}(z)$, that makes the problem (4.22) solvable. Indeed, the function $g^{[k]}(z)$ contains the information on the original $f^{[k]}(z, v)$ and a condition on the first corresponds to a constraint on the second.

Instead, if we want to check the possibility of the transformation, we must determine the existence of $\xi^{[k]}(z)$, $\mu^{[k]}(z, v)$, $\phi^{[k]}(z)$ and $v^{[k]}(z)$ solving the problem.

Conclusions

The manuscript dealt with the Andronov-Leontovich-Hopf bifurcation. This is a pretty common phenomenon, that has been widely studied in literature, since the early work of H. Poincaré. The proof by Andronov and Leontovich [1937] solved the second order problem, while the general n-dimensional case was first proved by Hopf some years later [Hopf, 1942]. The classic rigorous approach in the state space has been developed by several authors [Marsden and McCracken, 1976; Arnold, 1983; Guckenheimer and Holmes, 1983; Farkas, 1994], while the frequency method is essentially due to the works of Allwrigth and Mees [Allwright, 1977; Mees, 1981; Moiola and Chen, 1996].

The results related to the study of the Hopf bifurcation lead to exact mathematical tools, but unfortunately they turn out to be extremely complex even in the low dimensional case and usually their application to real life systems can be performed only numerically. Indeed, the standard state space approach is based on the Center Manifold Theorem [Kuznetsov, 1998; Wiggins, 2003] and it requires the computation of the related tangent eigenspace and the identification of its local dynamics. In turn, the graphical tools of the frequency approach are not feasible to find a direct relation between the parametric set of the system and the properties of its bifurcation, which is a central point to study an entire class of models.

Therefore, our aim has been the definition of exact mathematical tools, which could be analytically applied to real life systems.

We have followed the frequency approach, that is based on the Harmonic Balance method. In particular, we have derived a local second order HB problem, such that its solvability is directly connected to the existence of the real limit cycle. Then, the super or subcritical nature of the bifurcation can be determined as well. Moreover, the second HB solution turns out to be a local approximation of the periodic regime arising at the Hopf bifurcation and, thus, it provides useful information about the features of the real limit cycle. Since the HB method is particularly effective in the study of the differential equation systems, this class has been considered and the related second HB problem has been locally solved. To compare these results with the classic methods, we have derived the form of one of the most effective state space technique [Howard, 1979] for the differential equation systems class, obtaining the explicit relation between the coefficient of curvature of the bifurcation and the system parameters. In this case, it turns out that the computational burden grows rapidly with the system dimension and the analytical approach becomes unfeasible even for low dimensional models. In turn, our method results in the solution of a linear problem of five equations in five unknowns, independently from the dimension of the original system. However, when the latter grows, the coefficients of such a linear problem become more complex, but their computation can be systematically tackled, since they derive from powers of the periodic solution and its derivatives.

Our method turned out to be suitable to solve a control problem, since it can be employed to state both the bifurcation nature and the second order harmonic approximation of the real limit cycle. Therefore, we focused our attention in the transformations from the general state space model to the differential equation form, because our tools are specifically designed for this class. Unfortunately, the necessary and sufficient constraints, based on differential geometry, turn out to be unfeasible in the analytical approach and their application can be performed only numerically. Thus, we employed only sufficient conditions, which in turn can be analytically handled. This choice limits the number of state space systems which can be studied with our method, but this class can be extended observing that the Hopf bifurcation is a local phenomenon and that it can be completely disclosed just analyzing a local description of the original process.

Thus, we have been concerned with the problem of the local transformations. In particular, to develop our results in a uniform and general framework, we have resorted to the *extended controller normal forms* [Kang and Krener, 1992; Kang, 1994; Kang and Krener, 2005]. Indeed, according to such a theory, every control system with a scalar input can be represented in the controller normal form of order k with an error of order $(k+1)$. Therefore, we have focused our attention on the sufficient conditions under which a "quadratic plus cubic" normal form can be transformed into the differential equation model. Moreover, this local approach has been developed further, since the normal form theory does not spend any hypothesis on the nature of the control input. Hence, we have exploited the state feedback control law to perform an improvement of the degrees of freedom of the transformation process.

In conclusion, we have developed a rigorous theory for the local analysis and control of the Hopf bifurcation by means of exact mathematical tools, introducing some original ideas oriented to the analytical approach. These results may find useful application in the standard Hopf bifurcation control problems. For example, they could be employed to design controllers for the suppression of vibrations in mechanical systems or in bodies moving into fluids. Also in the biological field it exists the need to control the nature of the Hopf bifurcation. In such a case, the effort is usually in avoiding the subcritical case, so to preserve the system survival. In the telecommunication field, instead, the attention is focused on time delayed processes. Our approach is not originally designed to handle this kind of systems, but the feedback block diagram representation suggests the chance to extend some of our original ideas to this situation.

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