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Oscillation of a class of differential equations with generalized phi-Laplacian

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The oscillation of the nonlinear differential equation

$$(a(t)\Phi(x'))' + b(t)F(x) = 0,$$

where Φ is an increasing odd homeomorphism, is considered when the weight b is not summable near infinity. We extend previous results, stated for equations with the classical p-Laplacian, by obtaining necessary and sufficient conditions of integral type for the oscillation. The role of the boundedness of $\operatorname{Im}\Phi$ [$\operatorname{Dom}\Phi$] is analysed in detail. Our results includes the case $\Phi^* \circ F$ linear near zero or near infinity, where Φ^* is the inverse of Φ. Several examples, concerning the curvature or relativity operator, illustrate our results.

1. Introduction

Consider the second-order nonlinear differential equation

$$(a(t)\Phi(x'))' + b(t)F(x) = 0, \quad t \geqslant T_0, \tag{1.1}$$

where Φ is an increasing odd homeomorphism defined on an open interval $(-\rho, \rho)$, $0 < \rho \leqslant \infty$, and $\operatorname{Im}\Phi = (-\sigma, \sigma)$, $0 < \sigma \leqslant \infty$, F is a continuous non-decreasing function on $\mathbb{R}$ such that $F(u)u > 0$ for $u \neq 0$, a and b are positive continuous functions for $t \geqslant T_0$ and

$$\int_{T_0}^{\infty} b(t)\,\mathrm{d}t = \infty.$$

By a solution of (1.1) we mean a differentiable function x on $[T_x, \infty)$, $T_x \geqslant T_0$, such that $a(\cdot)\Phi(x'(\cdot))$ is continuously differentiable on $[T_x, \infty)$ and satisfies (1.1) on

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$[T_x, \infty)$. We shall consider only those solutions x of (1.1) which satisfy

$$\sup\{|x(t)|\colon t \geqslant T\} > 0 \quad \text{for all } T \geqslant T_x$$

and assume that (1.1) possesses such a solution.

As usual, a solution of (1.1) is said to be *oscillatory* if it has a sequence of zeros converging to infinity; otherwise, it is said to be *non-oscillatory*. Equation (1.1) is said to be *oscillatory* if any of its solutions are oscillatory.

A prototype of (1.1) is the Emden–Fowler equation

$$(a(t)\Phi_p(x'(t)))' + b(t)\Phi_q(x) = 0, \quad p \neq q, \tag{1.2}$$

where Φ_p is the classical p-Laplacian, i.e. $\Phi_p(u) = |u|^{p-1}u$, and p is a positive constant. If

$$\int_{T_0}^{\infty} \left(\frac{1}{a(t)}\right)^{1/p} \mathrm{d}t = \infty, \tag{1.3}$$

then (1.2) is oscillatory. When (1.3) does not hold, it is well known (see, for example, [14, theorem 6.6]) that (1.2) is oscillatory if and only if either

$$\int_{T_0}^{\infty} \left(\frac{1}{a(t)} \int_{T_0}^{t} b(s)\,\mathrm{d}s\right)^{1/p} \mathrm{d}t = \infty, \quad p > q \tag{1.4}$$

or

$$\int_{T_0}^{\infty} b(t)\biggl(\int_{t}^{\infty} \left(\frac{1}{a(s)}\right)^{1/p} \mathrm{d}s\biggr)^{q} \mathrm{d}t = \infty, \quad p < q. \tag{1.5}$$

The oscillation of more general equations, such as

$$(a(t)\Phi_p(x'(t)))' + b(t)F(x) = 0, \tag{1.6}$$

has been widely considered in the literature, from different points of view (see, for example, [1, 3, 5, 10, 11, 14–21, 23, 24] and the references therein). Other results can be obtained from [6, 9, 12, 13], in which coupled systems of the form

$$x' = A(t)G_1(y), \qquad y' = -B(t)G_2(x) \tag{1.7}$$

are considered.

Denote by Φ_p^* the inverse of Φ_p, i.e. $\Phi_p^*(u) = \Phi_{1/p}(u) = |u|^{1/p}\operatorname{sgn} u$. Most of the quoted results give various improvements of conditions (1.4) or (1.5), stated for (1.2), and often it is assumed, in the case of (1.6), that the function $H(u) = 1/\Phi_p^*(F(u))$ is summable near infinity or near zero. Similarly, for the system (1.7), the role of H is played by $\tilde{H}(u) = 1/G_1(G_2(u))$. Observe that for (1.2) these assumptions are equivalent to conditions $q > p$ or $q < p$, respectively.

When Φ is not the classical p-Laplacian Φ_p, the map Φ and its inverse Φ^* do not satisfy the homogeneity property

$$\Phi(uv) = \Phi(u)\Phi(v) \tag{1.8}$$

(see, for example, [8, proposition 2.1]). Thus, equations involving a general map Φ cannot be written in the form (1.7) and all the results quoted above cannot be applied to studying the oscillation or the asymptotic behaviour of non-oscillatory solutions of (1.1).

Prototypes of cases in which $\sigma < \infty$ or $\rho < \infty$ are the maps Φ_{C} and Φ_{R}, respectively, given by

$$\Phi_{\mathrm{C}}(u) = \frac{u}{\sqrt{1+|u|^2}}, \qquad \Phi_{\mathrm{R}}(u) = \frac{u}{\sqrt{1-|u|^2}}, \tag{1.9}$$

which arise in the study of the radially symmetric solutions of partial differential equations with the curvature operator or the relativity operator (see [8] and references therein for more details).

Equations (1.1) with a general map Φ have been considered in [7, 8] when the weight b is summable, i.e. $b \in L^1[T_0, \infty)$. More precisely, in [8] a classification of all non-oscillatory solutions is given with respect to the quasi-derivative

$$x^{[1]}(t) = a(t)\Phi(x'(t))$$

and conditions for the existence of all types of non-oscillatory solutions are obtained. In [7] the case $\liminf_{t\to\infty} a(t) = 0$ and $\sigma < \infty$ is considered, and oscillatory and asymptotic behaviour of all non-oscillatory solutions of (1.1) are studied, particularly for the map Φ_{C}.

Motivated by [7, 8], here we continue the study of the oscillation of (1.1) when the weight $b \notin L^1[T_0, \infty)$. We obtain for the oscillation necessary and sufficient conditions of integral type that extend previous results, stated for equations with the classical p-Laplacian. The 'necessity' part is proved by using the Tychonov fixed point theorem in the Fréchet space of continuous functions in a non-compact interval. This seems an appropriate choice for our asymptotic problem, because it permits us to avoid some difficulties related to the compactness test on non-compact intervals (see, for example, [2, 4] and the references therein).

Observe that, in view of the lack of the homogeneity property (1.8), the case $b \notin L^1[T_0, \infty)$ cannot be treated by the change of variable $z = x^{[1]}$, as is possible for equations (1.2) and (1.6); see, for example, [5, 6]. Our oscillation criteria cover all possibilities with regards to the growth of the weight a. Moreover, our results do not require the summability of the function

$$H(u) = \frac{1}{\Phi^*(F(u))}$$

near infinity or near zero. Thus, they also cover the case when $\Phi^*(F(u)) \sim u$ near zero, or near infinity when $\operatorname{Dom}\Phi^* \circ F = \mathbb{R}$; see Examples 4.4 and 4.5.

Some interesting applications for equations with the curvature and relativity operator are also given. A comparison of oscillation criteria with those for the classical p-Laplacian is given and illustrated by examples.

2. Leighton-type oscillation criteria

Any non-oscillatory solution x of (1.1) satisfies

$$x(t)x'(t) < 0$$

for large t. Indeed, let x be a non-oscillatory solution of (1.1), with $x(t) > 0$ for $t \geqslant t_0 \geqslant T_0$. Since $x^{[1]}$ is decreasing for $t \geqslant t_0$, if $x^{[1]}(t) > 0$ for $t \geqslant t_0$, then x is increasing for $t \geqslant t_0$. Integrating (1.1), we obtain

$$x^{[1]}(t) - x^{[1]}(t_0) \leqslant -F(x(t_0)) \int_{t_0}^{t} b(s)\,\mathrm{d}s,$$

which gives a contradiction with the positiveness of $x^{[1]}$ as $t \to \infty$.

Thus, setting $x_\infty = \lim_{t\to\infty} x(t)$, we can classify non-oscillatory solutions x of (1.1) as *solutions of class* $\mathbb{M}_0^-$ or $\mathbb{M}_\ell^-$, according to whether $x_\infty = 0$ or $x_\infty = \ell_x \neq 0$, respectively.

Set

$$\Lambda = \bigcap_{t \geqslant T_0} (0, \sigma a(t)).$$

If $\sigma = \infty$, then $\Lambda = (0, \infty)$. If $\sigma < \infty$ and $\liminf_{t\to\infty} a(t) > 0$, then Λ is a bounded non-empty interval. Finally, if $\sigma < \infty$ and $\liminf_{t\to\infty} a(t) = 0$, then Λ is empty.

In studying the asymptotic properties of (1.1) when Λ is non-empty, i.e. when either of the assumptions

$$\left.\begin{aligned} &\sigma < \infty \quad \text{and} \quad \liminf_{t\to\infty} a(t) > 0, \\ &\sigma = \infty \end{aligned}\right\} \tag{2.1}$$

are satisfied, an important role is played by the integral

$$I_\lambda = \int_{T_0}^{\infty} \Phi^*\left(\frac{\lambda}{a(t)}\right) \mathrm{d}t,$$

where $\lambda \in \Lambda$ (see [8]). Observe that, in view of (2.1), I_λ is well defined.

Concerning the convergence of I_λ when (2.1) holds, the possible cases are as follows:

(C1) $I_\lambda = \infty$ for any $\lambda \in \Lambda$;

(C2) there exist $\lambda_1, \lambda_2 \in \Lambda$, $0 < \lambda_1 < \lambda_2$, such that $I_{\lambda_1} < \infty$, $I_{\lambda_2} = \infty$;

(C3) $I_\lambda < \infty$ for any $\lambda \in \Lambda$.

Obviously, if Φ^* satisfies the homogeneity property (1.8), then the convergence of I_λ does not depend on the parameter λ, i.e. the case (C2) cannot occur for (1.6). Examples satisfying (C2) have been given in [8, examples 2.2 and 2.3]; in both cases $\sigma < \infty$ and $\rho < \infty$.

The following result can be considered as an extension of the Leighton oscillation criterion stated for the linear case in [22, theorem 2.24].

THEOREM 2.1. *Equation (1.1) is oscillatory if any of the following conditions are satisfied:*

(i) $\liminf_{t\to\infty} a(t) = 0$ *and* $\sigma < \infty$*;*

(ii) $\liminf_{t\to\infty} a(t) > 0$*,* $\sigma < \infty$ *and (C1) holds;*

(iii) $\sigma = \infty$ *and (C1) holds.*

Proof. By contradiction, let x be a non-oscillatory solution of (1.1), $x(t) > 0$ for $t \geqslant t_0 \geqslant T_0$. Thus, $x(t)x'(t) < 0$ for large t, i.e. $x^{[1]}$ becomes negative at some $t_1 \geqslant t_0$. Since $x^{[1]}$ is decreasing on $[t_0, \infty)$, we have

$$\Phi(x'(t)) \leqslant x^{[1]}(t_1)\frac{1}{a(t)}. \tag{2.2}$$

If (i) is verified, we obtain a contradiction with the boundedness of Φ as $t \to \infty$. If (ii) or (iii) is verified, from (2.2) we obtain

$$x(t) < x(t_1) - \int_{t_1}^{t} \Phi^*\left(\frac{|x^{[1]}(t_1)|}{a(s)}\right) \mathrm{d}s,$$

which contradicts the boundedness of x as $t \to \infty$. □

The following example illustrates theorem 2.1.

EXAMPLE 2.2. Consider the differential equation

$$(\tfrac{1}{2}\sqrt{4+\cos^2 t}\,\Phi_{\mathrm{C}}(x'))' + x = 0, \tag{2.3}$$

where Φ_{C} is given in (1.9). A direct calculation shows that $x(t) = (\sin t)/2$ is a solution of (2.3). Moreover, $\Lambda = (0,1)$ and (C1) holds. Thus, by theorem 2.1, any solution of (2.3) is oscillatory.

3. The case (C2)

When (C2) holds, a necessary and sufficient condition for the oscillation of (1.1) is given by the following.

THEOREM 3.1. *Assume (2.1) and (C2) hold. Equation (1.1) is oscillatory if and only if*

$$K_\lambda = \int_{T_0}^{\infty} b(t) F\left(\int_t^{\infty} \Phi^*\left(\frac{\lambda}{a(s)}\right) \mathrm{d}s\right) \mathrm{d}t = \infty \tag{3.1}$$

for any $\lambda \in \Lambda$ *such that* $I_\lambda < \infty$.

The necessary part of theorem 3.1 is given by the following existence result, which can also be applied when (C3) occurs.

THEOREM 3.2. *Assume (2.1) holds. If there exists* $\lambda \in \Lambda$ *such that* $I_\lambda < \infty$ *and* $K_\lambda < \infty$, *then (1.1) has solutions* x *in the class* $\mathbb{M}_0^-$.

Proof. Let $\lambda \in \Lambda$ and choose $t_0 \geqslant T_0$ large so that

$$\int_{t_0}^{\infty} b(t) F\left(\int_t^{\infty} \Phi^*\left(\frac{\lambda}{a(s)}\right) \mathrm{d}s\right) \mathrm{d}t < \frac{\lambda}{2}. \tag{3.2}$$

In the Fréchet space $C[t_0, \infty)$ of all continuous functions on $[t_0, \infty)$, endowed with the topology of uniform convergence on compact subintervals of $[t_0, \infty)$, consider the set Ω given by

$$\Omega = \left\{u \in C[t_0, \infty)\colon 0 \leqslant u(t) \leqslant \int_t^{\infty} \Phi^*\left(\frac{\lambda}{a(s)}\right) \mathrm{d}s\right\}.$$

Define in Ω the operator Γ as follows:

$$\Gamma(u)(t) = \int_t^\infty \Phi^*\left(\frac{1}{a(s)}\left(\frac{\lambda}{2} - \int_s^\infty b(r)F(u(r))\,\mathrm{d}r\right)\right)\mathrm{d}s.$$

From (3.2) we have

$$\begin{aligned}\frac{\lambda}{2} - \int_s^\infty b(r)F(u(r))\,\mathrm{d}r &\geqslant \frac{\lambda}{2} - \int_{t_0}^\infty b(r)F(u(r))\,\mathrm{d}r\\ &\geqslant \frac{\lambda}{2} - \int_{t_0}^\infty b(r)F\left(\int_r^\infty \Phi^*\left(\frac{\lambda}{a(\sigma)}\right)\mathrm{d}\sigma\right)\mathrm{d}r\\ &> 0.\end{aligned} \tag{3.3}$$

Thus,

$$0 \leqslant \Gamma(u)(t) \leqslant \int_t^\infty \Phi^*\left(\frac{\lambda}{2}\frac{1}{a(s)}\right)\mathrm{d}s$$

and so $\Gamma(\Omega) \subset \Omega$. Let us show that $\Gamma(\Omega)$ is relatively compact, i.e. $\Gamma(\Omega)$ consists of functions equibounded and equicontinuous on every compact interval of $[t_0,\infty)$. Because $\Gamma(\Omega) \subset \Omega$, the equiboundedness follows. Moreover, in view of the above estimates, we have, for any $u \in \Omega$,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\Gamma(u)(t)\right| \leqslant \Phi^*\left(\frac{\lambda}{a(t)}\right),$$

which yields the equicontinuity of the elements in $\Gamma(\Omega)$. Now we prove the continuity of Γ in Ω. Let $\{u_n\}$, $n \in \mathbb{N}$, be a sequence in Ω that uniformly converges on every compact interval of $[t_0,\infty)$ to $\bar{u} \in \Omega$. Because $\Gamma(\Omega)$ is relatively compact, the sequence $\{\Gamma(u_n)\}$ admits a subsequence $\{\Gamma(u_{n_j})\}$ converging, in the topology of $C[t_0,\infty)$, to $\bar{z}_u \in \overline{\Gamma(\Omega)}$. Since

$$|\Gamma(u_{n_j})(t)| \leqslant \int_t^\infty \Phi^*\left(\frac{\lambda}{a(s)}\right)\mathrm{d}s,$$

from the Lebesgue dominated convergence theorem, the sequence $\{\Gamma(u_{n_j})(t)\}$ converges pointwise to $\Gamma(\bar{u})(t)$. In view of the uniqueness of the limit, $\Gamma(\bar{u}) = \bar{z}_u$ is the only cluster point of the compact sequence $\{\Gamma(u_n)\}$, i.e. the continuity of Γ in the topology of $C[t_0,\infty)$. Hence, by the Tychonov fixed point theorem, the operator Γ has a fixed point x, which, clearly, is a solution of (1.1). Moreover, in view of (3.3), we have

$$x^{[1]}(t) = -\frac{\lambda}{2} + \int_t^\infty b(r)F(x(r))\,\mathrm{d}r < 0,$$

i.e. x is a non-trivial solution and belongs to $\mathbb{M}_0^-$. □

Proof of theorem 3.1. Necessity: the assertion follows from theorem 3.2.

Sufficiency: let x be a non-oscillatory solution of (1.1) such that $x(t) > 0, x'(t) < 0$ for $t \geqslant t_0 \geqslant T_0$. Let $\tilde{\lambda} \in \Lambda$ be such that $I_{\tilde{\lambda}} = \infty$. If $x \in \mathbb{M}_\ell^-$, integrating (1.1), we obtain

$$x^{[1]}(t) \leqslant x^{[1]}(t_0) - F(x_\infty)\int_{t_0}^t b(s)\,\mathrm{d}s,$$

where $x_\infty = \lim_{t\to\infty} x(t)$. Thus, $\lim_{t\to\infty} x^{[1]}(t) = -\infty$. Choose $T \geqslant t_0$ large so that $x^{[1]}(T) < -\tilde\lambda$. Since $x^{[1]}$ is decreasing for $t \geqslant t_0$, we obtain, for $t \geqslant T$,

$$x'(t) < -\Phi^*\left(\frac{\tilde\lambda}{a(t)}\right).$$

Integrating this inequality as $t \to \infty$, we get a contradiction with the positiveness of x.

Let us show that $\mathbb{M}_0^- = \emptyset$. By contradiction, assume $x \in \mathbb{M}_0^-$. Since $x^{[1]}$ is negative decreasing for $t \geqslant t_0$, we have $x^{[1]}(t) \leqslant x^{[1]}(t_0)$. Thus, setting $\lambda_0 = -x^{[1]}(t_0)$, we have

$$x(t) \geqslant \int_t^\infty \Phi^*\left(\frac{\lambda_0}{a(s)}\right) \mathrm{d}s \tag{3.4}$$

and so $I_{\lambda_0} < \infty$. Moreover, $\lambda_0 \in \Lambda$. Indeed, if $\lambda_0 \notin \Lambda$, we get $\sup \Lambda \leqslant \lambda_0$, and so, because (C2) holds, there exists $\lambda_2 < \lambda_0$, $\lambda_2 \in \Lambda$, such that $I_{\lambda_2} = \infty$, which contradicts $I_{\lambda_0} < \infty$. Hence, in view of (3.4), we have

$$-(x^{[1]}(t))' = b(t)F(x(t)) \geqslant b(t)F\left(\int_t^\infty \Phi^*\left(\frac{\lambda_0}{a(s)}\right) \mathrm{d}s\right).$$

Integrating this inequality, since $K_{\lambda_0} = \infty$, we obtain $\lim_{t\to\infty} x^{[1]}(t) = -\infty$. So, reasoning as above, the assertion follows. □

Remark 3.3. If F is not increasing, theorem 3.2 continues to hold if we replace in (3.1) the function F with the function G given by

$$G(u) = \max\{F(v)\colon 0 \leqslant v \leqslant u\}.$$

The details are left to the reader.

From the proof of theorem 3.1 we get the following.

Corollary 3.4. *Assume (2.1) and (C2) hold. Then any non-oscillatory solution of (1.1) tends to zero. Moreover, such solutions exist if and only if $K_\lambda < \infty$ for some $\lambda \in \Lambda$ such that $I_\lambda < \infty$.*

4. The case (C3)

In this section we study oscillation of (1.1) in the case (C3), i.e. when $I_\lambda < \infty$ for any $\lambda \in \Lambda$.

In this case, theorem 3.1 can fail, as the following example shows.

Example 4.1. Consider the following equation:

$$\left(t\sqrt{1+t^4}\Phi_{\mathrm{C}}(x')\right)' + \sqrt{\frac{t}{t+1}}\sqrt{|x|}\operatorname{sgn} x = 0, \quad t \geqslant 1, \tag{4.1}$$

where Φ_{C} is the function given by (1.9). A standard calculation shows that $x(t) = (t+1)/t$ is a non-oscillatory solution of (4.1). Moreover, $\Lambda = (0, \sqrt{2})$ and $I_\lambda < \infty$, $K_\lambda = \infty$ for any $\lambda \in \Lambda$.

Our main result is the following.

THEOREM 4.2. *Assume (2.1) and (C3) hold. If, for any $\lambda \in \Lambda$ sufficiently small and any $T > T_0$,*

$$\int_T^\infty \Phi^*\left(\frac{1}{a(t)}\int_T^t b(s)F\left(\int_s^\infty \Phi^*\left(\frac{\lambda}{a(\sigma)}\right)\mathrm{d}\sigma\right)\mathrm{d}s\right)\mathrm{d}t = \infty, \tag{4.2}$$

then (1.1) is oscillatory.

To prove theorem 4.2, the following auxiliary result is needed.

LEMMA 4.3. *Assume (2.1). If $\mathbb{M}_\ell^- \neq \emptyset$, then there exist $\mu > 0$ and $T \geqslant T_0$ such that*

$$J_\mu = \int_T^\infty \Phi^*\left(\frac{\mu}{a(t)}\int_T^t b(s)\,\mathrm{d}s\right)\mathrm{d}t < \infty. \tag{4.3}$$

Proof. Let x be a solution of (1.1) in the class $\mathbb{M}_\ell^-$ and let $x(t) > 0$, $x'(t) < 0$ for $t \geqslant t_0 \geqslant T_0$ and $x_\infty = \lim_{t\to\infty} x(t)$. Integrating (1.1), we have

$$x^{[1]}(t) = x^{[1]}(t_0) - \int_{t_0}^t b(s)F(x(s))\,\mathrm{d}s \leqslant -F(x_\infty)\int_{t_0}^t b(s)\,\mathrm{d}s$$

or

$$x(t) - x(t_0) \leqslant -\int_{t_0}^t \Phi^*\left(\frac{F(x_\infty)}{a(r)}\int_{t_0}^r b(s)\,\mathrm{d}s\right)\mathrm{d}r,$$

which gives the assertion. □

Proof of theorem 4.2. Let x be a non-oscillatory solution of (1.1). First, suppose $x \in \mathbb{M}_0^-$, and, without loss of generality, assume $x(t) > 0, x'(t) < 0$ for $t \geqslant t_0$. Integrating (1.1), we obtain

$$x'(t) \leqslant -\Phi^*\left(\frac{1}{a(t)}\int_{t_0}^t b(s)F(x(s))\,\mathrm{d}s\right). \tag{4.4}$$

Setting λ_0 sufficiently small such that $\lambda_0 \leqslant |x^{[1]}(t_0)|$, $\lambda_0 \in \Lambda$, because $x^{[1]}$ is decreasing for $t \geqslant t_0$ and $\lim_{t\to\infty} x(t) = 0$, we obtain

$$x'(t) \leqslant -\Phi^*\left(\frac{\lambda_0}{a(t)}\right).$$

Thus, from (4.4) we get

$$x'(t) \leqslant -\Phi^*\left(\frac{1}{a(t)}\int_{t_0}^t b(s)F\left(\int_s^\infty \Phi^*\left(\frac{\lambda_0}{a(\sigma)}\right)\mathrm{d}\sigma\right)\mathrm{d}s\right).$$

Integrating this inequality on $[t_0, \infty)$, we get a contradiction with (4.2).

Now, let $x \in \mathbb{M}_\ell^-$. By virtue of lemma 4.3, there exist $\mu > 0$ and $T \geqslant T_0$ such that

$$\int_T^\infty \Phi^*\left(\frac{\mu}{a(t)}\int_T^t b(s)\,\mathrm{d}s\right)\mathrm{d}t < \infty.$$

Let $t \geqslant t_0 \geqslant T$. Thus, we obtain, for any $t_0 \geqslant T$,

$$\int_{t_0}^\infty \Phi^*\left(\frac{\mu}{a(t)}\int_{t_0}^t b(s)\,\mathrm{d}s\right)\mathrm{d}t < \infty. \tag{4.5}$$

Let λ be sufficiently small and $\lambda \in \Lambda$. Since $I_\lambda < \infty$, we can choose t_0 large so that, for $s \geqslant t_0$,

$$F\biggl(\int_s^\infty \Phi^*\biggl(\frac{\lambda}{a(\sigma)}\biggr)\,\mathrm{d}\sigma\biggr) \leqslant \mu.$$

Thus,

$$\Phi^*\biggl(\frac{1}{a(t)}\int_{t_0}^t b(s)F\biggl(\int_s^\infty \Phi^*\biggl(\frac{\lambda}{a(\sigma)}\biggr)\,\mathrm{d}\sigma\biggr)\,\mathrm{d}s\biggr) \leqslant \Phi^*\biggl(\frac{\mu}{a(t)}\int_{t_0}^t b(s)\,\mathrm{d}s\biggr),$$

and from (4.5) we obtain a contradiction with (4.2). □

The following example illustrates theorem 4.2.

EXAMPLE 4.4. The following equation:

$$(a(t)\Phi_{\mathrm{R}}(x'))' + b(t)x = 0, \quad t \geqslant \pi, \tag{4.6}$$

where Φ_{R} is given by (1.9) and

$$a(t) = \sqrt{t^4 - (t\cos t - \sin t)^2} \quad \text{and} \quad b(t) = t^2,$$

has the oscillatory solution

$$x(t) = \frac{\sin t}{t}.$$

Let us show that, by theorem 4.2, all solutions of (4.6) are oscillatory.

Since $\sigma = \infty$, we have $\Lambda = (0, \infty)$. Thus, we get for $\lambda \in \Lambda$ and for large t

$$\frac{\lambda}{t^{5/2}} \leqslant \Phi_{\mathrm{R}}^*\biggl(\frac{\lambda}{a(t)}\biggr) \leqslant \frac{\lambda}{t^{3/2}}. \tag{4.7}$$

Hence, $I_\lambda < \infty$ for $\lambda \in \Lambda$, i.e. (C3) occurs. Moreover, from (4.7) we have

$$b(s)\int_s^\infty \Phi_{\mathrm{R}}^*\biggl(\frac{\lambda}{a(\sigma)}\biggr)\,\mathrm{d}\sigma \geqslant \frac{2\lambda}{3}s^{1/2}.$$

Then, we obtain, for any large T,

$$\int_T^\infty \Phi_{\mathrm{R}}^*\biggl(\frac{1}{a(t)}\int_T^t b(s)\int_s^\infty \Phi_{\mathrm{R}}^*\biggl(\frac{\lambda}{a(\sigma)}\biggr)\,\mathrm{d}\sigma\,\mathrm{d}s\biggr)\,\mathrm{d}t \geqslant \int_T^\infty \Phi_{\mathrm{R}}^*\biggl(\frac{4\lambda}{9}\frac{t^{3/2} - T^{3/2}}{a(t)}\biggr)\,\mathrm{d}t.$$

By a direct computation we obtain

$$\Phi_{\mathrm{R}}^*\biggl(\frac{4\lambda}{9}\frac{t^{3/2} - T^{3/2}}{a(t)}\biggr) \geqslant \frac{4\lambda}{9}\frac{t^{3/2} - T^{3/2}}{\sqrt{t^4 + (4\lambda/9)^2(t^3 - T^3)}}.$$

Since

$$\frac{t^{3/2} - T^{3/2}}{\sqrt{t^4 + (4\lambda/9)^2(t^3 - T^3)}} \sim \frac{1}{\sqrt{t}} \quad \text{as } t \to \infty,$$

the condition (4.2) is satisfied and theorem 4.2 yields the oscillation of (4.6).

Observe that for (4.6) we have $\Phi_{\mathrm{R}}^*(F(u)) \sim u$ as $u \to 0$. With a minor change, we can produce an example in which $\Phi^* \circ F$ is linear for any $u \in \mathbb{R}$.

EXAMPLE 4.5. The following equation:

$$(a(t)\Phi_{\mathrm{C}}(x'))' + b(t)\Phi_{\mathrm{C}}(x) = 0, \quad t \geqslant \pi, \tag{4.8}$$

where Φ_{C} is given by (1.9) and

$$a(t) = \sqrt{t^4 + (t\cos t - \sin t)^2} \quad \text{and} \quad b(t) = t\sqrt{t^2 + \sin^2 t},$$

has the oscillatory solution

$$x(t) = \frac{\sin t}{t}.$$

Let us show that by theorem 4.2 all solutions of (4.6) are oscillatory.

Since $\min_{t\geqslant\pi} a(t) = \pi\sqrt{1+\pi^2}$, we have $\Lambda = (0, \pi\sqrt{1+\pi^2})$. As in example 4.4, we get $I_\lambda < \infty$ for $\lambda \in \Lambda$, i.e. (C3) occurs. Moreover, since, for $t \geqslant \pi$,

$$(t\cos t - \sin t)^2 \leqslant (1+t)^2,$$

we obtain, for large t,

$$\Phi^*_{\mathrm{C}}\biggl(\frac{\lambda}{a(t)}\biggr) = \frac{\lambda}{\sqrt{a^2(t) - \lambda^2}} \geqslant \frac{\lambda}{\sqrt{t^4 + (1+t)^2}} \geqslant \frac{\lambda}{\sqrt{2}(1+t)^2}.$$

Thus, we have, for large s,

$$b(s)\Phi_{\mathrm{C}}\biggl(\int_s^\infty \Phi^*_{\mathrm{C}}\biggl(\frac{\lambda}{a(\sigma)}\biggr)\,\mathrm{d}\sigma\biggr) \geqslant b(s)\Phi_{\mathrm{C}}\biggl(\frac{\lambda}{\sqrt{2}(1+s)}\biggr) \geqslant \frac{\lambda s^2}{2(1+s)} \geqslant \frac{\lambda}{4}s.$$

Hence,

$$\frac{1}{a(t)}\int_T^t b(s)\Phi_{\mathrm{C}}\biggl(\int_s^\infty \Phi^*_{\mathrm{C}}\biggl(\frac{\lambda}{a(\sigma)}\biggr)\,\mathrm{d}\sigma\biggr)\,\mathrm{d}s \geqslant \frac{\lambda(t^2 - T^2)}{8\sqrt{t^4 + (t\cos t - \sin t)^2}}.$$

Since $\Phi^*_{\mathrm{C}}(u) \geqslant u$ for any $u \in (0,1)$, choosing λ sufficiently small, $\lambda \in \Lambda$, we get that the condition (4.2) is satisfied. Thus, theorem 4.2 yields the oscillation of (4.8).

For the half-linear equation

$$(a(t)\Phi_p(x'(t)))' + \mu b(t)\Phi_p(x) = 0, \quad \mu > 0, \tag{4.9}$$

theorem 4.2 reads as follows.

COROLLARY 4.6. *If*

$$\int_{T_0}^\infty \frac{1}{a^{1/p}(t)}\,\mathrm{d}t < \infty$$

and

$$\int_{T_0}^\infty \biggl(\frac{1}{a(t)}\int_{T_0}^t b(s)\biggl(\int_s^\infty \frac{1}{a^{1/p}(\sigma)}\,\mathrm{d}\sigma\biggr)^p \mathrm{d}s\biggr)^{1/p} \mathrm{d}t = \infty, \tag{4.10}$$

then (4.9) is oscillatory for any $\mu > 0$.

REMARK 4.7. A necessary and sufficient criterion for the oscillation of (4.9) for any $\mu > 0$ (the so-called strong oscillation of (4.9)) can be obtained from [11, theorem 4.2]. A detailed discussion and some existing results for half-linear equations are given in the last section.

When (C2) holds, $\mathbb{M}_\ell^- = \emptyset$. Nevertheless, when (C3) holds, (1.1) can admit solutions x in the class $\mathbb{M}_\ell^-$, as the following result shows.

THEOREM 4.8. *Assume (2.1) and (C3) hold. Equation (1.1) has solutions in the class* $\mathbb{M}_\ell^-$ *if and only if (4.3) holds for some* $\mu > 0$ *and* $T \geqslant T_0$.

Proof. Necessity: this follows from lemma 4.3.

Sufficiency: fix $L > 0$ such that

$$0 < F(2L) \leqslant \mu \tag{4.11}$$

and choose $t_0 \geqslant T$ large so that

$$\int_{t_0}^{\infty} \Phi^* \left(\frac{\mu}{a(t)} \int_{t_0}^{t} b(s)\, \mathrm{d}s \right) \mathrm{d}t < L. \tag{4.12}$$

Consider in the Fréchet space $C[t_0, \infty)$ the set Ω given by

$$\Omega = \{u \in C[t_0, \infty) \colon L \leqslant u(t) \leqslant 2L\}.$$

Define in Ω the operator Γ as follows:

$$\Gamma(u)(t) = L + \int_{t}^{\infty} \Phi^* \left(\frac{1}{a(s)} \left(\int_{t_0}^{s} b(r) F(u(r))\, \mathrm{d}r \right) \right) \mathrm{d}s.$$

Clearly, $\Gamma(u)(t) \geqslant L$. Moreover, from (4.11) and (4.12) we get

$$\Gamma(u)(t) \leqslant L + \int_{t_0}^{\infty} \Phi^* \left(\frac{F(2L)}{a(s)} \left(\int_{t_0}^{s} b(r)\, \mathrm{d}r \right) \right) \mathrm{d}s \leqslant 2L$$

and so $\Gamma(\Omega) \subset \Omega$. Using an argument similar to the one given in the proof of theorem 3.2, with minor changes, we get that $\Gamma(\Omega)$ is relatively compact and Γ is continuous on Ω. Then, by the Tychonov fixed point theorem, the assertion follows. □

REMARK 4.9. Theorem 4.8 continues to hold even if the function F is not increasing. Minor changes are needed in the proof.

From theorem 4.8 we get the following result.

COROLLARY 4.10. *Assume (2.1) holds. If*

$$\liminf_{t \to \infty} \frac{1}{a(t)} \int_{T_0}^{t} b(s)\, \mathrm{d}s > 0, \tag{4.13}$$

then $\mathbb{M}_\ell^- = \emptyset$.

Proof. Assume (2.1). Thus, the assertion follows from theorem 2.1 or corollary 3.4, according to the case (C1), or (C2) holds. If the case (C3) occurs, assume, by contradiction, $\mathbb{M}_\ell^- \neq \emptyset$. From theorem 4.8 we have for some $\mu > 0$ and $T \geqslant T_0$

$$\liminf_{t \to \infty} \Phi^* \left(\frac{\mu}{a(t)} \int_{T}^{t} b(s)\, \mathrm{d}s \right) = 0,$$

which gives a contradiction with (4.13). Finally, if (2.1) is not satisfied, the assertion follows from theorem 2.1. □

5. Concluding remarks

5.1. Non-oscillatory decaying-to-zero solutions

When (C2) occurs, corollary 3.4 gives a necessary and sufficient condition for the existence of decaying to zero solutions. When (C3) occurs, a sufficient condition is given by theorem 3.2.

The following example shows that conditions in theorem 3.2 are not necessary for having $\mathbb{M}_0^- \neq \emptyset$.

EXAMPLE 5.1. Consider the half-linear equation

$$(t^{4/3}|x'|^{1/3}\operatorname{sgn} x')' + \sqrt[3]{\tfrac{81}{256}}|x|^{1/3}\operatorname{sgn} x = 0, \quad t \geqslant 1. \tag{5.1}$$

For (5.1) we have $K_\lambda = \infty, I_\lambda < \infty$ for any $\lambda > 0$, and so theorem 3.2 cannot be applied. Nevertheless,

$$x(t) = \tfrac{27}{64}t^{-3/4}$$

is a solution of (5.1) in the class $\mathbb{M}_0^-$.

Hence, it is an open problem to find better conditions for having $\mathbb{M}_0^- \neq \emptyset$ also in case (C3).

5.2. Necessary conditions for oscillation

When (C3) holds, theorem 4.2 gives a sufficient condition for oscillation of (1.1). Observe that, if (C3) holds, then (4.2) implies that $K_\lambda = \infty$ for $\lambda \in \Lambda$ and $J_\mu = \infty$ for $\mu > 0$. Nevertheless, the conditions

$$J_\mu = K_\lambda = \infty \tag{5.2}$$

cannot be sufficient for the oscillation of (1.1), as the following example illustrates.

EXAMPLE 5.2. Consider the linear equation

$$(8t^2x')' + x = 0, \quad t \geqslant 1. \tag{5.3}$$

Thus, $K_\lambda = J_\mu = \infty$ for any $\lambda > 0$, $\mu > 0$ and $T \geqslant 1$. Setting $y(t) = 8t^2x'(t)$, from (5.3) we obtain the Euler equation

$$y'' + \tfrac{1}{8}t^{-2}y = 0. \tag{5.4}$$

Since (5.4) is not oscillatory (see, for example, [22, theorem 2.1]), (5.3) is also not oscillatory.

On the other hand, for the Emden–Fowler equation (1.2), conditions (5.2) become (1.4) and (1.5), respectively, which, as claimed, are necessary and sufficient for the oscillation of (1.2). So, it is natural to ask under which additional assumptions conditions (5.2) also become sufficient for the oscillation of (1.1) when (2.1) is satisfied and (C3) holds.

5.3. Oscillation of half-linear equations

From [11, theorem 4.2], using the transformation $y(t) = x^{[1]}(t)$, it is easy to show that (4.9) oscillates for any $\mu > 0$ if and only if

$$\limsup_{t\to\infty} \left(\int_T^t b(s)\,\mathrm{d}s \right) \left(\int_t^\infty \frac{1}{a^{1/p}(s)}\,\mathrm{d}s \right)^p = \infty. \tag{5.5}$$

Observe that in the linear case, i.e. for $p = 1$, (5.5) is the Hille–Nehari condition [22, theorem 2.9]. Thus, corollary 4.6 can be considered as a complement of Hille–Nehari-type oscillation criteria. It seems to be interesting also in view of a Hille–Wintner-type comparison theorem for a pair of half-linear equations.

Consider the half-linear equation

$$(a(t)\Phi_p(x'(t)))' + \mu B(t)\Phi_p(x) = 0, \tag{5.6}$$

where B is a positive continuous function for $t \geqslant T_0$ such that $B \notin L^1[T_0, \infty)$. If

$$\int_{T_0}^t b(s) \left(\int_s^\infty \frac{1}{a^{1/p}(\sigma)}\,\mathrm{d}\sigma \right)^p \mathrm{d}s \leqslant \int_{T_0}^t B(s) \left(\int_s^\infty \frac{1}{a^{1/p}(\sigma)}\,\mathrm{d}\sigma \right)^p \mathrm{d}s$$

for large t and (4.10) holds, then (5.6) is oscillatory for any $\mu > 0$. This result complements a Hille-Wintner-type comparison theorem in [23, theorem 5]. Moreover, via the transformation $y(t) = x^{[1]}(t)$, the above result can also be formulated for half-linear equations of type (4.9) when the weight $a^{1/p}$ is not summable on $[T_0, \infty)$ (see, for example, [10, theorem 2] and [20, theorem 4.1]). Finally, other results in this direction can be found in [19, theorem 1.1], in which a perturbed Euler-type equation is considered. In this case, corollary 4.6 cannot be applied because the equation in [19] is not oscillatory for any positive value of the parameter.

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