Riv. Mat. Univ. Parma, Vol. 5 (2014), 93-111

FATMA GAMZE DÜZGÜN
PAOLO MARCELLINI and VINCENZO VESPRI

Space expansion for a solution of an anisotropic *p* -Laplacian equation by using a parabolic approach

In memory of our friend Vitali Liskevich

Abstract. In this study we show that a technique introduced in the parabolic setting works also in the elliptic context. More precisely we prove a space expansion of positivity for solutions of an elliptic equation with anisotropic growth.

Keywords. Degenerate elliptic equations, anisotropic p-Laplacian, quantitative estimates.

Mathematics Subject Classification (2010): 35J70, 35J92, 35B65.

1 - Introduction

Let Ω be an open set in $\mathbb{R}^n (n \geq 2)$. We consider the elliptic differential equation

$$\frac{\partial}{\partial x} A_p(z, u, Du) + \sum_{i=1}^{n-1} \frac{\partial}{\partial y_i} A_{q,i}(z, u, Du) = 0. \tag{1.1}$$

We assume that the functions $A_p(z, u, Du)$, $A_{q,i}(z, u, Du)$ are defined for $z = (x, y) = (x, y_1, \ldots, y_{n-1}) \in \Omega$, $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ and $q > p > 1$. Let these functions satisfy the following structural conditions with some positive constants C_0, C_1 and a nonnegative constant C_2:

Received: April 29, 2013; accepted in revised form: September 5, 2013.

$$
(1.2)\qquad \begin{cases} A_p(z,u,Du)\cdot D_x u \geq C_0|D_x u|^p - C_2 \\ \sum_{i=1}^{n-1} A_{q,i}(z,u,Du)\cdot D_{y_i} u \geq C_0|D_y u|^q - C_2 \\ |A_p(z,u,Du)| \leq C_1|D_x u|^{p-1} + C_2 \\ |A_{q,i}(z,u,Du)| \leq C_1|D_y u|^{q-1} + C_2 \end{cases}
$$

for any $i = 1,\ldots,n-1$. Here $|D_y u| = \left(\sum_{i=1}^{n-1} |D_{y_i} u|^2\right)^{\frac{1}{2}}$.

We use the notation

$$
W^{1,[p,q]}(\Omega) = \{u \in L_1(\Omega) : D_x u \in L_p(\Omega), D_{y_i} u \in L_q(\Omega), \forall i = 1,\ldots,n-1\}.
$$

Moreover, $W_0^{1,[p,q]}(\Omega) = W^{1,[p,q]}(\Omega) \cap W_0^{1,1}(\Omega)$.

A function $u \in W^{1,[p,q]}(\Omega)$ is called a weak solution of equation (1.1) if it satisfies the equality

$$
(1.3)\qquad \int_\Omega A_p(z,u,Du)\frac{\partial\psi}{\partial x}\,dz + \int_\Omega \sum_{i=1}^{n-1} A_{q,i}(z,u,Du)\frac{\partial\psi}{\partial y_i}\,dz = 0
$$

for all test function $\psi \in W_0^{1,[p,q]}(\Omega)$.

Recently Liskevich and Skrypnik ([**10**]) proved Hölder regularity for weak solutions of equation (1.1) assuming $p = 2$ and $A_2(z,u,Du) = \frac{\partial u}{\partial x}$. Moreover they assumed that

$$
(1.4)\qquad 2 < q \leq \frac{n\bar{p}}{n-\bar{p}}
$$

where $\frac{1}{\bar{p}} = \frac{1}{n}\left(\frac{1}{2} + \frac{n-1}{q}\right)$ and $\bar{p} < n$.

We recall that under such assumptions the solutions are locally bounded (see [**1**], [**2**], [**13**]) and this condition is sharp (see [**1**], [**8**], [**11**], [**12**] and [**14**]). In [**10**] a so-called lemma of the expansion of positivity in space is crucial. An expansion of positivity in time was first proved in [**4**] for degenerate parabolic equations and it plays an essential role to prove Hölder regularity and Harnack inequality (see for instance [**5**], [**6**], [**7**] and [**9**]). The aim of this note is to use in the elliptic context the same approach used in the parabolic approach.

In this note we focus our attention only on the expansion of positivity in the space variables (see Lemma 1.1 below). We consider local bounded solutions of equation (1.1) with general operators $A_p(z,u,Du)$, $A_{q,i}(z,u,Du)$ for any couple $q > p > 1$. Moreover the assumption that u is bounded allows us to avoid the condition (1.4) that is a necessary condition in [**10**].

We follow a slightly different approach with respect to [**10**], hoping to clarify this difficult subject of Hölder regularity for anisotropic elliptic equations.

Assume that u is a bounded weak solution of (1.1) in a bounded domain Ω. Without loss of generality, we may assume that the origin belongs to Ω.

In the sequel we say that a constant depends only upon the data if it depends only upon n, C_0, C_1, C_2, p, q and $\|u\|_{L_\infty(\Omega)}$.

Let B_r be the ball of radius r in the y variables,

$$B_r = \{y \in \mathbb{R}^{n-1} : |y| < r\}. \tag{1.5}$$

Denote by ω the oscillation of u in Ω and let μ_- be the ess inf of u in Ω.

Assume that L and r are such that

$$[-Lr^{\frac{q}{p}}, Lr^{\frac{q}{p}}] \times B_r \subset \Omega.$$

Lemma 1.1 (Space expansion of positivity). *Let u be a weak solution of the equation* (1.1). *Assume that u is bounded and*

$$u \geq \mu_- + \eta \tag{1.6}$$

in $[-\varepsilon, \varepsilon] \times B_r$ *where* ε *is a positive constant and* $0 < \eta < \omega$. *Then there are two positive constants* C_3 *and* C_4, *depending only upon the data such that either*

$$\eta < C_3 r \tag{1.7}$$

or, for any (x, y) *satisfying* $|y| < \frac{r}{2}$ *and* $|x| < \frac{1}{2}Lr^{\frac{q}{p}}$ *we have that*

$$u(x, y) \geq \mu_- + C_4\eta(|x| + 1)^{\frac{p}{p-q}}. \tag{1.8}$$

This lemma says us what is the sharp decay of the solution in the direction of x. We recall that from this result Liskevich and Skrypnik deduced the Hölder regularity of the solution. In a forthcominig paper we intend to study the Harnack estimates for such kind of operators.

Remark. With the respect to the parabolic case there is an important difference: even if the initial operator is homogeneous, we have the presence of an alternative, (i.e. either the oscillation is small or the decay (1.8) occurs). In the proof we will point out in detail the differences with respect to the parabolic case.

Remark. Assumption (1.6) is not so strong. By the DiBenedetto's approach, the first step to prove the regularity of a solution, is to find a region where the solution is strictly positive (for more details see the monograph [**6**]).

The following regularity result is a consequence of the expansion of positivity lemma, i.e. Lemma 1.1 (see [**10**], see also [**4**]).

Theorem 1.2. *Assume that condition* (1.2) *holds. Let u be a locally bounded weak solution of* (1.1)*, then u is Hölder continuous.*

More precisely, in analogy with the parabolic case (see Chapter 3 of [**3**]) it is possible to prove that there are two positive constants $\beta<1$ and K depending only upon the data such that for any compact Γ strictly contained in Ω and for any $z_1=(x_1,y_1)$, $z_2=(x_2,y_2)\in\Gamma$

$$|u(z_1)-u(z_2)|\leq K\|u\|_{L_\infty(\Omega)}\left(\frac{|y_1-y_2|+\|u\|^{\frac{q-p}{q}}_{L_\infty(\Omega)}|x_1-x_2|^{\frac{p}{q}}}{(p,q)-dist(\Gamma,\partial\Omega)}\right)^{\beta} \tag{1.9}$$

where $(p,q)-dist(\Gamma,\partial\Omega)$ is the infimum of $|y_1-y_2|+\|u\|^{\frac{q-p}{q}}_{L_\infty(\Omega)}|x_1-x_2|^{\frac{p}{q}}$ ranging $(x_1,y_1)\in\Gamma$ and $(x_2,y_2)\in\partial\Omega$.

In Section 2 we state some preliminary results and in Section 3 we prove Lemma 1.1.

2 - Preliminary results

In this section we collect some results that we will use in the sequel.

Define the following cylinders in Ω,

$$Q_{R,r}=[-R,R]\times B_r$$

where B_r is defined in (1.5) and $R>0$.

Let $Q_1:=Q_{R,r}$ such that $Q_1\subset\Omega$ and $Q_2:=Q_{R_1,r_1}$ where $R_1<R$ and $r_1<r$.

Now define the function

$$G(u):=\left[\frac{1}{(u-\mu_-+a\omega H)^{p-1}}-\frac{1}{(\omega H)^{p-1}}\right]_+.$$

Here $0<a<1$, $0<H<1$ and $G(u)=0$ if $u>\mu_-+(1-a)\omega H$.

Lemma 2.1 (Logarithmic lemma). *Let u be a locally bounded weak solution of equation* (1.1) *and assume that condition* (1.2) *is satisfied. Then there exists a* *constant* $C>0$ (*depending only upon the data*) *such that*

$$\int_{Q_2\cap A}\left|D_x \ln_+ \frac{H\omega}{u-\mu_- + a\omega H}\right|^p dz \le C\int_{Q_1\cap A}|D_x\xi|^p dz \tag{2.1}$$

$$+ C\int_{Q_1\cap A}\left|D_y\xi\right|^q (u-\mu_-+a\omega H)^{q-p}dz + C\int_{Q_1\cap A}\frac{1}{(u-\mu_-+a\omega H)^p}dz,$$

here $A=\{z\in\Omega : u<\mu_- + (1-a)\omega H\}$ *and* $\xi\in C^\infty$ *is a function such that* $\xi=1$ *in* Q_2, $\xi=0$ *in* $\Omega - Q_1$.

Proof. Let take $G(u)\xi^q$ as a test function. Then using equation (1.1), we have

$$\int_\Omega A_p(z,u,Du)\frac{\partial}{\partial x}(G(u)\xi^q)dz + \int_\Omega \sum_{i=1}^{n-1} A_{q,i}(z,u,Du)\frac{\partial}{\partial y_i}(G(u)\xi^q)dz = 0.$$

If we use the definition of $G(u)$ and condition (1.2), then

$$(p-1)C_0\int_{Q_1\cap A}|D_x u|^p\frac{1}{(u-\mu_-+a\omega H)^p}\xi^q dz$$

$$+(p-1)C_0\int_{Q_1\cap A}\left|D_y u\right|^q\frac{1}{(u-\mu_-+a\omega H)^p}\xi^q dz$$

$$\le qC_1\int_{Q_1\cap A}|D_x u|^{p-1}\frac{1}{(u-\mu_-+a\omega H)^{p-1}}\xi^{q-1}|D_x\xi|dz$$

$$+qC_2\int_{Q_1\cap A}\frac{1}{(u-\mu_-+a\omega H)^{p-1}}\xi^{q-1}|D_x\xi|dz$$

$$+q(n-1)C_1\int_{Q_1\cap A}\left|D_y u\right|^{q-1}\frac{1}{(u-\mu_-+a\omega H)^{p-1}}\xi^{q-1}\left|D_y\xi\right|dz$$

$$+q(n-1)C_2\int_{Q_1\cap A}\frac{1}{(u-\mu_-+a\omega H)^{p-1}}\xi^{q-1}\left|D_y\xi\right|dz$$

$$+2(p-1)C_2\int_{Q_1\cap A}\frac{1}{(u-\mu_-+a\omega H)^p}\xi^q dz.$$

Using Young inequality in the right hand side above, we obtain

$$(p-1-\varepsilon_1)C_0 \int\limits_{Q_1\cap A} |D_x u|^p \frac{1}{(u-\mu_- + a\omega H)^p}\xi^q dz$$

$$+(p-1-\varepsilon_2)C_0 \int\limits_{Q_1\cap A} |D_y u|^q \frac{1}{(u-\mu_- + a\omega H)^p}\xi^q dz$$

$$\leq C(\varepsilon_1) \int\limits_{Q_1\cap A} |D_x \xi|^p dz + C(\varepsilon_2) \int\limits_{Q_1\cap A} |D_y \xi|^q (u-\mu_- + a\omega H)^{q-p} dz$$

$$+\varepsilon_3 \int\limits_{Q_1\cap A} \frac{1}{(u-\mu_- + a\omega H)^p}\xi^{\frac{p(q-1)}{p-1}} dz + C(\varepsilon_3) \int\limits_{Q_1\cap A} |D_x \xi|^p dz$$

$$+\varepsilon_4 \int\limits_{Q_1\cap A} \frac{1}{(u-\mu_- + a\omega H)^p}\xi^q dz + C(\varepsilon_4) \int\limits_{Q_1\cap A} |D_y \xi|^q (u-\mu_- + a\omega H)^{q-p} dz$$

$$+2(p-1)C_2 \int\limits_{Q_1\cap A} \frac{1}{(u-\mu_- + a\omega H)^p}\xi^q dz$$

ε_1 and ε_2 can be chosen to make the constants positive in the left hand side of the inequality above. If we consider the properties of ξ and the fact that $Q_2 \subset Q_1$, then (2.1) can be obtained from the last inequality. □

Remark. If the operator is homogeneous, the following inequality holds:

$$\int\limits_{Q_2\cap A} \left| D_x \ln_+ \frac{H\omega}{u-\mu_- + a\omega H}\right|^p dz$$

$$\leq C \int\limits_{Q_1\cap A} |D_x \xi|^p dz + C \int\limits_{Q_1\cap A} |D_y \xi|^q (u-\mu_- + a\omega H)^{q-p} dz.$$

Lemma 2.2 (Sobolev-Troisi inequality [**16**], see also [**15**], [**17**]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and consider $u \in W_0^{1,[p_1,\dots,p_n]}(\Omega)$, $p_i \geq 1$ for all $i = 1,\dots,n$. Let*

$$\frac{1}{\bar{p}} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}, \quad \bar{p}^* = \frac{n\bar{p}}{n-\bar{p}}.$$

Then there exists c depending on $n, p_1, .., p_n$ *if* $\bar{p} < n$ *such that*

$$\|u\|^n_{L_{\bar{p}^*}(\Omega)} \leq c \prod_{i=1}^{n} \|D_{x_i} u\|_{L_{p_i}(\Omega)}.$$

Lemma 2.3 (Algebrical lemma [3]). *Let* $\{Y_m\}, m = 0, 1, 2, \ldots,$ *be a sequence of positive numbers, satisfying the recursive inequalities*

$$Y_{m+1} \leq C_5 b^m Y_m^{1+\lambda}$$

where $C_5, b > 1$ *and* $\lambda > 0$ *are given numbers. If*

$$Y_0 \leq C_5^{\frac{-1}{\lambda}} b^{\frac{-1}{\lambda^2}},$$

then $\{Y_m\}$ *converges to zero as* $m \to \infty$.

Lemma 2.4 (Generalized of Caccioppoli's inequality). *Let u be a locally bounded weak solution of* (1.1)*. Then there exists a constant* $C_6 > 0$ (*depending only upon the data except u*) *such that for every test function* $\theta \in C_0^1(\Omega)$, *we have*

$$\int_\Omega \theta^q (|D_x(u-k)_-|^p + |D_y(u-k)_-|^q) dz$$

$$\leq C_6 \int_\Omega [\theta^{q-p} |D_x\theta|^p |(u-k)_-|^p + |D_y\theta|^q |(u-k)_-|^q + \theta^q] dz$$

for any constant $k > 0$.

Note that an analogous result also holds if we deal with $(u-k)_+$.

Proof. Let consider (1.3) with the test function

$$\psi = \psi(z) = \theta(z)^q (u-k)_-, \quad \theta \in C_0^1(\Omega).$$

Since

$$D_x\psi = q\theta^{q-1}(u-k)_- D_x\theta + \theta^q D_x u, \quad D_{y_i}\psi = q\theta^{q-1}(u-k)_- D_{y_i}\theta + \theta^q D_{y_i} u,$$

and recalling that these functions are defined in the set $\{x \in \Omega : u(x) < k\}$, we have

$$0 = \int_\Omega q\theta^{q-1}(u-k)_- A_p(z,u,Du) D_x\theta dz + \int_\Omega \theta^q A_p(z,u,Du) D_x(u-k)_- dz$$

$$+ \int_\Omega q\theta^{q-1} \sum_{i=1}^{n-1} (u-k)_- A_{q,i}(z,u,Du) D_{y_i}\theta dz + \int_\Omega \theta^q \sum_{i=1}^{n-1} A_{q,i}(z,u,Du) D_{y_i}(u-k)_- dz.$$

If we take into account (1.2), we obtain the following inequality

$$
\begin{aligned}
&C_0 \int_\Omega \theta^q (|D_x(u-k)_-|^p + |D_y(u-k)_-|^q) dz \\
&\leq qC_1 \int_\Omega \theta^{q-1} |(u-k)_-||D_x\theta||D_x(u-k)_-|^{p-1} dz \\
&+ qC_2 \int_\Omega \theta^{q-1} |(u-k)_-||D_x\theta| dz \\
&+ (n-1)qC_1 \int_\Omega \theta^{q-1} |(u-k)_-||D_y\theta||D_y(u-k)_-|^{q-1} dz \\
&+ (n-1)qC_2 \int_\Omega \theta^{q-1} |(u-k)_-||D_y\theta| dz + 2C_2 \int_\Omega \theta^q dz.
\end{aligned}
$$

Writing $\theta^{q-1} = \theta^{q\frac{p-1}{p}} \theta^{\frac{q-p}{p}}$ and applying Young inequality to the last expression, we obtain

$$
\begin{aligned}
&C_0 \int_\Omega \theta^q (|D_x(u-k)_-|^p + |D_y(u-k)_-|^q) dz \\
&\leq \varepsilon_1 \int_\Omega \theta^q |D_x(u-k)_-|^p dz + C(\varepsilon_1) \int_\Omega \theta^{q-p} |D_x\theta|^p |(u-k)_-|^p dz \\
&+ \varepsilon_2 \int_\Omega \theta^q |D_y(u-k)_-|^q dz \\
&+ C(\varepsilon_2) \int_\Omega |D_y\theta|^q |(u-k)_-|^q dz + \varepsilon_3 \int_\Omega \theta^{q-p} |D_x\theta|^p |(u-k)_-|^p dz \\
&+ C(\varepsilon_3) \int_\Omega \theta^q dz \\
&+ \varepsilon_4 \int_\Omega |D_y\theta|^q |(u-k)_-|^q dz + C(\varepsilon_4) \int_\Omega \theta^q dz + 2C_2 \int_\Omega \theta^q dz.
\end{aligned}
$$

Then the required inequality can be obtained from the last estimate. □

Remark. If the operator is homogeneous, the following inequality holds:

$$\int_\Omega \theta^q(|D_x(u-k)_-|^p+|D_y(u-k)_-|^q)dz$$

$$\leq C_6\int_\Omega [\theta^{q-p}|D_x\theta|^p|(u-k)_-|^p+|D_y\theta|^q|(u-k)_-|^q]dz.$$

In the next lemma we will consider a De Giorgi type lemma. It is necessary to introduce the intrinsic geometry related to the anisotropic elliptic equation. It is a geometry that is induced by the anisotropy of the operator itself (for more details about the intrinsic geometry see [3] and [18]).

Let $R_0, k_0 > 0$ be given numbers and $R_x = R_0^{\frac{q}{p}} k_0^{\frac{p-q}{p}}$. Define

$$Q_{R_j} := \{z=(x,y) : |x|<r_j, \quad |y|<R_j\}, \quad r_j = \frac{R_x}{2}+\frac{R_x}{2^{j+1}}, R_j = \frac{R_0}{2}+\frac{R_0}{2^{j+1}}$$

with

$$|Q_{R_j}| = c_j R_0^{\frac{q}{p}} k_0^{\frac{p-q}{p}} R_0^{n-1},$$

where c_j are equibounded positive constants converging to strictly positive constant c_∞.

Define

$$D_{s,j} := \{z=(x,y)\in Q_{R_s} : u(z)\leq k_j\}, \quad k_j = \frac{k_0}{2}+\frac{k_0}{2^{j+1}}+\mu_-$$

and $Z_j := \dfrac{|D_{2j,j}|}{|Q_{R_{2j}}|}$.

Let denote D_∞ as the intersection of the sets $D_{2j,j}$, i.e,

$$D_\infty = \bigcap_j D_{2j,j} = \left\{ z\in Q_{R_\infty} : u(z)\leq \frac{k_0}{2}+\mu_-\right\}.$$

Lemma 2.5 (De Giorgi type lemma). *There is a number $v>0$ depending only upon the data (but not depending on u, R_0, and k_0) such that either*

$$k_0 < R_0 \tag{2.2}$$

or if $Z_0 < v$ then $\{Z_j\}$ converges to zero as j goes to infinity.

Proof. If we consider the set $D_{2j+2,j+1}$, we can write

$$\int_{D_{2j+2,j+1}} |u-k_j|^{\bar p}dz \geq (k_j-k_{j+1})^{\bar p}|D_{2j+2,j+1}|.$$

Then

$$
\begin{aligned}
|D_{2j+2,\,j+1}| &\le \frac{1}{(k_j-k_{j+1})^{\bar p}} \int\limits_{D_{2j+2,\,j+1}} |u-k_j|^{\bar p} dz \\
&\le \frac{1}{(k_j-k_{j+1})^{\bar p}} \int\limits_{D_{2j+1,\,j}} \theta_{2j+1}^{\bar p} |u-k_j|^{\bar p} dz,
\end{aligned}
$$

here $\theta_{2j+1} \in C^\infty$ is a function such that $\theta_{2j+1} = 1$ in $Q_{R_{2j+2}}$ and $\theta_{2j+1} = 0$ out of $Q_{R_{2j+1}}$ with satisfying $|D_x\theta_{2j+1}| \le \tilde c \dfrac{4^j}{R_x}$, $|D_y\theta_{2j+1}| \le \tilde c \dfrac{4^j}{R_0}$ for a positive constant $\tilde c$ and for all $(x,y) \in Q_{R_{2j+1}}$.

If we use Hölder inequality, we obtain

$$
|D_{2j+2,\,j+1}| \le \frac{1}{(k_j-k_{j+1})^{\bar p}} \left[\int\limits_{D_{2j+1,\,j}} (\theta_{2j+1}|u-k_j|)^{\bar p^*} dz \right]^{\frac{\bar p}{\bar p^*}} |D_{2j+1,\,j}|^{1-\frac{\bar p}{\bar p^*}},
$$

$\bar p, \bar p^*$ are defined as in Lemma 2.2.

Now if we use Lemma 2.2, we have

$$
\begin{aligned}
|D_{2j+2,\,j+1}| &\le c \frac{1}{(k_j-k_{j+1})^{\bar p}} |D_{2j+1,\,j}|^{1-\frac{\bar p}{\bar p^*}} \qquad (2.3)\\
&\left[\int\limits_{D_{2j+1,\,j}} |D_x\theta_{2j+1}(u-k_j)|^p dz \right]^{\frac{\bar p}{pn}} \prod_{i=1}^{n-1} \left[\int\limits_{D_{2j+1,\,j}} |D_{y_i}\theta_{2j+1}(u-k_j)|^q dz \right]^{\frac{\bar p}{qn}} \\
&\le c \frac{1}{(k_j-k_{j+1})^{\bar p}} |D_{2j+1,\,j}|^{1-\frac{\bar p}{\bar p^*}} \\
&\left[\int\limits_{D_{2j+1,\,j}} |D_x\theta_{2j+1}(u-k_j)|^p dz \right]^{\frac{\bar p}{pn}} \left[\int\limits_{D_{2j+1,\,j}} |D_y\theta_{2j+1}(u-k_j)|^q dz \right]^{\frac{\bar p(n-1)}{qn}}.
\end{aligned}
$$

Let estimate $\displaystyle\int\limits_{D_{2j+1,\,j}} |D_x\theta_{2j+1}(u-k_j)|^p dz$,

$$
\begin{aligned}
&\int\limits_{D_{2j+1,\,j}} |D_x\theta_{2j+1}(u-k_j)|^p dz \\
&\le C_7 \int\limits_{D_{2j+1,\,j}} \left(|D_x\theta_{2j+1}|^p |(u-k_j)|^p + \theta_{2j+1}^p |D_x(u-k_j)|^p \right) dz.
\end{aligned}
$$

Considering the properties of function θ_{2j+1}, we have

$$\int\limits_{D_{2j+1,j}} \left|D_x\theta_{2j+1}(u-k_j)\right|^p dz$$

$$\leq C_8\frac{4^{jp}}{R_x^p}\int\limits_{D_{2j+1,j}} \left|(u-k_j)\right|^p dz + C_8\int\limits_{D_{2j+1,j}} \left|D_x(u-k_j)\right|^p dz.$$

If we pass to the larger set $D_{2j,j}$ and use function θ_{2j} with exponent q, we obtain

$$\int\limits_{D_{2j+1,j}} \left|D_x\theta_{2j+1}(u-k_j)\right|^p dz \leq C_8\frac{4^{jp}}{R_x^p}\int\limits_{D_{2j,j}} \left|(u-k_j)\right|^p dz \;+\; C_8\int\limits_{D_{2j,j}} \theta_{2j}^q\left|D_x(u-k_j)\right|^p dz.$$

Here we can apply Caccioppoli's inequality of Lemma 2.4 to the second integral on the right hand side and we get,

$$\int\limits_{D_{2j+1,j}} \left|D_x\theta_{2j+1}(u-k_j)\right|^p dz \leq C_8\frac{4^{jp}}{R_x^p}\int\limits_{D_{2j,j}} \left|(u-k_j)\right|^p dz \;+\; C_9\int\limits_{D_{2j,j}} \left|D_x\theta_{2j}\right|^p\left|u-k_j\right|^p dz$$

$$+\,C_9\int\limits_{D_{2j,j}} \left|D_y\theta_{2j}\right|^q\left|u-k_j\right|^q dz + C_9\int\limits_{D_{2j,j}} \theta_{2j}^q dz.$$

If we use the properties of θ_{2j},

$$\int\limits_{D_{2j+1,j}} \left|D_x\theta_{2j+1}(u-k_j)\right|^p dz \leq C_{10}\frac{4^{jp}}{R_x^p}\int\limits_{D_{2j,j}} \left|(u-k_j)\right|^p dz + C_{10}\frac{4^{jp}}{R_x^p}\int\limits_{D_{2j,j}} \left|u-k_j\right|^p dz$$

$$+\,C_{10}\frac{4^{jq}}{R_0^q}\int\limits_{D_{2j,j}} \left|u-k_j\right|^q dz + C_{10}|D_{2j,j}|.$$

Then we can write,

$$\int\limits_{D_{2j+1,j}} \left|D_x\theta_{2j+1}(u-k_j)\right|^p dz$$

$$\leq C_{10}\frac{4^{jp}}{R_x^p}\int\limits_{D_{2j,j}} \left|(u-k_j)\right|^p dz + C_{10}\frac{4^{jq}}{R_0^q}\int\limits_{D_{2j,j}} \left|u-k_j\right|^q dz + C_{10}|D_{2j,j}|.$$

Similar estimate can be obtained for also

$$\int\limits_{D_{2j+1,j}} \left|D_y\theta_{2j+1}(u-k_j)\right|^q dz.$$

Then since $\frac{\bar{p}}{pn} + \frac{\bar{p}(n-1)}{qn} = 1$ we obtain from (2.3),

$$|D_{2j+2,j+1}| \leq C_{10} \frac{1}{(k_j - k_{j+1})^{\bar{p}}} |D_{2j+1,j}|^{1-\frac{\bar{p}}{\bar{p}^*}} \tag{2.4}$$

$$\left[\frac{4^{jp}}{R_x^p} \int_{D_{2j,j}} \left|(u-k_j)\right|^p dz + \frac{4^{jq}}{R_0^q} \int_{D_{2j,j}} \left|u-k_j\right|^q dz + |D_{2j,j}| \right].$$

Note that $|D_{2j+1,j}| \leq |D_{2j,j}|$. Moreover, noting that in $D_{2j,j}$ we have $\mu_- \leq u \leq k_j < \mu_- + k_0$, we get $\left|u - k_j\right| \leq k_0$. Therefore from (2.4)

$$|D_{2j+2,j+1}| \leq C_{10} \frac{2^{(j+2)\bar{p}}}{k_0^{\bar{p}}} |D_{2j,j}|^{1+(1-\frac{\bar{p}}{\bar{p}^*})} \left[\frac{4^{(j+1)p} k_0^p}{R_x^p} + \frac{4^{(j+1)q} k_0^q}{R_0^q} + 1 \right].$$

By the definition of R_x and recalling $q > p$ we can write the last expression as

$$|D_{2j+2,j+1}| \leq C_{11} \frac{2^{(j+2)\bar{p}}}{k_0^{\bar{p}}} |D_{2j,j}|^{1+(1-\frac{\bar{p}}{\bar{p}^*})} 4^{(j+1)q} \frac{k_0^q}{R_0^q} \left[1 + \left(\frac{R_0}{k_0} \right)^q \right].$$

Assume $k_0 > R_0$ (otherwise (2.2) holds) and divide both sides by the measure of $Q_{R_{2j}}$ to have the following inequality (using the fact $|Q_{R_{2j}}| \leq |Q_{R_{2j+2}}| 2^{2n}$)

$$Z_{j+1} \leq C_{12} 2^{(j+2)\bar{p}} 4^{(j+1)q} Z_j^{1+\lambda}$$

where $\lambda := \left(1 - \frac{\bar{p}}{\bar{p}^*}\right)$.

Then the required statement follows by Lemma 2.3. □

Remark. If the operator is homogeneous, the alternative does not occur, i.e. it is possible to prove that there is an absolute number $\nu > 0$ depending only upon the data (and not depending on u, R_0, and k_0) such that if $Z_0 < \nu$ then $\{Z_j\}$ converges to zero as j goes to infinity.

3 - Proof of Lemma 1.1

In order to simplify the proof, instead of (1.6), assume that u is continuous so that we can ask that $u(0, y) \geq \mu_- + \eta$ in $\{0\} \times B_r$, where B_r is defined as in (1.5). The general case can be treated analogously.

Let s_0 be an integer and define a set A^0 in $B_{\frac{r}{2}}$,

$$A^0 = \left\{ y \in B_{\frac{r}{2}} : \exists x \in \left[0, \frac{1}{2} L r^{\frac{q}{p}}\right], u(x,y) - \mu_- \leq \frac{L^{\frac{p}{p-q}}}{e^{s_0}} \right\},$$

with assuming $L^{\frac{p}{p-q}} < \dfrac{\eta}{2}$.

Lemma 3.1. *For any positive constant* $\nu \in (0,1)$*, there exists a positive integer* s_0 *such that either*

$$\omega \leq C_{13} r \tag{3.1}$$

for a positive constant C_{13} *depending upon the data,* s_0 *and* L *(but not depending upon* u*), or*

$$|A^0| \leq \nu |B_r|. \tag{3.2}$$

We prove this result only in the cylinder $[0, Lr^{\frac{q}{p}}] \times B_r$. The case $[-Lr^{\frac{q}{p}}, 0] \times B_r$ is analogous.

Proof. Let $y \in A^0$. Then $\exists x \in \left[0, \frac{1}{2} L r^{\frac{q}{p}}\right]$ such that $u(x,y) - \mu_- \leq \dfrac{L^{\frac{p}{p-q}}}{e^{s_0}}$. Therefore,

$$s_0 - 1 \leq \ln_+ \frac{u(0,y) - \mu_- + L^{\frac{p}{p-q}} e^{-s_0}}{u(x,y) - \mu_- + L^{\frac{p}{p-q}} e^{-s_0}}$$

$$\ln_+ \frac{L^{\frac{p}{p-q}}}{u(x,y) - \mu_- + L^{\frac{p}{p-q}} e^{-s_0}} - \ln_+ \frac{L^{\frac{p}{p-q}}}{u(0,y) - \mu_- + L^{\frac{p}{p-q}} e^{-s_0}}$$

$$= \int_0^x D_x \ln_+ \frac{L^{\frac{p}{p-q}}}{u(t,y) - \mu_- + L^{\frac{p}{p-q}} e^{-s_0}} dt$$

$$\leq \int_0^{\frac{1}{2} L r^{\frac{q}{p}}} \left| D_x \ln_+ \frac{L^{\frac{p}{p-q}}}{u(t,y) - \mu_- + L^{\frac{p}{p-q}} e^{-s_0}} \right| dt.$$

Now if we integrate the inequality above over the set A^0, we obtain

$$(s_0 - 1)|A^0| \leq \int_{B_r} \int_0^{\frac{1}{2} L r^{\frac{q}{p}}} \left| D_x \ln_+ \frac{L^{\frac{p}{p-q}}}{u - \mu_- + L^{\frac{p}{p-q}} e^{-s_0}} \right| dz.$$

Using Hölder inequality, we have

$$(s_0-1)|A^0| \le \left[\int_{B_r} \int_0^{\frac{1}{2}Lr^{\frac{q}{p}}} \left| D_x \ln_+ \frac{L^{\frac{p}{p-q}}}{u-\mu_- + L^{\frac{p}{p-q}} e^{-s_0}} \right|^p dz \right]^{\frac{1}{p}} \left[Lr^{\frac{q}{p}} |B_r| \right]^{\frac{p-1}{p}}.$$

Here we apply Lemma 2.1 choosing $\xi = 1$ in $\left[0, \frac{1}{2} Lr^{\frac{q}{p}}\right] \times B_r$ and $\xi = 0$ out of $[0, Lr^{\frac{q}{p}}] \times B_r$, and putting $H\omega = L^{\frac{p}{p-q}}$ and $a = e^{-s_0}$. Note that in order to apply Lemma 2.1 with $H\omega = L^{\frac{p}{p-q}}$, it is necessary that $H = \frac{L^{\frac{p}{p-q}}}{\omega} < 1$ but this condition is satisfied since $L^{\frac{p}{p-q}} \le \frac{\eta}{2} \le \frac{\omega}{2}$. Hence by Lemma 2.1

$$(s_0-1)|A^0|$$

$$\le C_{14} \left[\int_{B_r} \int_0^{Lr^{\frac{q}{p}}} |D_x \xi|^p dz + \int_{B_r} \int_0^{Lr^{\frac{q}{p}}} |D_y \xi|^q L^{-p} dz \int_{B_r} \int_0^{Lr^{\frac{q}{p}}} \left(\frac{e^{s_0}}{L^{\frac{p}{p-q}}} \right)^p dz \right]^{\frac{1}{p}} \left[Lr^{\frac{q}{p}} |B_r| \right]^{\frac{p-1}{p}}.$$

Let estimate the right hand side of the last inequality taking

$$I := \left[\int_{B_r} \int_0^{Lr^{\frac{q}{p}}} |D_x \xi|^p dz \right]^{\frac{1}{p}} \left[Lr^{\frac{q}{p}} |B_r| \right]^{\frac{p-1}{p}}$$

$$II := \left[\int_{B_r} \int_0^{Lr^{\frac{q}{p}}} |D_y \xi|^q L^{-p} dz \right]^{\frac{1}{p}} \left[Lr^{\frac{q}{p}} |B_r| \right]^{\frac{p-1}{p}}$$

$$III := \left[\int_{B_r} \int_0^{Lr^{\frac{q}{p}}} \left(\frac{e^{s_0}}{L^{\frac{p}{p-q}}} \right)^p dz \right]^{\frac{1}{p}} \left[Lr^{\frac{q}{p}} |B_r| \right]^{\frac{p-1}{p}}.$$

If we consider $|D_x \xi| = \frac{d_1}{Lr^{\frac{q}{p}}}$ and $|D_y \xi| = \frac{d_2}{r}$ for some constants d_1, d_2, we get $I \le \nu_1 |B_r|$ and $II \le \nu_2 |B_r|$ (with positive constants ν_1, ν_2). Moreover,

$$III \le \nu_3 |B_r| L^{\frac{q}{q-p}} r^{\frac{q}{p}}.$$

Choose the integer s_0 sufficiently large to have $\nu_1 + \nu_2 < \frac{\nu}{2}(s_0 - 1)$.

Then either $\nu_3 L^{\frac{q}{q-p}} r^{\frac{q}{p}} < \frac{\nu}{2}(s_0 - 1)$ (and (3.2) holds) or $L^{\frac{q}{p-q}} \le C' r^{\frac{q}{p}}$ for a constant C'. Considering Lemma 2.1, note that we set in the proof $H\omega = L^{\frac{p}{p-q}}$, therefore we get $\omega \le C_{13} r$, i.e. the alternative (3.1) occurs. Note that C_{13} depends upon H, i.e. also upon the length of the cylinder. Note that if the operator is homogeneous, the alternative does not occur. □

The previous lemma says that, if one defines $A^x = \{y \in B_r : u(x,y) < \mu_- + C_4 \eta(|x|+1)^{\frac{p}{p-q}}\}$, we have that $|A^x| < \nu |B_r|$. Therefore Lemma 1.1 holds except in a set of a small measure. In order to prove Lemma 1.1, we have to show that this estimate holds everywhere. The idea is to apply Lemma 2.5, but we can not apply directly to equation (1.1). Actually to be applied we need that the solution is bigger than a constant k_0 (with the exception of a small set) in a suitable set given by the intrinsic geometry and depending on k_0. In this new set the solution could be smaller and we need another larger set and so on. To overcome this difficulty, we need an auxiliary equation as in [**4**] and [**10**].

Now let define the function

$$v(x,y) := (1+|x|)^{\alpha}(u(x,y) + \mu_-)$$

where $\alpha = \dfrac{p}{(q-p)}$.

Denote $\tilde{A_p}(z, v, Dv) := A_p(z, (1+|x|)^{-\alpha} v - \mu_-, D((1+|x|)^{-\alpha} v)) = A_p(z,u,Du)$, $\tilde{A}_{q,i}(z, v, Dv) := A_{q,i}(z, (1+|x|)^{-\alpha} v - \mu_-, D((1+|x|)^{-\alpha} v)) = A_{q,i}(z,u,Du)$.

Since $D_y v = (1+|x|)^{\alpha} D_y u$ and $D_x v = (1+|x|)^{\alpha} D_x u + \sigma(x)\alpha(1+|x|)^{\alpha-1}(u+\mu_-)$, by (1.2) (where with $\sigma(x)$ we denoted the signus of x) we have

$$\left|\tilde{A_p}(z,v,Dv)\right| = \left|A_p(z,u,Du)\right| \le C_1 |D_x u|^{p-1} + C_2$$

$$= C_1 \left|(1+|x|)^{-\alpha} D_x v - \sigma(x)\alpha v (1+|x|)^{-\alpha-1}\right|^{p-1} + C_2;$$

$$\tilde{A_p}(z,v,Dv)\cdot D_x v = A_p(z,u,Du)\left[(1+|x|)^{\alpha} D_x u + \sigma(x)\alpha \frac{v}{1+|x|}\right]$$

$$\ge C_0 (1+|x|)^{\alpha} |D_x u|^p - C_2 (1+|x|)^{\alpha} - \left|A_p(z,u,Du)\right| \frac{\alpha v}{1+|x|}$$

$$\ge C_0 (1+|x|)^{\alpha} |D_x u|^p - C_2 (1+|x|)^{\alpha} - (C_1 |D_x u|^{p-1} + C_2)\frac{\alpha v}{1+|x|}$$

$$\ge C_0 (1+|x|)^{\alpha} |D_x u|^p - C_2 (1+|x|)^{\alpha} - \varepsilon |D_x u|^p - C(\varepsilon)\left(\frac{v}{1+|x|}\right)^p - C_2 \frac{\alpha v}{1+|x|}$$

$$= [C_0(1+|x|)^\alpha - \varepsilon]\Big|(1+|x|)^{-\alpha}D_x v - \sigma(x)\alpha v(1+|x|)^{-\alpha-1}\Big|^p$$

$$- C_2(1+|x|)^\alpha - C(\varepsilon)\left(\frac{v}{1+|x|}\right)^p - C_2\frac{\alpha v}{1+|x|};$$

$$\Big|\tilde{A}_{q,i}(z,v,Dv)\Big| = \big|A_{q,i}(z,u,Du)\big| \le C_1\big|D_y u\big|^{q-1} + C_2 = C_1\frac{\big|D_y v\big|^{q-1}}{(1+|x|)^{\alpha(q-1)}} + C_2;$$

$$\sum_{i=1}^{n-1}\tilde{A}_{q,i}(z,v,Dv)\cdot D_{y_i}v = \sum_{i=1}^{n-1}A_{q,i}(z,u,Du)(1+|x|)^\alpha D_{y_i}u \ge (1+|x|)^\alpha(C_0\big|D_y u\big|^q - C_2)$$

$$= C_0(1+|x|)^{\alpha(1-q)}\big|D_y v\big|^q - C_2(1+|x|)^\alpha.$$

As

$$\frac{\partial}{\partial x}A_p(z,u,Du) + \sum_{i=1}^{n-1}\frac{\partial}{\partial y_i}A_{q,i}(z,u,Du) = 0$$

we have that v satisfies the following equation

$$\frac{\partial}{\partial x}\tilde{A}_p(z,v,Dv) + \sum_{i=1}^{n-1}\frac{\partial}{\partial y_i}\tilde{A}_{q,i}(z,v,Dv) = 0.$$

Formally, multiplying the equation by $(1+|x|)^{\alpha(q-1)}$, we have that v satisfies the equation

$$\frac{\partial}{\partial x}((1+|x|)^{\alpha(q-1)}\tilde{A}_p(z,v,Dv)) + \sum_{i=1}^{n-1}\frac{\partial}{\partial y_i}((1+|x|)^{\alpha(q-1)}\tilde{A}_{q,i}(z,v,Dv))$$

$$-\alpha(q-1)(1+|x|)^{\alpha(q-1)-1}\sigma(x)\tilde{A}_p(z,v,Dv) = 0.$$

where we recall that with $\sigma(x)$ we denoted the sign of x.

Hence:

$$\begin{cases} (1+|x|)^{\alpha q-\alpha}\tilde{A}_p(z,v,Dv)\cdot D_x v \ge \tilde{C}_0(1+|x|)^p|D_x v|^p - \tilde{C}_2 \\ (1+|x|)^{\alpha q-\alpha}\sum_{i=1}^{n-1}\tilde{A}_{q,i}(z,v,Dv)\cdot D_{y_i}v \ge \tilde{C}_0\big|D_y v\big|^q - \tilde{C}_2 \\ (1+|x|)^{\alpha q-\alpha}\Big|\tilde{A}_p(z,v,Dv)\Big| \le \tilde{C}_1(1+|x|)^p|D_x v|^{p-1} + \tilde{C}_2 \\ (1+|x|)^{\alpha q-\alpha}\Big|\tilde{A}_{q,i}(z,v,Dv)\Big| \le \tilde{C}_1\big|D_y v\big|^{q-1} + \tilde{C}_2 \\ (1+|x|)^{\alpha(q-1)-1}\Big|\tilde{A}_p(z,v,Dv)\Big| \le \tilde{C}_1(1+|x|)^{p-1}|D_x v|^{p-1} + \tilde{C}_2 \end{cases}$$

where $\tilde{C_0}, \tilde{C_1}, \tilde{C_2}$ are bounded constants because v is bounded (this comes from the boundness of u and the definition of v) and $(1+|x|)$ in the domain Ω is also bounded (since Ω is a bounded domain).

Note that v satisfies a nonlinear elliptic equation of Euler type, i.e. $(1+|x|)^{\alpha q-\alpha}\tilde{A_p}(z, v, Dv)\cdot D_x v$ behaves as $(1+|x|)^p|D_x v|^p$.

Remark. Note that even if the original operator A_p is homogeneous, the transformed operator $\tilde{A_p}$ is not more homogeneous. So either we assume stronger coercivity assumption on A_p or we are compelled to deal with alternatives of the type (1.7) and (1.8).

Sketch of the end of the proof of Lemma 1.1

The proof, from now to the follow, is similar to the one of [**4**] and [**10**], to which we refer the reader for more details.

Consider first the case of the cylinder $[0, Lr^{\frac{q}{p}}]\times B_r$. Translate this cylinder in $[1, 1+Lr^{\frac{q}{p}}]\times B_r$.

As already noticed, the function v satisfies a nonlinear elliptic equation of Euler type.

Therefore, in order to have structure conditions (1.2), as in [**4**] and [**10**] we need to change the variables and introduce the new function $\Phi(e^x, y) = v(x, y)$.

Reasoning as in [**4**] and [**10**], it is possible to prove that Φ is a weak solution of an equation similar to (1.1) with the structure conditions (1.2).

Define $\tilde{A^x} = \left\{y\in B_r : \Phi(x, y) < \mu_- + \frac{\gamma}{e^{s_0}}\right\}$ where γ is a suitable positive constant. By the result of Lemma 3.1 and by the definition of Φ either estimate (3.1) holds or for any $x\in[0,\, \ln(1+Lr^{\frac{q}{p}})]$ we have $|\tilde{A^x}| < \nu|B_r|$.

Repeating the same argument in the cylinder $[-Lr^{\frac{q}{p}}, 0]\times B_r$ we have that either estimate (3.1) holds or for any $x\in[-\ln(1+Lr^{\frac{q}{p}}),\, \ln(1+Lr^{\frac{q}{p}})]$ we have $|\tilde{A^x}| < \nu|B_r|$.

Now we have the conditions to apply Lemma 2.5. For any $x_0\in\Big[-\frac{1}{2}\ln(1+Lr^{\frac{q}{p}}), \frac{1}{2}\ln(1+Lr^{\frac{q}{p}})\Big]$ consider the intrinsic cylinder $Q_r = \left\{z=(x, y) : |x-x_0| < r^{\frac{q}{p}}\left(\frac{\gamma}{e^{s_0}}\right)^{\frac{p-q}{p}},\ |y|<r\right\}$ and apply Lemma 2.5 to get

$$\Phi(x_0, y) \geq \mu_- + \frac{\gamma}{2e^{s_0}}, \tag{3.3}$$

for any $y\in B_{\frac{r}{2}}$.

The proof of Lemma 1.1 is therefore a consequence of the definition of Φ and of estimate (3.3).

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FATMA GAMZE DÜZGÜN
Department of Mathematics
Hacettepe University
06800, Beytepe, Ankara, Turkey
e-mail: gamzeduz@hacettepe.edu.tr

PAOLO MARCELLINI, VINCENZO VESPRI
Dipartimento di Matematica e Informatica "U. Dini"
Università degli Studi di Firenze
Viale Morgagni 67/A, 50134 Firenze, Italy
e-mail: marcellini@math.unifi.it
e-mail: vespri@math.unifi.it